Unambiguous 1-uniform morphisms

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Unambiguous 1-Uniform Morphisms ☆

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Abstract

A morphism σ is unambiguous with respect to a word α if there is no other morphism τ that maps α to the same image as σ. In the present paper we study the question of whether, for any given word, there exists an unambiguous 1-uniform morphism, i. e., a morphism that maps every letter in the word to an image of length 1.

Keywords: Morphisms, Ambiguity, Fixed points of morphisms

1. Introduction

If, for a morphism $\sigma : \Delta^* \to \Sigma^*$ (where Δ and Σ are arbitrary alphabets) and a word $\alpha \in \Delta^*$, there exists another morphism τ mapping α to $\sigma(\alpha)$, then σ is called *ambiguous* with respect to α; if such a τ does not exist, then σ is *unambiguous*. For example, the morphism $\sigma_0 : \{A, B, C\}^* \to \{a, b\}^*$ – given by $\sigma_0(A) := a$, $\sigma_0(B) := a$, $\sigma_0(C) := b$ – is ambiguous with respect to the word $\alpha_0 := ABCACB$, since the morphism τ_0 – defined by $\tau_0(A) := \varepsilon$ (i. e., τ_0 maps A to the empty word), $\tau_0(B) := a$, $\tau_0(C) := ab$ – satisfies $\tau_0(\alpha_0) = \sigma_0(\alpha_0)$ and, for a symbol X occuring in α, $\tau_0(X) \neq \sigma_0(X)$:

$$\sigma_0(\alpha_0) = \underbrace{\overbrace{a}^{\sigma_0(A)}}_{\tau_0(B)} \underbrace{\overbrace{a}^{\sigma_0(B)} \overbrace{b}^{\sigma_0(C)}}_{\tau_0(C)} \underbrace{\overbrace{a}^{\sigma_0(A)} \overbrace{b}^{\sigma_0(C)}}_{\tau_0(C)} \underbrace{\overbrace{a}^{\sigma_0(B)}}_{\tau_0(B)} = \tau_0(\alpha_0)\,.$$

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In contrast to this, e. g., the morphism $\sigma_1 : \{A, B, C\}^* \to \{a, b\}^*$ – given by $\sigma_1(A) := a$, $\sigma_1(B) := ab$, $\sigma_1(C) := b$ – is unambiguous with respect to α_0, as can be verified with moderate effort.

The potential ambiguity of morphisms is relevant to various concepts in the combinatorial theory of morphisms, such as pattern languages (see, e. g., Mateescu and Salomaa [10]), equality sets (see, e. g., Harju and Karhumäki [7]) and word equations (see, e. g., Choffrut [3]). This relation is best understood for inductive inference of pattern languages, where it has been shown that a preimage can be computed from some of its morphic images if these images have been generated by morphisms with a restricted ambiguity (see, e. g., Reidenbach [12]). Hence, intuitively speaking, unambiguous morphisms have a desirable, namely structure-preserving, property in such a context, and therefore previous literature on the ambiguity of morphisms mainly studies the question of the *existence* of unambiguous morphisms for arbitary words. In the initial paper, Freydenberger, Reidenbach and Schneider [6] show that there exists an unambiguous nonerasing morphism with respect to a word α if and only if α is not a *fixed point* of a nontrivial morphism, i. e., there is no morphism ϕ satisfying $\phi(\alpha) = \alpha$ and, for a symbol x in α, $\phi(x) \neq x$. Freydenberger and Reidenbach [5] study those sets of words with respect to which so-called segmented morphisms are unambiguous, and these results lead to a refinement of the techniques used in [6]. Schneider [15] and Reidenbach and Schneider [14] investigate the existence of unambiguous erasing morphisms – i. e., morphisms that may map symbols to the empty word. Finally, Freydenberger, Nevisi and Reidenbach [4] study a definition of unambiguity that is completely restricted to nonerasing morphisms[1], and they provide a characterisation of those words with respect to which there exist unambiguous morphisms $\sigma : \Delta^+ \to \Sigma^+$ in such a context (this characterisation does not hold for binary target alphabets Σ, though).

In the present paper, we study the existence of unambiguous 1-uniform morphisms for arbitrary words, i. e., just as our initial example σ_0, these mor-

[1]Note that [6, 5] also deal with unambiguous *nonerasing* morphisms, but they use a stronger notion of unambiguity that is based on arbitrary monoid morphisms. Hence, they call a morphism σ unambiguous only if there is no other – erasing or nonerasing – morphism τ satisfying $\tau(\alpha) = \sigma(\alpha)$. In contrast to this, and in contrast to the present paper, [4] disregards erasing morphisms τ. Consequently, in the definition of unambiguity studied by [4], our initial example σ_0 is considered ("weakly") *unambiguous* with respect to α_0, since all morphisms τ with $\tau(\alpha_0) = \sigma_0(\alpha_0)$ are erasing morphisms.

phisms map every symbol in the preimage to an image of length 1. In order to obtain unrestricted results, we wish to consider words over an unbounded alphabet Δ as morphic preimages. Therefore, we assume $\Delta := \mathbb{N}$; in accordance with the existing literature in the field, we call any word $\alpha \in \mathbb{N}^*$ a *pattern*, and we call any symbol $x \in \mathbb{N}$ occurring in α a *variable*. Thus, more formally, we wish to investigate the following problem:

Problem 1. *Let $\alpha \in \mathbb{N}^*$ be a pattern, and let Σ be an alphabet. Does there exist a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α, i. e., there is no morphism $\tau : \mathbb{N}^* \to \Sigma^*$ satisfying $\tau(\alpha) = \sigma(\alpha)$ and, for a variable x occurring in α, $\tau(x) \neq \sigma(x)$?*

There are two main reasons why we study this question: Firstly, any insight into the existence of unambiguous 1-uniform morphisms improves the construction by Freydenberger et al. [6], which provides comprehensive results on the existence of unambiguous nonerasing morphisms, but is based on morphisms that are often much more involved than required. This can be illustrated using our above example pattern α_0 (now interpreted as $\alpha_0 := 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2$ in order to fit with the definition of patterns as words over $\mathbb{N}$). Here, the unambiguous morphism σ_1 – which is not 1-uniform, but still very simple – produces a morphic image of length 8, whereas the unambiguous morphism for α_0 defined in [6] leads to a morphic image of length 162. This substantial complexity of known unambiguous morphisms has a severe effect on the runtime of certain inductive inference procedures for pattern languages, which, as mentioned above, are necessarily based on such morphisms. Thus, any insight into the existence of uncomplex unambiguous morphisms is not only of intrinsic interest and helps to clarify to which extent the complexity of morphisms contributes to their unambiguity, but is also important from a more applied point of view. Secondly, as shown by $\sigma_0(\alpha_0)$, the images under 1-uniform morphisms have a structure that is very close to that of their preimages. More precisely, the only difference between the pre-image and the image under a 1-uniform morphism is that some letters that are different in the pre-image might be identical in the image. Thus, a 1-uniform morphism can be interpreted as a *morphic simplification* of a word, and its potential ambiguity is a very basic phenomenon that – as to be demonstrated by some of the results and techniques in this paper – is related to a number of well-known concepts and problems in combinatorics on words.

2. Definitions and Preliminary Results

For the definitions of *patterns*, *variables*, *1-uniform morphisms*, *ambiguous* and *unambiguous morphisms*, and *fixed points* of nontrivial morphisms, Section 1 can be consulted.

Let A be an *alphabet*, i. e., an enumerable set of symbols. A *word (over A)* is a a finite sequence of symbols taken from A. We write ε for the *empty word*, i. e., the word of length 0. The set A^* is the set of all words over A, and $A^+ := A^* \setminus \{\varepsilon\}$. For the *concatenation* of two words w_1, w_2, we write $w_1 \cdot w_2$ or simply $w_1 w_2$. The notion $|x|$ stands for the size of a set x or the length of a word x. For any word $w \in A^*$, the notation $|w|_x$ stands for the number of occurrences of the letter x in w. The symbol $[\ldots]$ is used to omit some canonically defined parts of a given word, e. g., $\alpha = 1 \cdot 2 \cdot [\ldots] \cdot 5$ stands for $\alpha = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$. We call a word $v \in A^*$ a *factor* of a word $w \in A^*$ if, for some $u_1, u_2 \in A^*$, $w = u_1 v u_2$; moreover, if v is a factor of w then we say that w contains v and denote this by $v \sqsubseteq w$ or $w = \cdots v \cdots$. If $v \neq w$, then we say that v is a *proper* factor of w and denote this by $v \sqsubset w$. If $u_1 = \varepsilon$, then v is a *prefix* of w, and if $u_2 = \varepsilon$, then v is a *suffix* of w. For every letter x in w, $L_x := \{y \in A \mid w = \cdots y \cdot x \cdots\} \cup L'_x$ and $R_x := \{y \in A \mid w = \cdots x \cdot y \cdots\} \cup R'_x$, where $L'_x = \{\varepsilon\}$ if $w = x \cdots$ and $L'_x = \emptyset$ if $w \neq x \cdots$, and $R'_x = \{\varepsilon\}$ if $w = \cdots x$ and $R'_x = \emptyset$ if $w \neq \cdots x$. We refer to the sets L_x and R_x as *neighbourhood sets*.

For alphabets A, B, a mapping $h : A^* \to B^*$ is a *morphism* if h is compatible with the concatenation, i. e., for all $v, w \in A^*$, $h(v) \cdot h(w) = h(vw)$. We call B the *target alphabet* of h. The morphism h is said to be *nonerasing* if, for every $x \in A$, $h(x) \neq \varepsilon$. A morphism is called a *renaming* if it is injective and 1-uniform. We additionally call any word v a renaming of a word w if there is a morphism h that is a renaming and satisfies $h(w) = v$.

With regard to an arbitrary pattern $\alpha \in \mathbb{N}^*$, $\mathrm{var}(\alpha)$ denotes the set of all variables occurring in α. We say that α is *in canonical form* if α is lexicographically minimal among all its renamings (where the lexicographic order is derived from the usual order on $\mathbb{N}$, i. e., $1 < 2 < 3 < \ldots$).

The question of whether a pattern α is a fixed point of a nontrivial morphism (which can be decided in polynomial time, see Holub [8]) is equivalent to a number of other concepts in combinatorics on words. More precisely, α is a fixed point of a nontrivial morphism iff α is *prolix* iff α is *morphically imprimitive* iff there exist a certain characteristic factorisation of α; these equivalences are explained by Reidenbach and Schneider [13] in more detail.

Results on unambiguous morphisms have been stated using any of these concepts. In the present paper, our presentation shall focus on the notion of fixed points. Therefore, we can now paraphrase a simple yet fundamental insight by Freydenberger et al. [6] – which implies that an answer to Problem 1 is trivial for those patterns that are fixed points of nontrivial morphisms – as follows:

Theorem 1 (Freydenberger et al. [6])**.** *Let $\alpha \in \mathbb{N}^*$ be a fixed point of a nontrivial morphism, and let Σ be any alphabet. Then every nonerasing morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ is ambiguous with respect to α.*

Hence, we can safely restrict our subsequent considerations to those patterns that are not fixed points.

3. Fixed Target Alphabets

In the the present section, we describe a number of conditions on the existence of unambiguous 1-uniform morphisms $\sigma : \mathbb{N}^* \to \Sigma^*$ with a *fixed* target alphabet Σ, i. e., the size of Σ does not depend on the number of variables occurring in α. While the main result by Freydenberger et al. [6] demonstrates that the set of patterns with an unambiguous *nonerasing* morphisms is independent of the size of Σ (provided that $|\Sigma| \geq 2$), our initial example α_0 and all patterns $\alpha_m := 1 \cdot 1 \cdot 2 \cdot 2 \cdot [\ldots] \cdot m \cdot m$ with $m \geq 4$ do not have an unambiguous *1-uniform* morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ for binary alphabets Σ. In contrast to this, such morphisms can be given for ternary (and, thus, larger) alphabets:

Theorem 2. *Let $m \in \mathbb{N}$, $m \geq 4$, let Σ be an alphabet, and let $\alpha_m := 1 \cdot 1 \cdot 2 \cdot 2 \cdot [\ldots] \cdot m \cdot m$. There exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α_m if and only if $|\Sigma| \geq 3$.*

Proof. Since squares cannot be avoided over unary and binary alphabets, it can be shown with very limited effort that there is no unambiguous 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ with respect to any α_m if Σ does not contain at least three letters.

According to Thue [16], there exists an infinite square-free word over a ternary alphabet. Let this word be w. Thus,

$$w = abcacbabcbacabcacbaca \cdots .$$

We define the word w' by repeating every letter of w twice. Consequently,

$$w' = aabbccaaccbbaabbccbbaaccaabbccaaccbbaaccaa \cdots .$$

We now define a 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b, c\}^*$ such that $\sigma(\alpha_m)$ is a prefix of w'. Since w is square-free, the only square factors of w' are aa, bb and cc. Hence, it can be easily verified that σ is unambiguous with respect to α_m. □

Thus – and just as for the equivalent problem on unambiguous *erasing* morphisms (see Schneider [15]) – any characteristic condition on the existence of unambiguous 1-uniform morphisms needs to incorporate the size of Σ, which suggests that such criteria might be involved. This is further strengthened by the following result, which establishes an analogous phenomenon for the transition from $|\Sigma| = 3$ to $|\Sigma| \geq 4$:

Theorem 3. *There exists an $\alpha \in \mathbb{N}^+$ such that*

- *every 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b, c\}^*$ is ambiguous with respect to α and*
- *there is a 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b, c, d\}^*$ that is unambiguous with respect to α.*

Proof. Let $\alpha := 1 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 \cdot 6^2 \cdot 1 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 \cdot 6^2 \cdot 2^2$. We begin with the first statement of Theorem 3: Assume to the contrary that there is a 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b, c\}^*$ that is unambiguous with respect to α. If $\sigma(3 \cdot 4 \cdot 5 \cdot 6)$ contains at most two different symbols, then there exists a morphism τ with $\tau(\alpha) = \sigma(\alpha)$ and $\tau(x) = \varepsilon$ for an $x \in \{3, 4, 5, 6\}$ (since $\sigma(3 \cdot 4 \cdot 5 \cdot 6)$ then necessarily contains a square), which is a contradiction.

Hence, $\sigma(3 \cdot 4 \cdot 5 \cdot 6)$ must be a word over $\{a, b, c\}$. This implies that there is an $x \in \{3, 4, 5, 6\}$ satisfying $\sigma(x) = \sigma(2)$. We now consider the morphism $\tau : \mathbb{N}^* \to \{a, b, c\}^*$ given by

$$\tau(i) := \begin{cases} \sigma(1 \cdot 2^2 \cdot [\ldots] \cdot (x-1)^2), & i = 1, \\ \sigma(i), & i = 2 \text{ or } x + 1 \leq i \leq 6, \\ \varepsilon, & 3 \leq i \leq x. \end{cases}$$

Hence, $\tau(\alpha) = \sigma(\alpha)$ and $\tau(x) = \varepsilon \neq \sigma(x)$. Thus, σ is ambiguous with respect to α, which is a contradiction.

Regarding the second statement of Theorem 3, we define a morphism $\sigma_{a,b,c,d} : \mathbb{N}^* \to \{a, b, c, d\}^*$ by

$$\sigma_{a,b,c,d}(i) := \begin{cases} a, & i \in \{1, 4, 6\}, \\ b, & i = 2, \\ c, & i = 3, \\ d, & i = 5. \end{cases}$$

An exhaustive search demonstrates that there is no morphism $\tau : \mathbb{N}^* \to \{a, b, c, d\}^*$ with $\tau(\alpha) = \sigma_{a,b,c,d}(\alpha)$ and $\tau(x) \neq \sigma_{a,b,c,d}(x)$ for an $x \in \text{var}(\alpha)$. Thus, $\sigma_{a,b,c,d}$ is unambiguous with respect to α. $\square$

Due to Theorems 2 and 3 and, thus, the expected intricacy of characteristic conditions on the existence of unambiguous 1-uniform morphisms, our further results in this section are restricted to *sufficient* conditions.

Our first criterion is based on (un)avoidable patterns and is, thus, related to the above-mentioned property of the patterns α_m:

Theorem 4. *Let $n \in \mathbb{N}$, $\beta := r_1 \cdot r_2 \cdot [\ldots] \cdot r_{\lceil n/2 \rceil}$ and $\alpha := 1^{r_1} \cdot 2^{r_1} \cdot 3^{r_2} \cdot 4^{r_2} \cdot [\ldots] \cdot n^{(r_{\lceil n/2 \rceil})}$ with $r_i \geq 2$ for every i, $1 \leq i \leq \lceil n/2 \rceil$. If β is square-free, then there exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b\}^*$ that is unambiguous with respect to α.*

Proof. For any $n \in \mathbb{N}$, let $A := \{1, 2, 3, \ldots, n\}$. For every $q \in A$, we define the 1-uniform morphism σ by $\sigma(q) := a$ if q is odd and $\sigma(q) := b$ if q is even. Thus, $\sigma(\alpha) = a^{r_1} b^{r_1} \cdot a^{r_2} b^{r_2} \cdot [\ldots] \cdot x^{(r_{\lceil n/2 \rceil})}$ with $x \in \{a, b\}$. We claim that σ is unambiguous with respect to α if β is square-free. Assume to the contrary that σ is ambiguous. Consequently, there is a morphism $\tau : A^* \to \{a, b\}^*$ with $\tau(\alpha) = \sigma(\alpha)$ and, for an $i \in A$, $\tau(i) \neq \sigma(i)$. Without loss of generality, we assume that for any $i' < i$, $\tau(i') = \sigma(i')$. Thus, we can define $u \in \{a, b\}^*$ such that $\sigma(\alpha) = u \cdot \tau(i) \cdot \cdots$. Let $B := \{r_1, r_2, \ldots, r_{\lceil n/2 \rceil}\}$ and assume that y is the maximum number in B.

Claim. $\sigma(\alpha)$ does not contain any factor v^2 such that $v \in \{a^p b^p | p \in B\}^+$.

Proof (Claim). Since β is square-free, every subpattern of it is square-free. So, by considering the structure of $\sigma(\alpha)$, this implies that $\sigma(\alpha)$ does not contain any factor v^2 such that $v \in \{a^p b^p | p \in B\}^+$.

Let, $\tau(i) = a^j \cdot b^k \cdot v \cdot a^l \cdot b^m$, $v \in \{a^p b^p | p \in B\}^*$, $0 \leq j \leq y$, $0 \leq k \leq y$, $0 \leq l \leq y$ and $0 \leq m \leq y$. Furthermore, since $r_i \geq 2$, $\tau(i)^2$ is a factor of

$\tau(\alpha)$. Hence,
$$\tau(\alpha) = u \cdot (a^j b^k v a^l b^m)^2 \cdots ,$$
$u \neq \cdots a$ if $j \neq 0$ and, $u \neq \cdots b$ if $j = 0$. We now consider the following cases:

1. $j \neq k$, $j \neq 0$ and $k \neq 0$.
   (a) $v \neq \varepsilon$. So, $\tau(\alpha) = u \cdot a^j b^k v \cdots$. However, the factor $u \cdot a^j b^k v$ does not occur in $\sigma(\alpha)$, because $j \neq k$.
   (b) $v = \varepsilon$.
      i. $l = m = 0$. Hence, $\tau(\alpha) = u \cdot a^j b^k a^j b^k \cdots$. However, due to $j \neq k$, $\sigma(\alpha)$ does not have the factor $u \cdot a^j b^k a^j$, and this contradicts the assumption $\tau(\alpha) = \sigma(\alpha)$.
      ii. $l = 0$ and $m \neq 0$. We have $\tau(\alpha) = u \cdot a^j b^k b^m a^j b^k b^m \cdots$; in other words, $\tau(\alpha)$ contains the factor $u \cdot a^j b^{k+m} a^j b^{k+m}$. Let $v' = a^j b^{k+m}$. Since $\tau(\alpha) = \sigma(\alpha)$, $j = (k + m)$ and $v' \in \{a^p b^p | p \in B\}$. So, $\tau(\alpha)$ contains the factor $v'v'$ which contradicts the Claim.
      iii. $l \neq 0$. So, $\tau(\alpha) = u \cdot a^j b^k a^l b^m a^j b^k a^l b^m \cdots$. However, the factor $u \cdot a^j b^k a^l$ does not occur in $\sigma(\alpha)$, because $j \neq k$. Hence, $\tau(\alpha) \neq \sigma(\alpha)$ and this again contradicts the assumption.
2. $j = k \neq 0$.
   (a) $l \neq m$, $l \neq 0$ and $m \neq 0$. Thus, $\tau(\alpha) = u \cdot a^j b^j v a^l b^m \cdot a^j b^j v a^l b^m \cdots$. This means that $\tau(\alpha)$ contains the factor $b^j v \cdot a^l b^m \cdot a^j$. Due to $l \neq m$, this factor does not occur in $\sigma(\alpha)$, and this contradicts the assumption $\sigma(\alpha) = \tau(\alpha)$.
   (b) $l = m = 0$. Hence, $\tau(\alpha) = u \cdot a^j b^j v \cdot a^j b^j v \cdots$. Let $v' = a^j b^j v$. Thus, $v'v'$ is a factor of $\tau(\alpha)$ which implies that $v' \in \{a^p b^p | p \in B\}^+$. However, this contradicts the above mentioned Claim.
   (c) $l = m \neq 0$ and $l \neq 1$. We can conclude that $\tau(\alpha) = u \cdot a^j b^j v a^l b^l \cdot a^j b^j v a^l b^l \cdots$. We can infer from the factor $b^j v \cdot a^l b^l \cdot a^j$ that $a^l b^l \in \{a^p b^p | p \in B\}$. Let $v' = a^j b^j v a^l b^l$. So, $v'v'$ is a factor of $\tau(\alpha)$ while $v' \in \{a^p b^p | p \in B\}^+$, which again contradicts the mentioned Claim.
   (d) $l = m = 1$. So, $\tau(\alpha)$ contains the factor $b^j v a^1 b^1 \cdot a^j$ which does not occur in $\sigma(\alpha)$.
   (e) $l \neq 0$ and $m = 0$. Hence, $\tau(\alpha) = u \cdot a^j b^j v \cdot a^{l+j} b^j \cdot v a^l \cdots$. However, this contradicts the assumption $\sigma(\alpha) = \tau(\alpha)$, because of $(l + j) \neq j$.

(f) $l = 0$ and $m \neq 0$. This means that $\tau(\alpha)$ has the factor $u \cdot a^j b^j v b^m \cdot a^j$.

i. $v = \varepsilon$. So, $u \cdot a^j b^{j+m} \cdot a^j$ is a factor of $\tau(\alpha)$, and this contradicts $\sigma(\alpha) = \tau(\alpha)$ due to $j \neq (j + m)$.

ii. $v \neq \varepsilon$. Thus, we have the factor $b^j \cdot v b^m \cdot a^j$ in $\tau(\alpha)$. However, the number of repetitions of the last b in v plus m is larger than the repetitions of its previous a, and such a factor does not occur in $\sigma(\alpha)$.

3. $j \neq 0$ and $k = 0$.

(a) $v \neq \varepsilon$. So, $a^j v a^l b^m \cdot a^j$ is a factor of $\tau(\alpha)$. However, the number of repetitions of the first a in v plus j is larger than the number of the subsequent b, and this contradicts the structure of $\sigma(\alpha)$.

(b) $v = \varepsilon$. This implies $\tau(\alpha) = u \cdot a^{j+l} b^m \cdot a^{j+l} b^m \cdots$.

i. $m \neq 0$. Since $\tau(\alpha) = \sigma(\alpha)$, we can infer from the factor $u \cdot a^{j+l} b^m \cdot a^{j+l}$ that $j + l = m$ and as a result $a^{j+l} b^m \in \{a^p b^p | p \in B\}$. Let $v' = a^{j+l} b^m$. So, $v'v'$ is a factor of $\tau(\alpha)$; however, this again contradicts the mentioned Claim.

ii. $m = 0$. We can conclude that $(a^{j+l})^{r_i}$ is a factor of $\tau(\alpha)$. However, $\sigma(\alpha)$ does not contain this factor, because we know that after r_i occurrences of a in $\sigma(\alpha)$, we have b or ε.

4. $j = 0$ and $k \neq 0$.

(a) $l \neq 0$. Hence, $\tau(\alpha) = u \cdot b^k v a^l b^m \cdot b^k v a^l b^m \cdots$ and, consequently, $\tau(\alpha)$ contains the factor $b^k v a^l b^{m+k} v a^l$. Because of $\tau(\alpha) = \sigma(\alpha)$, we can conclude that $l = (m + k)$ and $a^l b^{m+k} \in \{a^p b^p | p \in B\}$ and also $\tau(\alpha) = u \cdot (b^k v a^l b^{m+k} v a^l b^m) \cdot (b^k) \cdot \cdots$. Let $v' = v a^l b^{m+k}$. So, $v'v'$ is a factor of $\tau(\alpha)$ while $v' \in \{a^p b^p | p \in B\}^+$. This again contradicts the mentioned Claim.

(b) $l = 0$. So, $\tau(\alpha) = u \cdot b^k v b^m \cdot b^k v b^m \cdots$.

i. $v \neq \varepsilon$. As a result, $b^k \cdot v b^{k+m} \cdot v b^m$ is a factor of $\tau(\alpha)$. However, the number of repetitions of the last b in v plus $k+m$ is larger than the repetitions of its previous a, and such a factor does not occur in $\sigma(\alpha)$.

ii. $v = \varepsilon$. We can conclude that $(b^{k+m})^{r_i}$ is a factor of $\tau(\alpha)$. However, $\sigma(\alpha)$ does not contain this factor, because we know that after r_i occurrences of b in $\sigma(\alpha)$, we have a or ε.

5. $\tau(i) = \varepsilon$. Due to $\tau(\alpha) = \sigma(\alpha)$, there exists an $i' > i$ with $|\tau(i')| > 1$. So, we can consider $\tau(i') = a^j \cdot b^k \cdot v \cdot a^l \cdot b^m$, which leads to the above cases.

Consequently, in all cases, assuming the existence of a morphism τ with $\tau(\alpha) = \sigma(\alpha)$ and, for an $i \in \mathrm{var}(\alpha)$, $\tau(i) \neq \sigma(i)$ leads to a contradiction. Thus, σ is unambiguous with respect to α. $\square$

Our second criterion again holds for binary (and, thus, all larger) alphabets Σ. It features a rather restricted class of patterns, which, however, are minimal with regard to their length.

Theorem 5. *Let $n \in \mathbb{N}$, $n \geq 2$. If n is even, then let*

$$\alpha := 1 \cdot 2 \cdot [\ldots] \cdot n \cdot (n/2+1) \cdot 1 \cdot (n/2+2) \cdot 2 \cdot [\ldots] \cdot n \cdot n/2,$$

and if n is odd, then let

$$\alpha := 1 \cdot 1 \cdot 2 \cdot 3 \cdot [\ldots] \cdot n \cdot (\lceil n/2 \rceil + 1) \cdot 2 \cdot (\lceil n/2 \rceil + 2) \cdot 3 \cdot [\ldots] \cdot n \cdot \lceil n/2 \rceil.$$

Then α is a shortest pattern with $|\mathrm{var}(\alpha)| = n$ that is not a fixed point of a nontrivial morphism, and there exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b\}^*$ that is unambiguous with respect to α.

Proof. We first briefly explain why any pattern α' with $|\mathrm{var}(\alpha')| = n$ and $|\alpha'| < |\alpha|$ must be a fixed point of a nontrivial morphism: If $|\mathrm{var}(\alpha')| = n$ and $|\alpha'| < |\alpha|$, then α' must contain at least one variable z with just a single occurrence, because all variables in α have exactly two occurrences. We can then define a morphism $\phi : \mathbb{N}^* \to \mathbb{N}^*$ by $\phi(z) := \alpha'$ and $\phi(z') := \varepsilon$ for all $z' \in \mathrm{var}(\alpha') \setminus \{z'\}$. Since $n \geq 2$, ϕ is nontrivial, and obviously $\phi(\alpha') = \alpha'$. Hence, α' is a fixed point of ϕ. At the end of this proof, we shall show that α is not a fixed point of a nontrivial morphism, which will then complete the proof of the first statement of the theorem.

We now consider the second statement of the theorem. We define the morphism σ by

$$\sigma(x) := \begin{cases} a, & \text{if } 1 \leq x \leq \lceil n/2 \rceil, \\ b, & \text{else .} \end{cases}$$

Assume to the contrary that σ is ambiguous with respect to α. Consequently, there exists a morphism $\tau : \mathbb{N}^* \to \{a, b\}^*$ satisfying $\tau(\alpha) = \sigma(\alpha)$ and, for some $q \in \mathrm{var}(\alpha)$, $\tau(q) \neq \sigma(q)$.
Let n be even. So,

$$\alpha := 1 \cdot 2 \cdot [\ldots] \cdot n \cdot (n/2+1) \cdot 1 \cdot (n/2+2) \cdot 2 \cdot [\ldots] \cdot n \cdot n/2.$$

As a result,

$$\sigma(\alpha) = a^{n/2} \cdot b^{n/2} \cdot (ba)^{n/2}.$$

Assume that $\alpha = \beta_1\beta_2$ with

$$\beta_1 = 1 \cdot 2 \cdot [\ldots] \cdot n$$

and,

$$\beta_2 = (n/2+1) \cdot 1 \cdot (n/2+2) \cdot 2 \cdot [\ldots] \cdot n \cdot n/2.$$

Due to the structure of α and $|\tau(\alpha)| = |\sigma(\alpha)|$, it is easily verified that $|\tau(\beta_1)| = |\sigma(\beta_1)|$ and, hence, $\tau(\beta_1) = \sigma(\beta_1)$. Besides, $\tau(\beta_1) = \sigma(\beta_1)$ implies that $\tau(\beta_2) = \sigma(\beta_2)$. Since σ is a 1-uniform morphism, there exists a $q \in \mathrm{var}(\alpha)$ such that $|\tau(q)| \geq 2$. Due to $\tau(\beta_1) = \sigma(\beta_1)$, we have one of the following cases:

1. $a^k \sqsubseteq \tau(q)$ with $k \geq 2$. Since q has an occurrence in β_2 and $a^k \not\sqsubseteq \sigma(\beta)$, $\tau(\beta) \neq \sigma(\beta)$, and as a result, $\tau(\alpha) \neq \sigma(\alpha)$, which is a contradiction.
2. $b^k \sqsubseteq \tau(q)$ with $k \geq 2$. Using the same reasoning as above, this leads to a contradiction.
3. $\tau(q) = ab$. We consider the following cases:
   - $q < n/2$. Then, due to $\tau(\beta_1) = \sigma(\beta_1)$, there exists a $q' < q$ satisfying $\tau(q') = a^k$ with $k \geq 2$, which according to Case 1 leads to a contradiction.
   - $q = n/2$. Due to the facts that $n/2$ is the last variable occurring in α and ba should be a suffix of $\tau(\alpha)$, this leads to a contradiction.
   - $q = n/2+1$. Since $\tau(\beta_2) = \sigma(\beta_2)$, ba should be a prefix of $\tau(\beta_2)$. However, the variable $n/2+1$ is the first variable of β_2. Consequently, this contradicts $\tau(\alpha) = \sigma(\alpha)$.
   - $q > n/2+1$. Then, due to $\tau(\beta_1) = \sigma(\beta_1)$, there exists a $q' > q$ satisfying $\tau(q') = b^k$ with $k \geq 2$, which according to Case 2 leads to a contradiction.

Hence, all above cases contradict the assumption of $\tau(\alpha) = \sigma(\alpha)$.
However, if n is odd,

$$\alpha := 1 \cdot 1 \cdot 2 \cdot 3 \cdot [\ldots] \cdot n \cdot (\lceil n/2 \rceil + 1) \cdot 2 \cdot (\lceil n/2 \rceil + 2) \cdot 3 \cdot [\ldots] \cdot n \cdot \lceil n/2 \rceil.$$

Thus,

$$\sigma(\alpha) = aa \cdot a^{\lfloor n/2 \rfloor} \cdot b^{\lfloor n/2 \rfloor} \cdot (ba)^{\lfloor n/2 \rfloor}.$$

Due to the structure of α and $\tau(\alpha) = \sigma(\alpha)$, it is easily verified that $\tau(1) = \sigma(1) = a$. This implies that an analogous reasoning to the case when n is even can also be used for the case that n is odd. Consequently, we can conclude that σ is unambiguous with respect to α.

What remains to explain is why α is not a fixed point of a nontrivial morphism. Since σ is nonerasing and, as shown above, unambiguous with respect to α, this directly follows from the contraposition of Theorem 1. $\square$

The following examples illustrates Theorem 5 and its proof: For $n := 6$, $\alpha := 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot4\cdot1\cdot5\cdot2\cdot6\cdot3$, and the 1-uniform morphism $\sigma : \mathbb{N}^* \to \{a, b\}^*$ with $\sigma(1) := \sigma(2) := \sigma(3) := a$ and $\sigma(4) := \sigma(5) := \sigma(6) := b$ is unambiguous with respect to α. For $n := 5$, $\alpha := 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 4 \cdot 2 \cdot 5 \cdot 3$, and the respective unambiguous morphism is given by $\sigma(1) := \sigma(2) := \sigma(3) := a$ and $\sigma(4) := \sigma(5) := b$.

From Theorem 5 we can conclude that patterns α with unambiguous 1-uniform morphisms using a binary target alphabet exist for every cardinality of $\mathrm{var}(\alpha)$ and that corresponding examples can be given where every variable occurs just twice.

4. Variable Target Alphabets

In order to continue our examination of Problem 1, we now relax one of the requirements of Section 3: We no longer investigate criteria on the existence of unambiguous 1-uniform morphisms for a fixed target alphabet Σ, but we permit Σ to depend on the number of variables in the pattern α in question. Regarding this question, we conjecture the following statement to be true:

Conjecture 1. *Let α be a pattern with $|\mathrm{var}(\alpha)| \geq 4$. There exists an alphabet Σ satisfying $|\Sigma| < |\mathrm{var}(\alpha)|$ and a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α if and only if α is not a fixed point of a nontrivial morphism.*

This conjecture would be trivially true if we allowed Σ to satisfy $|\Sigma| \geq |\mathrm{var}(\alpha)|$. That explains why we exclusively study the case where the number of letters in the target alphabet is smaller than the number of variables in the pattern. From Theorem 2, it directly follows that an analogous conjecture would not be true if we considered fixed binary target alphabets (as is done in

Section 3), since none of the patterns α_m is a fixed point of a nontrivial morphism – this can be easily verified using tools discussed by Reidenbach and Schneider [13] and Holub [8]. Hence, characteristic criteria must necessarily look different in such a context. It can also be effortlessly understood that Conjecture 1 would be incorrect if we dropped the condition that α needs to contain at least 4 distinct variables, since not only σ_0, but all 1-uniform morphisms $\sigma : \mathbb{N}^* \to \Sigma^*$ with $|\Sigma| \leq 2$ are ambiguous with respect to our example pattern $\alpha_0 = 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2$ discussed in Section 1.

Technically, many of our subsequent technical considerations are based on the following generic morphisms:

Definition 1. *Let Σ be an infinite alphabet, and let $\rho : \mathbb{N}^* \to \Sigma^*$ be a renaming. For any $i, j \in \mathbb{N}$ with $i \neq j$ and for every $x \in \mathbb{N}$, let the morphism $\sigma_{i,j}$ be given by*

$$\sigma_{i,j}(x) := \begin{cases} \rho(i), & \text{if } x = j\,, \\ \rho(x), & \text{if } x \neq j\,. \end{cases}$$

Thus, $\sigma_{i,j}$ maps exactly two variables to the same image, and therefore, for any pattern α with at least two different variables, $\sigma_{i,j}(\alpha)$ is a word over $|\operatorname{var}(\alpha)| - 1$ distinct letters. Using this definition, we can now state a more specific version of Conjecture 1:

Conjecture 2. *Let α be a pattern with $|\operatorname{var}(\alpha)| \geq 4$. There exist $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, such that $\sigma_{i,j}$ is unambiguous with respect to α if and only if α is not a fixed point of a nontrivial morphism.*

Before we study Conjectures 1 and 2 in more detail, we establish that they are equivalent. To this end, we need the following concept:

Definition 2. *Let $\alpha \in \mathbb{N}^*$ and let i, j, $i \neq j$, be arbitrary variables of α. Let the morphism $\psi_{i,j} : \mathbb{N}^* \to \mathbb{N}^*$ be given by $\psi_{i,j}(j) := i$ and $\psi_{i,j}(x) = x$, $x \in \mathbb{N} \setminus \{j\}$. We then define the pattern $\alpha_{i,j}$ by $\alpha_{i,j} := \psi_{i,j}(\alpha)$.*

For example, let $\alpha := 1 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 4$. If we consider $i := 2$ and $j := 4$, then $\alpha_{i,j} = 1 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 2$.

Using Definition 2, we can now address the relation between Conjectures 1 and 2:

Proposition 1. *Let α be a pattern with $|\operatorname{var}(\alpha)| \geq 4$. There exists an alphabet Σ satisfying $|\Sigma| < |\operatorname{var}(\alpha)|$ and a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$ that is unambiguous with respect to α if and only if there exist $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, such that $\sigma_{i,j}$ is unambiguous with respect to α.*

Proof. Since the size of the target alphabet of $\sigma_{i,j}$ equals $|\operatorname{var}(\alpha)| - 1$, the *if* direction is trivially true.

We now prove the *only if* direction. So, we assume that there exists a 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$, $|\Sigma| < |\operatorname{var}(\alpha)|$, that is unambiguous with respect to α. This means that there does not exist any morphism $\tau : \mathbb{N}^* \to \Sigma^*$ satisfying $\tau(\alpha) = \sigma(\alpha)$ and, for a variable q occurring in α, $\tau(q) \neq \sigma(q)$. Let $V := \{v \in \operatorname{var}(\alpha) \mid |\sigma(\alpha)|_{\sigma(v)} \neq |\alpha|_v\}$. If $|V| = 2$, then the *only if* direction holds immediately. Otherwise, we choose two arbitrary variables i, j from V satisfying $\sigma(i) = \sigma(j)$. We define a morphism $\phi : \Sigma^* \to \mathbb{N}^*$ by

$$\phi(x) := \begin{cases} i, & \text{if } x = \sigma_{i,j}(i), \\ \sigma_{i,j}^{-1}(x), & \text{else.} \end{cases}$$

The morphism ϕ exists due to the definition of $\sigma_{i,j}$, and we can directly conclude the correctness of the following statement:

Claim 1. $\phi \circ \sigma_{i,j}(\alpha) = \alpha_{i,j}$.

We consider the morphism $\psi_{i,j} : \mathbb{N}^* \to \mathbb{N}^*$, given by

$$\psi_{i,j}(x) := \begin{cases} i, & \text{if } x = j, \\ x, & \text{else.} \end{cases}$$

Since, by Definition 2, $\alpha_{i,j}$ equals $\psi_{i,j}(\alpha)$, we can prove the following vital fact:

Claim 2. $\sigma(\alpha) = \sigma(\alpha_{i,j})$.

Proof (Claim 2). Due to our choice of i and j, we know that $\sigma(i) = \sigma(j)$ is satisfied. Furthermore, $\psi_{i,j}(i) = \psi_{i,j}(j) = i$, and therefore $\sigma(\psi_{i,j}(i)) = \sigma(\psi_{i,j}(j)) = \sigma(i) = \sigma(j)$. Hence, and since the definition of $\psi_{i,j}$ directly implies $\sigma(x) = \sigma(\psi_{i,j}(x))$ for every $x \in \operatorname{var}(\alpha) \setminus \{i, j\}$, we can conclude $\sigma(\alpha) = \sigma(\psi_{i,j}(\alpha))$. Since $\psi_{i,j}(\alpha) = \alpha_{i,j}$, this proves $\sigma(\alpha) = \sigma(\alpha_{i,j})$. $\square$ *(Claim 2)*

We now assume to the contrary that $\sigma_{i,j}(\alpha)$ is ambiguous. Hence, there is a morphism $\tau_{i,j} : \mathbb{N}^* \to \Sigma^*$ satisfying $\tau_{i,j}(\alpha) = \sigma_{i,j}(\alpha)$ and, for a variable q occurring in α, $\tau_{i,j}(q) \neq \sigma_{i,j}(q)$. Since $\sigma_{i,j}$ is 1-uniform, this implies that there exists a variable $q' \in \operatorname{var}(\alpha)$ with $\tau_{i,j}(q') = \varepsilon$.

The following diagram illustrates all morphisms, patterns and words introduced so far:

$$
\begin{tikzcd}
\alpha \arrow[r, "\sigma"] \arrow[d, shift right, "\tau_{i,j}"'] \arrow[d, shift left, "\sigma_{i,j}"] \arrow[dr, "\psi_{i,j}"] & \sigma(\alpha) \\
\sigma_{i,j}(\alpha) \arrow[r, "\phi"] & \alpha_{i,j} \arrow[u, "\sigma"']
\end{tikzcd}
$$

We now define the morphism $\tau : \mathrm{var}(\alpha)^* \to \Sigma^*$ by

$$\tau := \sigma \circ \phi \circ \tau_{i,j}.$$

Since we assume that $\tau_{i,j}(\alpha)$ equals $\sigma_{i,j}(\alpha)$, Claims 1 and 2 facilitate the following reasoning:

$$
\begin{aligned}
\tau(\alpha) &= \sigma \circ \phi \circ \tau_{i,j}(\alpha) \\
&= \sigma \circ \phi \circ \sigma_{i,j}(\alpha) \\
&= \sigma(\alpha_{i,j}) \\
&= \sigma(\alpha).
\end{aligned}
$$

Consequently, $\tau(\alpha) = \sigma(\alpha)$. As stated above, there exists a variable $q' \in \mathrm{var}(\alpha)$ that satisfies $\tau_{i,j}(q') = \varepsilon$, and therefore $\tau(q') = \sigma \circ \phi \circ \tau_{i,j}(q') = \varepsilon$. On the other hand, σ is 1-uniform, and therefore $\sigma(q') \neq \varepsilon$. Hence, the existence of τ implies that σ is ambiguous with respect to α, and this is a contradiction to the initial assumption of our proof for the *only if* direction. Thus, $\sigma_{i,j}$ is unambiguous with respect to α. □

Thus, our two conjectures are equivalent:

Corollary 1. *Conjecture 1 is true if and only if Conjecture 2 is true.*

Proof. Directly from Proposition 1. □

As a side note, we consider it worth mentioning that Conjecture 2 shows connections to another conjecture from the literature. In order to state the latter, we define, for any $i \in \mathbb{N}$, the morphism $\delta_i : \mathbb{N}^* \to \mathbb{N}^*$ by $\delta_i(i) := \varepsilon$ and, for every $j \in \mathbb{N} \setminus \{i\}$, $\delta_i(j) := j$.

Conjecture 3 (Billaud [1], Levé and Richomme [9]). *Let α be a pattern with $|\operatorname{var}(\alpha)| \geq 3$. If, for every $i \in \operatorname{var}(\alpha)$, $\delta_i(\alpha)$ is a fixed point of a nontrivial morphism, then α is a fixed point of a nontrivial morphism.*

In general, the correctness of Conjecture 3 has not been established yet. The problem is intensively studied by Levé and Richomme [9], where it is shown to be correct for certain subclasses of $\mathbb{N}^*$.

Due to Theorem 1, the *only if* directions of Conjectures 1 and 2 hold true immediately. In the remainder of this section, we shall therefore exclusively study those patterns that are not fixed points. Our corresponding results yield large classes of such patterns that have an unambiguous 1-uniform morphism, but we have to leave the overall correctness of our conjectures open.

Conjecture 2 suggests that the examination of the existence of unambiguous 1-uniform morphisms for a pattern α may be reduced to finding suitable variables i and j such that $\sigma_{i,j}$ is unambiguous with respect to α. In this regard, one particular choice can be ruled out immediately:

Proposition 2. *Let α be a pattern, and let $i, j \in \operatorname{var}(\alpha)$, $i \neq j$. If $\sigma_{i,j}(\alpha)$ is a fixed point of a nontrivial morphism, then $\sigma_{i,j}$ is ambiguous with respect to α.*

Proof. If $\sigma_{i,j}(\alpha)$ is a fixed point of a nontrivial morphism, then, by definition, there is a morphism ϕ satisfying $\phi(\sigma_{i,j}(\alpha)) = \sigma_{i,j}(\alpha)$ and, for a letter a in $\sigma_{i,j}(\alpha)$, $\phi(a) \neq a$. This implies that there must be a letter in α that is mapped by ϕ to the empty word; without loss of generality, we simply assume $\phi(a) := \varepsilon$. If we now define $\tau := \phi \circ \sigma_{i,j}$, then $\tau(\alpha) = \sigma_{i,j}(\alpha)$ and $\tau(x) = \varepsilon \neq \sigma_{i,j}(x)$, where x is a variable in α satisfying $\sigma_{i,j}(x) = a$. Thus, $\sigma_{i,j}$ is ambiguous with respect to α. $\square$

For example, if we consider the pattern $\alpha_1 := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 4 \cdot 3 \cdot 2$ (which is not a fixed point) and define $\Sigma := \{a, b, c\}$, then $\sigma_{2,4}(\alpha_1)$ equals *abcbabcb* (or any renaming thereof), which is a fixed point of the morphism ϕ given by $\phi(a) := abcb$ and $\phi(b) := \phi(c) := \varepsilon$. Thus, $\sigma_{2,4}$ is ambiguous with respect to α_1. However, Proposition 2 does not provide a characteristic condition on the ambiguity of $\sigma_{i,j}$, since $\sigma_{2,3}(\alpha_1) = abbcacbb$ is not a fixed point, but still $\sigma_{2,3}$ is ambiguous with respect to α_1. Furthermore, while the ambiguity of $\sigma_{2,3}$ results from the fact that α_1 contains the factors $2 \cdot 3$ and $3 \cdot 2$, and is therefore easy to comprehend, there are more difficult examples of morphisms $\sigma_{i,j}$ that are ambiguous although they do not lead to a morphic image that is a fixed point. This is illustrated by the example $\alpha_2 := 1{\cdot}2{\cdot}3{\cdot}3{\cdot}4{\cdot}4{\cdot}1{\cdot}2{\cdot}3{\cdot}3{\cdot}4{\cdot}4{\cdot}2$. Here, $\sigma_{2,4}(\alpha_1) = abccbbabccbbb$ again is not a fixed point, but $\sigma_{2,4}$ is nevertheless ambiguous with respect to α_2, since the morphism τ given by $\tau(1) := abccb$, $\tau(2) := b$ and $\tau(3) := \tau(4) := \varepsilon$ satisfies $\tau(\alpha_2) = \sigma_{2,4}(\alpha_2)$. We therefore conclude that it seems not to be a straightforward task to find amendments that could turn Proposition 2 into a characteristic condition.

We now show that Conjecture 2 is correct for several types of patterns. To this end, we need the following simple sufficient condition on a pattern being a fixed point:

Lemma 1. *Let $\alpha \in \mathbb{N}^+$. If there exists a variable $i \in \mathrm{var}(\alpha)$ such that*

1. *$\varepsilon \notin L_i$ and, for every $k \in L_i$, $R_k = \{i\}$, or*
2. *$\varepsilon \notin R_i$ and, for every $k \in R_i$, $L_k = \{i\}$,*

then α is a fixed point of a nontrivial morphism.

Proof. Assume that Condition 1 of the lemma is satisfied. So, without loss of generality, let

$$\alpha := \alpha_1 \cdot l_1 \cdot i_1 \cdot \alpha_2 \cdot l_2 \cdot i_2 \cdot \alpha_3 \cdot [\ldots] \cdot \alpha_n \cdot l_n \cdot i_n \cdot \alpha_{n+1}$$

where $i_1, i_2, \ldots, i_n$ are all occurrences of the variable i in α and, for every j, $1 \leq j \leq n$, $\alpha_j \in \mathbb{N}^*$, $\alpha_{n+1} \in \mathbb{N}^*$ and $l_j \in \mathbb{N}$. Also, Condition 1 implies that, for every j, $1 \leq j \leq n$ and for every j', $1 \leq j' \leq n+1$, $l_j \neq i$, $l_j \not\sqsubseteq \alpha_{j'}$. We define the morphism $\phi : \mathbb{N}^+ \to \mathbb{N}^*$ by:

$$\phi(x) := \begin{cases} l_j i, & \text{if } x = l_j,\ 1 \leq j \leq n, \\ \varepsilon, & \text{if } x = i, \\ x, & \text{else.} \end{cases}$$

Hence, $\phi(\alpha) = \alpha$ which means that α is a fixed point of a nontrivial morphism ϕ. Using an analogous reasoning as above, we can show that the lemma also holds true when Condition 2 is satisfied. □

Using this lemma, we can now establish a class of patterns for which Conjecture 2 holds true. All variables in these patterns have the same number of occurrences, and for one pair of variables they do not contain any factors as discussed above with respect our example $\sigma_{2,3}$:

Theorem 6. *Let $m \in \mathbb{N}$, $m \geq 2$. Let $\alpha \in \mathbb{N}^+$ be a pattern that is not a fixed point of a nontrivial morphism and satisfies, for every $x \in \mathrm{var}(\alpha)$, $|\alpha|_x = m$. If there are $i, j \in \mathrm{var}(\alpha)$, $i \neq j$, such that*

- *there is no $k \in \mathrm{var}(\alpha)$ with $\{i, j\} \subseteq L_k$ or $\{i, j\} \subseteq R_k$, and*
- *$\alpha \neq \alpha_1 \cdot \mathbf{i} \cdot \mathbf{j} \cdot \alpha_2 \cdot \mathbf{j} \cdot \mathbf{i} \cdot \alpha_3$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}^*$,*

then $\sigma_{i,j}$ is unambiguous with respect to α.

Proof. Assume to the contrary that $\sigma_{i,j}$ is ambiguous. So, there exists a morphism $\tau : \mathbb{N}^+ \to \Sigma^*$ such that $\tau(\alpha) = \sigma_{i,j}(\alpha)$ and, for some $x \in \mathrm{var}(\alpha)$, $\tau(x) \neq \sigma_{i,j}(x)$. Since $\sigma_{i,j}$ is a 1-uniform morphism, there exists a $k \in \mathrm{var}(\alpha)$ with $|\tau(k)| \geq 2$. Let $uv \sqsubseteq \tau(k)$, $u, v \in \Sigma$. Due to the fact that k occurs m times in α, $\sigma_{i,j}(\alpha) = \tau(\alpha) = w_1 \cdot uv \cdot w_2 \cdot uv \cdot [\ldots] \cdot w_m \cdot uv \cdot w_{m+1}$ with, for every q, $1 \leq q \leq m+1$, $w_q \in \Sigma^*$. We now consider the following cases:

- $\sigma_{i,j}(i) \neq u$ and $\sigma_{i,j}(i) \neq v$. This implies that there exist the variables $x_1, x_2 \in \mathrm{var}(\alpha)$, $x_1, x_2 \neq i$ and $x_1, x_2 \neq j$, such that $\alpha = \alpha_1 \cdot x_1 x_2 \cdot \alpha_2 \cdot x_1 x_2 \cdot [\ldots] \cdot \alpha_m \cdot x_1 x_2 \cdot \alpha_{m+1}$, for every q, $1 \leq q \leq m+1$, $\alpha_q \in \mathbb{N}^*$, and $\sigma_{i,j}(x_1) = u$ and $\sigma_{i,j}(x_2) = v$. Due to $|\alpha|_{x_1} = |\alpha|_{x_2} = m$, $x_1, x_2 \not\sqsubseteq \alpha_q$, for every q, $1 \leq q \leq m+1$. This implies that $R_{x_1} = \{x_2\}$ and $L_{x_2} = \{x_1\}$. Then, according to Lemma 1, α is a fixed point of a nontrivial morphism which is a contradiction to the assumption of the theorem.

- $\sigma_{i,j}(i) = \sigma_{i,j}(j) = u$, and $u \neq v$. So, we assume that $\alpha = \alpha_1 \cdot x_1 x' \cdot \alpha_2 \cdot x_2 x' \cdot [\ldots] \cdot \alpha_m \cdot x_m x' \cdot \alpha_{m+1}$ with, $x' \in \mathrm{var}(\alpha)$ and, for every q, $1 \leq q \leq m+1$, $x_q \in \mathrm{var}(\alpha)$, $\alpha_q \in \mathbb{N}^*$, and $\sigma_{i,j}(x_q) = u$ and $\sigma_{i,j}(x') = v$. Additionally, since $\sigma_{i,j}(x') = v$ and $u \neq v$, we can conclude that $x' \neq i$ and $x' \neq j$. We now consider the following cases:

  1. For every q, $1 \leq q \leq m$, $x_q = i$. This implies, using the same reasoning as above, that α is a fixed point of a nontrivial morphism which is a contradiction.
  2. There exists q, q', $1 \leq q, q' \leq m$ and $q \neq q'$, such that $x_q = i$ and $x_{q'} = j$. This means that $\{i, j\} \subseteq L_{x_2}$, which contradicts the first condition of the theorem.

- $\sigma_{i,j}(i) = v$, and $u \neq v$. The reasoning is analogous to that in the previous case.

- $\sigma_{i,j}(i) = \sigma_{i,j}(j) = u$ and $v = u$. Hence, we may assume that $\alpha = \alpha_1 \cdot x_1 x'_1 \cdot \alpha_2 \cdot x_2 x'_2 \cdot [\ldots] \cdot \alpha_m \cdot x_m x'_m \cdot \alpha_{m+1}$ with, for every q, $1 \leq q \leq m+1$, $\alpha_q \in \mathbb{N}^*$, $x_q, x'_q \in \text{var}(\alpha)$ and $\sigma_{i,j}(x_q) = \sigma_{i,j}(x'_q) = u$. Due to the conditions of the theorem, the factors $i \cdot i \cdot j$, $i \cdot j \cdot j$, $j \cdot i \cdot i$ and $j \cdot j \cdot i$ could not be the factors of α. Moreover, it can be observed that $u \cdot u \cdot u \not\sqsubseteq \tau(k)$; otherwise, since $\tau(\alpha) = \sigma_{i,j}(\alpha)$, then $|\alpha|_i > m$ or $\alpha_j > m$. This implies that $i \cdot j \cdot i$ and $j \cdot i \cdot j$ are not the factors of α. We now consider the following cases:

    1. For every q, $1 \leq q \leq m$, $x_q = i$ and $x'_q = j$. As a result, $R_i = \{j\}$ and $L_j = \{i\}$. According to Lemma 1, α is a fixed point of a nontrivial morphism.
    2. For every q, $1 \leq q \leq m$, $x_q = j$ and $x'_q = i$. As a result, $R_j = \{i\}$ and $L_i = \{j\}$. Referring to Lemma 1, α is a fixed point of a nontrivial morphism.
    3. There exists a q, q', $1 \leq q, q' \leq m$ and $q \neq q'$, such that $x_q \cdot x'_q = i \cdot j$ and $x_{q'} \cdot x'_{q'} = j \cdot i$. This case contradicts the second condition of the theorem.
    4. There exists a q, q', $1 \leq q, q' \leq m$ and $q \neq q'$, such that $x_q \cdot x'_q = i \cdot j$ and, $x_{q'} \cdot x'_{q'} = i \cdot i$ or $x_{q'} \cdot x'_{q'} = j \cdot j$. This means that $\{i, j\} \subseteq R_i$ or $\{i, j\} \subseteq L_j$ which is a contradiction to the first condition of the theorem.
    5. There exists a q, q', $1 \leq q, q' \leq m$ and $q \neq q'$, such that $x_q \cdot x'_q = j \cdot i$ and, $x_{q'} \cdot x'_{q'} = i \cdot i$ or $x_{q'} \cdot x'_{q'} = j \cdot j$. This implies that $\{i, j\} \subseteq L_i$ or $\{i, j\} \subseteq R_j$ which contradicts the first condition of the theorem.
    6. There exist q, q', $1 \leq q, q' \leq m$, $q' \neq q$, such that $x_q \cdot x'_q = i \cdot i$ and $x_{q'} \cdot x'_{q'} = j \cdot j$. Since $uu \sqsubseteq \tau(k)$ and due to the conditions of the theorem, it results from $\tau(\alpha) = \sigma_{i,j}(\alpha)$ that $k \neq i$ and $k \neq j$, in other words, $\tau(i) \neq uu$ and $\tau(j) \neq uu$; otherwise, $|\tau(\alpha)|_u > |\sigma_{i,j}(\alpha)|_u$. Moreover, we may observe that if $\sigma_{i,j}(k) \sqsubseteq \tau(k)$, then this implies that there exists an $x \in \text{var}(\alpha) \backslash \{i, j\}$, with $\{i, j\} \subseteq L_x$ or $\{i, j\} \subseteq R_x$, which is a contradiction. Thus, $\sigma_{i,j}(k) \not\sqsubseteq \tau(k)$. Since $\tau(\alpha) = \sigma_{i,j}(\alpha)$, there should be a $k' \in \text{var}(\alpha)$, $k' \neq k, i, j$, such that $\sigma_{i,j}(k) \sqsubseteq \tau(k')$, which means that $|\tau(k')| \geq 2$ or we can extend the reasoning over the other variables. Consequently, since $\tau(\alpha) = \sigma(\alpha)$, this argumentation implies the existence of a $k'' \in \text{var}(\alpha)$, $k'' \neq k, i, j$, such that $|\tau(k'')| \geq 2$, which, according to the above cases, leads to a contradiction.

Hence, in all cases, our assumption leads to a contradiction, and this proves the theorem. □

We wish to point out that Theorem 6 does not only demonstrate the correctness of Conjecture 2 for the given class of patterns, but additionally provides an efficient way of finding an unambiguous morphism $\sigma_{i,j}$. For example, we can immediately conclude from it that $\sigma_{1,4}$ is unambiguous with respect to our above example pattern α_1. Furthermore, the theorem also holds for patterns with less than four different variables.

We now consider those patterns that are not a fixed point and, moreover, contain all of their variables exactly twice (note that some of these "shortest" patterns that are not fixed points are also studied in Theorem 5). We wish to demonstrate that Theorem 6 implies the existence of an unambiguous $\sigma_{i,j}$ for *every* such pattern. This insight is based on the following lemma:

Lemma 2. *Let $\alpha \in \mathbb{N}^+$ be a pattern with $|\operatorname{var}(\alpha)| > 6$ and, for every $x \in \operatorname{var}(\alpha)$, $|\alpha|_x = 2$. Then there exist $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, such that*

- *there is no $k \in \operatorname{var}(\alpha)$ with $\{i, j\} \subseteq L_k$ or $\{i, j\} \subseteq R_k$, and*
- $\alpha \neq \alpha_1 \cdot \mathbf{i} \cdot \mathbf{j} \cdot \alpha_2 \cdot \mathbf{j} \cdot \mathbf{i} \cdot \alpha_3$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}^*$.

Proof. Let $n := |\operatorname{var}(\alpha)|$. Since every variable occurs exactly twice in α, it directly follows that, for every $x \in \operatorname{var}(\alpha)$, $|R_x| \leq 2$ and $|L_x| \leq 2$. By omitting the neighbourhood sets containing ε, we have at most $2n - 2$ sets of size 2. Besides, it can be verified with little effort that α contains at most $n-1$ different factors $i \cdot j$, $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, such that $\alpha = \alpha_1 \cdot i \cdot j \cdot \alpha_2 \cdot j \cdot i \cdot \alpha_3$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}^*$ (e. g., for $n := 4$, $\alpha := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ has 3 different factors $i \cdot j$, $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, satisfying the mentioned condition). Assume to the contrary that, for every $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, one of the following cases is satisfied:

- there exists $k \in \operatorname{var}(\alpha)$ with $\{i, j\} \subseteq L_k$ or $\{i, j\} \subseteq R_k$, or
- $\alpha = \alpha_1 \cdot \mathbf{i} \cdot \mathbf{j} \cdot \alpha_2 \cdot \mathbf{j} \cdot \mathbf{i} \cdot \alpha_3$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}^*$.

As mentioned above, the maximum number of pairs that are covered by the first case is $2n - 2$, and for the second case it is $n - 1$. On the other hand, since $|\operatorname{var}(\alpha)| = n$, there exist $\binom{n}{2}$ different pairs of variables. However, for $n > 6$, we have

$$\binom{n}{2} > (2n - 2) + (n - 1),$$

which contradicts the assumption. □

Hence, whenever a pattern α is not a fixed point, the conditions of Theorem 6 are automatically satisfied if α contains at least seven distinct variables and all of its variables occur exactly twice. Using a less elegant reasoning than the one on Lemma 2, we can extend this insight to all such patterns over at least four distinct variables. This yields the following result:

Theorem 7. *Let $\alpha \in \mathbb{N}^+$ be a pattern with $|\operatorname{var}(\alpha)| > 3$ and, for every $x \in \operatorname{var}(\alpha)$, $|\alpha|_x = 2$. If α is not a fixed point of a nontrivial morphism, then there exist $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, such that $\sigma_{i,j}$ is unambiguous with respect to α.*

Proof. Let $n := |\operatorname{var}(\alpha)|$. For $n > 6$, it directly follows from Theorem 6 and Lemma 2 that Theorem 7 is satisfied. Hence, we consider the following cases:

- $|\operatorname{var}(\alpha)| = 4$. The only patterns that do not satisfy the conditions of Theorem 6 are:

$$\begin{aligned}
&\alpha_1 := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 1 \cdot 3 \cdot 2,\ \alpha_2 := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 1 \cdot 3,\\
&\alpha_3 := 1 \cdot 2 \cdot 3 \cdot 4 \cdot 2 \cdot 1 \cdot 4 \cdot 3,\ \alpha_4 := 1 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 1,\\
&\alpha_5 := 1 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 1 \cdot 4 \cdot 2,\ \alpha_6 := 1 \cdot 2 \cdot 3 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 4,\\
&\alpha_7 := 1 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 2 \cdot 1 \cdot 4,\ \alpha_8 := 1 \cdot 2 \cdot 3 \cdot 2 \cdot 4 \cdot 4 \cdot 1 \cdot 3,\\
&\alpha_9 := 1 \cdot 2 \cdot 1 \cdot 3 \cdot 4 \cdot 4 \cdot 2 \cdot 3,\ \alpha_{10} := 1 \cdot 2 \cdot 3 \cdot 1 \cdot 4 \cdot 4 \cdot 3 \cdot 2.
\end{aligned}$$

  It can be verified with little effort that

  - $\sigma_{3,4}$ is unambiguous with respect to α_1, α_2, α_5, α_9 and α_{10},
  - $\sigma_{2,3}$ is unambiguous with respect to α_3, α_6 and α_7,
  - $\sigma_{1,4}$ is unambiguous with respect to α_4, α_8.

- $|\operatorname{var}(\alpha)| \in \{5, 6\}$. Assume to the contrary that for every $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, $\sigma_{i,j}$ is ambiguous with respect to α. This implies that the conditions of Theorem 6 are not satisfied. Consequently, for every $i, j \in \operatorname{var}(\alpha)$, one of the following cases is satisfied:

  - there is a $k \in \operatorname{var}(\alpha)$ with $\{i, j\} \subseteq L_k$ or $\{i, j\} \subseteq R_k$, or
  - $\alpha = \cdots i \cdot j \cdots j \cdot i \cdots$.

It directly follows from the proof of Lemma 2 that, if $\mathrm{var}(\alpha) = n$, then the maximum number of pairs of variables satisfying the first case is $2n - 2$. On the other hand, the number of different pairs i, j which should satisfy the above cases is $\binom{n}{2}$, consequently, there exist

$$\binom{n}{2} - (2n - 2) \quad n \geq 5$$

pairs which should satisfy the second case. So, for $n = 5$, since

$$\binom{5}{2} - (2 * 5 - 2) = 2,$$

there exist at least two different pairs of i, j satisfying $\alpha = \cdots i \cdot j \cdots j \cdot i \cdots$. For $n = 6$, that amount increases to 5, due to:

$$\binom{6}{2} - (2 * 6 - 2) = 5.$$

By investigating the all patterns α with $\mathrm{var}(\alpha) = 5$ which are containing 2 different pairs of i, j, we can conclude that there exist $i, j \in \mathrm{var}(\alpha)$, $i \neq j$, such that $\sigma_{i,j}$ is unambiguous with respect to α. Moreover, the only pattern α with $|\mathrm{var}(\alpha)| = 6$ that is not a fixed point of a nontrivial morphism and contains 5 different pairs of i, j is $\alpha = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, with respect to which there exists an unambiguous 1-uniform morphisms $\sigma_{1,6}$.

Hence, in both cases, the results contradict the assumption. □

Theorem 7 does not only directly prove the correctness of Conjecture 2 for all patterns that contain all their variables exactly twice, but it also allows a large set of patterns to be constructed for which the conjecture holds true as well. This construction is specified as follows:

Theorem 8. *Let $\alpha := \alpha_1 \cdot \beta \cdot \alpha_2$ and $\gamma := \alpha_1 \cdot \alpha_2$ be patterns with $\alpha_1, \alpha_2, \beta \in \mathbb{N}^*$, such that*

- *γ and β are not a fixed point of a nontrivial morphism,*
- *$|\mathrm{var}(\gamma)| > 3$ and, for every $x \in \mathrm{var}(\gamma)$, $|\gamma|_x = 2$, or $|\mathrm{var}(\beta)| > 3$ and, for every $x \in \mathrm{var}(\beta)$, $|\beta|_x = 2$, and*

- $\operatorname{var}(\gamma) \cap \operatorname{var}(\beta) = \emptyset$.

Then there exist $i, j \in \operatorname{var}(\alpha)$, $i \neq j$, such that $\sigma_{i,j}$ is unambiguous with respect to α.

Proof. Assume that $|\operatorname{var}(\gamma)| > 3$ and, for every $x \in \operatorname{var}(\gamma)$, $|\gamma|_x = 2$. So, since γ satisfies the conditions of Theorem 7, there exist $i, j \in \operatorname{var}(\gamma)$, $i \neq j$, such that $\sigma_{i,j}$ with target alphabet Σ_1 is unambiguous with respect to γ. Also, due to β not being a fixed point of a nontrivial morphism, there is an unambiguous 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma_2^*$, $|\Sigma_2| = |\operatorname{var}(\beta)|$, with respect to β. Let $\Sigma_1 \cap \Sigma_2 := \emptyset$.

We now assume to the contrary that $\sigma_{i,j}$ with target alphabet $\Sigma_1 \cup \Sigma_2$ is ambiguous with respect to α. This implies that there is a morphism $\tau : \mathbb{N}^* \to \Sigma^*$ satisfying $\tau(\alpha) = \sigma_{i,j}(\alpha)$ and, for some $q \in \operatorname{var}(\alpha)$, $\tau(q) \neq \sigma(q)$.

Claim 1. There does not exist an $x \in \operatorname{var}(\alpha)$ satisfying $|\tau(x)| \geq 2$ and $v_1 v_2 \sqsubseteq \tau(x)$, $v_1 \in \Sigma_1$ and $v_2 \in \Sigma_2$, or $v_1 \in \Sigma_2$ and $v_2 \in \Sigma_1$.

Proof of Claim 1. Assume to the contrary that there is an $x \in \operatorname{var}(\alpha)$ such that $|\tau(x)| \geq 2$ and $v_1 v_2 \sqsubseteq \tau(x)$, $v_1 \in \Sigma_1$ and $v_2 \in \Sigma_2$, or $v_1 \in \Sigma_2$ and $v_2 \in \Sigma_1$. Since x occurs at least twice in α, $\tau(\alpha) = \cdots \cdot v_1 v_2 \cdot \cdots \cdot v_1 v_2 \cdot \cdots$. However, because of $\alpha := \alpha_1 \cdot \beta \cdot \alpha_2$ and $\operatorname{var}(\gamma) \cap \operatorname{var}(\beta) = \emptyset$, this contradicts $\sigma_{i,j}(\alpha) = \tau(\alpha)$. □(Claim 1)

Claim 2. There exists an $x \in \operatorname{var}(\beta)$ such that $\tau(x) \in \Sigma_1^+$.

Proof of Claim 2. Assume to the contrary that, for every $x \in \operatorname{var}(\beta)$, $\tau(x) \notin \Sigma_1^+$. Due to Claim 1, it results from $\tau(\alpha) = \sigma_{i,j}(\alpha)$ and $\sigma_{i,j}$ being unambiguous with respect to γ and β that there exist some $x' \in \operatorname{var}(\gamma)$ such that $\tau(x') \in \Sigma_2^+$. Let $A \subseteq \operatorname{var}(\gamma)$ be the set of all variables x' with $\tau(x') \in \Sigma_2^+$. We can now define a morphism $\sigma' : \mathbb{N}^* \to \Sigma_1^*$ such that, for every $k \in \operatorname{var}(\gamma) \setminus A$, $\sigma'(k) = \tau(k)$ and, for every $x' \in A$, $\sigma'(x') = \varepsilon$. Consequently, due to the fact that there is no $k \in \operatorname{var}(\gamma)$ with $\tau(k) \in (\Sigma_1 \cup \Sigma_2)^* \setminus (\Sigma_1^* \cup \Sigma_2^*)$, $\sigma'(\gamma) = \sigma_{i,j}(\alpha)$, which means that $\sigma_{i,j}$ is ambiguous with respect to γ. This is a contradiction. □(Claim 2)

Claim 3. There exists an $x \in \operatorname{var}(\gamma)$ satisfying $\tau(x) \in \Sigma_2^+$.

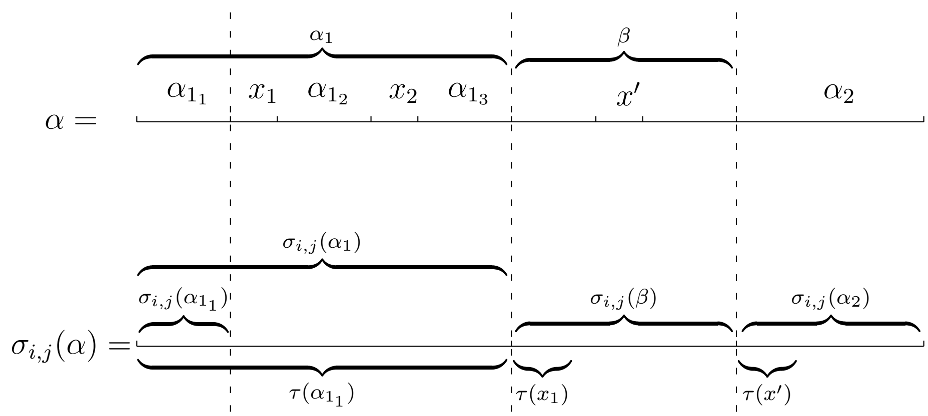
Proof of Claim 3. Assume to the contrary that, for every $x \in \operatorname{var}(\gamma)$, $\tau(x) \notin \Sigma_2^+$. Because of Claim 1, $\tau(\alpha) = \sigma_{i,j}(\alpha)$ and $\sigma_{i,j}$ being unambiguous with respect to γ and β imply that there exists a nonempty set $A \subseteq \operatorname{var}(\beta)$

such that, for every $x' \in A$, $\tau(x') \in \Sigma_1^+$. We can now define a morphism $\sigma' : \mathbb{N}^* \to \Sigma_2^*$ such that, for every $k \in \text{var}(\beta) \setminus \{x'\}$, $\sigma'(k) = \tau(k)$ and, for every $x' \in A$, $\sigma'(x') = \varepsilon$. Consequently, due to the fact that there is no $k \in \text{var}(\beta)$ with $\tau(k) \in (\Sigma_1 \cup \Sigma_2)^* \setminus (\Sigma_1^* \cup \Sigma_2^*)$, $\sigma'(\beta) = \sigma(\beta)$, which contradicts σ being unambiguous with respect to β. □(Claim 3)

Claim 4. If $|\tau(q)| \geq 2$, $q \in \text{var}(\gamma)$, and $\tau(q) \in \Sigma_1^+$, then $\sigma_{i,j}(i) \sqsubseteq \tau(q)$.

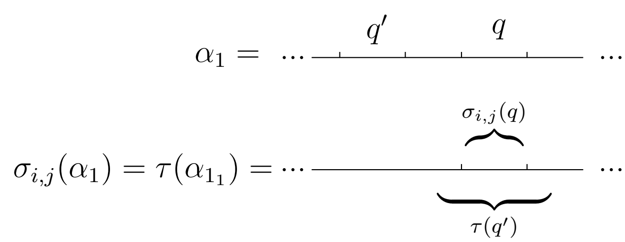
Proof of Claim 4. Assume to the contrary that $\sigma_{i,j}(i) \not\sqsubseteq \tau(q)$. Let $v_1 v_2 \sqsubseteq \tau(q)$, $v_1, v_2 \in \Sigma_1 \setminus \{\sigma_{i,j}(i)\}$. Due to $|\gamma|_q = 2$, $\tau(\alpha) = \cdots v_1 v_2 \cdot \cdots \cdot v_1 v_2 \cdot \cdots$. Since $\Sigma_1 \cap \Sigma_2 := \emptyset$ and $\tau(\alpha) = \sigma_{i,j}(\alpha)$, we can conclude that $\gamma = \cdots \cdot x_1 x_2 \cdot \cdots \cdot x_1 x_2 \cdot \cdots$, $x_1, x_2 \in \text{var}(\gamma) \setminus \{i, j\}$. Because of $|\gamma|_{x_1} = 2$ and $|\gamma|_{x_2} = 2$, Lemma 1 implies that γ is a fixed point of a nontrivial morphism, which contradicts the assumption. □(Claim 4)

According to Claims 1, 2 and 3, there exists an $x \in \text{var}(\gamma)$ such that $\tau(x) \in \Sigma_2^+$, and there exists an $x' \in \text{var}(\beta)$ with $\tau(x') \in \Sigma_1^+$. The two occurrences of x are both in α_1 or both in α_2; otherwise, there does not exist an $x' \in \text{var}(\beta)$ such that $\tau(x') \in \Sigma_1^+$. Without loss of generality, we assume that both occurrences of x are in α_1, and we also assume that x is the leftmost variable in α_1 satisfying $\tau(x) \in \Sigma_2^+$ and x' is the leftmost variable in β with $\tau(x') \in \Sigma_1^+$. Let x_1 be the first occurrence of x, and let x_2 be the second occurrence of x. So, $\alpha_1 = \alpha_{1_1} \cdot x_1 \cdot \alpha_{1_2} \cdot x_2 \cdot \alpha_{1_3}$, $\alpha_{1_1}, \alpha_{1_2}, \alpha_{1_3} \in \mathbb{N}^*$. Consequently, $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$.

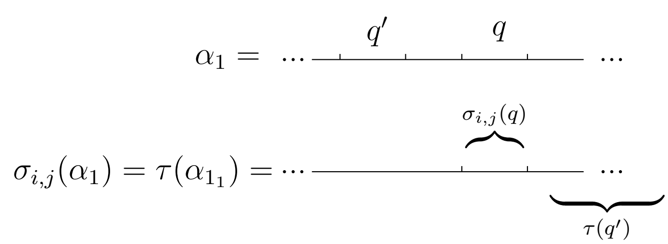


Before we proceed with our proof, we define two notations. If, for variables q, q' in α_1 (q and q' have a same position or q' occurs to the left of q in α_1)

$\sigma_{i,j}(q) \sqsubseteq \tau(q')$ and $\tau(q')$ in $\tau(\alpha_{1_1})$ is located at the position of $\sigma_{i,j}(q)$ in $\sigma_{i,j}(\alpha_1)$, then we write $\sigma_{i,j}(q) \downarrow \tau(q')$. This is illustrated by the following diagram (where we assume that the occurrence of q' is to the left of the occurrence of q):



If the position of $\tau(q')$ in $\tau(\alpha_{1_1})$ is located to the right of the position of $\sigma_{i,j}(q)$ in $\sigma_{i,j}(\alpha_1)$, then we write $\sigma_{i,j}(q) \mapsto \tau(q')$. We again give a diagram (assuming that the occurrence of q' is to the left of the occurrence of q) that illustrates the setting where we use this notation:

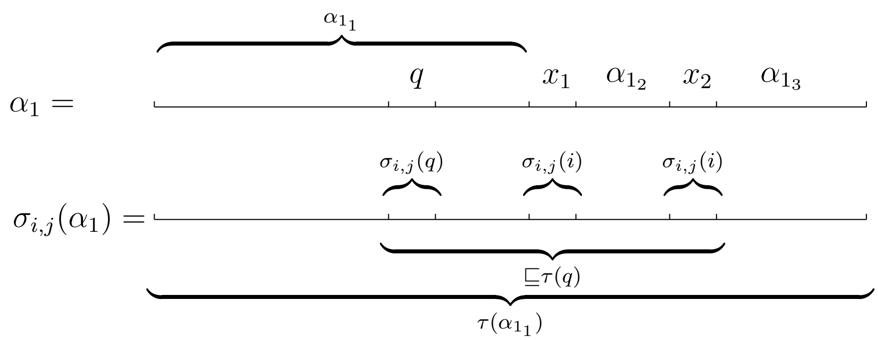


We return to our proof and recollect that $\alpha_1 = \alpha_{1_1} \cdot x_1 \cdot \alpha_{1_2} \cdot x_2 \cdot \alpha_{1_3}$, $x_1 = x_2 = x$, and $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$. This implies that we have to consider the following cases:

Case 1. $x = i$ or $x = j$

Due to $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$, one of the following cases holds true:

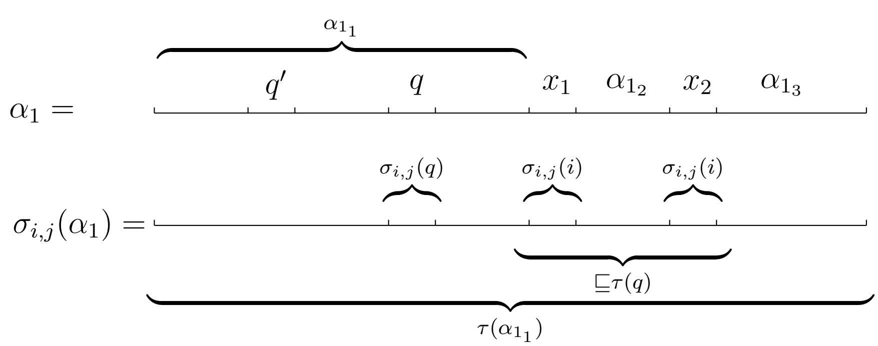
Case 1.1. There exists a variable $q \in \mathrm{var}(\alpha_{1_1})$ to the left of x_1 satisfying $|\tau(q)|_{\sigma_{i,j}(i)} \geq 2$ and $\sigma_{i,j}(i) \downarrow \tau(q)$.
Assume that $\sigma_{i,j}(q) \downarrow \tau(q)$.



Let A be a set of those variables $k \in \mathrm{var}(\gamma) \setminus \{q\}$ satisfying $\sigma_{i,j}(k) \sqsubseteq \tau(q)$. We define a morphism $\sigma' : \mathbb{N}^* \to \Sigma_1^*$ such that, for every $k' \in \mathrm{var}(\gamma)$,

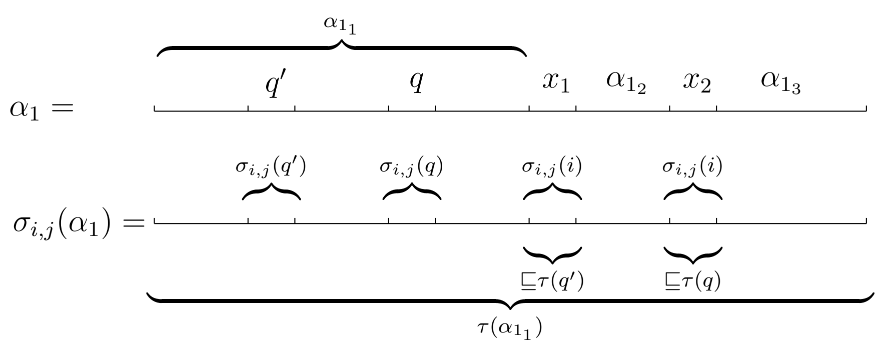
$$\sigma'(k') := \begin{cases} \varepsilon, & \text{if } k' \in A, \\ \tau(q), & \text{if } k' = q, \\ \sigma_{i,j}(k'), & \text{else.} \end{cases}$$

Due to the facts that, for all $k, k' \in \mathrm{var}(\gamma)$, $k \neq k'$, $|\gamma|_k = 2$, and if $k \neq i$ and $k' \neq j$, then $\sigma_{i,j}(k) \neq \sigma_{i,j}(k')$, it can be verified that $\sigma'(\gamma) = \sigma_{i,j}(\gamma)$, which is a contradiction to $\sigma_{i,j}$ being unambiguous with respect to γ.
If $\sigma_{i,j}(q) \mapsto \tau(q)$, then, due to $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$, there exists a variable $q' \in \mathrm{var}(\alpha_{1_1})$ to the left of q satisfying $|\tau(q')| \geq 2$.



According to Claim 4, $\sigma_{i,j}(i) \sqsubseteq \tau(q')$. Besides, $|\gamma|_{q'} = 2$. On the other hand, $|\gamma|_q = 2$ and we assume $|\tau(q)|_{\sigma_{i,j}(i)} \geq 2$ in the present case. Consequently, $|\tau(\alpha)|_{\sigma_{i,j}(i)} > 4$, which contradicts $\tau(\alpha) = \sigma_{i,j}(\alpha)$.

Case 1.2. There exist variables $q, q' \in \mathrm{var}(\alpha_{1_1})$ to the left of x_1 satisfying $\sigma_{i,j}(i) \downarrow \tau(q)$ and $\sigma_{i,j}(i) \downarrow \tau(q')$.



Therefore, due to $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$ and $\tau(x) \in \Sigma_2^+$, we can conclude that $\sigma_{i,j}(q) \mapsto \tau(q)$. If $\sigma_{i,j}(q') \downarrow \tau(q')$, then $\sigma_{i,j}(q) \downarrow \tau(q')$. This implies that $\sigma_{i,j}(q') \cdot w \cdot \sigma_{i,j}(q) \cdot w' \cdot \sigma_{i,j}(i) \sqsubseteq \tau(q')$, $w, w' \in \Sigma_1^*$. Due to $|\gamma|_{q'} = 2$, it can be verified that $\gamma = \gamma_1 \cdot q' \cdot \gamma_2 \cdot q \cdot \gamma_3 \cdot q' \cdot \gamma_2 \cdot q \cdot \gamma_4$ with $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{N}^*$ and $\sigma_{i,j}(\gamma_2) = w$. Without loss of generality, we assume that $x = i$. This implies that $q \neq i$, $q' \neq i$ and $i \notin \mathrm{var}(\gamma_2)$. Also, for every $k \in \mathrm{var}(\gamma)$, $|\gamma|_k = 2$. Consequently, $(\{q, q'\} \cup \mathrm{var}(\gamma_2)) \cap (\mathrm{var}(\gamma_1) \cup \mathrm{var}(\gamma_3) \cup \mathrm{var}(\gamma_4)) = \emptyset$. So, the structure of γ satisfies Lemma 1, which implies that γ is a fixed point of a nontrivial morphism. This is a contradiction. As a result, $\sigma_{i,j}(q') \mapsto \tau(q')$; in addition, as mentioned, $\sigma_{i,j}(q) \mapsto \tau(q)$. Therefore, and again because of $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$, there exists a variable $q'' \in \mathrm{var}(\alpha_{1_1})$ to the left of q' satisfying $|\tau(q'')| \geq 2$. According to Claim 4, $\sigma_{i,j}(i) = \sigma_{i,j}(j) \sqsubseteq \tau(q'')$. Without loss of generality, assume that $x = i$. Hence, it results from $\sigma_{i,j}(j) \sqsubseteq \tau(q'')$, $|\tau(q'')| \geq 2$ and $|\gamma|_{q''} = 2$ that there is a factor $k \cdot j \sqsubseteq \gamma$ or $j \cdot k \sqsubseteq \gamma$, $k \in \mathrm{var}(\alpha_1)$, $k \neq i$ and $k \neq j$, which occurring twice in γ. Consequently, we can assume $\gamma = \gamma_1 \cdot k \cdot j \cdot \gamma_2 \cdot k \cdot j \cdot \gamma_3$ or $\gamma = \gamma_1 \cdot j \cdot k \cdot \gamma_2 \cdot j \cdot k \cdot \gamma_3$ where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}^*$ and $k, j \notin \mathrm{var}(\gamma_1\gamma_2\gamma_3)$. According to Lemma 1, this implies that γ is a fixed point of a nontrivial morphism, which is a contradiction.

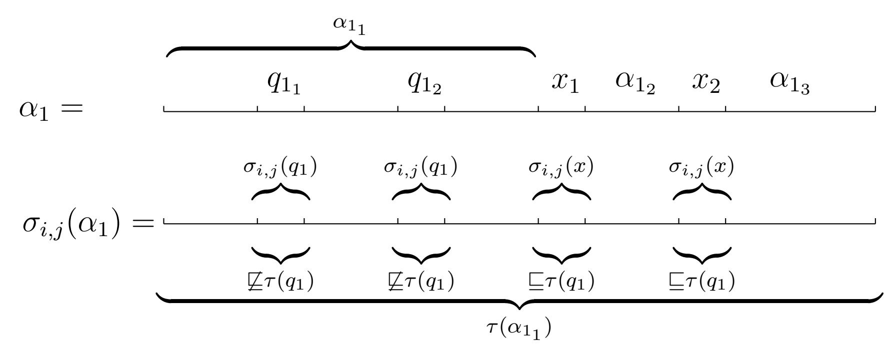
Case 2. $x \neq i$ and $x \neq j$.

Since $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$, one of the following cases holds true:

Case 2.1. There exists a variable $q \in \text{var}(\alpha_{1_1})$ to the left of x_1 satisfying $|\tau(q)|_{\sigma_{i,j}(x)} = 2$. Since $|\gamma|_q = 2$, $|\tau(\alpha)|_{\sigma_{i,j}(x)} > 2$, which contradicts $\tau(\alpha) = \sigma_{i,j}(\alpha)$.

Case 2.2. There exist variables $q, q' \in \text{var}(\alpha_{1_1})$, $q \neq q'$, to the left of x_1 satisfying $\sigma_{i,j}(x) \downarrow \tau(q)$ and $\sigma_{i,j}(x) \downarrow \tau(q')$. It results from $|\gamma|_q = 2$ and $|\gamma|'_q = 2$ that $|\tau(\alpha)|_{\sigma_{i,j}(x)} > 2$, which is a contradiction to $\tau(\alpha) = \sigma_{i,j}(\alpha)$.

Case 2.3. There exists a variable $q_1 \in \text{var}(\alpha_{1_1})$, with two occurrences named q_{1_1} and q_{1_2}, to the left of x_1 satisfying $|\tau(q_1)|_{\sigma_{i,j}(x)} = 1$, $\sigma_{i,j}(x_1) \downarrow \tau(q_{1_1})$ and $\sigma_{i,j}(x_2) \downarrow \tau(q_{1_2})$. Due to $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$, $\tau(x) \in \Sigma_2^+$ and the two occurrences of q_1 being to the left of x_1, we can conclude that $\sigma_{i,j}(q_1) \mapsto \tau(q_1)$.



We first demonstrate that the overall condition of Case 2 does not only hold for x, but also for q_1:

Claim 5. $q_1 \neq i$ and $q_1 \neq j$.

Proof of Claim 5. Assume to the contrary that $q_1 = i$ or $q_1 = j$. Without loss of generality let $q_1 := i$. Thus, $q_{1_1} = q_{1_2} = i$. On the other hand, as mentioned, $\sigma_{i,j}(q_1) \mapsto \tau(q_1)$. Thus, again because of $\tau(\alpha_{1_1}) = \sigma_{i,j}(\alpha_1)$, there exists a variable $k \in \text{var}(\alpha_{1_1})$ to the left of q_{1_1} satisfying $|\tau(k)| \geq 2$. According to Claim 4, $\sigma_{i,j}(j) \sqsubseteq \tau(k)$. This implies that due to $|\gamma|_k = 2$ there is a factor $k' \cdot j \sqsubseteq \gamma$ or $j \cdot k' \sqsubseteq \gamma$, $k' \in \text{var}(\alpha_1)$, $k' \neq i$ and $k' \neq j$, which occurs twice in γ. Consequently, we can assume $\gamma = \gamma_1 \cdot k' \cdot j \cdot \gamma_2 \cdot k' \cdot j \cdot \gamma_3$ or $\gamma = \gamma_1 \cdot j \cdot k' \cdot \gamma_2 \cdot j \cdot k' \cdot \gamma_3$, where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}^*$ and $k', j \notin \text{var}(\gamma_1\gamma_2\gamma_3)$.

According to Lemma 1, this implies that γ is a fixed point of a nontrivial morphism, which is a contradiction.
If we assume to the contrary that $q_1 = j$, then the same reasoning as above leads to a contradiction. □(Claim 5)

The following statement shall be the core argument of our reasoning on Case 2.3.

Claim 6. There exists a variable to the left of q_{1_1} *in* α_{1_1} *satisfying the condition of Case 2.3.*

Proof of Claim 6. According to Claim 5, $q_1 \neq i$ and $q_1 \neq j$. Besides, as mentioned in Case 2.3, $\sigma_{i,j}(q_1) \mapsto \tau(q_1)$. Consequently, applying Case 2 leads to the existence of a variable q_2 to the left of q_{1_1} satisfying $\sigma_{i,j}(q_1) \downarrow \tau(q_2)$. However, a same reasoning as in Cases 2.1, 2.2 (considering q_1 instead of x) leads to a contradiction. As a result, q_2 must satisfy the condition of Case 2.3. □(Claim 6)

Therefore, according to Claim 6 and Case 2.3, there exists a $q_2 \in \alpha_{1_1}$ with two occurrences named q_{2_1} and q_{2_2}, to the left of q_{1_1} with $|\tau(q_2)|_{\sigma(i,j)(q_1)} = 1$ and $\sigma_{i,j}(q_2) \mapsto \tau(q_2)$. Furthermore, due to a same reasoning as in Claim 5, $q_2 \neq i$ and $q_2 \neq j$. Hence, we can again apply Claim 6. Consequently, this reasoning finally leads to a contradiction based on Case 2.1 or 2.2 since the length of α_1 is finite, which means that, by a continued application of Claim 6, there is a $q_n \in \mathrm{var}(\alpha_{1_1})$ not satisfying Case 2.3.

Now, assume the case that $|\mathrm{var}(\beta)| > 3$ and, for every $x \in \mathrm{var}(\beta)$, $|\beta|_x = 2$. It can be verified that this case satisfies Claims 1, 2 and 3. Consequently, using an analogous reasoning as previous case leads to a contradiction again.

Hence, there is no morphism τ satisfying $\tau(\alpha) = \sigma_{i,j}(\alpha)$ and $\tau(x) \neq \sigma_{i,j}(x)$, for an $x \in \mathrm{var}(\alpha)$, and this implies that $\sigma_{i,j}$ is unambiguous with respect to α. □

In order to illustrate the above statement, we consider the following example. Let

$$\alpha := 1 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 5 \cdot 7 \cdot 8 \cdot 6 \cdot 8 \cdot 4 \cdot 2 \cdot 9 \cdot 3 \cdot 9 \cdot 2.$$

We now define

$$\begin{aligned}
\alpha_1 &:= 1 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 4, \\
\alpha_2 &:= 4 \cdot 2 \cdot 9 \cdot 3 \cdot 9 \cdot 2, \\
\beta &:= 5 \cdot 6 \cdot 7 \cdot 5 \cdot 7 \cdot 8 \cdot 6 \cdot 8,
\end{aligned}$$

which implies $\alpha = \alpha_1 \cdot \beta \cdot \alpha_2$. Referring to, e. g., Holub [8] or Reidenbach and Schneider [13], it can be effortlessly verified that both β and $\gamma = \alpha_1 \cdot \alpha_2$ are not a fixed point of a nontrivial morphism. Furthermore, β contains four different variables, and every $x \in \mathrm{var}(\beta)$ satisfies $|\beta|_x = 2$. Therefore, we can apply Theorem 8, which says that there are $i, j \in \mathrm{var}(\alpha)$ such that $\sigma_{i,j}$ is unambiguous with respect to α; from the proofs of Theorems 6, 7 and 8, we can conclude that, for example, $i := 5$ and $j := 7$ are a suitable choice for the definition of $\sigma_{i,j}$.

In the remainder of this section, we shall not directly address the morphism $\sigma_{i,j}$ any longer. Hence, we focus on Conjecture 1, and we use an approach that differs quite significantly from those above: We consider words that cannot be morphic images of a pattern under any ambiguous 1-uniform morphism, and we construct suitable morphic preimages from these words. This method yields another major set of patterns for which Conjecture 1 is satisfied.

Our corresponding technique is based on the well-known concept of *de Bruijn sequences*. Since de Bruijn sequences are cyclic, which does not fit with our subject, we introduce a non-cyclic variant:

Definition 3. *A* non-cyclic De Bruijn sequence *(of order n) is a word over a given alphabet Σ (of size k) for which all possible words of length n in Σ^* appear exactly once as factors of this sequence. We denote the set of all non-cyclic De Bruijn sequences of order n by $B'(k, n)$. A $w \in B'(k, n)$ is said to be* in canonical form *if it is lexicographically minimal (with regard to any fixed order on Σ) among all its renamings in $B'(k, n)$.*

For example, the word $w_0 := aabacbbcca$ is a non-cyclic de Bruijn sequence in $B'(3, 2)$ if we assume $\Sigma := \{a, b, c\}$. Furthermore, w_0 is in canonical form if we assume Σ to be ordered alphabetically. The introduction of a canonical form is needed at the end of this section, where we shall provide a lower bound on the number of patterns with unambiguous 1-uniform morphisms that can be derived from de Bruijn sequences.

It can now be easily understood that a non-cyclic de Bruijn sequence cannot be a morphic image of any pattern under ambiguous 1-uniform morphisms:

Theorem 9. *Let Σ be an alphabet, and let $\alpha \in \mathbb{N}^+$ be a pattern satisfying, for every $x \in \mathrm{var}(\alpha)$, $|\alpha|_x \geq 2$. Let $\sigma : \mathbb{N}^* \to \Sigma^*$ be a 1-uniform morphism such that, for every $u_1u_2 \sqsubseteq \sigma(\alpha)$, $u_1, u_2 \in \Sigma$, the factor u_1u_2 occurs in $\sigma(\alpha)$ exactly once. Then σ is unambiguous with respect to α.*

Proof. Assume to the contrary that σ is ambiguous with respect to α. Consequently, there exists a morphism $\tau : \mathbb{N}^* \to \Sigma^*$ satisfying $\tau(\alpha) = \sigma(\alpha)$ and, for some $q \in \mathrm{var}(\alpha)$, $\tau(q) \neq \sigma(q)$. Since σ is a 1-uniform morphism, there exists a $q \in \mathrm{var}(\alpha)$ satisfying $|\tau(q)| \geq 2$. Hence, let $v_1v_2 \sqsubseteq \tau(q)$, $v_1, v_2 \in \Sigma$. Due to $|\alpha|_q \geq 2$, this implies that $\tau(\alpha) = \cdots \cdot v_1v_2 \cdot \cdots \cdot v_1v_2 \cdot \cdots$. However, this contradicts the condition of the theorem stating that, for every $u_1u_2 \sqsubseteq \sigma(\alpha)$, $u_1, u_2 \in \Sigma$, the factor u_1u_2 occurs in $\sigma(\alpha)$ exactly once. So, σ is unambiguous with respect to α. □

This insight implies that if a pattern can be mapped by a 1-uniform morphism to a de Bruijn sequence and has at least two occurrences of each of its variables, then this pattern necessarily is not a fixed point. Thus, for such patterns, Conjecture 1 holds true:

Corollary 2. *Let Σ be an alphabet, and let $\alpha \in \mathbb{N}^+$ be a pattern satisfying, for every $x \in \mathrm{var}(\alpha)$, $|\alpha|_x \geq 2$. Let $\sigma : \mathbb{N}^* \to \Sigma^*$ be a 1-uniform morphism such that, for every $u_1u_2 \sqsubseteq \sigma(\alpha)$, $u_1, u_2 \in \Sigma$, the factor u_1u_2 occurs in $\sigma(\alpha)$ exactly once. Then α is not a fixed point of a nontrivial morphism.*

Proof. According to Theorem 9, σ is unambiguous with respect to α. Since σ, by definition, is nonerasing, the corollary directly follows from Theorem 1. □

We now show how we can construct patterns that satisfy the conditions of Theorem 9 and Corollary 2:

Definition 4. *Let $\Sigma := \{a_1, a_2, \ldots, a_k\}$. Let $B'(k, 2)$ be the set of non-cyclic de Bruijn sequences of order 2 over Σ. Then $\Pi_{DB}(k) \subseteq \mathbb{N}^*$ is the set of all patterns that can be constructed as follows: For every $w \in B'(k, 2)$ and every letter a_j in w, all n_j occurrences of a_j are replaced by $\lfloor n_j/2 \rfloor$ different variables from a set $N_j := \{x_{j_1}, x_{j_2}, \ldots, x_{j_{\lfloor n_j/2 \rfloor}}\} \subseteq \mathbb{N}$, such that the following conditions are satisfied:*

- *for every* $x \in N_j$, $|\alpha|_x > 1$,
- *for all* i, i', $1 \leq i, i' \leq k$, *with* $i \neq i'$, $N_i \cap N_{i'} = \emptyset$, *and*
- *for all* i, $1 \leq i \leq k$, *the variables in* N_i *are assigned to occurrences of* a_i *in a way such that the resulting pattern is in canonical form.*

For instance, with regard to our above example word $w_0 = aabacbbcca \in B'(3,2)$, Definition 4 says that, e. g., the pattern $1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \cdot 3$ is contained in $\Pi_{DB}(3)$.

From this construction, it directly follows that Conjecture 1 holds true for every pattern in $\Pi_{DB}(k)$:

Theorem 10. *Let* $\Sigma := \{a_1, a_2, \ldots, a_k\}$, $k \geq 3$. *Then, for every* $\alpha \in \Pi_{DB}(k)$,

- $\mathrm{var}(\alpha)$ *contains at least* $k+1$ *elements, and*
- *there exists a 1-uniform morphism* $\sigma : \mathbb{N}^* \to \Sigma^*$ *that is unambiguous with respect to* α.

Proof. We begin this proof with the first statement of the theorem: It is obvious that there are k^2 different words of length 2 over Σ. The shortest word that contains k^2 factors of length 2 has length k^2+1, which means that this is the length of any word $w \in B'(k,2)$. Thus, there must be at least one letter in w that has at least $\lceil (k^2+1)/k \rceil$ occurrences. Since we assume $k \geq 3$, this means that this letter has at least 4 occurrences. From Definition 4 it then follows that this letter is replaced by at least two different variables when a pattern $\alpha \in \Pi_{DB}(k)$ is generated from w. Since all other letters in w must be replaced by at least one variable, this shows that $|\mathrm{var}(\alpha)| \geq k+1$. Note that from the proof of Theorem 11 it can be derived that, more precisely, $|\mathrm{var}(\alpha)| = (k-1)\lfloor k/2 \rfloor + \lfloor (k+1)/2 \rfloor$.

Concerning the second statement, we define σ by, for every j, $1 \leq j \leq k$, and for every $x \in N_j$, $\sigma(x) := a_j$. Thus, σ is 1-uniform, and $\sigma(\alpha) \in B'(k,2)$. This implies that, for every $u_1 u_2 \sqsubseteq \sigma(\alpha)$, $u_1, u_2 \in \Sigma$, the factor $u_1 u_2$ occurs in $\sigma(\alpha)$ exactly once. Consequently, according to Theorem 9, σ is unambiguous with respect to α. $\square$

We conclude this paper with a statement on the cardinality of $\Pi_{DB}(k)$, demonstrating that the use of de Bruijn sequences indeed leads to a rich class of patterns α with unambiguous 1-uniform morphisms, and that these morphisms, in general, can even have a target alphabet of size much less than $\mathrm{var}(\alpha) - 1$ (as featured by Theorem 10):

Theorem 11. *Let $k \in \mathbb{N}$. Then $|\Pi_{DB}(k)| \geq k!^{(k-1)}$, and, for every $\alpha \in \Pi_{DB}(k)$,*

$$|\operatorname{var}(\alpha)| = (k-1)\lfloor k/2 \rfloor + \lfloor (k+1)/2 \rfloor .$$

Proof. Let $B(k,n)$ be the set of all distinct De Bruijn sequences of order n over alphabet Σ, and let $B'(k,n)$ be the set of all distinct non-cyclic De Bruijn sequences over Σ.

Claim 1. Every element of $B'(k,n)$ has length $k^n + n - 1$, and $|B'(k,n)| = k!^{k^{n-1}}$.

Proof (Claim 1). According to [2],

- every element of $B(k,n)$ has length k^n, and
- $|B(k,n)| = k!^{k^{n-1}}/k^n$.

Let $w \in B(k,n)$. Therefore, $|w| = k^n$. Assume that $w = a_1a_2[\ldots]a_m$, $m = k^n$. Since all words of length n over alphabet Σ appear exactly once in the cyclic sequence w, this implies that, for every v,

$$\begin{aligned} v \in \quad & \{a_{m-(n-2)}a_{m-(n-3)}[\ldots]a_m a_1, a_{m-(n-3)}a_{m-(n-4)}[\ldots]a_m a_1 a_2, [\ldots], \\ & a_m a_1 a_2[\ldots]a_{n-1}\}, \end{aligned}$$

$v \not\sqsubseteq w$. Consequently, by defining $w' := a_1a_2[\ldots]a_m a_1 a_2 \cdots a_{n-1}$, w' satisfies Definition 3, and as a result, $w' \in B'(k,n)$. Thus, $|w'| = |w| + (n-1)$, and this implies that, for every $w' \in B'(k,n)$,

$$|w'| = k^n + (n-1)$$

Besides, since w is a cyclic sequence, all words in

$$W := \{a_1a_2[\ldots]a_{k^n}, a_2a_3[\ldots]a_{k^n}a_1, \ldots, a_{k^n}a_1a_2, \ldots, a_{k^n-1}\}$$

are equivalent, and they are counted as one sequence of $B(k,n)$. Consequently, to find the number of distinct non-cyclic De Bruijn sequences $B'(k,n)$, it is sufficient to multiply $|W| = k^n$ to the number of distinct De Bruijn sequences $B(k,n)$. Thus,

$$|B'(k,n)| = k^n \, \frac{k!^{k^{n-1}}}{k^n} = k!^{k^{n-1}}.$$

Now, let $B''(k,n)$ be the set of non-cyclic De Bruijn sequences in canonical form of order n.

Claim 2. $|B''(k,n)| = k!^{(k^{n-1}-1)}$

Proof (Claim 2). Let $\Sigma := \{a_1, a_2, \dots, a_k\}$ and let $w \in B'(k,n)$. According to Definition 3, w is in canonical form if it is lexicographically minimal with regard to Σ, $a_1 < a_2 < \ldots < a_k$. However, by renaming w, it can be verified that there exist $k! - 1$ other sequences in $B'(k,n)$; in other words, we can consider w as a representative of $k!$ elements of $B'(k,n)$. So, it directly follows from Claim 1 that the number of non-cyclic De Bruijn sequences in canonical form of order n over Σ is

$$\frac{k!^{k^{n-1}}}{k!} = k!^{(k^{n-1}-1)}.$$

Consequently, according to Definition 4,

$$|\Pi_{DB}(k)| \geq k!^{(k-1)}.$$

We continue to prove the second part of Theorem 11 by the following claim:
Claim 3. Let $\Sigma := \{a_1, a_2, \dots, a_k\}$. Let $B''(k,2)$ be the set of non-cyclic De Bruijn sequences in canonical form of order 2 over Σ. Then, for every $w \in B''(k,2)$, $|w|_{a_1} = k+1$ and, for every j, $2 \leq j \leq k$, $|w|_{a_j} = k$.

Proof (Claim 3). Let a_i, $i \neq 1$, be an arbitrary element of Σ. According to Definition 3, for every $w \in B''(k,2)$, $a_ia_1, a_ia_2, \dots, a_ia_i, a_ia_{i+1}, \dots, a_ia_k \sqsubseteq w$. Hence, without loss of generality regarding to order of letters in Σ , we can assume one of the following cases to be satisfied:

- $w = w_1a_ia_1 \cdot w_2a_ia_2 \cdot [\ldots] \cdot w_ia_ia_i \cdot w_{i+1}a_ia_{i+1} \cdot [\ldots] \cdot w_ka_ia_k \cdot w_{k+1}$, or
- $w = w_1a_ia_1 \cdot w_2a_ia_2 \cdot [\ldots] \cdot w_ia_ia_ia_{i+1} \cdot w_{i+1}a_ia_{i+2} \cdot [\ldots] \cdot w_{k-1}a_ia_k \cdot w_k$,

where, for every j, $1 \leq j \leq k+1$, $w_j \in \Sigma^*$ and $a_i \not\sqsubseteq w_j$. Since $i \neq 1$ and w is in canonical form, then $w_1 \neq \varepsilon$.
In the first case, a_i occurs $k+1$ times. Since $w_1 \neq \varepsilon$ and every word of length 2 over Σ appears exactly once in w, $|L_{a_i}| = k+1$, $\varepsilon \notin L_{a_i}$. Consequently, we can conclude that there exist a sequence ua_i, $u \in \Sigma$, occurring more than once in w. This contradict the fact that $w \in B''(k,2)$. Thus, in accordance with the second case, $|w|_{a_i} = k$. As a result, for every j, $2 \leq j \leq k$, $|w|_{a_j} = k$. Hence, for every $w \in B''(k,2)$, $|w| - |w|_{a_1} = (k-1)k$. On the other hand, Claim 1 implies that, for every $w \in B''(k,2)$, $|w| = k^2 + 1$. This means that

$$|w|_{a_1} = (k^2+1) - ((k-1)k) = k+1.$$

Consequently, according to Definition 4, for every $\alpha \in \Pi_{DB}(k)$, $k+1$ occurrences of a_1 are replaced by $\lfloor (k+1)/2 \rfloor$ different variables from N_1 and, for every j, $2 \leq j \leq k$, k occurrences of a_j are replaced by $\lfloor k/2 \rfloor$ different variables from N_j. Therefore,

$$|\operatorname{var}(\alpha)| = (k-1)\lfloor k/2 \rfloor + \lfloor (k+1)/2 \rfloor \,,$$

and this proves the theorem. □

5. Conclusions

In the present paper we have investigated the question of whether, for a given pattern in $\mathbb{N}^*$, there exists an unambiguous 1-uniform morphism $\sigma : \mathbb{N}^* \to \Sigma^*$. To this end, we have considered two different settings: in Section 3 we have assumed Σ to be *fixed*, i. e., $|\Sigma|$ does not depend on the number of variables in the pattern, and in Section 4 we have allowed Σ to be *arbitrarily chosen*, subject to the number of different variables in the pattern α in question (provided that $|\Sigma| < |\operatorname{var}(\alpha)|$). Our results in Section 3 have revealed that, for fixed alphabets Σ, the task of characterising those patterns that have unambiguous 1-uniform morphisms might be quite involved, as the sets of these patterns differ for $|\Sigma| = 2$, $|\Sigma| = 3$ and $|\Sigma| = 4$. With regard to variable alphabets Σ, we have given two equivalent conjectures in Section 4, which say that such morphisms exist if and only if the pattern is not a fixed point of a nontrivial morphism. Our corresponding results have established major sets of patterns for which these conjectures hold true.

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