Automorphisms Generating Disjoint Hamilton Cycles in Star Graphs

by

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Parisa Derakhshan

6th May 2014

تقدیم با بوسه بر دستان پدرم.

I dedicate this thesis to my father for his constant support and unconditional love. I hope you are proud of me.

Abstract

In the first part of the thesis we define an automorphism ϕ_n for each star graph St_n of degree $n-1$, which yields permutations of labels for the edges of St_n taken from the set of integers $\{1, \ldots, \lfloor n/2 \rfloor\}$. By decomposing these permutations into permutation cycles, we are able to identify edge-disjoint Hamilton cycles that are automorphic images of a known two-labelled Hamilton cycle $H_{1\ 2}(n)$ in St_n. Our main result is an improvement from the existing lower bound of $\lfloor \varphi(n)/10 \rfloor$ to $\lfloor 2\varphi(n)/9 \rfloor$, where φ is Euler's totient function, for the known number of edge-disjoint Hamilton cycles in St_n for all odd integers n. For prime n, the improvement is from $\lfloor n/8 \rfloor$ to $\lfloor n/5 \rfloor$. We extend this result to the cases when n is the power of a prime other than 3 and 7.

The second part of the thesis studies 'symmetric' collections of edge-disjoint Hamilton cycles in St_n, i.e. collections that comprise images of $H_{1\ 2}(n)$ under general label-mapping automorphisms. We show that, for all even n, there exists a symmetric collection of $\lfloor \varphi(n)/2 \rfloor$ edge-disjoint Hamilton cycles, and St_n cannot have symmetric collections of greater than $\lfloor \varphi(n)/2 \rfloor$ such cycles for any n. Thus, St_n is not symmetrically Hamilton decomposable if n is not prime. We also give cases of even n, in terms of Carmichael's reduced totient function λ, for which 'strongly' symmetric collections of edge-disjoint Hamilton cycles, which are generated from $H_{1\ 2}(n)$ by a single automorphism, can and cannot attain the optimum bound $\lfloor \varphi(n)/2 \rfloor$ for symmetric collections. In particular, we show that if n is a power of 2, then St_n has a spanning subgraph with more than half of the edges of St_n, which is strongly symmetrically Hamilton decomposable. For odd n, it remains an open problem as to whether the $\lfloor \varphi(n)/2 \rfloor$ can be achieved for symmetric collections, but we are able to show that, for certain odd n, a $\varphi(n)/4$ bound is achievable and optimal for strongly symmetric collections.

The search for edge-disjoint Hamilton cycles in star graphs is important for the design of interconnection network topologies in computer science. All our results improve on the known bounds for numbers of any kind of edge-disjoint Hamilton cycles in star graphs.

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Contents

Chapter 1

Introduction

The interconnection network is an essential element of designed multiprocessor systems and it is critical to determine its performance. An interconnection network is a system formed by nodes and links among the nodes. There are two classes of interconnection network topologies, static and dynamic. Networks with static linking are used where the communications among nodes are known or can be estimated. Otherwise, a dynamic topology is used if there are changes of connections among processors. An interconnection network of a multiprocessor architecture can be represented by an undirected graph where vertices of the graph represent computing nodes and edges of the graph represent communication links between the nodes. Hypercubes, complete binary trees, butterflies and torus are examples of interconnection network topologies. Much research has been carried out to optimize the network in various approaches by designing new network topologies.

1.1 Star graph and hypercube

The n-dimensional star graph St_n is a graph whose vertex set is the set of all permutations on $\{1, 2, ..., n\}$. Two vertices, $u_1 \dots u_i \dots u_n$ and $v_1 \dots v_i \dots v_n$, are adjacent if $u_1 = v_i$, $v_1 = u_i$, and $u_j = v_j$ for $j \in \{1, 2, ..., n\} - \{1, i\}$. The star graph has been proposed [1] as an alternative interconnection network topology to the hypercube, because of its ability to connect a greater number of nodes with lower degree. This allows a reduction in the number of interconnections and therefore cost, whilst maintaining high connectivity and fault tolerance. An n-dimensional hypercube Q_n or n-cube consists of 2^n nodes and $n \times 2^{n-1}$ edges. The hypercube is a bipartite graph with vertex set consisting of all binary vectors of length n, and with edges between two vertices whenever they differ in exactly one coordinate. The degree and diameter of Q_n are n. The attractive properties of a hypercube are node and edge symmetry, a simple recursive structure, and an efficient embedding into other interconnection networks such as ring, tree, pyramid, and mesh networks. However, as the dimension of a hypercube increases, the degree of the hypercube also increases. Relative to the degree, the hypercube has a rather large diameter and average distance between nodes. Akers et al. [1] introduced the star graph as an attractive alternative to the hypercube. An n-dimensional star graph St_n consists of $n!$ nodes and $n!(n-1)/2$ edges. St_n has node and edge symmetry, a smaller degree and diameter than the hypercube. In [1], Akers, Harel, and Krishnamurthy have shown that the diameter of the n-star graph is $\lfloor 3/2(n-1) \rfloor$.

1.2 (n, k)-star graph

In spite of all the advantages of an n-star over the hypercube, a major drawback is its lack of scalability. There exists a large gap between $n!$ and $(n+1)!$ for expanding an St_n to an St_{n+1}. To relax the restriction of the numbers of vertices $n!$ in an St_n, a generalized version of the star graph, the (n, k)-star graph, was proposed in 1995 [15].

Definition 1.1. *The (n, k)-star graph, denoted by $St_{n,k}$, is an undirected graph with vertex-set $P(n, k) = \{p_1p_2 \dots p_k : p_i \in J_n, p_i \neq p_j, 1 \leq i \neq j \leq k\}$ where $J_n = \{1, 2, \dots, n\}$ and $P(n, k)$ be the set of k-permutations (permutatuons of k elements) on J_n for $1 \leq k \leq n-1$. The adjacency is defined as follows: a vertex $p_1p_2 \dots p_i \dots p_k$ is adjacent to a vertex*

(i) $p_ip_2 \dots p_{i-1}p_1p_{i+1} \dots p_k$, where $2 \leq i \leq k$ (swap p_1 with p_i)

(ii) $xp_2 \dots p_k$ where $x \in J_n - \{p_i : 1 \leq i \leq k\}$ (replace p_1 by x)

In consequence, $St_{n,k}$ has $n!/(n-k)!$ nodes and $((n-1)/2) \times (n!/(n-k)!)$ edges. A $St_{n,k}$ graph preserves many attractive properties of a St_n graph, such as vertex symmetry, hierarchical structure, maximal fault tolerance, and simple shortest routing. Because of good topological properties of $St_{n,k}$, properties such as diameter and connectivity [40, 15], independent number and dominating number [14] and so on have been researched. The work [16] examines various topological properties of the (n, k)-star graph. It is shown that two different types of edges in the (n, k)-star graph prevent it from being edge-symmetric, but edges in each class are essentially symmetric with respect to each other. Also, the diameter and the exact average distance of the (n, k)-star graph are derived.

Proposition 1.2 (Chiang et al. [16])**.** *The diameter $D(St_{n,k})$ of the (n, k)-star graph is:*

$$D(St_{n,k}) = \begin{cases} 2k-1 & \textit{if } 1 \le k \le \lfloor n/2 \rfloor \\ k + \lfloor (n-1)/2 \rfloor & \textit{if } \lfloor n/2 \rfloor + 1 \le k < n \end{cases}$$

Moreover, It is proved that (n, k)-star graph is a Hamilton graph.

Definition 1.3. *A Hamilton cycle in a graph is a cycle that includes all the vertices of the graph exactly once.*

If a graph has a Hamilton cycle, we call such graph a 'Hamilton' graph. It is well known that a star graph is a Hamilton graph. Hsu et al. proved that $St_{n,k}$ is also a Hamilton graph for $n \ge 3$:

Proposition 1.4 (Hsu et al. [23])**.** *$St_{n,k}$ with $n \ge 3$, is a Hamilton graph.*

This proposition directly follows from the structure of the Hamilton cycle and the distance between vertices of a (n, k)-star graph. In [23], the authors also consider the fault Hamiltonicity, and the fault Hamilton connectivity of the (n, k)-star graph $St_{n,k}$. Other important properties of the (n, k)-star graph, are given in [23].

1.3 Edge-disjoint Hamilton cycles of interconnection networks

The presence of edge-disjoint Hamilton cycles is a desirable feature for an interconnection network topology. The reason for this is that in multiport systems, where nodes communicate with neighbours in unit time, messages can be broken down into small units and sent along disjoint Hamilton cycles to improve performance. The Hamilton decomposition of a k-regular graph G is the partitioning of its edge set into Hamiltonian cycles, i.e. if k is even, the edge set can be partitioned into $k/2$ Hamiltonian cycles, and if k is odd, the edge set can be partitioned into $(k-1)/2$ Hamiltonian cycles and a perfect matching. Several results concerning the existence of disjoint Hamilton cycles on graphs, in particular hypercubes, are known. One of the most interesting properties of the hypercube is that it is Hamilton decomposable [35]. It is known that there are $\lfloor n/2 \rfloor$ disjoint Hamiltonian cycles on a hypercube of dimension n:

Theorem 1.5 (Alspach et al. [5])**.** *The binary n-cube with even n, or equivalently the product of $n/2$ cycles, $C_4 \times C_4 \times \ldots \times C_4$, can be partitioned into $n/2$ Hamiltonian cycles.*

Note that $C_4 \times C_4 \times \ldots \times C_4$ is the Cartesian product of cycles of length 4 which we define in the next section. The proof of this result, however, does not lead to any simple algorithm to construct the disjoint Hamilton cycles. In [37], Song presents ideas towards a simple and interesting method to this problem. He first decomposes the hypercube into cycles of length 16, C_{16}, and then applies a merge operator to join the C_{16} cycles into larger Hamilton cycles. The case of dimension $n = 6$ (a 64-node hypercube) is illustrated. He conjectures the method can be generalized for any even n. In [35], the authors generalize the first phase of that method, decomposition of the hypercube into C_{16}, for any even n and prove its correctness. Also they show a merge operator for the case of $n = 8$ (a 256-node hypercube). This result can be viewed as a step toward the general merge operator, thus proving the conjecture.

Whilst the hypercube is Hamilton decomposable, much less is known about Hamilton cycles in the star graph. Edge-disjoint Hamilton cycles have been studied in various graph topologies. In [6, 36], multiple disjoint Hamilton cycles are constructed in various tori and in deBruijn networks. Micheneau [34] studies disjoint Hamilton cycles in recursive circulant graphs. Hamilton decompositions have been found by many authors in bipartite graphs. These include extended results such as [26, 27] where the authors generalise to bipartite hypergraphs and prove the Hamilton decomposability of complete bipartite hypergraphs. The work [20] examines the Hamilton decomposition of random bipartite regular graphs by proving equivalence of two probabilistic models of 4-regular bipartite graphs. For the recently introduced locally twisted cube [41], the existence of a Hamilton cycle is shown in [42]. In [22], the authors have investigated the edge-fault tolerant Hamiltonicity of an n-dimensional locally twisted cube, and in [24] two edge-disjoint Hamilton cycles have been constructed for the locally twisted cube.

There have also been a handful of results for star graphs concerning Hamilton cycles and paths. Most work has studied the existence of Hamilton paths with certain properties, notably the Hamilton laceability of star graphs in [21] and the mutually independent Hamilton laceability in [30].

Definition 1.6. *Suppose G is a bipartite graph with two partite sets of equal size. G is said to be strongly Hamilton-laceable if there is a Hamilton path between every two vertices that belong to different partite sets, and there is a path of (maximal) length $N - 2$ between every two vertices that belong to the same partite set, where N is the order of G.*

The star graph is known to be bipartite. The work [21], shows that the n-dimensional star graph, where $n \geq 4$, is strongly Hamilton-laceable:

Theorem 1.7 (Hsieh et al. [21]). *The star graph St_n, with $n \geq 4$ is strongly Hamilton-laceable.*

Two Hamilton paths $P_1 = \{u_1, u_2, \ldots, u_{n(G)}\}$ and $P_2 = \{v_1, v_2, \ldots, v_{n(G)}\}$ of G, with $n(G)$ nodes, from u to v are 'independent' if $u = u_1 = v_1, v = v_{n(G)} = u_{n(G)}$, and $v_i \neq u_i$ for every $1 < i < n(G)$. A set of Hamilton paths, $\{P_1, P_2, ..., P_k\}$, of G from u to v are 'mutually' independent if any two different Hamilton paths are independent.

Definition 1.8. *A bipartite graph is k -mutually independent Hamilton laceable if there exists k-mutually independent Hamilton paths between any two nodes from distinct partite sets.*

There is an interesting result in [30] with regard to the mutually independent Hamilton-laceability of star graphs. We state this result below where $IHP_L(G)$ denotes the maximum integer k such that G is k-mutually independent Hamilton-laceable.

Theorem 1.9 (Lin et al. [30])**.** *Let St_n denote the n-dimensional star graph. Then $IHP_L(St_2) = 1$, $IHP_L(St_3) = 0$, and $IHP_L(St_n) = n - 2$ if $n \geq 4$.*

1.4 Cartesian product of star graphs

As a method for combining desirable properties of component networks, the Cartesian product of interconnection networks has been investigated recently. The Cartesian product method is a very effective method of building larger networks from several specified small-scale networks. Many popular networks can be constructed by the Cartesian product.

Definition 1.10. *Given any two undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where V_1 and V_2 are the sets of vertices, and E_1 and E_2 are the sets of edges, the Cartesian product of G_1 and G_2 is an undirected graph $G_1 \otimes G_2 = (V, E)$, where*

- $V = \{\langle x_1, x_2\rangle \mid x_1 \in V_1 \text{ and } x_2 \in V_2\}$
- $E = \{(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) \mid \text{ if } (x_1, y_1) \in E_1 \text{ and } x_2 = y_2 \text{ or } (x_2, y_2) \in E_2 \text{ and } x_1 = y_1\}$

Here, G_1 and G_2 are referred to as the 'factor' graphs and $G_1 \otimes G_2$ is referred to as the 'product' graph. The hyperstar is an undirected graph constructed by repeatedly applying Definition 1.10 on a set of star graphs. Properties of product graphs are obtained from those of the factor graphs. Such properties can include symmetry, recursive structure, attractive topological metrics (size, degree, diameter, etc.), optimal routing, and optimal broadcasting. These features make the Cartesian product of star graphs interesting and suitable for designing practical algorithms. The work [3] investigates topological properties of the hyperstar, and compares the hyperstar with the hypercube and the star graph. The authors also show that the hyperstar uses a smaller number of links for fixed minimum size requirements and scales better than the star graph and the hypercube. They also discuss that the broadcasting cost incurred by the hyperstar is lower than the corresponding cost of the star graph for some graph size choices and is lower than the corresponding cost of the hypercube for all graph sizes. So, they propose the hyperstar as an improvement over the star graph and the hypercube.

1.5 Star graph and Cayley graph

A Cayley graph is a graph defined from a pair (G, S) where G is a group and S is a set of group elements.

Definition 1.11. *Let G be a group, and let $S \subset G$ be a set of group elements such that $S^{-1} = S$. The Cayley graph (G, S) is a graph in which the vertices are the elements of G and there is an edge between g and gx for all $g \in G$ and $x \in S$.*

The Cayley graph may depend on the choice of a generating set, and is connected if and only if S generates G. It is known that any connected Cayley graph on an abelian group is a Hamilton graph. Alspach [4] conjectured that every $2k$-regular connected Cayley graph on a finite abelian group has a decomposition into k edge-disjoint Hamilton cycles. Bermond, Favaron, and Maheo prove this conjecture for a special case, where G is a Cayley graph of degree 4:

Theorem 1.12 (Bermond et al. [7])**.** *Every 4-regular connected Cayley graph on a finite abelian group can be decomposed into two hamiltonian cycles.*

Liu, then, investigated some other cases for Alspach's conjecture in [31]. In the following theorems, Liu proves that the Cayley graph (G, S) can be decomposed into Hamilton cycles provided that (G, S) is $2m$-regular, G is an abelian group and $S = \{s_1, s_2, ..., s_k\}$ is either a generating set of G such that $gcd(ord(s_i), ord(s_j)) = 1$ for $i \neq j$ or a minimal generating set of G with $k = 3$ and with either two elements of order 2 or one element of prime order:

Theorem 1.13 (Liu [31])**.** *Let G be a finite abelian group and $S = \{s_1, s_2, ..., s_k\}$ be a generating set of G with $gcd(ord(s_i), ord(s_j)) = 1$ for $i \neq j$. If Cayley graph (G, S) is $2m$-regular, then it can be decomposed into m Hamilton cycles.*

Theorem 1.14 (Liu [31])**.** *Let G be finite abelian group and $S = \{s_1, s_2, s_3\}$ be a minimal generating set of G with either two elements of order 2 or one element* *of prime order. If Cayley graph (G, S) is 2m-regular, then it has a Hamilton decomposition.*

He also proves that, the conjecture is true for (G, S), where G is an abelian group of odd order and $S = \{s_1, s_2, s_3\}$ is a minimal generating set of G.

Theorem 1.15 (Liu [31]). *Let G be a finite abelian group of odd order and $S = \{s_1, s_2, s_3\}$ be a minimal generating set of G. Then the Cayley graph (G, S) has a Hamilton decomposition.*

There are other examples of work on Hamilton decompositions of Cayley graphs such as [39, 32, 33, 38], all concerning Cayley graphs over abelian groups.

The star graph St_n is the Cayley graph (S_n, X_0) where S_n is the symmetric group of permutations of order n, and $X_0 = \{(0, 1), (0, 2), \ldots, (0, n-1)\} \subset S_n$. As it is known that a Cayley graph over a symmetric group and any generating set of transpositions has a Hamiltonian cycle [29], star graphs of any degree have a hamiltonian cycle.

1.6 Star graph automorphisms and disjoint Hamilton cycles

To date, the only results on edge-disjoint Hamilton cycles in star graphs are the Hamilton decomposition of the star graph of degree 4, denoted here by St_5, in [25] and lower bounds for the number of edge-disjoint Hamilton cycles in star graphs of degree $n-1$, denoted here by St_n, given in [11]. In [25], a Hamilton cycle for St_5 is constructed by partitioning the vertices of St_5 into 6 pairwise disjoint cycles $C_1, ..., C_6$, and then producing a 7th cycle C_7 that meets each of the other cycles at exactly two vertices and a common edge. Then, the authors define an automorphism for the graph St_5, denoted Φ_5 and prove that the Hamilton cycle defined by means of $C_1, ..., C_6, C_7$, produces a Hamilton cycle when mapped by the automorphism Φ_5. By showing that the 2 Hamilton cycles are edge-disjoint, they prove there exist two edge-disjoint Hamilton cycles for St_5. As 5-star is of degree 4, this gives a Hamilton decomposition of St_5.

In [11], Cada, Kaiser, Rosenfeld, and Ryjacek continue the study on Hamilton cycles of star graphs and give new lower bounds for the number of edge-disjoint Hamilton cycles for St_n for general n.

Before stating the results of [11], we need to give some definitions. For $i \in \{1, \ldots, n\}$, let $C_i(n)$ be the set of all edges $\sigma\tau$ of St_n such that $\sigma(0) - \tau(0)$ is congruent to $\pm i$ modulo n and let $C_{1\ 2}(n)$ be the spanning subgraph of St_n with edge set $C_1(n) \cup C_2(n)$. Using the concept of path graph and doubly adjacent

Gray codes, the important outcome of [11] is the following lemma that we shall use in this thesis:

Lemma 1.16 (Cada et al. [11]). *For $n \geq 5$, the graph $C_{1\ 2}(n)$ is Hamiltonian.*

This paper then defines a permutation ψ on V, the set of vertices of St_n, by $\psi(i) = ij \bmod n$, where $j \in U \subset \{1, \ldots, n-1\}$, and U is the set of elements relatively prime to n, and then, introduces an automorphism of St_n, $\pi \mapsto \psi o \pi$, which carries each edge set $C_i(n)$ to $C_{i\ j}(n) = C_i(n) \cup C_j(n)$. From an application of Lemma 1.16, the authors derive that, $C_j(n) \cup C_{2j}(n)$ is a Hamilton subgraph of St_n for $j \in U$. The main result of [11] on edge-disjoint Hamilton cycles of star graphs shows that St_n contains $\lfloor n/8 \rfloor$ pairwise edge-disjoint Hamilton cycles when n is prime, and $\Omega(n/loglogn)$ such cycles for arbitrary n:

Theorem 1.17 (Cada et al. [11]). *(i) If n is a prime, then St_n contains $\lfloor n/8 \rfloor$ pairwise edge-disjoint Hamilton cycles.*

(ii) For arbitrary n, there are $\Omega(n/loglogn)$ pairwise disjoint Hamilton cycles in St_n.

1.7 Contribution

In this thesis, by defining automorphisms which produce edges with different labels incident at each vertex in the image of a known Hamilton cycle, the lower bounds of the number of edge-disjoint Hamilton cycles of star graphs are improved, and bounds are obtained for edge-disjoint Hamilton cycles with certain symmetric properties.

In Chapter 2, we introduce the basic definitions and notations that we shall use in this thesis. We calculate new lower bounds for the number of edge-disjoint Hamilton cycles in star graphs of odd dimension by defining an automorphism which produces edges with different labels. As the automorphism in that chapter is not defined for even integers and requires n to be odd, we distinguish the study of edge-disjoint Hamilton cycles in star graphs of odd dimension from the study of edge-disjoint Hamilton cycles in star graphs of even dimension. We also give improved results for the number of edge-disjoint Hamilton cycles of St_n for prime n in that chapter. A version of Chapter 2 of this thesis has been published in the International Journal of Computer Mathematics [19].

In Chapter 3, we define 'symmetric' and 'strongly' symmetric collections of edge-disjoint Hamilton cycles of star graphs and we produce optimal symmetric collections of disjoint Hamilton cycles for star graphs of even dimension.

We define directed versions of labels, distance maps, star graphs and label automorphisms, and considers the existence of strongly symmetric collections of disjoint Hamilton cycles for star graphs of even dimension in Chapter 4.

In Chapter 5, we study strongly symmetric collections of disjoint Hamilton cycles for star graphs of odd dimension.

Finally, we summarise the main results of the thesis in Chapter 6.

Chapter 2

Disjoint Hamilton cycles in odd dimensions

In the present chapter, we address the number of edge-disjoint Hamilton cycles in star graphs of odd dimension. In both [25] and [11], edge-disjoint Hamilton cycles in star graphs are obtained by labelling the edges of the graph in a certain way, and then defining automorphisms which produce edges with different labels incident at each vertex in the image of a known Hamilton cycle. If the vertices of St_n are permutations of symbols $a_1, \ldots, a_n$, and edges correspond to swapping the symbol a_i in the first position with some other symbol a_j, then the label of an edge (which we call the 'length' of the edge) is the distance between a_i and a_j on the cyclic graph whose vertices are $a_1, \ldots, a_n$ in which a_n is adjacent to a_{n-1} and a_1. In [11], it is shown that St_n has a Hamilton cycle whose edges are of length 1 or 2. There, other Hamilton cycles are produced by a set of automorphisms ψ_j, one for each integer j that is coprime to n, defined by:

$$\psi_j(a_i) = a_k,\ \textit{where}\ k = ij\ \textit{mod}\ n$$

The automorphism ψ_j produces a Hamilton cycle whose edges are of length j or $2j$ (modulo n). Some of these edges clash for different j's, and the calculation in [11] produces $\lfloor \varphi(n)/10 \rfloor$ (where φ is the Euler function) and $\lfloor n/8 \rfloor$ lower bounds for the number of edge-disjoint Hamilton cycles for general n and prime n respectively. In this chapter, we give a single automorphism ϕ_n for St_n, which is applied repeatedly to generate new edge lengths. By examining how ϕ_n permutes lengths, we obtain better organized clashes and can offer improved bounds on the numbers of edge-disjoint Hamilton cycles.

This Chapter is structured as follows. In Section 2.1, we give the definitions of edge length and our automorphism ϕ_n, and some basic results. In Section 2.2, we show how ϕ_n permutes the distances between the a_is and a_js exchanged at edges, and hence the lengths of the edges, in its image of the edges. We decompose these distance permutations into permutation cycles which we call ‘constituents’. Constituents play a central role in our study, and we give their properties in Section 2.2. In Section 2.3, we show how edge-disjoint Hamilton cycles are obtained from constituents. We prove that either all or none of the integers in constituents are coprime to n in Section 2.4. This result is used in Section 2.5, to calculate new lower bounds for the number of edge-disjoint Hamilton cycles in St_n of $\lfloor 2\varphi(n)/9 \rfloor$ for odd integers and $\lfloor n/5 \rfloor$ for prime numbers. In Section 2.6, we obtain $\lfloor n/4 \rfloor$ bounds for special cases of primes.

2.1 Basic definitions and results

Throughout the sections, we assume that all graphs are undirected and n is odd and greater than 2. The automorphism that we will define requires n to be odd. The problem for even n is discussed in later chapters.

Definition 2.1. *Let n be odd or even. The n-star graph St_n is the simple (n-1)-regular graph of order $|S_n|$ with a set V of vertices and a set E of edges, where S_n is the symmetric group of permutations of order n, given by:*

$$\begin{aligned} V(St_n) &= \{a_{\rho(1)} \cdots a_{\rho(n)} \mid \rho \in S_n\}, \\ E(St_n) &= \{e \mid e = \{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, \\ &\qquad a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\},\ \rho \in S_n\} \end{aligned}$$

The 4-star graph is shown in Figure 1.

Definition 2.2 ([25])**.** *We define the distance between two elements to be:*

$$\delta(a_i, a_j) = min\{|i-j|, n-|i-j|\}, \ \ (1 \le i, j \le n).$$

If $e = (a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)})$, the length of the edge e, denoted $\lambda(e)$, is defined to be $\delta(a_{\rho(1)}, a_{\rho(i)})$.

A *Hamilton* cycle is a cycle in a graph G which visits each vertex exactly once and also returns to the starting vertex. The graph G is a *Hamilton graph* if it has at least one Hamilton cycle.

Definition 2.3. *A Hamilton cycle in a graph G with a set of $n!$ vertices V and a set of edges E is a pair of sequences $(\overline{v}, \overline{e})$ of vertices $\overline{v} = v_1 ... v_{n!+1}$ and edges $\overline{e} = e_1 ... e_{n!}$ such that:*

(i) $e_i = (v_i, v_{i+1}) \in E \ \ (1 \le i \le n!)$,

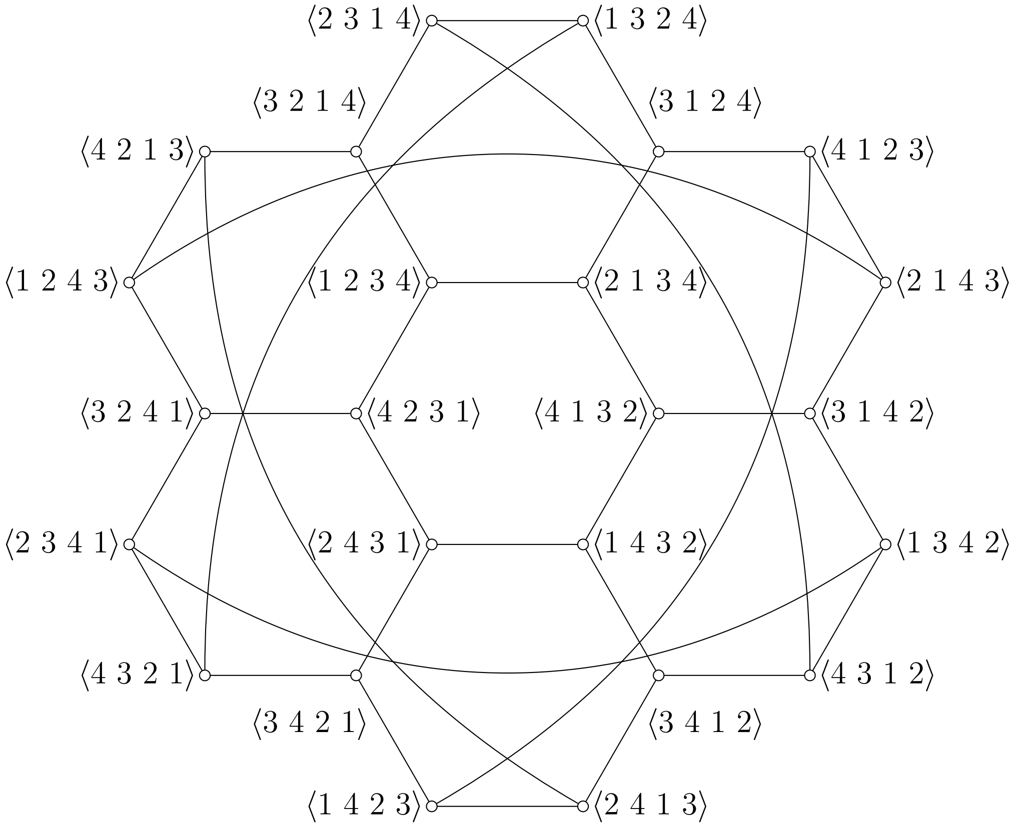

Figure 2.1: The 4-star graph.

(ii) $\{v_1, \ldots, v_{n!+1}\} = V$,

(iii) $v_1 = v_{n!+1}$.

Definition 2.4. *Let $G = (V, E)$ be a graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. Then, a mapping $\Phi : V \mapsto V$ is an automorphism iff:*

(i) Φ *is bijective.*

(ii) for all $v_1, v_2 \in V, (v_1, v_2) \in E$ if and only if $(\Phi(v_1), \Phi(v_2)) \in E$.

Lemma 2.5 ([25])**.** *Let $\phi : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ be a bijection. Then:*

(i) $\Phi : V(St_n) \mapsto V(St_n)$, given by $\Phi(a_{\rho(1)} \ldots a_{\rho(n)}) = \phi(a_{\rho(1)}) \ldots \phi(a_{\rho(n)})$, is an automorphism of the graph St_n,

(ii) if $\overline{v} = v_1, \ldots, v_{n!+1}$, $\overline{e} = (v_1, v_2) \ldots (v_{n!}, v_{n!+1})$and $(\overline{v}, \overline{e})$ is a Hamilton cycle in St_n, then the pair of sequences of vertices and edges $\Phi_H(\overline{v}, \overline{e})$ defined by

$$\Phi_H(\overline{v}, \overline{e}) = (\Phi(v_1) \ldots \Phi(v_{n!+1}), (\Phi(v_1), \Phi(v_2))...(\Phi(v_{n!}), \Phi(v_{n!+1})))$$

is also a Hamilton cycle,

(iii) if a spanning subgraph G of St_n is a Hamilton graph, then so is the spanning subgraph that is its image $\Phi(G)$.

Definition 2.6. *For $i \in \{1, ..., n\}$, we define the automorphism Φ_n to correspond to the bijection ϕ_n given by:*

$$\phi_n(a_i) = \begin{cases} a_{i/2}, & i \text{ even}, \\ a_{(n+i)/2}, & i \text{ odd}, \end{cases}$$

where $1 \leq i \leq n$.

Lemma 2.7. *ϕ_n is well-defined and bijective.*

Proof. First we prove that ϕ_n is well-defined. Let $a_i = a_j$ where $i, j \in \{1, \ldots, n\}$. Then, by Definition 2.1, $i = j$. So, $i/2 = j/2$, and $(n + i)/2 = (n + j)/2$. Thus, $a_{i/2} = a_{j/2}$, and $a_{(n+i)/2} = a_{(n+j)/2}$, and since $i \leq n$, $(n + i)/2 \leq n$. As a result, $\phi_n(a_i) = \phi_n(a_j)$.

To prove that a function is bijective, we need to show that, it is injective and surjective. Since ϕ_n has domain and codomain of the same cardinality, we just need to prove that it is injective.

Let $\phi_n(a_i) = \phi_n(a_j)$. If i is even and j is odd, then $i/2 = (n + j)/2$, and if j is even and i is odd, then $j/2 = (n + i)/2$. Therefore, $i > n$ or $j > n$ which is a contradiction with $i, j \in \{1, \ldots, n\}$. So, let $a_{i/2} = a_{j/2}$ where i, j are both even, or $a_{(n+i)/2} = a_{(n+j)/2}$ where i, j are both odd. If $a_{i/2} = a_{j/2}$, then, by Definition 2.1, $i/2 = j/2$. So, $i = j$, and $a_i = a_j$ where i, j are even. If $a_{(n+i)/2} = a_{(n+j)/2}$, then, by Definition 2.1, $(n+i)/2 = (n+j)/2$. So, $i = j$, and $a_i = a_j$ where i, j are odd. As a result, $a_i = a_j$ and ϕ_n is injective. □

The next lemma, which results from Definition 2.6, discusses the distance between elements, when the bijection ϕ_n acts on them.

Lemma 2.8. *Let r be an odd and s an even integer. Then the following hold:*

(i) If $\delta(a_i, a_j) = s$, then $\delta(\phi_n(a_i), \phi_n(a_j)) = s/2$

(ii) If $\delta(a_i, a_j) = r$, then $\delta(\phi_n(a_i), \phi_n(a_j)) = (n - r)/2$

Proof. Without loss of generality, we assume that $j > i$.

(i) Let $\delta(a_i, a_j) = s$. Consider two cases:

Case 1: If $j - i \leq (n-1)/2$, then, by Definition 2.2, $s = j - i$, and i and j are both even or both odd. If both are even, by Definitions 2.2 and 2.6, we have that $\delta(\phi_n(a_j), \phi_n(a_i)) = j/2 - i/2 = (j-i)/2 = s/2$. If both are odd, by Definitions 2.2 and 2.6, we have that $\delta(\phi_n(a_j), \phi_n(a_i)) = (n+j)/2 - (n+i)/2 = (j-i)/2 = s/2$.

Case 2: If $j - i > (n-1)/2$, then, by Definition 2.2, $s = n - (j - i)$, and $j - i$ is odd. Suppose j is odd and i even. Then, as $(n+j)/2 - i/2 = (n+j-i)/2 > j - i > (n-1)/2$, we have, by Definitions 2.2 and 2.6, $\delta(\phi_n(a_i), \phi_n(a_j)) = n - (n+j-i)/2 = s/2$. If j is even and i odd, then, as $j - i \geq 1$, and $(n+i-j)/2 \leq (n-1)/2$, we have that $\delta(\phi_n(a_i), \phi_n(a_j)) = (n+i)/2 - j/2 = (n+i-j)/2 = s/2$.

(ii) Let $\delta(a_i, a_j) = r$. Consider the following cases:

Case 1: If $j - i \leq (n-1)/2$, then, by Definition 2.2, $r = j - i$, and $j - i$ is odd. Suppose j is odd and i even. Since $j - i \geq 1$, and thus, $(n+j)/2 - i/2 > (n-1)/2$, we have that $\delta(\phi_n(a_i), \phi_n(a_j)) = n - (n+j-i)/2 = (n-r)/2$. If j is even and i odd, we have that $(n+i)/2 - j/2 \leq (n-1)/2$, and so, $\delta(\phi_n(a_i), \phi_n(a_j)) = (n+i-j)/2 = (n-r)/2$.

Case 2: If $j - i > (n-1)/2$, then, by Definition 2.2, $r = n - (j - i)$, and so, $j - i$ is even. Thus, i and j are either both even or both odd. In each of these cases, by Definitions 2.2 and 2.6, and as $(j-i)/2 \leq (n-1)/2$, we can check that $\delta(\phi_n(a_i), \phi_n(a_j)) = (j-i)/2 = (n-r)/2$.

□

2.2 Distance permutations and their constituents

We show that the automorphism ϕ_n yields a permutation of distances via the images of the a_is. An analysis of the nature of this permutation will subsequently be used to demonstrate the presence of edge-disjoint Hamilton cycles in St_n.

Definition 2.9. *We define the mapping:*

$$\pi_n : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}, \quad \pi_n(\delta(a_i, a_j)) = \delta(\phi_n(a_i), \phi_n(a_j)),$$

where ϕ_n is as in Definition 2.6.

We can check easily, using Lemma 2.8, that π_n is a well-defined bijective mapping and hence an element of $S_{\lfloor n/2 \rfloor}$ (symmetric group of permutations of order $\lfloor n/2 \rfloor$). Note that, as a result of Lemma 2.8, $\pi_n(x) = x/2$ if x is an even integer, and $\pi_n(x) = (n-x)/2$ if x is an odd integer. By elementary properties of permutation groups, the mapping π_n can be written as a product of permutation cycles

$$\pi_n = \pi_n^1 \ldots \pi_n^k = (d_1^1, \ldots, d_{n_1}^1) \ldots (d_1^k, \ldots, d_{n_k}^k)$$

where

$$\{d_1^1, \ldots, d_{n_1}^1, \ldots, d_1^k, \ldots, d_{n_k}^k\} = \{1, \ldots, \lfloor n/2 \rfloor\}$$

and, for $1 \leq i \leq k$, the expression $(d_1^i, \ldots, d_{n_i}^i)$ denotes the cycle π_n^i whose action is to map

$$d_1^i \mapsto d_2^i \mapsto \ldots \mapsto d_{n_i}^i \mapsto d_1^i$$

As such, there are n_i different ways to denote π_n^i:

$$(d_1^i, \ldots, d_{n_i}^i) = (d_2^i, \ldots, d_{n_i}^i, d_1^i) = \ldots = (d_{n_i}^i, d_1^i, \ldots, d_{n_i-1}^i)$$

We shall call $\pi_n^1, \ldots, \pi_n^k$ the *constituent cycles* or simply the *constituents* of π_n or n. A *coprime constituent* will be a constituent π_n^i all of whose elements are coprime to n.

Example 2.10. *Let $n = 17$. We calculate $\pi_{17}(1) = 8, \pi_{17}(8) = 4, \pi_{17}(4) = 2, \pi_{17}(2) = 1$ and $\pi_{17}(3) = 7, \pi_{17}(7) = 5, \pi_{17}(5) = 6, \pi_{17}(6) = 3$. Thus,*

$$\pi_{17} = \pi_{17}^1 \pi_{17}^2 \ \ \textit{where} \ \ \pi_{17}^1 = (1\ 8\ 4\ 2), \ \ \pi_{17}^2 = (3\ 7\ 5\ 6)$$

The next lemma considers a case where a certain constituent π_n^j has just one member, namely where the only member of π_n^j is $n/3$.

Lemma 2.11. *Let $n \in \mathbb{N}$ be odd and $\pi_n = \pi_n^1 \dots \pi_n^k$. Then, there exists $d \in \mathbb{N}$ satisfying $n = 3d$ if and only if $\pi_n^j = (d)$, for some j with $1 \leq j \leq k$.*

Proof. Let $\pi_n = \pi_n^1 \dots \pi_n^k$. We begin with the 'only if' direction. Assume that there exists $d \in \mathbb{N}$ with $n = 3d$. Since n is odd, so is d. As $d \leq \lfloor n/2 \rfloor$, there is $j \in \{1, \dots, k\}$ such that d is in π_n^j. Let π_n^j start with d. By Definition 2.9 and Lemma 2.8, $\pi_n(d) = (n-d)/2$. As $n = 3d$,

$$\pi_n(d) = (3d - d)/2 = d.$$

Thus $\pi_n^j = (d)$. For the 'if' direction, suppose there is $1 \leq j \leq k$ with $\pi_n^j = (d)$. This means that $\pi_n(d) = d$. If d is even, then $\pi_n(d) = d/2 = d$ which cannot be the case as $d \geq 1$. So, d is odd and it follows from $\pi_n(d) = (n-d)/2 = d$ that $n = 3d$. □

We now prove the existence of both an odd and even number in each constituent of π_n, other than the π_n^j of Lemma 2.11.

Lemma 2.12. *Let n be odd and $\pi_n = \pi_n^1 \dots \pi_n^k$. Then, one of the following cases holds:*

(i) If $n \mod 3 \neq 0$, then there is at least one even number and one odd number in π_n^i, for all i with $1 \leq i \leq k$.

(ii) If $n \mod 3 = 0$ and $\pi_n^1 = (n/3)$, then there is at least one even number and one odd number in π_n^i, for all i with $2 \leq i \leq k$.

Proof. We need to prove that, if $\pi_n^i = (d_1^i, \dots, d_{n_i}^i)$ where $n_i \geq 2$, then $\{d_1^i, \dots, d_{n_i}^i\}$ has at least one even and one odd number. Assume, on the contrary, that either all numbers in π_n^i are even or all are odd. If all the numbers in π_n^i are even, then, by Lemma 2.8, $\pi_n^i = (d_1^i, d_1^i/2 \dots, d_1^i/2^{n_i-1})$. Since $\pi_n(d_1^i/2^{n_i-1}) = d_1^i/2^{n_i} = d_1^i$, then either $n_i = 0$ or $d_1^i = 0$ (as equality in Lemma 2.8 is absolute and not modulo n), which are both contradictions. Suppose, on the other hand, that all the numbers in π_n^i are odd. Let $\beta = n_i - 1$. Using Lemma 2.8, we can compute $\pi_n^i = (d_1^i, d_2^i, \dots, d_{n_i}^i)$ to be equal to:

$$(d_1^i, (n - d_1^i)/2, \dots, [(2^{\beta-1} - 2^{\beta-2} + \dots + 2^{\beta-\beta}(-1)^{\beta-1})n + (-1)^{\beta} d_1^i]/2^{\beta})$$

As $\pi_n(d_{n_i}^i) = d_1^i$, i.e. $(n - d_{n_i}^i)/2 = d_1^i$, we have that

$$[(2^{\beta} - 2^{\beta-1} + \dots + (-1)^{\beta})n + (-1)^{\beta+1} d_1^i]/2^{\beta+1} = d_1^i$$

Thus,

$$[(2^{\beta} - 2^{\beta-1} + \dots + (-1)^{\beta})/(2^{\beta+1} + (-1)^{\beta})]n = d_1^i \tag{2.1}$$

Since $(2^\beta - 2^{\beta-1} + \ldots + (-1)^\beta)$ is a geometric series, we have that

$$(2^\beta - 2^{\beta-1} + \ldots + (-1)^\beta) = (2^{\beta+1} + (-1)^\beta)/3 \tag{2.2}$$

From (2.1) and (2.2), $d_1^i = n/3$. So, by Lemma 2.11, $\pi_n^i = (n/3)$ and $n_i = 1$ which contradicts the assumption that $n_i \geq 2$. □

2.3 Edge-disjoint Hamilton cycles

For $i \in \{1, \ldots, \lfloor n/2 \rfloor\}$, let $C_i(n)$ be the set of all edges e of St_n such that $\lambda(e) = i$. Let $C_{i\ j}(n)$ be the spanning subgraph of St_n with edge set $C_i(n) \cup C_j(n)$ (see[11]). We have the following lemma:

Lemma 2.13 ([11]). *Let $U \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}$ be the set of elements coprime to n. Then, for all $j \in U$, $C_{j\ 2j}(n)$ is a Hamilton graph.*

Definition 2.14. *Let $\pi_n = \pi_n^1 \ldots \pi_n^k$. For $i \in \{1, \ldots, k\}$ we define*

$$\overline{\pi}_n^i = (C_{d_1^i\ d_{n_i}^i}, C_{d_2^i\ d_1^i}, \ldots, C_{d_j^i\ d_{j-1}^i}, \ldots, C_{d_{n_i}^i\ d_{n_i-1}^i})$$

and

$$\overline{\pi}_n = \overline{\pi}_n^1 \ldots \overline{\pi}_n^k,$$

where $C_{p\ q}$ stands for $C_{p\ q}(n)$ for all $p, q \in \{d_1^i, \ldots, d_{n_i}^i\}$. Also, denote by $\sigma(\overline{\pi}_n^i)$ the number of subgraphs in $\overline{\pi}_n^i$ (so that, if $\pi_n^i = (d_1^i, \ldots, d_{n_i}^i), \sigma(\overline{\pi}_n^i) = n_i$).

Example 2.15. *Let $n = 17$. Then,*

$$\overline{\pi}_{17} = \overline{\pi}_n^1\ \overline{\pi}_n^2 = (C_{1\ 2}, C_{8\ 1}, C_{4\ 8}, C_{2\ 4})(C_{3\ 6}, C_{7\ 3}, C_{5\ 7}, C_{6\ 5}), \quad \sigma(\overline{\pi}_{17}^1) = \sigma(\overline{\pi}_{17}^2) = 4.$$

Notice that, if $\pi_n = \pi_n^1 \ldots \pi_n^k$, where

$$\pi_n^i = (d_1^i, \ldots, d_j^i, d_{j+1}^i, d_{j+2}^i, \ldots, d_{n_i}^i),$$

and if $C_{d_1^i\ d_{n_i}^i}$ is a Hamilton graph, then, by Definition 2.9 and Lemma 2.5(iii),

$$\Phi_H(C_{d_1^i\ d_{n_i}^i}) = C_{d_2^i\ d_1^i}, \ldots, \Phi_H(C_{d_{n_i-1}^i\ d_{n_i-2}^i}) = C_{d_{n_i}^i\ d_{n_i-1}^i} \tag{2.3}$$

are also Hamilton graphs. We have the following result for the case of coprime constituents.

Lemma 2.16. *Let $\pi_n = \pi_n^1 \ldots \pi_n^k$. Then, for $i \in \{1, \ldots, k\}$, if all elements of π_n^i are coprime to n, all subgraphs in $\overline{\pi}_n^i$ are Hamilton graphs.*

Proof. As all elements of π_n^i are coprime to n, it follows from Lemma 2.12 that π_n^i contains an even integer $2m$. By Definition 2.9 and Lemma 2.8, $\pi_n(2m) = \pi_n^i(2m) = m$. Thus, by Definition 2.14, there exists a subgraph of the form $C_{2m\ m} = C_{m\ 2m}$ in $\overline{\pi}_n^i$, where $m \in \{1, \dots, \lfloor n/2 \rfloor\}$, and m is coprime to n. By Lemma 2.13, $C_{m\ 2m}$ is a Hamilton graph. Arguing as in (2.3), all subgraphs in $\overline{\pi}_n^i$ are therefore Hamilton graphs. $\square$

Example 2.17. *Let $n = 31$. According to Definition 2.9 and Definition 2.14,*

$$\begin{aligned}
\pi_{31} &= (1\ 15\ 8\ 4\ 2)(3\ 14\ 7\ 12\ 6)(5\ 13\ 9\ 11\ 10) \\
\overline{\pi}_{31} &= (C_{1\ 2}C_{15\ 1}C_{8\ 15}C_{4\ 8}C_{2\ 4})(C_{3\ 6}C_{14\ 3}C_{7\ 14}C_{12\ 7}C_{6\ 12}) \\
&\quad (C_{5\ 10}C_{13\ 5}C_{9\ 13}C_{11\ 9}C_{10\ 11})
\end{aligned}$$

As, 31 is prime, all elements of the constituents of π_{31} are coprime to 31. So, all 15 subgraphs in $\overline{\pi}_{31}$ are Hamilton graphs.

We are interested in edge-disjoint Hamilton cycles. Coprime constituents correspond to certain sets of (pairwise) edge-disjoint Hamilton cycles.

Lemma 2.18. *Let $\pi_n = \pi_n^1 \dots \pi_n^k$ and let $\pi_n^i = (d_1^i, \dots, d_{n_i}^i)$, where $1 \le i \le k$, be a coprime constituent. If n_i is even, then there are at least $n_i/2$ edge-disjoint Hamilton cycles in $\overline{\pi}_n^i$, and, if n_i is odd, then the number of edge-disjoint Hamilton cycles in $\overline{\pi}_n^i$ is at least $(n_i - 1)/2$.*

Proof. Two subgraphs of the form $C_{i_1\ j_1}$ and $C_{i_2\ j_2}$, where $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, are obviously edge-disjoint. Since in $\overline{\pi}_n^i$ the only common edges occur in $C_{d_j^i\ d_{j-1}^i}$ and $C_{d_{j+1}^i\ d_j^i}$ where $1 \le j \le n_i$ ($d_0^i = d_{n_i}^i$, and $d_{n_i+1}^i = d_1^i$), it follows from Lemma 2.16 that, if n_i is odd, there are $(n_i - 1)/2$, and, if n_i is even, there are $n_i/2$ edge-disjoint Hamilton cycles in $\overline{\pi}_n^i$. $\square$

2.4 Multiples of constituents

In this section we show that for any $i < n$, that is coprime to n, the constituent of π_n that contains i is a coprime constituent. We multiply all elements of constituents of π_n by m, and see whether the resulting permutation cycle is a constituent of π_{mn}.

Definition 2.19. *Let m be an odd integer, $\pi_n = \pi_n^1 \dots \pi_n^k$, and let $\pi_n^i = (d_1^i, \dots, d_{n_i}^i)$ be one of these constituents of π_n. We denote the permutation cycle $(md_1^i, \dots, md_{n_i}^i)$ by $m\pi_n^i$, and say that $m\pi_n^i$ is π_n^i multiplied by m and is a multiple of π_n^i.*

The next lemma shows that a constituent of π_n multiplied by m, is a constituent of π_{mn}.

Lemma 2.20. *Let m be an odd integer and let $\pi_n = \pi_n^1 \dots \pi_n^k$. If $\pi_{mn} = \pi_{mn}^1 \dots \pi_{mn}^\ell$, then, for every $i \in \{1, \dots, k\}$, there exists a $j \in \{1, \dots, \ell\}$ such that $\pi_{mn}^j = m\pi_n^i$.*

Proof. Let $\pi_n = \pi_n^1 \dots \pi_n^k$ and (without loss of generality) d be an odd integer in the first position of π_n^i. As $1 \le d \le \lfloor n/2 \rfloor$, we have that $1 \le md \le \lfloor mn/2 \rfloor$ and so there exists j, with $1 \le j \le \ell$ such that md is (without loss of generality) in the first position of π_{mn}^j. We need to show that

$$(md_1^i, \dots, md_{n_i}^i) = (d_1^j, \dots, d_{(mn)_j}^j)$$

It suffices to show that, if $1 \le f \le n_i$ and $1 \le g \le (mn)_j$, and

$$md_f^i = d_g^j \tag{2.4}$$

then

$$m\pi_n(d_f^i) = \pi_{mn}(d_g^j) \tag{2.5}$$

The lemma will then follow from (2.5) by an inductive argument. Suppose, then, that (2.4) holds. As m is odd, d_f^i and d_g^j are either both even or both odd. If d_f^i and d_g^j are both even, then, by Definition 2.9, Lemma 2.8, and (2.4),

$$m\pi_n(d_f^i) = m(d_f^i/2) = d_g^j/2 = \pi_{mn}(d_g^j),$$

and if d_f^i and d_g^j are both odd, then, by Definition 2.9, Lemma 2.8, and (2.4),

$$m\pi_n(d_f^i) = m(n - d_f^i)/2 = (mn - md_f^i)/2 = (mn - d_g^j)/2 = \pi_{mn}(d_g^j).$$

□

The above result relating multiples of constituents of n and constituents of corresponding multiples of n, has implications for how factors of n and integers that are coprime to n are distributed amongst the constituents.

Lemma 2.21. *Let $\pi_n = \pi_n^1 \dots \pi_n^k$. Then:*

(i) If $m > 1$ is a factor of n and $\pi_m = \pi_m^1 \dots \pi_m^\ell$, then, for every constituent π_m^i of π_m $(1 \le i \le \ell)$, there is a constituent π_n^j of π_n $(1 \le j \le k)$ such that $\pi_n^j = (n/m)\pi_m^i$.

(ii) If a constituent π_n^j $(1 \le j \le k)$ of π_n is not of the form $(n/m)\pi_m^i$ for some constituent π_m^i of π_m for some factor m of n, then all elements of π_n^j are coprime to n.

Proof. The proof of (i) follows directly from Lemma 2.20. For (ii), it is straightforward to show that, if some elements of π_n^j have a common factor p with n, then we will have $\pi_n^j = p\pi_{n/p}^i$ for some i. $\square$

Corollary 2.22. *Let $\pi_n = \pi_n^1 \dots \pi_n^k$. Then there exists $W \subseteq \{1, \dots, k\}$ such that, for all $i \in W$, all elements of π_n^i have a common factor with n, and, for all $j \notin W$, all elements of π_n^j are coprime to n.*

The following examples clarify multiples of constituents.

Example 2.23. *Let $n = 17$ and $m = 3$. According to Definition 2.9,*

$$\begin{aligned}
\pi_{17} &= \pi_{17}^1\ \pi_{17}^2 = (1\ 8\ 4\ 2)(3\ 7\ 5\ 6)\\
\pi_{3\times 17} &= \pi_{51} = \pi_{51}^1\ \pi_{51}^2\ \pi_{51}^3\ \pi_{51}^4\ \pi_{51}^5 = (1\ 25\ 13\ 19\ 16\ 8\ 4\ 2)\\
&\qquad (5\ 23\ 14\ 7\ 22\ 11\ 20\ 10)\\
&\qquad (3\ 24\ 12\ 6)(9\ 21\ 15\ 18)(17)
\end{aligned}$$

As, $(3\ 24\ 12\ 6) = 3(1\ 8\ 4\ 2)$, and $(9\ 21\ 15\ 18) = 3(3\ 7\ 5\ 6)$, then $\pi_{51}^3 = 3\pi_{17}^1$, and $\pi_{51}^4 = 3\pi_{17}^2$. This implies that for $i \in \{1, 2\}$, there exists a $j \in \{1, 2, 3, 4, 5\}$ such that $\pi_{51}^j = 3\pi_{17}^i$.

Example 2.24. *Let $n = 85$. Then,*

$$\begin{aligned}
\pi_{85} = \pi_{85}^1\ \pi_{85}^2 \dots \pi_{85}^7 = {} & (1\ 42\ 21\ 32\ 16\ 8\ 4\ 2)(3\ 41\ 22\ 11\ 37\ 24\ 12\ 6)\\
& (9\ 38\ 19\ 33\ 26\ 13\ 36\ 18)(7\ 39\ 23\ 31\ 27\ 29\ 28\ 14)\\
& (5\ 40\ 20\ 10)(15\ 35\ 25\ 30)(17\ 34)
\end{aligned}$$

This shows that all the elements in $\pi_{85}^1, \pi_{85}^2, \pi_{85}^3$, and π_{85}^4 are coprime to 85 and all the elements of π_{85}^5, π_{85}^6, and π_{85}^7 have a common factor with 85.

Example 2.25. *Let $n = 45$. Then,*

$$\begin{aligned}
\pi_{45} = \pi_{45}^1\ \pi_{45}^2\ \pi_{45}^3\ \pi_{45}^4\ \pi_{45}^5 = {} & (1\ 22\ 11\ 17\ 14\ 7\ 19\ 13\ 16\ 8\ 4\ 2)(3\ 21\ 12\ 6)\\
& (5\ 20\ 10)(9\ 18)(15)
\end{aligned}$$

As, $45 = 3^2 \times 5$, we consider the constituents of π_3, π_9, and π_{15}:

$$\begin{aligned}
\pi_3 &= \pi_3^1 = (1)\\
\pi_9 &= \pi_9^1\ \pi_9^2 = (1\ 4\ 2)(3)\\
\pi_{15} &= \pi_{15}^1\ \pi_{15}^2\ \pi_{15}^3 = (1\ 7\ 4\ 2)(3\ 6)(5)
\end{aligned}$$

This implies that,

$$\pi_{45}^{5} = (15) = 3(5) = 3\pi_{15}^{3}$$
$$\pi_{45}^{5} = (15) = 5(3) = 5\pi_{9}^{2}$$
$$\pi_{45}^{5} = (15) = 15(1) = 15\pi_{3}^{1}$$

Moreover,

$$\pi_{45}^{4} = (9\ 18) = 3(3\ 6) = 3\pi_{15}^{2}$$
$$\pi_{45}^{3} = (5\ 20\ 10) = 5(1\ 4\ 2) = 5\pi_{9}^{1}$$
$$\pi_{45}^{2} = (3\ 21\ 12\ 6) = 3(1\ 7\ 4\ 2) = 3\pi_{15}^{1}$$

All the integers of π_{45}^{1} *are coprime to* $n = 45$.

2.5 Bounds for the number of edge-disjoint Hamilton cycles

Our calculation of a lower bound for the number of edge-disjoint Hamilton cycles in star graphs, will use the observation (2.6) and Lemma 2.26 below, repeatedly. Let $\varphi(n)$ be the Euler totient function, i.e. $\varphi(n)$ is the number of positive integers less than or equal to n that are coprime to n. We observe that, if d and n are coprime, where $1 \le d < n$, then $n-d$ and n are also coprime, and, if $1 \le d \le (n-1)/2$, then $(n+1)/2 \le n-d \le n-1$. It follows that

$$|\{d \in \mathbb{N}|\ 1 \le d \le (n-1)/2 \text{ and } d \text{ is coprime to } n\}| = \varphi(n)/2 \tag{2.6}$$

Lemma 2.26. *Let a constituent* $\pi_n^i = (d_1^i, \ldots, d_{n_i}^i)$ *of* n *be such that* d_1^i *is odd, and let* f *be an integer greater than or equal to* 2*. Then,*

$$(\pi_n^i)^f(d_1^i) = (cn \pm d_1^i)/2^f, \tag{2.7}$$

where $(\pi_n^i)^f(d_1^i)$ *applies* π_n^i f *times to* d_1^i*, and* c *is an integer such that* $1 \le c \le 2^{f-1} - 1$.

Proof. By induction. For $f = 2$, by Definition 2.9 and Lemma 2.8, $\pi_n^i(d_1^i) = (n - d_1^i)/2$ and $(\pi_n^i)^2(d_1^i) = ((n-d_1^i)/2)/2$ or $(n - (n-d_1^i)/2)/2$ both of which satisfy (2.7). Assume (2.7) holds for some integer $f \ge 2$. If $(\pi_n^i)^f(d_1^i)$ is even, then, by Definition 2.9 and Lemma 2.8, $(\pi_n^i)^{f+1}(d_1^i) = \pi_n^i((\pi_n^i)^f(d_1^i)) = ((\pi_n^i)^f(d_1^i))/2 = (cn \pm d_1^i)/2^{f+1}$ (inductively) where $1 \le c \le 2^{f-1} - 1 < 2^{(f+1)-1} - 1$. If $(\pi_n^i)^f(d_1^i)$

is odd, then, by Definition 2.9 and Lemma 2.8, $(\pi_n^i)^{f+1}(d_1^i) = (n - (\pi_n^i)^f(d_1^i))/2 = ((2^f - c)n \pm d_1^i)/2^{f+1}$ and $1 \leq 2^f - c \leq 2^{(f+1)-1} - 1$. □

The required bounds are achieved by obtaining edge-disjoint Hamilton cycles by means of coprime constituents, and counting them using Lemma 2.18. For all but finitely many n: specifically, for those $n \notin X$ where

$$X = \{3, 5, 7, 9, 11, 13, 15, 17, 21, 31, 33, 43, 51, 63, 65, 85, 127, 129, 255, 257\}, \quad (2.8)$$

we use the fact that coprime constituents have at least 9 elements (proved in Theorem 2.27 below). For the finitely many $n \in X, n \neq 127$, the bounds hold by enumeration of all constituents (Lemma 2.28). For the case $n = 127$, the number of edge-disjoint Hamilton cycles produced by the constituents is 1 short of our target bound. We demonstrate that another edge-disjoint Hamilton cycle does exist for this case, by use of another automorphism (Lemma 2.29).

For the remainder of the chapter, let $EDH(n)$ denote the number of edge-disjoint Hamilton cycles in St_n.

Theorem 2.27. *If n is odd and $n \notin X$, then St_n contains at least $\lceil 2\varphi(n)/9 \rceil$ pairwise edge-disjoint Hamilton cycles, where $\varphi(n)$ is the Euler function.*

Proof. Let $\pi_n = \pi_n^1 \dots \pi_n^k$. By Corollary 2.22, there exists a $W \subseteq \{1, \dots, k\}$ such that, for all $i \in W$, all elements of π_n^i are coprime to n. By Lemma 2.16, all subgraphs in each such π_n^i are Hamilton graphs. We first show that

$$\sigma(\overline{\pi}_n^i) \geq 9 \text{ for all } i \in W \quad (2.9)$$

(using the notation σ of Definition 2.14). Let $i \in W$, so we know that all the elements of π_n^i are coprime to n. Assume, on the contrary, that the number of elements in π_n^i $(= \sigma(\overline{\pi}_n^i))$ is less than 9. This means that $\sigma(\overline{\pi}_n^i) \in \{1, \dots, 8\}$. We will consider each case.

Case $\sigma(\overline{\pi}_n^i) = 1$. Here, $\pi_n^i = (d_1^i)$. If d_1^i is even, then $\pi_n(d_1^i) = d_1^i/2$ by Definition 2.9 and Lemma 2.8, and this must equal d_1^i, which implies that $d_1^i = 0$. Thus, d_1^i must be odd and $\pi_n(d_1^i) = (n - d_1^i)/2$ by Definition 2.9 and Lemma 2.8. So, $d_1^i = n/3$ and d_1^i divides n. Since $n \neq 3, d_1^i$ and n are not coprime which contradicts our assumption.

Case $\sigma(\overline{\pi}_n^i) = 2$. Without loss of generality, by Lemma 2.12 we can write $\pi_n^i = (d_1^i, d_2^i)$ where d_1^i is odd. By Lemma 2.26, $d_1^i = (\pi_n^i)^2(d_1^i) = (n + d_1^i)/4$ or $(n - d_1^i)/4$. Thus, $n = 3d_1^i$ or $n = 5d_1^i$. Both cases contradict the assumption that d_1^i and n are coprime.

Case $\sigma(\overline{\pi}^i_n) = 3$. Let $\pi^i_n = (d^i_1, d^i_2, d^i_3)$ where d^i_1 is odd. By Lemma 2.26, $d^i_1 = (cn + d^i_1)/2^3$ or $d^i_1 = (cn - d^i_1)/2^3$ where $1 \leq c \leq 2^2 - 1$. So,

$$cn = 7d^i_1 \text{ or } cn = 3 \times 3d^i_1$$

Since $n > 2$ is coprime to d^i_1, we have $n|7$ in the first case, or $n|9$ in the second case, contradicting $n \notin X$.

Case $\sigma(\overline{\pi}^i_n) = 4$. Let $\pi^i_n = (d^i_1, d^i_2, d^i_3, d^i_4)$ where d^i_1 is odd. By Lemma 2.26, $d^i_1 = (cn + d^i_1)/2^4$ or $d^i_1 = (cn - d^i_1)/2^4$ where $1 \leq c \leq 2^3 - 1$. So,

$$cn = 3 \times 5d^i_1 \text{ or } cn = 17d^i_1$$

Since $n > 2$ and d^i_1 are coprime, we have $n|15$ in the first case, or $n|17$ in the second case, contradicting $n \notin X$.

Case $\sigma(\overline{\pi}^i_n) = 5$. Let $\pi^i_n = (d^i_1, d^i_2, d^i_3, d^i_4, d^i_5)$ where d^i_1 is odd. By Lemma 2.26, $d^i_1 = (cn + d^i_1)/2^5$ or $d^i_1 = (cn - d^i_1)/2^5$ where $1 \leq c \leq 2^4 - 1$. So,

$$cn = 3 \times 11d^i_1 \text{ or } cn = 31d^i_1$$

As $n > 2$ is coprime to d^i_1, we have $n|33$ in the first case, or $n|31$ in the second case, contradicting $n \notin X$.

Case $\sigma(\overline{\pi}^i_n) = 6$. Let $\pi^i_n = (d^i_1, d^i_2, d^i_3, d^i_4, d^i_5, d^i_6)$ where d^i_1 is odd. By Lemma 2.26, $d^i_1 = (cn + d^i_1)/2^6$ or $d^i_1 = (cn - d^i_1)/2^6$ where $1 \leq c \leq 2^5 - 1$. So,

$$cn = 3^2 \times 7d^i_1 \text{ or } cn = 5 \times 13d^i_1$$

Since $n > 2$ is coprime to d^i_1, we have $n|63$ in the first case, or $n|65$ in the second case, contradicting $n \notin X$.

Case $\sigma(\overline{\pi}^i_n) = 7$. Let $\pi^i_n = (d^i_1, d^i_2, d^i_3, d^i_4, d^i_5, d^i_6, d^i_7)$ where d^i_1 is odd. By Lemma 2.26, $d^i_1 = (cn + d^i_1)/2^7$ or $d^i_1 = (cn - d^i_1)/2^7$ where $1 \leq c \leq 2^6 - 1$. So,

$$cn = 3 \times 43d^i_1 \text{ or } cn = 127d^i_1$$

As $n > 2$ and d^i_1 are coprime, we have $n|129$ in the first case, or $n|127$ in the second case, contradicting $n \notin X$.

Case $\sigma(\overline{\pi}^i_n) = 8$. Let $\pi^i_n = (d^i_1, d^i_2, d^i_3, d^i_4, d^i_5, d^i_6, d^i_7, d^i_8)$ where d^i_1 is odd. By Lemma 2.26, $d^i_1 = (cn + d^i_1)/2^8$ or $d^i_1 = (cn - d^i_1)/2^8$ where $1 \leq c \leq 2^7 - 1$. So,

$$cn = 3 \times 5 \times 17d^i_1 \text{ or } cn = 257d^i_1$$

Since $n > 2$ is coprime to d^i_1, then we have $n|255$ in the first case, or $n|257$ in the second case, contradicting $n \notin X$.

We have thus shown that (2.9) holds. Now, the coprime constituents of n are exactly the constituents π_n^i where $i \in W$. By (2.9), each has at least nine elements. Therefore, counting the edge-disjoint Hamilton cycles produced by each, we have, by Lemma 2.18,

$$EDH(n) \geq \sum_{i \in W} (\sigma(\overline{\pi}_n^i) - 1)/2 \geq 1/2\ [\sum_{i \in W} \sigma(\overline{\pi}_n^i) - (\sum_{i \in W} \sigma(\overline{\pi}_n^i))/9] \tag{2.10}$$

Since the π_n^i, $(i \in W)$ contain all integers $\leq (n-1)/2$ that are coprime to n, it follows, by (2.6), that

$$\sum_{i \in W} \sigma(\overline{\pi}_n^i) = \varphi(n)/2 \tag{2.11}$$

Therefore, by (2.10) and (2.11),

$$EDH(n) \geq 1/2(\varphi(n)/2 - (\varphi(n)/2)/9) = 2\varphi(n)/9$$

and so the integer $EDH(n) \geq \lceil 2\varphi(n)/9 \rceil$. This completes the proof. □

Lemma 2.28. *If $n \in X$ and $n \neq 127$, then St_n contains at least $\lfloor 2\varphi(n)/9 \rfloor$ pairwise edge-disjoint Hamilton cycles, where $\varphi(n)$ is the Euler function.*

Proof. This lemma is proved by enumerating the constituents for all $n \in X$ in (2.8), other than for $n = 127$. We consider the following cases:

- Let $n = 3$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 2)/9 \rfloor = 0$.
- Let $n = 5$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 4)/9 \rfloor = 0$.
- Let $n = 7$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 6)/9 \rfloor = 1$. As $\pi_7 = (1\ 3\ 2)$, $EDH(7) \geq \lfloor 3/2 \rfloor = 1 \geq 1$.
- Let $n = 9$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 6)/9 \rfloor = 1$. We have that $\pi_9 = (1\ 4\ 2)(3)$. As $9 = 3^2$, then $(1\ 4\ 2)$ is a coprime constituent. So, $EDH(9) \geq \lfloor 3/2 \rfloor = 1 \geq 1$.
- Let $n = 11$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 10)/9 \rfloor = 2$. As $\pi_{11} = (1\ 5\ 3\ 4\ 2)$, $EDH(9) \geq \lfloor 5/2 \rfloor = 2 \geq 2$.
- Let $n = 13$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 12)/9 \rfloor = 2$. As $\pi_{13} = (1\ 6\ 3\ 5\ 4\ 2)$, $EDH(13) \geq 6/2 = 3 \geq 2$.
- Let $n = 15$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 8)/9 \rfloor = 1$. We have that $\pi_{15} = (1\ 7\ 4\ 2)(3\ 6)(5)$. As $15 = 3 \times 5$, then $(1\ 7\ 4\ 2)$ is a coprime constituent. So, $EDH(15) \geq 4/2 = 2 \geq 1$.

- Let $n = 17$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 16)/9 \rfloor = 3$. We have that $\pi_{17} = (1\ 8\ 4\ 2)(3\ 7\ 5\ 6)$. Since 17 is prime, both $(1\ 8\ 4\ 2)$ and $(3\ 7\ 5\ 6)$ are coprime constituents. Thus, $EDH(17) \geq 4/2 + 4/2 = 4 \geq 3$.

- Let $n = 21$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 12)/9 \rfloor = 2$. We have that

$$\pi_{21} = (1\ 10\ 5\ 8\ 4\ 2)(3\ 9\ 6)(7)$$

As $21 = 3 \times 7$, $(1\ 10\ 5\ 8\ 4\ 2)$ is a coprime constituent. So, $EDH(21) \geq 6/2 = 3 \geq 2$.

- Let $n = 31$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 30)/9 \rfloor = 6$. We have that

$$\pi_{31} = (1\ 15\ 8\ 4\ 2)(3\ 14\ 7\ 12\ 6)(5\ 13\ 9\ 11\ 10)$$

Since 31 is prime, $(1\ 15\ 8\ 4\ 2), (3\ 14\ 7\ 12\ 6)$, and $(5\ 13\ 9\ 11\ 10)$ are coprime constituents. So, $EDH(31) \geq \lfloor 5/2 \rfloor + \lfloor 5/2 \rfloor + \lfloor 5/2 \rfloor = 6 \geq 6$.

- Let $n = 33$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 20)/9 \rfloor = 4$. We have that

$$\pi_{33} = (1\ 16\ 8\ 4\ 2)(3\ 15\ 9\ 12\ 6)(5\ 14\ 7\ 13\ 10)(11)$$

As $33 = 3 \times 11$, then $(1\ 16\ 8\ 4\ 2)$ and $(5\ 14\ 7\ 13\ 10)$ are coprime constituents. So, $EDH(33) \geq \lfloor 5/2 \rfloor + \lfloor 5/2 \rfloor = 4 \geq 4$.

- Let $n = 43$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 42)/9 \rfloor = 9$. We have that

$$\pi_{43} = (1\ 21\ 11\ 16\ 8\ 4\ 2)(3\ 20\ 10\ 5\ 19\ 12\ 6)(7\ 18\ 9\ 17\ 13\ 15\ 14)$$

Since 43 is prime, $(1\ 21\ 11\ 16\ 8\ 4\ 2), (3\ 20\ 10\ 5\ 19\ 12\ 6)$, and $(7\ 18\ 9\ 17\ 13\ 15\ 14)$ are coprime constituents. So, $EDH(43) \geq \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor = 9 \geq 9$.

- Let $n = 51$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 32)/9 \rfloor = 7$. We have that

$$\begin{aligned}\pi_{51} &= (1\ 25\ 13\ 19\ 16\ 8\ 4\ 2)(5\ 23\ 14\ 7\ 22\ 11\ 20\ 10)\\ &\quad (3\ 24\ 12\ 6)(9\ 21\ 15\ 18)(17)\end{aligned}$$

As $51 = 3 \times 17$, $(1\ 25\ 13\ 19\ 16\ 8\ 4\ 2)$ and $(5\ 23\ 14\ 7\ 22\ 11\ 20\ 10)$ are coprime constituents. So, $EDH(51) \geq 8/2 + 8/2 = 8 \geq 7$.

- Let $n = 63$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 36)/9 \rfloor = 8$. We have that

$$\begin{aligned}\pi_{63} &= (1\ 31\ 16\ 8\ 4\ 2)(3\ 30\ 15\ 24\ 12\ 6)(5\ 29\ 17\ 23\ 20\ 10)\\ &\quad (11\ 26\ 13\ 25\ 19\ 22)(7\ 28\ 14)(9\ 27\ 18)\end{aligned}$$

As $63 = 7\times 3^2$, then (1 31 16 8 4 2), (5 29 17 23 20 10), and (11 26 13 25 19 22) are coprime constituents. So, $EDH(63) \geq 6/2 + 6/2 + 6/2 = 9 \geq 8$.

- Let $n = 65$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2\times 48)/9 \rfloor = 10$. We have that

$$\begin{aligned}\pi_{65} \;=\;& (1\ 32\ 16\ 8\ 4\ 2)(3\ 31\ 17\ 24\ 12\ 6)(7\ 29\ 18\ 9\ 28\ 14)\\ & (11\ 27\ 19\ 23\ 21\ 22)(5\ 30\ 15\ 25\ 20\ 10)(13\ 26)\end{aligned}$$

As $65 = 5\times 13$, then (1 32 16 8 4 2), (3 31 17 24 12 6), (7 29 18 9 28 14), and (11 27 19 23 21 22) are coprime constituents. So, $EDH(65) \geq 6/2 + 6/2 + 6/2 + 6/2 = 12 \geq 10$.

- Let $n = 85$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2\times 64)/9 \rfloor = 14$. We have that

$$\begin{aligned}\pi_{85} \;=\;& (1\ 42\ 21\ 32\ 16\ 8\ 4\ 2)(3\ 41\ 22\ 11\ 37\ 24\ 12\ 6)\\ & (7\ 39\ 23\ 31\ 27\ 29\ 28\ 14)(9\ 38\ 19\ 33\ 26\ 13\ 36\ 18)\\ & (5\ 40\ 20\ 10)(25\ 30\ 15\ 35)(17\ 34)\end{aligned}$$

As $85 = 5\times 17$, then (1 42 21 32 16 8 4 2), (3 41 22 11 37 24 12 6), (7 39 23 31 27 29 28 14), and (9 38 19 33 26 13 36 18) are coprime constituents. So, $EDH(85) \geq 8/2 + 8/2 + 8/2 + 8/2 = 16 \geq 14$.

- Let $n = 129$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2\times 84)/9 \rfloor = 18$. We have that

$$\begin{aligned}\pi_{129} \;=\;& (1\ 64\ 32\ 16\ 8\ 4\ 2)(3\ 63\ 33\ 48\ 24\ 12\ 6)\\ & (5\ 62\ 31\ 49\ 40\ 20\ 10)(7\ 61\ 34\ 17\ 56\ 28\ 14)\\ & (9\ 60\ 30\ 15\ 57\ 36\ 18)(11\ 59\ 35\ 47\ 41\ 44\ 22)\\ & (13\ 58\ 29\ 50\ 25\ 52\ 26)(19\ 55\ 37\ 46\ 23\ 53\ 38)\\ & (21\ 54\ 27\ 51\ 39\ 45\ 42)(43)\end{aligned}$$

As $129 = 3\times 43$, then the coprime constituents are (1 64 32 16 8 4 2), (5 62 31 49 40 20 10), (7 61 34 17 56 28 14), (11 59 35 47 41 44 22), (13 58 29 50 25 52 26), and (19 55 37 46 23 53 38). So, $EDH(129) \geq \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor + \lfloor 7/2 \rfloor = 18 \geq 18$.

- Let $n = 255$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 128)/9 \rfloor = 28$. We have that

$$\begin{aligned}\pi_{255} &= (1\ 127\ 64\ 32\ 16\ 8\ 4\ 2)(3\ 126\ 63\ 96\ 48\ 24\ 12\ 6)\\ &\quad (5\ 125\ 65\ 95\ 80\ 40\ 20\ 10)(7\ 124\ 62\ 31\ 112\ 56\ 28\ 14)\\ &\quad (9\ 123\ 66\ 33\ 111\ 72\ 36\ 18)(11\ 122\ 61\ 97\ 79\ 88\ 44\ 22)\\ &\quad (13\ 121\ 67\ 94\ 47\ 104\ 52\ 26)(19\ 118\ 59\ 98\ 49\ 103\ 76\ 38)\\ &\quad (21\ 117\ 69\ 93\ 81\ 87\ 84\ 42)(23\ 116\ 58\ 29\ 113\ 71\ 92\ 46)\\ &\quad (25\ 115\ 70\ 35\ 110\ 55\ 100\ 50)(27\ 114\ 57\ 99\ 78\ 39\ 108\ 54)\\ &\quad (37\ 109\ 73\ 91\ 82\ 41\ 107\ 74)(43\ 106\ 53\ 101\ 77\ 89\ 83\ 86)\\ &\quad (15\ 120\ 60\ 30)(17\ 119\ 68\ 34)(45\ 105\ 75\ 90)(51\ 102)(85)\end{aligned}$$

As $255 = 3 \times 5 \times 17$, then

$$\begin{aligned}&(1\ 127\ 64\ 32\ 16\ 8\ 4\ 2), (7\ 124\ 62\ 31\ 112\ 56\ 28\ 14),\\ &(11\ 122\ 61\ 97\ 79\ 88\ 44\ 22), (13\ 121\ 67\ 94\ 47\ 104\ 52\ 26),\\ &(19\ 55\ 37\ 46\ 23\ 53\ 38), (23\ 116\ 58\ 29\ 113\ 71\ 92\ 46),\\ &(37\ 109\ 73\ 91\ 82\ 41\ 107\ 74),\ \text{and}\ (43\ 106\ 53\ 101\ 77\ 89\ 83\ 86)\end{aligned}$$

are coprime constituents. So, $EDH(255) \geq 8/2 + 8/2 + 8/2 + 8/2 + 8/2 + 8/2 + 8/2 + 8/2 = 32 \geq 28$.

- Let $n = 257$. Then, $\lfloor 2\varphi(n)/9 \rfloor = \lfloor (2 \times 256)/9 \rfloor = 56$. We have that

$$\begin{aligned}\pi_{257} &= (1\ 128\ 64\ 32\ 16\ 8\ 4\ 2)(3\ 127\ 65\ 96\ 48\ 24\ 12\ 6)\\ &\quad (5\ 126\ 63\ 97\ 80\ 40\ 20\ 10)(7\ 125\ 66\ 33\ 112\ 56\ 28\ 14)\\ &\quad (9\ 124\ 62\ 31\ 113\ 72\ 36\ 18)(11\ 123\ 67\ 95\ 81\ 88\ 44\ 22)\\ &\quad (13\ 122\ 61\ 98\ 49\ 104\ 52\ 26)(15\ 121\ 68\ 34\ 17\ 120\ 60\ 30)\\ &\quad (19\ 119\ 69\ 94\ 47\ 105\ 76\ 38)(21\ 118\ 59\ 99\ 79\ 89\ 84\ 42)\\ &\quad (23\ 117\ 70\ 35\ 111\ 73\ 92\ 46)(25\ 116\ 58\ 29\ 114\ 57\ 100\ 50)\\ &\quad (27\ 115\ 71\ 93\ 82\ 41\ 108\ 54)(37\ 110\ 55\ 101\ 78\ 39\ 109\ 74)\\ &\quad (43\ 107\ 75\ 91\ 83\ 87\ 85\ 86)(45\ 106\ 53\ 102\ 51\ 103\ 77\ 90)\end{aligned}$$

As 257 is prime, all the constituents are coprime constituents. So, $EDH(257) \geq 16(8/2) = 64 \geq 56$.

$\square$

Now, the constituents for $n = 127$ are as follows:

$$\begin{aligned}\pi_{127} &= (1\ 63\ 32\ 16\ 8\ 4\ 2)(3\ 62\ 31\ 48\ 24\ 12\ 6)(5\ 61\ 33\ 47\ 40\ 20\ 10)\\ &\quad (7\ 60\ 30\ 15\ 56\ 28\ 14)(9\ 59\ 34\ 17\ 55\ 36\ 18)(11\ 58\ 29\ 49\ 39\ 44\ 22)\\ &\quad (13\ 57\ 35\ 46\ 23\ 52\ 26)(19\ 54\ 27\ 50\ 25\ 51\ 38)(21\ 53\ 37\ 45\ 41\ 43\ 42)\end{aligned}$$

Since 127 is a prime number, $EDH(127) \geq 3 \times 9 = 27$ arguing as above. However, $\lfloor 2\varphi(127)/9 \rfloor = 28$. Thus, the $\lfloor 2\varphi(127)/9 \rfloor$ bound cannot be achieved for $n = 127$ by our existing methods alone. For the sake of completeness, we show that St_{127} has a 28-th edge-disjoint Hamilton cycle, and thus the $\lfloor 2\varphi(n)/9 \rfloor$ bound holds for all odd integers n.

Lemma 2.29. *If $n = 127$, St_n has at least $2\varphi(n)/9$, i.e. 28, edge-disjoint Hamilton cycles.*

Proof. First of all, by (2.3), we can obtain 27 edge-disjoint Hamilton cycles from the following subgraphs of the constituents of 127:

$$C_{63\ 1}, C_{16\ 32}, C_{4\ 8}, C_{31\ 62}, C_{24\ 48}, C_{6\ 12}, C_{61\ 5}, C_{47\ 33}, C_{20\ 40},$$
$$C_{7\ 14}, C_{30\ 60}, C_{56\ 15}, C_{9\ 18}, C_{34\ 59}, C_{55\ 17}, C_{11\ 22}, C_{29\ 58}, C_{39\ 49},$$
$$C_{57\ 13}, C_{23\ 46}, C_{26\ 52}, C_{54\ 19}, C_{50\ 27}, C_{51\ 25}, C_{53\ 21}, C_{41\ 45}, C_{42\ 43}.$$

The edge lengths that do not appear in any of these 27 Hamilton cycles are those in the set $\{2, 3, 10, 28, 36, 44, 35, 38, 37\}$. It suffices to find a Hamilton cycle whose edge lengths are in this set. We obtain such a Hamilton cycle as an automorphic image of the Hamilton cycle, which we denote by H_n, comprising edges of lengths 1 and 2, constructed in [11]. The automorphism Θ that we use is defined as in Lemma 2.5 by the bijection $\theta : \{a_1, \ldots, a_{127}\} \to \{a_1, \ldots, a_{127}\}$ given by:

$$\theta_n(a_i) = \begin{cases} a_{i-9}, & i \text{ even},\ i - 9 \geq 1, \\ a_{127+(i-9)-1}, & i \text{ even},\ i - 9 < 1, \\ a_{i+27}, & i \text{ odd},\ i + 27 < 127, \\ a_{(i+27)-127+1}, & i \text{ odd},\ i + 27 > 127,\ i \neq 127, \\ a_{127}, & i = 127. \end{cases}$$

Our interest is in the possible lengths of edges in the automorphic image of H_n for $n = 127$. We note, from the construction of H_n in Lemma 10 of [11], that edges in H_n of length 2 are of the form:

$$(a_{\rho(1)} \ldots a_{\rho(n)}, a_{\rho(i)} \ldots a_{\rho(n)}),\ \delta(a_{\rho(1)}, a_{\rho(i)}) = 2,\ 1 \leq i \leq n-1,\ \rho(n) = n$$

Thus, the set of possible edge lengths in the automorphic image of H_{127} is a subset of the set:

$$\{\delta(a_{\theta(i)}, a_{\theta(j)}) \mid \delta(a_i, a_j) = 2 \text{ and } i, j \neq 127,\ \text{or } \delta(a_i, a_j) = 1\}$$

We evaluate this set for θ. The possible cases are:

$\delta(a_i, a_j) = 2,\ i, j\ \textit{even}:\ \ \delta(a_{\theta(i)}, a_{\theta(j)}) = 2\ \textit{or}\ 3,$
$\delta(a_i, a_j) = 2,\ i, j\ \textit{odd},\ i, j \neq 127:\ \ \delta(a_{\theta(i)}, a_{\theta(j)}) = 2$ or $3,$
$\delta(a_i, a_j) = 1,\ i\ \textit{even},\ j\ \textit{odd},\ i < j,\ j \neq 127:\ \ \delta(a_{\theta(i)}, a_{\theta(j)}) = 37\ \textit{or}\ 38,$
$\delta(a_i, a_j) = 1,\ i\ \textit{odd},\ j\ \textit{even},\ i < j,\ i \neq 127:\ \ \delta(a_{\theta(i)}, a_{\theta(j)}) = 35\ \textit{or}\ 36,$
$\delta(a_i, a_j) = 1,\ i = 126,\ j = 127:\ \ \delta(a_{\theta(i)}, a_{\theta(j)}) = 10,$
$\delta(a_i, a_j) = 1,\ i = 127,\ j = 1:\ \ \delta(a_{\theta(i)}, a_{\theta(j)}) = 28.$

Thus, the edge lengths of $\Theta(H_{127}) \subseteq \{2, 3, 10, 28, 35, 36, 37, 38\}$ and so $\Theta(H_{127})$ is a 28-th edge-disjoint Hamilton cycle in St_{127}. □

Summarizing Theorem 2.27, and Lemmas 2.28, and 2.29 gives our main result:

Theorem 2.30. *For any odd integer n, St_n contains at least $\lfloor 2\varphi(n)/9 \rfloor$ pairwise edge-disjoint Hamilton cycles, where $\varphi(n)$ is the Euler function.*

We can obtain bounds for St_n which do not invoke the Euler function, when n is prime. The bounds are an almost twofold improvement on those in [11]. We show that, St_n contains at least $\lfloor 2(n-1)/9 \rfloor$ pairwise edge-disjoint Hamilton cycles for all prime n, and so, we can have a bound of the type $\lfloor n/c \rfloor$ for some constant c. Morever, we can extend these results to powers of primes greater than 7.

Corollary 2.31. *If n is prime, then St_n contains at least $\lfloor n/5 \rfloor$ pairwise edge-disjoint Hamilton cycles.*

Proof. By Theorem 2.30, the number of edge-disjoint Hamilton cycles of St_n is at least $\lfloor 2\varphi(n)/9 \rfloor$. Suppose that $n < 11$. We check that, if $n \in \{3, 5\}$, then $\lfloor 2\varphi(n)/9 \rfloor = 0$ and, if $n = 7$, then $\lfloor 2\varphi(n)/9 \rfloor = \lfloor n/5 \rfloor = 1$. Thus, for $n \in \{3, 5, 7\}$, St_n contains at least $\lfloor n/5 \rfloor$ pairwise edge-disjoint Hamilton cycles. Suppose $n \geq 11$. By Theorem 2.30, and the fact that $\varphi(n) = n - 1$ if n is prime,

$$EDH(n) \geq \lfloor 2\varphi(n)/9 \rfloor = \lfloor 2(n-1)/9 \rfloor \geq \lfloor n/5 \rfloor.$$

Therefore, St_n contains at least $\lfloor n/5 \rfloor$ pairwise edge-disjoint Hamilton cycles for all prime n. □

Corollary 2.32. *Let n be a prime such that $n \notin \{3, 5, 7\}$, and let $\beta > 1$. Then St_{n^β} contains at least $\lfloor n^\beta/5 \rfloor$ pairwise edge-disjoint Hamilton cycles.*

Proof. If n is prime, then $\varphi(n^\beta) = n^\beta - n^{\beta-1}$. Therefore, by Theorem 2.30,

$$EDH(n^\beta) \geq \lfloor 2\varphi(n^\beta)/9 \rfloor = \lfloor 2(n^\beta - n^{\beta-1})/9 \rfloor.$$

Since $n \geq 11$, $n^\beta \geq 10n^{\beta-1}$ and thus $10(n^\beta - n^{\beta-1}) \geq 9n^\beta$. This implies that

$$2(n^\beta - n^{\beta-1})/9 \geq n^\beta/5.$$

So, $EDH(n^\beta) \geq \lfloor n^\beta/5 \rfloor$ and the proof is complete. □

Finally, we have the following lower bounds for the number of edge-disjoint Hamilton cycles for the cases of powers of 3, 5 and 7.

Corollary 2.33. *Let $EDH(n)$ be the number of edge-disjoint Hamilton cycles of St_n and $\beta > 1$. Then*

(i) $EDH(3^\beta) \geq \lfloor 3^\beta/7 \rfloor$,

(ii) $EDH(7^\beta) \geq \lfloor 7^\beta/6 \rfloor$.

Proof. Note that, if $n \in \{3, 7\}$, then $\varphi(n^\beta) = n^\beta - n^{\beta-1}$. For (i), since $5 \times 3^\beta \geq 14 \times 3^{\beta-1}$, we have that $(14 \times 3^\beta) - (9 \times 3^\beta) \geq 14 \times 3^{\beta-1}$. It follows that $2\varphi(3^\beta)/9 = 2(3^\beta - 3^{\beta-1})/9 \geq 3^\beta/7$. Thus, by Theorem 2.30, $EDH(3^\beta) \geq \lfloor 3^\beta/7 \rfloor$.

For (ii), since $3 \times 7^\beta \geq 12 \times 7^{\beta-1}$ we have that $(12 \times 7^\beta) - (12 \times 7^{\beta-1}) \geq 9 \times 7^\beta$. Then, $2\varphi(7^\beta)/9 = 2(7^\beta - 7^{\beta-1})/9 \geq 7^\beta/6$, i.e. $EDH(7^\beta) \geq \lfloor 7^\beta/6 \rfloor$.

□

We can improve the bound $\lfloor 2\varphi(n)/9 \rfloor$ to $\lfloor \varphi(n)/4 \rfloor = \varphi(n)/4$ for St_n where n is odd and divisible by 5.

Lemma 2.34. *Let n be an odd integer divisible by 5 and $\pi_n = \pi_n^1 \ldots \pi_n^k$, and let $\sigma(\pi_n^i)$ be the number of elements of π_n^i. If π_n^i is a coprime constituent of n, then $\sigma(\pi_n^i)$ is even.*

Proof. Firstly, note that the last digit of the positive powers of 2 cycles through the digits $2, 4, 6$, and 8. If the power is odd, then the last digit is 2 or 8, and if the power is even, then the last digit is 4 or 6. So, only the even powers of 2, can be of the form $5K + 1$ or $5K - 1$ (integers which leave remainders 1 or -1 when divided by 5).

Let π_n^i be a coprime constituent where $\sigma(\pi_n^i) = e$, and let $d \in \pi_n^i$. According to Lemma 2.26,

$$(\pi_n^i)^f(d) = (cn \pm d)/2^f$$

where $1 \leq c \leq 2^{f-1} - 1$ and $f \geq 2$. Since $\sigma(\pi_n^i) = e$, $(\pi_n^i)^e(d) = (cn \pm d)/2^e = d$. Thus, $cn = d(2^e \pm 1)$, and then, $cn/d = 2^e \pm 1$. As $gcd(n, d) = 1$, c is divisible by d. Let $c/d = m$. So, $nm = 2^e \pm 1$. n is divisible by 5 implies that 2^e is of the form $5K + 1$ or $5K - 1$. Thus, e is even. □

Theorem 2.35. *Let n be odd and divisible by 5. Then, the number of edge-disjoint Hamilton cycles of St_n is at least $\varphi(n)/4$ where $\varphi(n)$ is the Euler function.*

Proof. Let n be odd and divisible by 5, and let $\pi_n = \pi_n^1 \dots \pi_n^k$. Then, by Lemma 2.34, if π_n^i is a coprime constituent, $\sigma(\pi_n^i)$ is even for $1 \le i \le k$. Thus, by Lemma 2.18, for every coprime constituent π_n^i, we have at least $\sigma(\pi_n^i)/2$ edge-disjoint Hamilton cycles. Let $\pi_n^1, \dots, \pi_n^\ell$ be all the coprime constituents of n. As, $1/2 \sum_{i=1}^{\ell} \sigma(\pi_n^i) = \varphi(n)/4$, the number of all edge-disjoint Hamilton cycles of St_n is at least $\varphi(n)/4$. □

Corollary 2.36. *Let $EDH(n)$ be the number of edge-disjoint Hamilton cycles of St_n, and let $n = 5^\beta$ for $\beta \ge 1$. Then, by Theorem 2.35,*

$$EDH(n) = EDH(5^\beta) \ge \varphi(n)/4 = 5^{\beta-1} = n/5$$

The cases of $n = 5^\beta$ are also considered in Chapter 5 from the point of view of symmetric properties of collections of edge-disjoint Hamilton cycles.

2.6 Bounds for special cases of primes

In this section, we improve the lower bounds for the number of edge-disjoint Hamilton cycles in star graphs St_n of prime dimensions. Throughout this section, we assume that $\sigma(\pi_n^i)$ denotes the number of elements in constituent π_n^i.

Lemma 2.37. *Let $\pi_n^i = (d_1^i, \dots, d_{n_i}^i)$ be a constituent of n such that d_1^i is odd, and let $(\pi_n^i)^g(d_1^i) = d_1^i$ for some $g \ge 2$. Then, $n = d_1^i(2^g \pm 1)/c'$ where $1 \le c' \le 2^{g-1} - 1$, and $n_i = \sigma(\pi_n^i) \mid g$.*

Proof. According to Lemma 2.26, if $\pi_n^i = (d_1^i, \dots, d_{n_i}^i)$ is a constituent of n such that d_1^i is odd, and f is an integer greater than or equal to 2, then

$$(\pi_n^i)^f(d_1^i) = (cn \pm d_1^i)/2^f \tag{2.12}$$

where $1 \le c \le 2^{f-1} - 1$. As $\sigma(\pi_n^i) = n_i$ and $\pi_n(d_{n_i}^i) = d_1^i$, then $(\pi_n^i)^{n_i}(d_1^i) = (cn \pm d_1^i)/2^{n_i} = d_1^i$, and so, $n = d_1^i(2^{n_i} \pm 1)/c$ where $1 \le c \le 2^{n_i-1} - 1$. Thus, we have that $(\pi_n^i)^{n_i+j}(d_1^i) = d_{j+1}^i$ for $0 \le j \le n_i - 1$, and so, $(\pi_n^i)^{2n_i}(d_1^i) = d_1^i$. As a result, by an easy inductive argument, $(\pi_n^i)^{kn_i}(d_1^i) = d_1^i$ where k is an integer greater than or equal to 1. As, $d_s^i \ne d_r^i$ for $1 \le s, r \le n_i$, we can conclude that, if $(\pi_n^i)^g(d_1^i) = d_1^i$, then $n = d_1^i(2^g \pm 1)/c'$ where $1 \le c' \le 2^{g-1} - 1$, and $n_i = \sigma(\pi_n^i) \mid g$. □

In the following lemma, we prove that the number of elements in the constituent that includes 1, is greater than or equal to the number of elements of all the other constituents.

Lemma 2.38. *Let n be odd and $\pi_n = \pi_n^1, \ldots, \pi_n^k$, and let $1 \in \pi_n^1$. Then, for every $i \in \{2, \ldots, n\}$, $\sigma(\pi_n^1) \geq \sigma(\pi_n^i)$.*

Proof. Let $\pi_n^i = (d_1^i \ldots d_{n_i}^i)$. By Lemma 2.12 (i), let d_1^i be an odd integer. By Lemma 2.26, $(\pi_n^i)^f(d_1^i) = (cn \pm d_1^i)/2^f$ where f is an integer greater than or equal to 2 and c is an integer such that $1 \leq c \leq 2^{f-1} - 1$. Since $\sigma(\pi_n^i) = n_i$, $(\pi_n^i)^{n_i}(d_1^i) = \pi_n(d_{n_i}^i) = d_1^i$. Thus, there exists an odd integer $1 \leq c \leq 2^{n_i-1} - 1$, where $(cn \pm d_1^i)/2^{n_i} = d_1^i$, i. e.,

$$n = d_1^i \times (2^{n_i} \pm 1)/c$$

Assume that $1 \in \pi_n^1$. This implies that for an odd c where $1 \leq c \leq 2^{n_1-1} - 1$, $n = (2^{n_1} \pm 1)/c \times 1$. Without loss of generality, let $b = d_1^i \in \pi_n^i$ be an odd integer greater than 1 where $2 \leq i \leq k$. So, $b \in \{3, \ldots, (n-1)/2\}$, and we have that $n = b \times (2^{n_i} \pm 1)/c'$ where $1 \leq c' \leq 2^{n_i-1} - 1$. We need to prove that $n_i \leq n_1$. As $n = (2^{n_1} \pm 1)/c \times 1$, then $n = (2^{n_1} \pm 1)/cb \times b$. cb is an odd integer as c and b are both odd. We show that if $1 < b \leq (n-1)/2$, then cb is such that $1 < cb \leq 2^{n_1-1} - 1$ in the following cases:

- Let $c = 1$. Then, $n = [(2^{n_1} \pm 1)/b] \times b$. So, $n = 2^{n_1} \pm 1$. As $b \leq (n-1)/2$, $b \leq (2^{n_1} \pm 1 - 1)/2$. Thus, $b \leq 2^{n_1-1} - 1$ or $b \leq 2^{n_1-1}$. Since b is odd, $b \leq 2^{n_1-1} - 1$. Therefore, $1 < cb \leq 2^{n_1-1} - 1$.

- Let $c \geq 3$. Since $b \leq (n-1)/2$, then $cb \leq cn/2 - c/2$. As $n = (2^{n_1} \pm 1)/c$, it follows that $cn = 2^{n_1} \pm 1$. So, $cb \leq 2^{n_1-1} - c/2 \pm 1/2$. $c \geq 3$ implies that $c/2 \pm 1/2 \geq 1$. Thus, $cb \leq 2^{n_1-1} - (c/2 \pm 1/2) \leq 2^{n_1-1} - 1$.

So, we have that $n = (2^{n_1} \pm 1)/cb \times b$ where $1 < cb \leq 2^{n_1-1} - 1$. As $b \in \pi_n^i$ and $\sigma(\pi_n^i) = n_i$, by Lemma 2.37, $n_i | n_1$, and so, $n_i \leq n_1$. As a result, if $1 \in \pi_n^1$, then for every $i \in \{2, \ldots, k\}$, $\sigma(\pi_n^1) \geq \sigma(\pi_n^i)$. ☐

We show that all constituents of n have the same number of elements if n is a prime integer.

Lemma 2.39. *Let n be prime and $\pi_n = \pi_n^1 \ldots \pi_n^k$. Then, for $1 \leq i, j \leq k$, $\sigma(\pi_n^i) = \sigma(\pi_n^j)$.*

Proof. Assume that $1 \in \pi_n^1$. We show that $\sigma(\pi_n^1) = \sigma(\pi_n^i)$ for $i \in \{2, \ldots, k\}$. Let $b \in \pi_n^i$ be an odd integer greater than 1 where $2 < i \leq k$, and let $\sigma(\pi_n^i) = n_i$.

Assume, on the contrary, that $\sigma(\pi_n^1) \neq \sigma(\pi_n^i)$. By Lemma 2.38, $\sigma(\pi_n^1) \geq \sigma(\pi_n^i)$. Thus, $n_1 > n_i$. As $\sigma(\pi_n^i) = n_i$ and $b \in \pi_n^i$, then $n = b \times (2^{n_i} \pm 1)/c$ where $1 \leq c \leq 2^{n_i - 1}$. As n is prime, then, by Lemma 2.11, $n_i > 1$, and c should be divisible by b. Let $c = h \times b$. Thus, $n = 1 \times (2^{n_i} \pm 1)/h$. As h is an odd integer and $1 \leq h < 2^{n_i - 1} - 1$, by Lemma 2.37, $\sigma(\pi_n^1) = n_1 | n_i$. So, $n_1 \leq n_i$ which is a contradiction. Consequently, for prime n, $\sigma(\pi_n^1) = \sigma(\pi_n^i)$ for every $i \in \{2, \ldots, k\}$. Therefore, $\sigma(\pi_n^i) = \sigma(\pi_n^j)$ for $1 \leq i, j \leq k$. □

We now define two special cases of prime integers, safe primes and Fermat primes, to obtain new lower bounds on the number of edge-disjoint Hamilton cycles.

Definition 2.40. *Let n be a prime number. Then, n is called a safe prime if $n = 2z + 1$ where z is prime.*

The first few safe primes are $5, 7, 11, 23, 47, 59, 83, 107$, and 167.

Definition 2.41. *A Fermat prime is a Fermat number $F_n = 2^{2^n} + 1$ that is prime.*

The only known Fermat primes are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$. We show that the number of elements in every constituent of π_{F_n} in St_{F_n} is equal to 2^n, where F_n is a Fermat prime.

Lemma 2.42. *Let F_n be a Fermat prime, and let $\pi_{F_n} = \pi_{F_n}^1 \ldots \pi_{F_n}^k$. Then, for $1 \leq i \leq k$, $\sigma(\pi_{F_n}^i) = 2^n$.*

Proof. Let F_n be a Fermat prime, and let $1 \in \pi_{F_n}^1$. Then, $F_n = 2^{2^n} + 1$. So, 2^n is the least integer which satisfies $F_n = (2^{2^n} \pm 1)/c$ where $1 \leq c \leq 2^{2^n - 1} - 1$. Therefore, $\sigma(\pi_{F_n}^1) = 2^n$. Since F_n is prime, by Lemma 2.39, $\sigma(\pi_{F_n}^i) = 2^n$ for $1 \leq i \leq k$. □

In the next theorems, we prove that there exist at least $\lfloor n/4 \rfloor$ edge-disjoint Hamilton cycles of star graph St_n for two special cases of primes.

Theorem 2.43. *Let n be a safe prime. Then, the number of edge-disjoint Hamilton cycles of St_n is at least $\lfloor n/4 \rfloor$.*

Proof. Let $n = 2z + 1$ be a safe prime, and let $\pi_n = \pi_n^1 \ldots \pi_n^k$. Since $\sum_{i=1}^k \sigma(\pi_n^i) = (n-1)/2$, then $\sum_{i=1}^k \sigma(\pi_n^i) = z$ which is a prime number. By Lemma 2.39, $\sigma(\pi_n^1) = \sigma(\pi_n^i)$ for $i \in \{2, \ldots, k\}$. So, $z = k \times \sigma(\pi_n^1)$. Since z is a prime integer and as, by Lemma 2.11, $\sigma(\pi_n^1) > 1$, it follows that $k = 1$. We conclude that, $\sigma(\pi_n^1) = (n-1)/2 = z$. By Lemma 2.18, there are at least $(z-1)/2 = (n-3)/4$ edge-disjoint Hamilton cycles in St_n. According to [25], for $n = 5$, there are two edge-disjoint Hamilton cycles in St_5. With the exception of 5, a safe prime is of the form $4K + 3$ (integers which leave a remainder 3 when divided by 4). So, $\lfloor (n-3)/4 \rfloor = \lfloor (4K+3-3)/4 \rfloor = K$, and $\lfloor n/4 \rfloor = \lfloor (4K+3)/4 \rfloor = K$. Therefore, $\lfloor (n-3)/4 \rfloor = \lfloor n/4 \rfloor$. As a result, if n is a safe prime, then St_n contains at least $\lfloor n/4 \rfloor$ edge-disjoint Hamilton cycles and the proof is complete. $\square$

Theorem 2.44. *Let F_n be a Fermat prime. Then, the number of edge-disjoint Hamilton cycles of St_{F_n} is at least $\lfloor F_n/4 \rfloor$.*

Proof. Let F_n be a Fermat prime, and let $\pi_{F_n} = \pi_{F_n}^1 \dots \pi_{F_n}^k$. Then, by Lemma 2.42, $\sigma(\pi_{F_n}^i) = 2^n$ for $1 \leq i \leq k$. According to Lemma 2.18, for every $1 \leq i \leq k$, we have at least $\sigma(\pi_{F_n}^i)/2$ edge-disjoint Hamilton cycles. As a result, if F_n is a Fermat prime, then St_{F_n} contains at least $\lfloor (F_n - 1)/4 \rfloor = \lfloor F_n/4 \rfloor$ edge-disjoint Hamilton cycles and the proof is complete. $\square$

Hence, we can improve the lower bound for the number of edge-disjoint Hamilton cycles in star graphs St_n of prime dimensions from $\lfloor n/5 \rfloor$ to $\lfloor n/4 \rfloor$ for some special cases of primes.

Chapter 3

Symmetric disjoint Hamilton cycles

Symmetric properties of Hamilton decompositions have been studied extensively in cases of complete graphs. These include ‘symmetric’ collections of Hamilton cycles as in [2] and [8], and ‘cyclic’ collections of Hamilton cycles as in [9] and [28]. The former require the existence of a single involutory automorphism fixing all Hamilton cycles in the collection, whereas the latter require a single automorphism which is a cyclic permutation of all the vertices of the graph such that the collection of Hamilton cycles is invariant under the application of the automorphism. Hamilton decompositions having both properties have also been studied [10]. In this paper, we are interested in general symmetric properties of edge-disjoint Hamilton cycles in star graphs St_n for the purposes of designing better fault tolerant interconnection network topologies. Star graphs are Cayley graphs over the symmetric group and not much was known about disjoint Hamilton cycles in star graphs until recently, with much of the work on Hamilton decompositions of Cayley graphs revolving around Alspach's longstanding conjecture for Cayley graphs over Abelian groups [4]. The first results were the Hamilton decomposition for the star graph St_5 of dimension 5 constructed in [25], and the multiple edge-disjoint Hamilton cycles for general n-dimensional star graphs St_n in [11]. Surprisingly, the constructions were symmetric in the sense that (the edges of) any two Hamilton cycles were images of each other under automorphisms of labelled versions of St_n, mapping labels consistently, and all of them were automorphic to a base 2-labelled Hamilton cycle constructed in [11]. Although asymptotic bounds for the number of disjoint Hamilton cycles in St_n were given in [11], and the stated $\varphi(n)/10$ bounds for all n in [11] were improved to $\varphi(n)/5$ for odd n in [19] (given in Chapter 2 of this thesis), it was not known what the optimum bounds are for obtaining Hamilton cycles in this way and, indeed, whether or not St_n is Hamilton decomposable by these means for any n other than 5. Furthermore, there has been no work on how many disjoint Hamilton cycles could be generated by repeated application of a single automorphism to the base 2-labelled Hamilton cycle, thus providing collections of Hamilton cycles invariant under application of a single automorphism as in the case of cyclic collections discussed above, and a greater degree of symmetry for the benefit of interconnection network design. In this chapter, we define *symmetric* collections of disjoint Hamilton cycles for labelled versions St_n to be those for which, given a Hamilton cycle in the collection, there is an automorphism mapping labels consistently such that the chosen Hamilton cycle is the image of the base 2-labelled Hamilton cycle in [11] (see Lemma 1.16 in Chapter 1 of this thesis). A collection of disjoint Hamilton cycles is *strongly symmetric* if a single such automorphism can generate all the Hamilton cycles from the base Hamilton cycle. We show in this chapter that there are at most $\varphi(n)/2$ symmetric disjoint Hamilton cycles, where φ is Euler's totient function, and that this bound is sharp for all even n. In Chapter 4 we give conditions, in terms of Carmichael's function [13], on cases of even n for which this bound can and cannot be achieved by strongly symmetric collections. We are unable to give optimum bounds for symmetric collections of disjoint Hamilton cycles for the case of odd n, but give cases of odd n for which $\varphi(n)/4$ is the optimum bound for strongly symmetric collections in Chapter 5. All the cases that we give, whether they are for symmetric or strongly symmetric collections, improve on the known number of Hamilton cycles in the corresponding (unlabelled) star graphs St_n.

Throughout Chapters 3, 4 and 5, all arithmetic will be modulo n when St_n is the star graph in context. Therefore, $x = y$ will mean $x = y \; mod \; n$. Mostly, the 'mod n' will be omitted, but may sometimes appear for emphasis. In arithmetic modulo n, we shall use n instead of 0 so that the set of integers modulo n will be $\{1, \ldots, n\}$. If non-positive integers result from a calculation then the corresponding positive integer will be meant: e.g. if $n = 9$ then $a_0 = a_9$ and $a_{-4} = a_5$. Integers which leave a remainder r when divided by an integer q other than n, will be referred to as 'of the form $qK + r$', so that, for example, 5, 13, 21, ... are of the form $8K + 5$. For graphs, whenever we refer to 'disjoint' Hamilton cycles, we will mean edge-disjoint Hamilton cycles. If G is a graph, H is a subgraph of G, and Φ an automorphism of G, $\Phi(H)$ will refer to the subgraph of G that is the image of the vertices and edges of H under Φ. Equality of subgraphs H and H', $H = H'$, will mean equality of both the sets of vertices and edges.

In this chapter we work with edge-labelled undirected star graphs. Their directed counterparts will be introduced in Chapter 4. We define an edge labelling for star graphs St_n and label automorphisms which are automorphisms that map these labels consistently. We show that St_n cannot have symmetric collections of greater than $\varphi(n)/2$ disjoint Hamilton cycles in Theorem 3.14 and that therefore St_n is not symmetrically Hamilton decomposable for non-prime n (Corollary 3.15). If n is even, we show that St_n does have a symmetric collection of $\varphi(n)/2$ Hamilton cycles in Theorem 3.18 and that such a collection cannot be enlarged to include further non-symmetric 2-labelled edge-disjoint Hamilton cycles (Theorem 3.19).

3.1 Labelled star graphs and label automorphisms

Definition 3.1. *The n-dimensional labelled star graph $St_n = (V, E, L)$ is the (n-1)-regular graph of order $|S_n|$, where S_n is the symmetric group of permutations of order n, with a set V of vertices, E of edges and a mapping of edges to integer labels $L : E \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}$, given by:*

$$\begin{aligned} V(St_n) &= \{a_{\rho(1)} \cdots a_{\rho(n)} \mid \rho \in S_n\}, \\ E(St_n) &= \{e \mid e = \{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, \\ &\qquad a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\},\ \rho \in S_n\} \end{aligned}$$

$$L(\{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\}) = \delta(a_{\rho(1)}, a_{\rho(i)})$$

where

$$\delta(a_i, a_j) = min\{|i-j|, n-|i-j|\} \ \ (1 \leq i, j \leq n)$$

is the distance between a_i and a_j on the cyclic graph whose vertices are $a_1, \ldots, a_n$ in which a_n is adjacent to a_{n-1} and a_1.

The class of automorphisms of St_n of interest are those which map labels consistently.

Definition 3.2. *A label map for St_n is a bijection*

$$\phi^\ell : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}$$

of labels. An automorphism Φ is a label automorphism if there exists a label map ϕ^ℓ such that:

$$L(\{\Phi(v_1), \Phi(v_2)\}) = \phi^\ell(L\{v_1, v_2\}), \ \textit{for all } v_1, v_2 \in V(St_n)$$

3.2 Pointwise maps and distance maps

We will generate automorphism 'pointwise' by means of a bijection of the elements $\{a_1, \ldots, a_n\}$.

Definition 3.3. *A pointwise map for St_n is a bijection ϕ as in Lemma 2.5. The corresponding automorphism is the automorphism Φ as defined in Lemma 2.5. If ϕ is such that there exists a bijection*

$$\phi^d : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\}$$

satisfying, for all $a_i, a_j \in \{a_1, \ldots, a_n\}$,

$$\delta(\phi(a_i), \phi(a_j)) = \phi^d(\delta(a_i, a_j)) \tag{3.1}$$

then Φ is trivially a label automorphism with $\phi^\ell = \phi^d$ in Definition 3.2 (iii). We shall call ϕ^d the corresponding distance map.

Distance maps allude to distances in the cyclic graph of the elements $\{a_1, ..., a_n\}$, and not to distances in St_n. The class of label automorphisms generated by a pointwise map and with a distance map as in Definition 3.3 will be denoted by $\mathcal{A}_n$.

3.3 Symmetry and strong symmetry

Our definitions of symmetry are with respect to this class of automorphisms and the Hamilton cycle with edge labels 1 and 2 constructed in [11] as the base Hamilton cycle with which all Hamilton cycles have to be symmetric via an automorphism $\Phi \in \mathcal{A}_n$. First of all, we introduce some notation.

Definition 3.4. *A vertex $v \in V(St_n)$ of the form $a_i \ldots$ (respectively $\ldots a_i$), where $a_i \in \{a_1, \ldots a_n\}$ will be denoted by $\overrightarrow{a}_i$ (respectively $\overleftarrow{a}_i$) or $\overrightarrow{a}^k_i$ (respectively $\overleftarrow{a}^k_i$) for some subscript k if several such vertices are under consideration. For a vertex $v = \overrightarrow{a}_i = \overleftarrow{a}_j$ we define $head(v) = head(\overrightarrow{a}_i) = a_i$ and $last(v) = last(\overleftarrow{a}_j) = a_j$.*

Definition 3.5. *The base Hamilton cycle $H_{1\ 2}(n)$ in St_n is the Hamilton cycle constructed in [11] consisting of alternate paths of $n(n-1)-1$ edges with label 1 and single edges with label 2:*

$$\ldots \underbrace{\bullet \overset{1}{\text{---}} \bullet \ldots\ldots\ldots \bullet \overset{1}{\text{---}} \bullet}_{n(n-1)-1 \ edges} \overset{2}{\text{---}} \underbrace{\bullet \overset{1}{\text{---}} \bullet \ldots\ldots\ldots \bullet \overset{1}{\text{---}} \bullet}_{n(n-1)-1 \ edges} \overset{2}{\text{---}} \bullet \ldots$$

where the total number of edges with label 1 *in* $H_{1\ 2}(n)$ *is* $n! - (n-2)!$ *which is greater than the number of remaining edges with label* 1 *(= n!-(n!-(n-2)!)=(n-2)!) in* St_n*, and such that*

$$last(v) = a_n$$

for all vertices v *in* $H_{1\ 2}(n)$ *of edges with label* 2*.*

A collection of edge-disjoint Hamilton cycles in St_n are 'symmetric' if any Hamilton cycle in the collection is the image of $H_{1\ 2}(n)$ under an automorphism in $\mathcal{A}_n$. It is 'strongly symmetric' if a single automorphism in $\mathcal{A}_n$ generates all the Hamilton cycles in the collection from $H_{1\ 2}(n)$.

Definition 3.6. *A collection* $\widetilde{H}$ *of edge-disjoint Hamilton cycles in* St_n *is symmetric if* $H_{1\ 2}(n) \in \widetilde{H}$ *and if, for all* $H^e, H^f \in \widetilde{H}$*, there is a label automorphism* $\Phi_{ef} \in \mathcal{A}_n$ *such that*

$$\Phi_{ef}(H^e) = H^f \tag{3.2}$$

The collection $\widetilde{H}$ *is strongly symmetric if there is a single* $\Phi \in \mathcal{A}_n$ *such that, for some* $r \in \mathbb{N}$*,*

$$H_{1\ 2}(n), \Phi(H_{1\ 2}(n)), \ldots, \Phi^r(H_{1\ 2}(n)) \tag{3.3}$$

are exactly the distinct Hamilton cycles in $\widetilde{H}$ *and* Φ^{r+1} *is the identity mapping.*

Hamilton cycles that are the image of automorphisms in $\mathcal{A}_n$ have a similar structure.

Lemma 3.7. *Let* $\Phi \in \mathcal{A}_n$ *be a label automorphism with corresponding distance map* ϕ^d*. Then,* $\Phi(H_{1\ 2}(n))$ *is a Hamilton cycle consisting of alternate paths of* $n(n-1)-1$ *edges with label* $\phi^d(1)$ *and single edges with label* $\phi^d(2)$*:*

$$\cdots \underbrace{\bullet \overset{\phi^d(1)}{\text{———}} \bullet \cdots\cdots\cdots \bullet \overset{\phi^d(1)}{\text{———}} \bullet}_{n(n-1)-1\ edges} \overset{\phi^d(2)}{\text{———}} \underbrace{\bullet \overset{\phi^d(1)}{\text{———}} \bullet \cdots\cdots\cdots \bullet \overset{\phi^d(1)}{\text{———}} \bullet}_{n(n-1)-1\ edges} \overset{\phi^d(2)}{\text{———}} \bullet \cdots$$

Proof. Follows from Definitions 3.3 and 3.5. □

From Lemma 3.7, we see that a Hamilton cycle which is the image of $H_{1\ 2}(n)$ under a label automorphism in $\mathcal{A}_n$, is a succession of edges the majority of which share the same label and the remaining minority of which share the same second label. This leads to the following definition.

Definition 3.8. *A Hamilton cycle which is the image of* $H_{1\ 2}(n)$ *under an automorphism as in Lemma 3.7, will be denoted by* $H_{i\ j}(n)$ *(or just* $H_{i\ j}$ *if* n *is clear* *from the context) where the subscript* $i = \phi^d(1)$ *is the label for the majority of the edges and the subscript* $j = \phi^d(2)$ *is the label for the minority of the edges. We shall call these two sets of edges the majority and minority edges of* $H_{i\ j}$ *and shall denote them by* $E_{maj}(H_{i\ j})$ *and* $E_{min}(H_{i\ j})$ *respectively.*

3.4 Upper bounds for symmetric collections

Not all labels can be majority or minority labels of images of $H_{1\ 2}$ under label automorphisms from $\mathcal{A}_n$. The underlying reason for this is the difference in the length of cycles of different labels.

Definition 3.9. *The spanning subgraph of* St_n *comprising edges with labels* i *and* j *where* $i, j \in \{1, \dots, \lfloor n/2 \rfloor\}$ *will be denoted by* $C_{i\ j}(n)$ *and the spanning subgraph comprising only edges with label* i *will be denoted* $C_i(n)$*. Each* $C_i(n)$ *is a union of disjoint cycles* $B_i^x(n)$ *of edges with label* i *[11]*

$$E(C_i(n)) = \bigcup_{x \in X} E(B_i^x(n)) \qquad \textit{(X is some index set)}$$

We shall call a cycle $B_i^x(n)$ *an* i*-ball. Again, we will abbreviate our notation to* $C_{i\ j}$*,* C_i *and* B_i^x *when* n *is clear from the context and will drop the* x *index in* B_i^x *when only one* i*-ball is under consideration. For an* i*-ball* B_i*,* $|B_i|$ *will denote the number of edges in* B_i*.*

Lemma 3.10. *Let* B_i *be an* i*-ball in* St_n*, where* $i \in \{1, \dots, \lfloor n/2 \rfloor\}$*. Then,*

(i) $|B_i| = n(n-1)$ *if* i *is coprime to* n*, and*

(ii) $|B_i| < n(n-1)$ *if* i *is not coprime to* n*.*

Proof. Let $n = dq_1$ and $i = dq_2$ where $d = gcd(n, i)$ and $gcd(q_1, q_2) = 1$. Without loss of generality, assume that the vertex

$$a_1 \dots a_n \in B_i$$

Now, the elements

$$a_1, a_{1+i}, \dots, a_{1+(q_1-1)i}$$

are distinct (else, for some r, s such that $0 \le r < s \le (q_1 - 1)$ and $K \in \mathbb{N}$, $Kn + (1 + ri) = (1 + si)$ and so $Kdq_1 = (s - r)dq_2$ and as $gcd(q_1, q_2) = 1$, q_1 divides $(s - r)$ which is a contradiction as $(s - r) \le (q_1 - 1)$). The path in B_i of the form

$$\overrightarrow{a}_1, \overrightarrow{a}_{1+i}, \dots, \overrightarrow{a}_{1+(q_1-1)i},$$

where $\overrightarrow{a}_1 = a_1 \dots a_n$, rotates the elements $a_1, \dots, a_{1+(q_1-1)i}$ within the vertex $a_1 \dots a_n$ thus:

$$a_1 \to a_{1+i} \to \dots a_{1+(q_1-1)i} \to a_1$$

After $q_1 - 1$ such rotations, the starting vertex $a_1 \dots a_n$ is reached again, i.e. there is a path in B_i of $(q_1 - 1)$ sets of q_1 vertices

$$\underbrace{\overrightarrow{a}_1, \overrightarrow{a}_{1+i}, \dots \overrightarrow{a}_{1+(q_1-1)i}}_{q_1 \ vertices}, \underbrace{\dots\dots\dots}_{q_1 \ vertices}, \quad \dots \quad , \underbrace{\dots\dots\dots}_{q_1 \ vertices}, \overrightarrow{a}_1$$

separated by edges with label i, and returning to $\overrightarrow{a}_1$ after $q_1(q_1 - 1)$ steps. If i is coprime to n, $q_1 = n$ and (i) follows. If i is not coprime to n, then $q_1 < n$ and (ii) follows. □

Lemma 3.11. *Let $\Phi \in \mathcal{A}_n$ and let B_i^x be an i-ball in St_n, where $1 \leq i \leq \lfloor n/2 \rfloor$. Then, there exists an i'-ball $B_{i'}^{x'}$ in St_n, for some i' with $1 \leq i' \leq \lfloor n/2 \rfloor$, such that*

$$\Phi(B_i^x) = B_{i'}^{x'} \textit{ and } (gcd(i, n) = 1 \textit{ iff } gcd(i', n) = 1)$$

Proof. As Φ is an automorphism, $\Phi(B_i^x)$ is a cycle such that $|\Phi(B_i^x)|$ equals $|B_i^x|$. Also, as Φ is a label automorphism all edges of $\Phi(B_i^x)$ must have the same label, and thus $\Phi(B_i^x)$ must be an i'-ball, $B_{i'}^{x'}$ say, for some i' where $1 \leq i' \leq \lfloor n/2 \rfloor$. Then, by Lemma 3.10,

$$gcd(i, n) = 1 \text{ iff } |B_i^x| = n(n-1) = |B_{i'}^{x'}| \text{ iff } gcd(i', n) = 1$$

□

As a result of Lemma 3.11, we are able to give constraints on how automorphisms $\Phi \in \mathcal{A}_n$ map labels. Indeed, we can characterize the pointwise maps ϕ that generate label automorphisms $\Phi \in \mathcal{A}_n$.

Lemma 3.12. *Let $\Phi \in \mathcal{A}_n$ be a label automorphism with corresponding pointwise and distance maps ϕ and ϕ^d respectively, as in Definition 3.3. Then:*

(i) for all labels $\ell \in \{1, \dots, \lfloor n/2 \rfloor\}$,

$$gcd(\ell, n) = 1 \quad \textit{iff} \quad gcd(\phi^d(\ell), n) = 1$$

(ii) there exist $i_0, j \in \{1, \dots, n\}$, where j is coprime to n, such that

$$\phi(a_i) = a_{i_0 + ji} \qquad (1 \leq i \leq n)$$

Proof. For (i), let B_ℓ^x be a ℓ-ball in St_n. As Φ is a label automorphism with distance map ϕ^d, $\Phi(B_\ell^x)$ is a $\phi^d(\ell)$-ball, $B_{\phi^d(\ell)}^{x'}$ in St_n. By Lemma 3.11, $gcd(\ell, n) = 1$ iff $gcd(\phi^d(\ell), n) = 1$.

For (ii), let $i_0, i_1 \in \{1, \ldots, n\}$ be such that

$$\phi(a_n) = a_{i_0} \text{ and } \phi(a_1) = a_{i_1}$$

where ϕ is the pointwise map of Φ. Put

$$j_p = \delta(\phi(a_n), \phi(a_1)) = min\{|i_0 - i_1|, n - |i_0 - i_1|\}$$

As $\delta(a_n, a_1) = 1$ and $\delta(\phi(a_n), \phi(a_1)) = j_p$, it follows that

$$\phi^d(1) = j_p \tag{3.4}$$

Let $a_i \in \{a_1, \ldots, a_n\}$ and consider the $a_g, a_h \in \{a_1, \ldots, a_n\}$ such that

$$\phi(a_i) = a_g \text{ and } \phi(a_{i+1}) = a_h$$

As $\delta(a_i, a_{i+1}) = 1$, by (3.1) of Definition 3.3 and (3.4) we have that

$$\delta(a_g, a_h) = j_p$$

Therefore,

$$g - h = j_p \ mod \ n \quad \text{or} \quad g - h = -j_p \ mod \ n$$

and so

$$h = g - j_p \ mod \ n \quad \text{or} \quad h = g + j_p \ mod \ n$$

As $\phi(a_n) = a_{i_0}$ and ϕ is injective, it is clear that either

$$\phi(a_n) = a_{i_0}, \phi(a_1) = a_{i_0 - j_p}, \ldots, \phi(a_{n-1}) = a_{i_0-(n-1)j_p} \tag{3.5}$$

or

$$\phi(a_n) = a_{i_0}, \phi(a_1) = a_{i_0 + j_p}, \ldots, \phi(a_{n-1}) = a_{i_0+(n-1)j_p} \tag{3.6}$$

hold. If (3.5) is the case put $j = -j_p$ and if (3.6) is the case put $j = j_p$ and the proof of (ii) is complete.

□

Definition 3.13. *Given a label automorphism $\Phi \in \mathcal{A}_n$ and corresponding pointwise map $\phi(a_i) = a_{i_0+ji}$, i_0 is called the offset and j the generator of ϕ.*

The constraints of label automorphisms in turn impose limits on the number of edge-disjoint Hamilton cycles in symmetric collections.

Theorem 3.14. *Let $\widetilde{H}$ be a symmetric collection of disjoint Hamilton cycles in St_n. Then $|\widetilde{H}| \leq \varphi(n)/2$, where $|\widetilde{H}|$ is the number of Hamilton cycles in $\widetilde{H}$.*

Proof. By Definition 3.6, as $\widetilde{H}$ is symmetric, any Hamilton cycle in $\widetilde{H}$ is the image of $H_{1\ 2}$ under a label automorphism and thus, by Lemma 3.7 and Definition 3.8, is of the form $H_{i\ j}$ with majority edge labels i and minority edge labels j. By Lemma 3.12 (i) with $\ell = 1$, $gcd(i, n) = 1$. Thus, the disjoint Hamilton cycles in $\widetilde{H}$ can be listed as

$$H_{i_1\ j_1}, H_{i_2\ j_2}, \ldots, H_{i_s\ j_s}$$

with majority edges with labels $i_1, \ldots, i_s$ respectively and minority edges with labels $j_1, \ldots, j_s$ respectively, and

$$gcd(i_r, n) = 1 \quad \text{(for all } r \text{ with } 1 \leq r \leq s)$$

Therefore, $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}$ is a set of edge labels coprime to n, and there are at most $\varphi(n)/2$ such integer labels. $\square$

An important corollary to Theorem 3.14 is that, if n is not a prime number, St_n is not symmetrically Hamilton decomposable.

Corollary 3.15. *If $n \geq 5$ is not a prime number, then there is no symmetric collection of disjoint Hamilton cycles $\widetilde{H}$ such that*

$$E(St_n) = \bigcup_{H \in \widetilde{H}} E(H),$$

where $E(H)$ denotes the set of edges in Hamilton cycle H.

Proof. If the edges $E(St_n)$ of St_n are partitioned into a collection $\widetilde{H}$ of disjoint Hamilton cycles, $\widetilde{H}$ will have $\lfloor n/2 \rfloor$ such cycles if n is odd and $n/2 - 1$ such cycles if n is even. However, if the non-prime n is odd then $\varphi(n) < n - 1$ and if n is even $\varphi(n) \leq n/2$. By Theorem 3.14, $\widetilde{H}$ cannot be symmetric. $\square$

3.5 Lower bounds in even dimensions

Although St_n is not symmetrically Hamilton decomposable for any even integer n, we will find an optimal symmetric collection of disjoint Hamilton cycles, i.e. a collection with $\varphi(n)/2$ Hamilton cycles, in Theorem 3.18 below. Constructing a symmetric collection involves finding a collection of label automorphisms which, when applied to $H_{1\ 2}$, generate disjoint Hamilton cycles as the images of $H_{1\ 2}$. Lemma 3.12 (ii) characterizes the pointwise maps of label automorphisms to be of the form $\phi(a_i) = a_{i_0+ji}$. In the following Lemma 3.16 (i) and (ii), the converse is given, i.e. that any pointwise map of the form $\phi(a_i) = a_{i_0+ji}$ consistently defines a distance map of edge labels

$$\phi^d : \{1, ..., \lfloor n/2 \rfloor\} \mapsto \{1, ..., \lfloor n/2 \rfloor\}$$

and therefore a label automorphism.

Lemma 3.16. *Let n be odd or even and $i_0, j \in \{1, \ldots, n\}$ be such that j is coprime to n. If the bijection $\phi_j : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ is defined by*

$$\phi_j(a_i) = a_{i_0+ji} \quad (1 \leq i \leq n)$$

then the following hold:

(i) for all $a_g, a_h \in \{a_1, ..., a_n\}$,

$$\delta(\phi_j(a_g), \phi_j(a_h)) = min\{|j(g-h) \ mod \ n|, n - |j(g-h) \ mod \ n|\},$$

(ii) there exists a bijection $\phi_j^d : \{1, ..., \lfloor n/2 \rfloor\} \mapsto \{1, ..., \lfloor n/2 \rfloor\}$ such that, for all $a_g, a_h \in \{a_1, ..., a_n\}$,

$$\delta(\phi_j(a_g), \phi_j(a_h)) = \phi_j^d(\delta(a_g, a_h)),$$

(iii) if $i_0 = n$, i.e. $\phi_j(a_i) = a_{ji}$, then for the label automorphism Φ_j corresponding to ϕ_j as in Definition 3.3, we have that, for all $\overleftarrow{a}_n \in V(St_n)$, there exists $\overleftarrow{a}'_n \in V(St_n)$ such that

$$\Phi_j(\overleftarrow{a}_n) = \overleftarrow{a}'_n,$$

i.e. vertices ending in a_n are mapped to vertices ending in a_n by Φ_j.

Proof. For (i), we have that (arithmetic expressions are evaluated modulo n):

$$\begin{aligned} \delta(\phi_j(a_g), \phi_j(a_h)) &= min\{|(i_0+jg)-(i_0+jh)|, n - |(i_0+jg)-(i_0+jh)|\} \\ &= min\{|j(g-h)|, n - |j(g-h)|\} \end{aligned}$$

To prove (ii), we need to show that if $a_g, a_h, a_{g'}, a_{h'} \in \{a_1, \ldots, a_n\}$, then $\delta(a_g, a_h) =$

$\delta(a_{g'}, a_{h'})$ implies that $\delta(\phi_j(a_g), \phi_j(a_h)) = \delta(\phi_j(a_{g'}), \phi_j(a_{h'}))$. We have that:

$$\begin{aligned}
\delta(a_g, a_h) = \delta(a_{g'}, a_{h'}) &\Rightarrow min\{|g-h|, n-|g-h|\} \\
&= min\{|g'-h'|, n-|g'-h'|\} \\
&\Rightarrow |g-h| = |g'-h'| \quad \text{or} \quad |g'-h'| = n-|g-h| \\
&\Rightarrow \{|g-h|, n-|g-h|\} = \{|g'-h'|, n-|g'-h'|\} \\
&\Rightarrow \{|j(g-h)|,\ n-|j(g-h)|\} \\
&= \{|j(g'-h')|,\ n-|j(g'-h')|\} \\
&\Rightarrow \delta(\phi_j(a_g), \phi_j(a_h)) = \delta(\phi_j(a_{g'}), \phi_j(a_{h'})) \quad \text{(by (i))}
\end{aligned}$$

Condition (iii) follows immediately from the definition of the corresponding label automorphism Φ_j, Lemma 2.5, and the fact that $\phi_j(a_n) = a_n$ if $i_0 = n$. □

The offset i_0 in pointwise maps $\phi(a_i) = a_{i_0+ji}$ is important for ensuring that there is no clash of minority edges. Lemma 3.16 (iii) above shows that, if i_0 is not used, then vertices ending in a_n are mapped to vertices ending in a_n. As, by Definition 3.5, minority edges have vertices ending in a_n, any collection of disjoint Hamilton cycles which use exclusively pointwise maps without i_0, would have all minority edges in the collection with vertices ending in a_n. This would lead to the possibility of the same edges belonging to different Hamilton cycles in the collection, as a clash of edge labels of minority edges is unavoidable for all even n. By use of i_0, we can ensure that even though different Hamilton cycles may share the same minority edge labels, different Hamilton cycles will not share the same edges as their vertices will end in a different $a_i \in \{a_1, \ldots, a_n\}$. The next lemma, Lemma 3.17, introduces the pointwise map ϕ_{+1} which just replaces a_i by a_{i+1}.

Lemma 3.17. *Let $\phi_{+1} : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ be the pointwise map defined by:*

$$\phi_{+1}(a_i) = a_{i+1} \quad (1 \leq i \leq n)$$

Then:

(i) ϕ_{+1} defines a corresponding distance map

$$\phi^d_{+1} : \{1, \ldots, \lfloor n/2 \rfloor\} \mapsto \{1, \ldots, \lfloor n/2 \rfloor\},$$

such that, for all $\ell \in \{1, \ldots, \lfloor n/2 \rfloor\}$,

$$\phi^d_{+1}(\ell) = \ell$$

(ii) if Φ_{+1} is the label automorphism corresponding to ϕ_{+1} then, for all

$\overleftarrow{a}_n \in V(St_n)$, there exists $\overleftarrow{a}_1 \in V(St_n)$ such that

$$\Phi_{+1}(\overleftarrow{a}_n) = \overleftarrow{a}_1$$

i.e. vertices ending in a_n are mapped to vertices ending in a_1 by Φ_{+1}.

Proof. If $a_g, a_h \in \{a_1, ..., a_n\}$ then (with arithmetic being modulo n)

$$\begin{aligned}\delta(\phi_{+1}(a_g), \phi_{+1}(a_h)) &= min\{|(g+1)-(h+1)|,\ n-|(g+1)-(h+1)|\}\\ &= min\{|g-h|,\ n-|g-h|\}\\ &= \delta(a_g, a_h)\end{aligned}$$

Thus, ϕ_{+1} defines the identity distance map $\phi^d_{+1} : L \mapsto L$. For (ii), we have that:

$$\begin{aligned}\Phi_{+1}(a_{g_1} \dots a_{g_{n-1}} a_n) &= \phi_{+1}(a_{g_1}) \dots \phi_{+1}(a_{g_{n-1}})\phi_{+1}(a_n)\\ &= a_{g_1+1} \dots a_{g_{n-1}+1} a_1\end{aligned}$$

□

We now prove that, for all even n, there are $\varphi(n)/2$ symmetric disjoint Hamilton cycles. The Hamilton cycles are generated by the label automorphisms of chosen pointwise maps, and make additional use of the pointwise map ϕ_{+1} of Lemma 3.17 to resolve any possible clashes of minority edges.

Theorem 3.18. *For all even n, St_n has a symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles $\widetilde{H}$.*

Proof. Let

$$i_1, \dots, i_{\varphi(n)/2}$$

be the $\varphi(n)/2$ integers less than $n/2$ which are coprime to n. First of all, for all $j \in \{i_1, \dots, i_{\varphi(n)/2}\}$ define $\phi_j : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ by

$$\phi_j(a_i) = a_{ji}$$

Then, by Lemma 3.16 (ii), ϕ_j defines a distance map ϕ^d_j and corresponding label automorphism Φ_j as in Definition 3.3. Consider the image of $H_{1\ 2}$ under Φ_j. From Lemma 3.16 (i) and as $j < n/2$, we have that:

$$\delta(a_2, a_1) = 1 \ \text{ and } \ \delta(\phi_j(a_2), \phi_j(a_1)) = min\{|j|, n-|j|\} = j$$

and

$$\delta(a_3, a_1) = 2 \ \text{ and } \ \delta(\phi_j(a_3), \phi_j(a_1)) = min\{|2j|, n-|2j|\}$$

Thus, $\phi_j^d(1) = j$ and $\phi_j^d(2) = \pm 2j \ mod \ n$. Taking the image $\Phi_j(H_{1\ 2})$ for each $j \in \{i_1, \ldots, i_{\varphi(n)/2}\}$ we produce a list of Hamilton cycles (with the majority and minority edge labels indicated in the subscripts):

$$H_{i_1\ \pm 2i_1}, \ldots, H_{i_{\varphi(n)/2}\ \pm 2i_{\varphi(n)/2}} \tag{3.7}$$

as in Definition 3.8. As $i_1, \ldots, i_{\varphi(n)/2}$ are distinct odd integers coprime to n, each majority edge in any Hamilton cycle in (3.7) only occurs in that Hamilton cycle as no other Hamilton cycle has the same edge label. However, it is possible that different Hamilton cycles in (3.7) share the same minority edge labels. We may have, for some distinct $i_r, i_s \in \{i_1, \ldots, i_{\varphi(n)/2}\}$,

$$min\{|2i_r \ mod \ n|, n - |2i_r \ mod \ n|\} = min\{|2i_s \ mod \ n|, n - |2i_s \ mod \ n|\}$$

when $2i_r = -2i_s \ mod \, n$, i.e.

$$2i_s = n - 2i_r \ \text{ and so } \ i_s = n/2 - i_r \tag{3.8}$$

From (3.8), it is clear that any minority edge label may be common to at most two Hamilton cycles in (3.7). To resolve this clash of minority edge labels, we replace one of the Hamilton cycles involved by one with the same labels but different vertices for minority edges. Suppose that the minority edges of $H_{i_r\ \pm 2i_r}$ and $H_{i_s\ \pm 2i_s}$ clash, so that $i_s = n/2 - i_r$. Consider the Hamilton cycles:

$$H_{i_r\ \pm 2i_r} = \Phi_{i_r}(H_{1\ 2}) \ \text{ and } \ H'_{i_s\ \pm 2i_s} = \Phi_{+1}(H_{i_s\ \pm 2i_s}) = \Phi_{+1}(\Phi_{i_s}(H_{1\ 2})) \tag{3.9}$$

By Definitions 3.5 and 3.8, all vertices of minority edges of $H_{1\ 2}$ are of the form $\overleftarrow{a}_n$, and so, by Lemma 3.16 (iii), all vertices of minority edges of $\Phi_{i_r}(H_{1\ 2})$ and $\Phi_{i_s}(H_{1\ 2})$ are also of the form $\overleftarrow{a}_n$. From the latter it follows, by Lemma 3.17 (ii), that all vertices of minority edges of $\Phi_{+1}(\Phi_{i_s}(H_{1\ 2}))$ are of the form $\overleftarrow{a}_1$. Thus, as the vertices of minority edges of $H_{i_r\ \pm 2i_r}$ are of the form $\overleftarrow{a}_n$ and those of $H'_{i_s\ \pm 2i_s}$ are of the form $\overleftarrow{a}_1$, $H_{i_r\ \pm 2i_r}$ and $H'_{i_s\ \pm 2i_s}$ are edge-disjoint despite having the same minority edge labels. By resolving all pairs of clashes in this way in (3.7) we produce a collection of $\varphi(n)/2$ symmetric and edge-disjoint cycles as required. □

Theorem 3.18 shows that, for all even n, there is a symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles $\widetilde{H}$ and Theorem 3.14 shows that this is the best that can be achieved for *symmetric* collections. Can this $\varphi(n)/2$ bound be improved by adding non-symmetric disjoint Hamilton cycles to the collection $\widetilde{H}$ in Theorem 3.18? The answer is negative for 2-labelled Hamilton cycles sharing labels with Hamilton cycles in $\widetilde{H}$. If an extra disjoint Hamilton cycle $H'_{j\ i}$ could be added, such that there is some Hamilton cycle $H_{i\ j} \in \widetilde{H}$, then the label automorphism that maps $H_{1\ 2}$ to $H_{i\ j}$ would also map $H'_{2\ 1}$ to $H'_{j\ i}$, where

$$H'_{2\ 1} = C_{1\ 2} - H_{1\ 2},$$

is the spanning subgraph of St_n comprising the edges with labels 1 and 2 that are not in $H_{1\ 2}$, and $H'_{2\ 1}$ would be also be hamiltonian. If $H'_{2\ 1}$ is hamiltonian then, even though it is not symmetric to $H_{1\ 2}$ (as there is no distance map of $\{a_1, ..., a_n\}$ mapping distances 1 to distances 2 and distances 2 to distances 1 for all n greater than 5) the symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles in Theorem 3.18 could be doubled in size to produce a non-symmetric collection of $\varphi(n)$ Hamilton cycles that are still edge-disjoint. Unfortunately, $H'_{2\ 1}$ is not hamiltonian as the following theorem shows.

Theorem 3.19. *The spanning subgraph $H'_{2\ 1}$ of St_n, comprising the edges of labels 1 and 2 that are not in $H_{1\ 2}$, is not a Hamilton cycle if n is even.*

Proof. It is clear from Definition 3.5 that the number of edges with label 2 in $H_{1\ 2}$ is (n-2)!. Therefore, $H_{1\ 2}$ meets at most (n-2)! 2-balls. The total number of 2-balls in $C_{1\ 2}$ is the number of vertices in $C_{1\ 2}$ (= n!) divided by the number of vertices in a 2-ball:

$$|C_{1\ 2}|/|B_2| \tag{3.10}$$

As n is even and hence 2 is not coprime to n, by Lemma 3.10(ii) the number of vertices in a 2-ball is less than n(n-1) and so, by (3.10), the number of 2-balls exceeds (n-2)!. Hence, there is some 2-ball B_2^k which $H_{1\ 2}$ does not meet. Clearly, the edges of this 2-ball B_2^k must belong to $H'_{2\ 1}$ which then cannot be hamiltonian as it contains a cycle with fewer than $n!$ vertices. □

3.6 Symmetric collections in odd dimensions

Whilst the $\varphi(n)/2$ upper bound, on the number of Hamilton cycles in a symmetric collection also holds for St_n if n is odd, it is not clear that this bound can be achieved for any odd n other than n equals 5 [25]. In the case of even n, the number of Hamilton cycles in a symmetric collection $\widetilde{H}$ is limited to $\varphi(n)/2$ because every majority edge label in $\widetilde{H}$ has to be coprime to n as the majority edge label 1 of the base Hamilton cycle $H_{1\ 2}$ is coprime to n. However, in the case of odd n, both the majority and minority edge labels of Hamilton cycles in symmetric collections have to be coprime to n as both the majority and minority edge labels of $H_{1\ 2}$, i.e. 1 and 2, are coprime to n. For this reason, it would appear that the upper bound for symmetric collections in the case of odd n should be $\varphi(n)/4$. To exceed this bound would require a symmetric collection of Hamilton cycles $\widetilde{H}$ containing Hamilton cycles

$$H_{i\ \ell}, H_{\ell\ j} \in \widetilde{H}$$

such that the minority edges of $H_{i\ \ell}$ are exactly the edges with label ℓ that are not present as majority edges in $H_{\ell\ j}$. This is a very tight restriction which is satisfied for n equals 5 [25] where there is a distance map which maps labels 1 to 2, and therefore 2 to 1 as there are no other labels, such that the 2 Hamilton cycles produced automorphically map minority edges with label 2 in one Hamilton cycle to the unused edges with label 1 as minority edges in the second Hamilton cycle. It seems unlikely that the same majority and minority edge labels can occur in symmetric collections for odd n if n is greater than 5 and labels 1 and 2 cannot map to each other, though this remains an open problem. However, if $\varphi(n)/4$ is the true bound, this is nearly achieved for all but one odd n by the construction in [19] (given in Chapter 2 of this thesis).

Theorem 3.20. *For all odd $n \neq 127$, St_n has a symmetric collection of $2\varphi(n)/9$ disjoint Hamilton cycles $\widetilde{H}$.*

Proof. See [19] and Lemma 2.28 in Chapter 2 of this thesis. □

In Chapter 5, we are able to achieve the $\varphi(n)/4$ bound for certain cases of n. Indeed, the $\varphi(n)/4$ bound is achieved for strongly symmetric collections for those cases of n.

Chapter 4

Strongly symmetric disjoint Hamilton cycles in even dimensions

We know from Chapter 3 that, for all even n, St_n has a symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles. In this chapter, we consider whether there are any cases where this optimal $\varphi(n)/2$ symmetric bound can be achieved by strongly symmetric collections. The closest that symmetric collections come to a Hamilton decomposition for even n, in terms of the proportion of edges used, are the cases where n=2^k. In those cases, more than half of the edges in St_n are present in a symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles. We show in Theorem 4.8 below that, for all such n, St_n has a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles. Other cases are also shown to have strongly symmetric collections of $\varphi(n)/2$ disjoint Hamilton cycles (Corollary 4.11). However, not all even n have strongly symmetric collections at the $\varphi(n)/2$ bound. We show that if n is the product of any power of 2 greater than 2 and the power of any other prime, then there does not exist a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles for St_n (Corollary 4.15).

4.1 Directed labels and directed labelled star graphs

We introduce directed distances in (4.1) below and then directed versions of labels, distance maps, star graphs and label automorphisms. To show strong symmetry, we need to find a label automorphism Φ which can be used to generate a collection of edge-disjoint Hamilton cycles starting with $H_{1\ 2}$ and applying Φ repeatedly until the power of Φ which is the identity map is reached. In order to prove that a certain power of Φ is the identity map, we need to consider the effect of Φ on a directed version of our labelled star graphs.

Definition 4.1. *Given $a_i, a_j \in \{a_1, ..., a_n\}$, the directed distance $\delta^{\pm}(a_i, a_j)$ between a_i and a_j is defined to be:*

$$\delta^{\pm}(a_i, a_j) = (i - j) \; mod \, n \tag{4.1}$$

The directed labelled n-star graph $St_n^{\leftrightarrows}$ is a triple (V, E, L), where $V(St_n^{\leftrightarrows}) = V(St_n)$ is the same set of vertices as in the corresponding undirected star graph, $E(St_n^{\leftrightarrows})$ has a pair of arcs, here called directed edges, in both directions for each (undirected) edge in $E(St_n)$,

$$\begin{aligned} E(St_n) \;\; = \;\; & \{e \mid e = (a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, \\ & a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}), \;\rho \in S_n\} \end{aligned}$$

and $L : E(St_n^{\leftrightarrows}) \to \{1, \ldots, n-1\}$ maps directed edges to integer directed labels as follows:

$$L(\{a_{\rho(1)} \cdots a_{\rho(i-1)} a_{\rho(i)} a_{\rho(i+1)} \cdots a_{\rho(n)}, a_{\rho(i)} \cdots a_{\rho(i-1)} a_{\rho(1)} a_{\rho(i+1)} \cdots a_{\rho(n)}\})$$

$$= \delta^{\pm}(a_{\rho(1)}, a_{\rho(i)})$$

Note that, from Definitions 3.1 and 4.1, we have that, for all $a_i, a_j \in \{a_1, ..., a_n\}$,

$$\delta^{\pm}(a_i, a_j) = \delta(a_i, a_j) \; mod \; n \quad \text{or} \quad \delta^{\pm}(a_i, a_j) = (-\delta(a_i, a_j)) \; mod \; n \tag{4.2}$$

A label automorphism on an undirected star graph, which defines a mapping of labels ϕ^d, also defines a mapping of directed labels $\phi^{d\pm}$, given below, of the corresponding directed star graph.

Lemma 4.2. *Let $\Phi \in \mathcal{A}_n$ be a label automorphism of St_n with corresponding pointwise and distance maps ϕ and ϕ^d respectively as in Definition 3.3. Let $\phi^{d\pm}$ be a mapping of directed labels*

$$\phi^{d\pm} : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$$

defined as

$$\phi^{d\pm}(x) = \delta^{\pm}(\phi(a_i), \phi(a_j)),$$

where $a_i, a_j \in \{a_1, ..., a_n\}$ are such that $\delta^{\pm}(a_i, a_j) = x$. Then,

(i) $\phi^{d\pm}$ is well defined,

(ii) for all $k \geq 0$,

$$(\phi^{d\pm})^k(\delta^\pm(a_i, a_j)) = \delta^\pm(\phi^k(a_i), \phi^k(a_j)) = (\phi^k)^{d\pm}(\delta^\pm(a_i, a_j))$$

(iii) for all $\ell \in \{1, \ldots, \lfloor n/2 \rfloor\}$, $\phi^d(\ell) = \phi^{d\pm}(\ell) \bmod n$ *or* $\phi^d(\ell) = -\phi^{d\pm}(\ell) \bmod n$.

Proof. For (i), by Lemma 3.12(ii), there exist $i_0, j \in \{1, \ldots, n\}$ such that, for all $a_i \in \{a_1, ..., a_n\}$,

$$\phi(a_i) = a_{i_0+ji} \tag{4.3}$$

We need to show that, if $a_g, a_h, a_{g'}, a_{h'} \in \{a_1, ..., a_n\}$ are such that $\delta^\pm(a_g, a_h) = \delta^\pm(a_{g'}, a_{h'})$, then $\delta^\pm(\phi(a_g), \phi(a_h)) = \delta^\pm(\phi(a_{g'}), \phi(a_{h'}))$. Let $a_g, a_h, a_{g'}, a_{h'} \in \{a_1, ..., a_n\}$ be such that $\delta^\pm(a_g, a_h) = \delta^\pm(a_{g'}, a_{h'})$, i.e. by Definition 4.1,

$$(g - h) \bmod n \quad = \quad (g' - h') \bmod n$$

Then,

$$\begin{array}{rcll} \delta^\pm(\phi(a_g), \phi(a_h)) & = & \delta^\pm(a_{i_0+jg}, a_{i_0+jh}) & \text{(by (4.3))} \\ & = & j(g-h) & \text{(by Definition 4.1)} \\ & = & j(g'-h') & \\ & = & \delta^\pm(\phi(a_{g'}), \phi(a_{h'})) & \end{array}$$

For (ii), we have inductively by (i),

$$(\phi^{d\pm})^k(\delta^\pm(a_i, a_j)) = (\phi^{d\pm})^{k-1}(\delta^\pm(\phi(a_i), \phi(a_j))) = \ldots$$

$$\ldots = \delta^\pm(\phi^k(a_i), \phi^k(a_j)) = (\phi^k)^{d\pm}(\delta^\pm(a_i, a_j))$$

For (iii), choose $i, j \in \{1, \ldots, n\}$ such that $i - j = \ell$. Then,

$$\begin{array}{rcll} \phi^d(\ell) & = & \phi^d(\delta(a_i, a_j)) & \text{(by Definition 3.1)} \\ & = & \delta(\phi(a_i), \phi(a_j)) & \text{(by (3.1) of Definition 3.3)} \\ & = & \pm\delta^\pm(\phi(a_i), \phi(a_j)) \bmod n & \text{(by (4.2))} \\ & = & \pm\phi^{d\pm}(\delta^\pm(a_i, a_j)) \bmod n & \text{(by (i))} \\ & = & \pm\phi^{d\pm}(\ell) \bmod n & \text{(by Definition 4.1)} \end{array}$$

□

We have the following special cases of Lemma 4.2.

Lemma 4.3. *Let* $\Phi \in \mathcal{A}_n$ *be a label automorphism with pointwise map* ϕ *defined by* $\phi(a_i) = a_{i_0+ji}$, *and let the mapping* $\phi^{d\pm}$ *be as in Lemma 4.2.*

(i) If $\delta^{\pm}(a_g, a_h) = 1$ *then, for all* $k \geq 0$, $\delta^{\pm}(\phi^k(a_g), \phi^k(a_h)) = j^k \bmod n$.

(ii) If $\delta^{\pm}(a_g, a_h) = 2$ *then, for all* $k \geq 0$, $\delta^{\pm}(\phi^k(a_g), \phi^k(a_h)) = 2j^k \bmod n$.

4.2 Mapping directed labels

Definition 4.4. *Let* Φ, ϕ, ϕ^d *and* $\phi^{d\pm}$ *be as in Lemma 4.2. Then,* $\phi^{d\pm}$ *is called the corresponding directed distance map. As* ϕ^d *and* $\phi^{d\pm}$ *are permutations of labels and directed labels respectively, they can be decomposed into permutation cycles. We shall call the permutation cycles of* ϕ^d *and* $\phi^{d\pm}$ *the constituents and directed constituents, respectively, of* ϕ *or* Φ.

Constituents were introduced in Chapter 2 for the automorphism studied in that chapter. Directed constituents are just cosets of a subgroup of the multiplicative group of integers coprime to n modulo n. We use the term 'directed constituents', instead, as a comparison with constituents. The following result will be the basis of proofs that some power of a label automorphism is the identity mapping.

Lemma 4.5. *Let* $\Phi \in \mathcal{A}_n$ *be a label automorphism of* St_n *with corresponding pointwise and directed distance maps* ϕ *and* $\phi^{d\pm}$ *respectively. Suppose that, for some* $k \geq 1$,

$$(\phi^{d\pm})^k(1) = 1 \quad \text{and} \quad \phi^k(a_n) = a_n \tag{4.4}$$

Then, for all $a_i \in \{a_1, ..., a_n\}$,

$$\phi^k(a_i) = a_i \tag{4.5}$$

i.e. Φ^k *is the identity automorphism.*

Proof. Clearly, the mapping $\Phi^k : V(St_n) \mapsto V(St_n)$ is a label automorphism with corresponding pointwise map ϕ^k and directed distance map $(\phi^k)^{d\pm}$ where, for all $\ell \in \{1, \ldots, n-1\}$,

$$(\phi^k)^{d\pm}(\ell) = (\phi^{d\pm})^k(\ell)$$

By Lemma 3.12(ii), as ϕ^k is the pointwise map of the label automorphism Φ^k, there exist $i_0, j \in \{1, \ldots, n\}$ such that

$$\phi^k(a_i) = a_{i_0+ji} \quad (1 \leq i \leq n) \tag{4.6}$$

Also, by Lemma 4.2(ii) and (4.4),

$$\delta^{\pm}(\phi^k(a_2), \phi^k(a_1)) = (\phi^k)^{d\pm}(\delta^{\pm}(a_2, a_1)) = (\phi^{d\pm})^k(1) = 1 \tag{4.7}$$

Thus, by Definition 4.1, (4.6) and (4.7),

$$(i_0 + 2j) - (i_0 + j) = j = 1 \bmod n \tag{4.8}$$

By (4.4), (4.6) and (4.8),

$$a_n = \phi^k(a_n) = a_{i_0+jn} = a_{i_0+n} \tag{4.9}$$

and so, by (4.9), $i_0 = n$ and hence $\phi^k(a_i) = a_i$ for all i in (4.6). □

4.3 Primitive roots and generators

In order to construct a strongly symmetric collection of disjoint Hamilton cycles starting from $H_{1\ 2}$ we need to find a pointwise map $\phi(a_i) = a_{i_0+ji}$, for suitable $i_0, j \in \{1, \ldots, n\}$, which defines the Φ that will generate the Hamilton cycles. The majority edge labels of all Hamilton cycles in the collection will be coprime to n as the majority edge label of $H_{1\ 2}$ is coprime to n and therefore, by repeated application of Lemma 3.12(i), $gcd((\phi^d)^k(1), n) = 1$ for all $k \geq 0$, i.e.

$$1, \phi^d(1), (\phi^d)^2(1), \ldots\ , \tag{4.10}$$

which are the majority edge labels of the Hamilton cycles in the collection, are all coprime to n. To find the largest strongly symmetric collection we need to maximize the list of distinct coprime integers in (4.10). It is tempting to consider instead the list

$$1, \phi^{d\pm}(1), (\phi^{d\pm})^2(1), \ldots\ , \tag{4.11}$$

as, by Lemma 4.2(iii), each entry in (4.11) is only plus or minus the corresponding entry in (4.10) and therefore represents the same (undirected) label. In order to maximize the size of the list (4.11) we note that, by Lemma 4.3(i), it is equal to (with all calculations being modulo n):

$$1, j, j^2, \ldots$$

It would seem, therefore, that choosing a primitive root modulo n as j, if such exists, would produce the largest possible list. This would suggest that maybe for all n that have primitive roots - in the case of even n this is known to be when $n = 2p^k$ where p is a prime number greater than 2 - a strongly symmetric collection of $\varphi(n)$ disjoint Hamilton cycles could be produced by putting j equal to the primitive root. This is not possible as Theorem 3.14 shows that a symmetric (and, a fortiori, a strongly symmetric) collection can have at most $\varphi(n)/2$ Hamilton cycles. The reason that putting j equal to the primitive root fails is that in the list of distinct directed labels

$$1, j, \ldots, j^{\varphi(n)-1}$$

every undirected label occurs twice and corresponding Hamilton cycles would have clashes of majority edge labels and would not be edge-disjoint. A better generator j would be one that generates $\varphi(n)/2$ integers coprime to n

$$1, j, \ldots, j^{\varphi(n)/2-1}, \tag{4.12}$$

such that $j^{\varphi(n)/2} = 1 \bmod n$, in which the corresponding undirected edge labels are distinct. We shall show that this can be achieved for $n = 2^k$, where $k \geq 3$, by using 'primitive lambda-roots' instead. Primitive lambda-roots stem from the Carmichael function $\lambda(n)$ [13] which is defined to be the smallest integer m such that

$$j^m = 1 \bmod n$$

for all integers j that are coprime to n. A *primitive lambda root* is any coprime j satisfying

$$j^{\lambda(n)} = 1 \bmod n, \quad j^k \neq 1 \bmod n \text{ if } k < \lambda(n) \tag{4.13}$$

If n has a primitive root then $\lambda(n) = \varphi(n)$, but the more interesting cases for us are when $\lambda(n) \neq \varphi(n)$. In particular, if $\lambda(n) = \varphi(n)/2$, j is a primitive lambda root, and $-1 \bmod n$ does not appear in (4.12), then the corresponding undirected edge labels of the directed edge labels in (4.12) will be distinct. This will follow from the following two lemmata (all arithmetic is modulo n).

Lemma 4.6. *Let j be coprime to n and such that either*

(i) $j^{\varphi(n)/2} = 1$ and $\mid j^q \mid \neq 1$ for any q with $1 \leq q < \varphi(n)/2$, or

(ii) $j^{\varphi(n)/2} = -1$ and $j^q \neq -1$ for any q with $1 \leq q < \varphi(n)/2$.

Then, for all r, s such that $1 \leq r < s \leq \varphi(n)/2$,

$$j^r + j^s \neq n$$

and therefore

$$1, j, \ldots, j^{\varphi(n)/2-1}$$

is a sequence of distinct undirected edge labels of St_n .

Proof. If $j^r + j^s = n$, then $j^r(1 + j^{s-r}) = n$ and, as j is coprime to n, $j^{s-r} = -1$ and thus neither (i) nor (ii) can be the case as $s - r < \varphi(n)/2$. □

Lemma 4.7. *For all $n = 2^k$ where $k \geq 3$, there is a primitive lambda root j satisfying Lemma 4.6(i).*

Proof. It is well known that $\lambda(n) = \varphi(n)/2$ if $n = 2^k$ and $k \geq 3$. Thus, there is some primitive lambda root j such that $j^{\varphi(n)/2} = 1$. By Theorems 8.8 and 8.9 in [12], there are no 'negating' primitive lambda-roots of n, i.e. no primitive lambda-roots j such that $j^q = -1$ for some q with $1 \leq q \leq \lambda(n)(= \varphi(n)/2)$. Thus, Lemma 4.6(i) is satisfied by any chosen primitive lambda root. □

4.4 Strong symmetry in dimensions that are powers of two

We can now show that St_n has a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles if $n = 2^k$ and $k \geq 3$.

Theorem 4.8. *Let $n = 2^k$ where $k \geq 3$. Then, St_n has a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles.*

Proof. Let the pointwise map ϕ be defined by

$$\phi(a_i) = a_{(j-1)+ji}, \tag{4.14}$$

where j is a primitive lambda root modulo n as in Lemma 4.7, and let $\widetilde{H}$ be the collection of Hamilton cycles comprising

$$H_{1\ 2}, \Phi(H_{1\ 2}), \ldots, \Phi^{\varphi(n)/2-1}(H_{1\ 2}), \tag{4.15}$$

where Φ is the label automorphism corresponding to ϕ. In order to prove strong symmetry of $\widetilde{H}$, Definition 3.6 requires that we prove the Hamilton cycles in (4.15) to be edge-disjoint and $\Phi^{\varphi(n)/2}$ to be the identity mapping, i.e. $\phi^{\varphi(n)/2}(a_i) = a_i$ for all $a_i \in \{a_1, ..., a_n\}$.

Firstly, we show that no two Hamilton cycles in (4.15) share the same label for their majority edges. Note that each undirected edge in St_n that is in $H_{1\ 2}$, has label equal to 1 or 2, and has two corresponding directed edges with directed labels +1 and -1 or +2 and -2 in $St_n^{\leftrightarrows}$. Hence, there is a Hamilton cycle $H^+_{1\ 2}$ in $St_n^{\leftrightarrows}$ which, for each undirected edge labelled 1 or 2 in $H_{1\ 2}$, has its corresponding positively labelled directed edge, i.e. a directed edge labelled by +1 or +2, as an edge of the Hamilton cycle $H^+_{1\ 2}$. (We remark here that if the vertices $v_1, \ldots, v_{n!}$ follow the path of the Hamilton cycle $H_{1\ 2}$, the directed edges of $H^+_{1\ 2}$ are not necessarily, and nor do we require them to be, the pairs $(v_1, v_2), \ldots, (v_{n!-1}, v_{n!})$.

Some of these pairs may be reversed in $H^+_{1\ 2}$.) This gives rise to a succession of Hamilton cycles in $St_n^{\leftrightarrows}$, obtained from those in St_n listed in (4.15),

$$H^+_{1\ 2}, \Phi(H^+_{1\ 2}), \ldots, \Phi^{\varphi(n)/2-1}(H^+_{1\ 2}),$$

which, by (4.14) and Lemma 4.3, can be written as

$$H^+_{1\ 2}, H_{j\ 2j}, \ldots, H_{j^{\varphi(n)/2-1}\ 2j^{\varphi(n)/2-1}}, \tag{4.16}$$

displaying majority and minority directed edge labels in the subscripts. If we can show that the two directed labels of the majority edges of any two Hamilton cycles in (4.16) are not equal to plus or minus of each other (modulo n), then the two undirected labels of the majority edges of the corresponding Hamilton cycles in (4.15) are different and so no two Hamilton cycles in (4.15) have majority edges in common. So, assume, on the contrary, that two Hamilton cycles in (4.16)

$$H_{j^r\ 2j^r} \text{ and } H_{j^s\ 2j^s}, \text{ where } 1 \leq r < s \leq \varphi(n)/2,$$

have majority edge labels which are equal to plus or minus of each other, i.e. $j^r = j^s$ or $j^r + j^s = n$. If $j^r = j^s$ then $j^{s-r} = 1$ which contradicts the fact that j is a primitive lambda root as $s - r < \varphi(n)/2 = \lambda(n)$ (see (4.13)). But, the chosen j satisfies that in Lemma 4.7 and therefore that in Lemma 4.6(i) and so, by Lemma 4.6(i), $j^r + j^s \neq n$. It follows that the majority edge labels of two Hamilton cycles in (4.15) are different.

Secondly, we note that majority edge labels cannot clash with minority edge labels in two Hamilton cycles in (4.15) as the majority edge label of $H_{1\ 2}$ (which equals 1) is coprime to n and therefore, by Lemma 3.12(i), the majority edge labels of all succeeding Hamilton cycles are coprime to n, whereas the minority edge labels of $H_{1\ 2}$ (which equals 2) is not coprime to n which is even and therefore all succeeding minority edge labels are not coprime to n.

Thirdly, we consider clashes of minority edge labels between two Hamilton cycles in (4.15). A clash of minority edge labels can occur in a similar way to that described in Theorem 3.18. As in that theorem, we switch our attention to the vertices of minority edges instead in order to prove that the same minority edge cannot occur in two Hamilton cycles in (4.15). We show that the sets of vertices of minority edges, in two Hamilton cycles in (4.15), are disjoint. By Definition 3.5, for all vertices v of minority edges in $H_{1\ 2}$, we have that $last(v) = a_n$. We compute such last(v)s for successive Hamilton cycles in (4.15) (below $V_{min}(H)$ denotes the set of vertices of the minority edges of Hamilton cycle H):

$$
\begin{aligned}
&last(v) = a_n \quad (v \in V_{min}(H_{1\ 2}))\\
&last(v) = \phi(a_n) = a_{(j-1)+jn} = a_{(j-1)} \quad (v \in V_{min}(\Phi(H_{1\ 2})))\\
&last(v) = \phi^2(a_n) = \phi(\phi(a_n)) = a_{(j-1)+j(j-1)} = a_{(j-1)(1+j)} \quad (v \in V_{min}(\Phi^2(H_{1\ 2})))\\
&last(v) = \phi^3(a_n) = \phi(\phi^2(a_n)) = a_{(j-1)+j(j-1)(j+1)} = a_{(j-1)(1+j+j^2)}\\
&(v \in V_{min}(\Phi^3(H_{1\ 2})))\\
&\dots\\
&last(v) = \phi^{\varphi(n)/2-1}(a_n) = a_{(j-1)(1+j+j^2+\dots j^{\varphi(n)/2-2})} \quad (v \in V_{min}(\Phi^{\varphi(n)/2-1}(H_{1\ 2})))
\end{aligned}
$$

Given that j is a primitive lambda root and thus $j^{\lambda(n)} = j^{\varphi(n)/2} = 1$, we have that

$$
\phi^{\varphi(n)/2}(a_n) = \phi(\phi^{\varphi(n)/2-1}(a_n)) = a_{(j-1)(1+j+j^2+\dots j^{\varphi(n)/2-1})}
$$

$$
= a_{(j-1)(j^{\varphi(n)/2}-1)/(j-1)} = a_n \tag{4.17}
$$

and so the last(v)s $\phi(a_n), \phi^2(a_n), \dots, \phi^{\varphi(n)/2-1}(a_n), \phi^{\varphi(n)/2}(a_n)$ are:

$$
a_{j-1}, \dots, a_{j^r-1}, \dots, a_{j^{\varphi(n)/2}-1}
$$

No two of these can be the same, else if

$$
a_{j^r-1} = a_{j^s-1}, \quad \text{such that } 1 \le r < s \le \varphi(n)/2,
$$

then $j^r = j^s$ and $j^{s-r} = 1$ where $s - r < \varphi(n)/2$ contrary to the fact that j is a primitive lambda root (4.13). As all the last(v)s are different, no two Hamilton cycles in (4.15) can have the same minority edge. We have now shown that the Hamilton cycles in (4.15) are edge-disjoint.

Finally,

$$
\begin{aligned}
(\phi^{d\pm})^{\varphi(n)/2}(1) &= (\phi^{d\pm})^{\varphi(n)/2}(\delta^{\pm}(a_2, a_1))\\
&= \delta^{\pm}(\phi^{\varphi(n)/2}(a_2), \phi^{\varphi(n)/2}(a_1)) \quad \text{(by Lemma 4.2(ii))}\\
&= j^{\varphi(n)/2} \quad \text{(by Lemma 4.3(i))}\\
&= 1,
\end{aligned}
$$

and $\phi^{\varphi(n)/2}(a_n) = a_n$ by (4.17). By Lemma 4.5, $\phi^{\varphi(n)/2}(a_i) = a_i$ for all $a_i \in \{a_1, ..., a_n\}$ and therefore $\Phi^{\varphi(n)/2}$ is the identity automorphism. Thus, the collection $\widetilde{H}$ at (4.15) is strongly symmetric. □

4.5 Strong symmetry in dimensions that are twice the power of a prime

Does St_n have a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles if the Carmichael function $\lambda(n)$ does not equal $\varphi(n)/2$? We consider the cases where $\lambda(n) > \varphi(n)/2$ (i.e. $\lambda(n) = \varphi(n)$ as $\lambda(n)$ divides $\varphi(n)$) in this section and where $\lambda(n) < \varphi(n)/2$ in the next section.

Lemma 4.9. *Let j be a primitive root modulo n and $\varphi(n)$ be even. Then, $j^{\varphi(n)/2} = -1$ and $j^r \neq -1$ if $1 \leq r < \varphi(n)/2$.*

Proof. As j is a primitive root modulo n, $j^r = -1$ for some unique r with $1 \leq r < \varphi(n)$. If $r < \varphi(n)/2$, then $j^{2r} = 1$ which cannot be the case as $2r < \varphi(n)$. If $\varphi(n)/2 < r < \varphi(n)$, then $j^{2r-\varphi(n)} = 1$ which cannot be the case as $2r - \varphi(n) < \varphi(n)$. □

Theorem 4.10. *Suppose that n is even, $\lambda(n) = \varphi(n)$, and that $\varphi(n)/2$ is odd. Then, St_n has a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles.*

Proof. Let j be a primitive root modulo n, so that

$$1 = j^{\varphi(n)}, j, j^2, \ldots, j^{\varphi(n)-1} \tag{4.18}$$

are all the distinct integers coprime to n. By Lemma 4.9, $j^{\varphi(n)/2} = -1$. Then, the sequence

$$1 = (-1)^{\varphi(n)/2}.j^{\varphi(n)/2} = (-j)^{\varphi(n)/2}, -j, (-j)^2, \ldots, (-j)^{\varphi(n)/2-1} \tag{4.19}$$

does not contain -1, else -1 or 1 would occur in the subsequence

$$j, j^2, \ldots, j^{\varphi(n)/2-1}$$

of the sequence in (4.18), which cannot happen by Lemma 4.9 and the fact that j is a primitive root. Thus, $-j$ satisfies the conditions for j of Lemma 4.6(i) and therefore, by that lemma, the integers in (4.19) are distinct. Put

$$\phi(a_i) = a_{((-j)-1)+(-j)i}$$

and consider the Hamilton cycles

$$H_{1\ 2}, \Phi(H_{1\ 2}), \ldots, \Phi^{\varphi(n)/2-1}(H_{1\ 2}), \tag{4.20}$$

where Φ is the corresponding label automorphism. The sequence of majority edge labels of the sequence of Hamilton cycles in (4.20) is exactly the sequence of (distinct) integers in (4.19) and thus there is no clash of majority edge labels. The proof that majority and minority edge labels do not clash, that minority edges do not clash and that $\Phi^{\varphi(n)/2}$ is the identity mapping is exactly the same as in Theorem 4.8. □

The set of even n for which $\lambda(n) = \varphi(n)$ and $\varphi(n)/2$ is odd, as specified in Theorem 4.10, is infinite. Corollary 4.11 gives an example of infinitely many such n.

Corollary 4.11. *Let $n = 2p^k$ where p is a prime number of the form $4K + 3$. Then, St_n has a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles.*

Proof. As p is an odd prime number, there is a primitive root modulo n and so $\lambda(n) = \varphi(n)$, Also,

$$\varphi(n) = p^{k-1}(p - 1)$$

which is of the form $p^{k-1}(4K + 2)$ and so $\varphi(n)/2$ is odd. The result follows by Theorem 4.10. □

4.6 Cases where symmetric exceed strongly symmetric bounds

In all cases where $\lambda(n) < \varphi(n)/2$, St_n does not have a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles. The result is also true for such odd n. For such even n, this means that fewer strongly symmetric disjoint Hamilton cycles are possible than merely symmetric disjoint Hamilton cycles.

Theorem 4.12. *Let n be odd or even and such that $\lambda(n) < \varphi(n)/m$ where $m \geq 1$. Then, St_n does not have a strongly symmetric collection of $\varphi(n)/m$ disjoint Hamilton cycles.*

Proof. Let Φ be any label automorphism. By Lemma 3.12(ii), Φ is defined by a pointwise map $\phi : \{a_1, ..., a_n\} \mapsto \{a_1, ..., a_n\}$ such that $\phi(a_i) = a_{i_0+ji}$ for some $i_0, j \in \{1, \ldots, n\}$ with j coprime to n. If $\phi^{d\pm}$ is the corresponding directed distance map then, by Lemma 4.2(ii) and Lemma 4.3(i) with $a_g = a_2$ and $a_h = a_1$, we have that:

$$\phi^{d\pm}(1) = j, (\phi^2)^{d\pm}(1) = j^2, \ldots, (\phi^{\lambda(n)})^{d\pm}(1) = j^{\lambda(n)} \tag{4.21}$$

By Carmichael's Theorem, as j is coprime to n, $j^{\lambda(n)} = 1$. From (4.21), it follows that the labels of the majority edges of the Hamilton cycles:

$$H_{1\ 2}, \Phi(H_{1\ 2}), \ldots, \Phi^{\lambda(n)}(H_{1\ 2}) \tag{4.22}$$

are, successively,

$$1, j, \ldots, j^{\lambda(n)} = 1 \tag{4.23}$$

As, by Definitions 3.8 and 3.5, no two edge-disjoint Hamilton cycles can both have majority edges with label 1, it follows from (4.23) that the Hamilton cycles (4.22) cannot be edge-disjoint. Thus, Φ cannot generate $\varphi(n)/m$ ($\varphi(n)/m > \lambda(n)$) edge-disjoint Hamilton cycles. As the chosen Φ was arbitrary, it follows that St_n does not have a strongly symmetric collection of $\varphi(n)/m$ disjoint Hamilton cycles. □

We now give an infinite set of even n for which St_n does not have a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles because $\lambda(n) < \varphi(n)/2$. The set we choose gives a partial converse to Theorem 4.8, in that it shows that if $n = 2^k p^r$ where $k \geq 3$, p is prime and $r \geq 1$, then St_n does not have a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles. The result is given in Corollary 4.15 and follows from Theorem 4.12 and the calculation of Carmichael's function in the following two lemmata.

Lemma 4.13.

(i) If $n = p_1^{\alpha_1} \ldots p_w^{\alpha_w}$ for distinct primes $p_1, \ldots, p_w$ and $\alpha_i > 0$ $(1 \leq i \leq w)$ then

$$\lambda(n) = lcm(\lambda(p_1^{\alpha_1}), \ldots, \lambda(p_w^{\alpha_w}))$$

(ii) If $n = 2^k$, where $k \geq 3$, then

$$\lambda(n) = \varphi(n)/2 = 2^{k-2}$$

Proof. See [12] Proposition 5.1, for example. □

Lemma 4.14. *Let $n = 2^k p^r$ where $k \geq 3$, p is prime and $r \geq 1$. Then,*

(i) if p is of the form $4K + 3$ then $\lambda(n) = \varphi(n)/4$, and

(ii) if p is of the form $4K + 1$ then $\lambda(n) \leq \varphi(n)/4$.

Proof. Firstly, note that $\varphi(n) = (2^k p^r - (2^k p^r)/p) * 1/2 = 2^{k-1}(p-1)p^{r-1}$. For (i), we have that $(p-1)/2$ is odd as p is of the form $4K + 3$. Thus, by repeated use of Lemma 4.13,

$$
\begin{aligned}
\lambda(n) &= lcm(\lambda(2^k), \lambda(p^r)) \\
&= lcm(2^{k-2}, \varphi(p^r)) && (p^r \text{ has a primitive root}) \\
&= lcm(2^{k-2}, (p-1)p^{r-1}) \\
&= lcm(2^{k-2}, ((p-1)/2) * 2p^{r-1}) \\
&= 2^{k-2} * (p-1)/2 * p^{r-1} && ((p-1)/2 \text{ is odd and coprime to } p) \\
&= \varphi(n)/4
\end{aligned}
$$

For (ii), we note that $(p-1)/2$ is even. By repeated use of Lemma 4.13,

$$
\begin{aligned}
\lambda(n) &= lcm(\lambda(2^k), \lambda(p^r)) \\
&= lcm(2^{k-2}, (p-1)p^{r-1}) \\
&= lcm(2^{k-2}, (4(p-1)/4)p^{r-1}) \\
&\leq 2^{k-1} * (p-1)/4 * p^{r-1} && ((p-1)/4 \text{ may have factors of } 2) \\
&= \varphi(n)/4
\end{aligned}
$$

□

Corollary 4.15. *Let $n = 2^k p^r$ where $k \geq 3$, p is prime and $r \geq 1$. Then, St_n does not have a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles.*

Proof. Follows from Theorem 4.12 and Lemma 4.14. □

Chapter 5

Strongly symmetric disjoint Hamilton cycles in odd dimensions

Can we find optimal strongly symmetric collections for any other odd n, even though, by the discussion in Chapter 3, optimality of symmetric collections for odd n is an unresolved problem? We consider the case of odd n having a factor of 5^k where $k \geq 1$. As 5^k is not prime if $k > 1$, we know from Corollary 3.15 that St_n is not symmetrically Hamilton decomposable if n is divisible by 5^k ($n > 5$). However, in this chapter, we are able to find a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles for n=5^k, for each $k \geq 1$, which we prove to be optimal. Although, this bound can be achieved for n=5^k, we show that, in general, this bound is not achievable if n has, additionally, a prime factor other than 5. We identify other cases of odd n where the $\varphi(n)/4$ bound is achievable, though cannot prove optimality in every case. The results do, however, improve, for those n, the best existing bounds for the known number of edge-disjoint Hamilton cycles in St_n, symmetric or otherwise. Cases where the $\varphi(n)/4$ bound is optimal are given in Theorem 5.3. These are shown to include the case n=5^k in Corollary 5.5. Cases where the $\varphi(n)/4$ is achievable are given in Theorem 5.7, and cases where the $\varphi(n)/4$ bound is both achievable and optimal are presented in Theorem 5.8. Corollary 5.9 shows that these include the cases n=5^k. Finally, Theorem 5.10 shows that not all n having factors of 5^k can attain the $\varphi(n)/4$ bound for strongly symmetric collections.

5.1 An upper bound on strongly symmetric collections

For certain cases of odd n we can establish an upper bound on the number of Hamilton cycles in strongly symmetric collections. The bounds arise from the fact that all labels of edges of Hamilton cycles in strongly symmetric collections must be coprime to n by Lemma 3.12, and the way in which coprime labels can be generated by a single automorphism.

Lemma 5.1. *Let n be odd and suppose that St_n has a symmetric collection of disjoint Hamilton cycles $\widetilde{H}$. Then the following hold.*

(i) All edge labels of Hamilton cycles in $\widetilde{H}$ must be coprime to n.

(ii) For any two distinct $H_1, H_2, \in \widetilde{H}$, the majority edge labels of H_1 and H_2 are different as are their minority edge labels.

Proof. The proof of (i) follows easily from Lemma 3.12(i) because if $H_{i\ j} \in \widetilde{H}$ and Φ is a label automorphism with distance map ϕ^d such that $\Phi(H_{1\ 2}) = H_{i\ j}$, then the majority and minority edge labels i and j respectively are such that $gcd(i, n) = gcd(\phi^d(1), n) = gcd(1, n) = 1$ and $gcd(j, n) = gcd(\phi^d(2), n) = gcd(2, n) = 1$.

For(ii), let

$$H_{i_1\ j_1}, \ldots, H_{i_{|\widetilde{H}|}\ j_{|\widetilde{H}|}} \tag{5.1}$$

list the distinct Hamilton cycles in $\widetilde{H}$. Certainly, no two Hamilton cycles in (5.1) can have the same majority edge label ℓ as each of these cycles would need greater than half of all edges with label ℓ. But, also, unlike the case of even n, two distinct Hamilton cycles in (5.1) cannot have the same minority edge label. To see this, suppose that $H_{i_g\ j_g}, H_{i_h\ j_h} \in \widetilde{H}$ are such that $j_g = j_h$ and let Φ_g and Φ_h be label automorphisms with corresponding pointwise maps ϕ_g and ϕ_h and distance maps ϕ_g^d and ϕ_h^d such that

$$\Phi_g(H_{1\ 2}) = H_{i_g\ j_g} \text{ and } \Phi_h(H_{1\ 2}) = H_{i_h\ j_h} \tag{5.2}$$

By Lemma 3.12(ii), ϕ_g and ϕ_h are defined by

$$\phi_g(a_i) = a_{i'_0+j'i} \text{ and } \phi_h(a_i) = a_{i''_0+j''i} \tag{5.3}$$

for some $i'_0, i''_0, j', j'' \in \{1, \ldots, n\}$ with j' and j'' coprime to n. Then, by (5.3) and (3.1) of Definition 3.3,

$$j_g = \phi_g^d(2) = \phi_g^d(\delta(a_3, a_1)) = \delta(\phi_g(a_3), \phi_g(a_1)) = min\{|2j'|, n - |2j'|\} \tag{5.4}$$

and, in the same way,

$$j_h = min\{|2j''|, n - |2j''|\} \tag{5.5}$$

We can show, similarly, that

$$i_g = min\{|j'|, n - |j'|\} \text{ and } i_h = min\{|j''|, n - |j''|\} \tag{5.6}$$

As $j_g = j_h$, from (5.4) and (5.5) we must have that either

$$j' = j'' \text{ or } j' = n - j'' \tag{5.7}$$

Both of the cases in (5.7) give $i_g = i_h$ in (5.6). This shows that if $j_g = j_h$ then $H_{i_g\ j_g}$ and $H_{i_h\ j_h}$ are the same Hamilton cycle in (5.1). □

Lemma 5.2. *If $\widetilde{H}$ is a strongly symmetric collection of disjoint Hamilton cycles, then $|\widetilde{H}|$ divides $\varphi(n)$.*

Proof. Let Φ be a label automorphism with pointwise map ϕ and directed distance map $\phi^{d\pm}$, such that $\widetilde{H}$ is listed as the distinct Hamilton cycles

$$H_{1\ 2}, \Phi(H_{1\ 2}), \ldots, \Phi^{|\widetilde{H}|-1}(H_{1\ 2}),$$

where $\Phi^{|\widetilde{H}|}$ is the identity mapping. The sequence of corresponding directed labels of majority edges starting at $+1$ is:

$$1, j, \ldots, j^{|\widetilde{H}|-1}, j^{|\widetilde{H}|} = 1$$

By Euler's Theorem, as $j^{|\widetilde{H}|} = 1$, $|\widetilde{H}|$ divides $\varphi(n)$. □

Theorem 5.3. *Suppose that n is odd, $\varphi(n)/2$ is even, and 2 is a primitive root modulo n. If $\widetilde{H}$ is a strongly symmetric collection of disjoint Hamilton cycles, then $|\widetilde{H}| \le \varphi(n)/4$.*

Proof. Assume, on the contrary, that $|\widetilde{H}| > \varphi(n)/4$. Then, as $\widetilde{H}$ is a symmetric collection, by Theorem 3.14 we have that $|\widetilde{H}| \le \varphi(n)/2$. Thus, by Lemma 5.2, $|\widetilde{H}| = \varphi(n)/2$ or $|\widetilde{H}| = \varphi(n)/3$. Let the Hamilton cycles of $\widetilde{H}$ be

$$H_{i_1\ j_1}(= H_{1\ 2}),\ H_{i_2\ j_2}(= \Phi(H_{1\ 2})),\ \ldots,\ H_{i_{|\widetilde{H}|}\ j_{|\widetilde{H}|}}(= \Phi^{|\widetilde{H}|-1}(H_{1\ 2})) \tag{5.8}$$

where Φ is a label automorphism such that $\Phi^{|\widetilde{H}|}$ is the identity mapping. By Lemma 5.1(i), all edge labels must be coprime to n and, by Lemma 5.1(ii), all majority and minority edge labels must be different. However, all edge labels in (5.8) cannot be different else there would be $2|\widetilde{H}| \ge 2\varphi(n)/3$ labels in total, which is more than the $\varphi(n)/2$ edge labels possible. Thus, some majority edge label is equal to a minority edge label in (5.8), say in the Hamilton cycles

$$H_{i_e\ \ell} \text{ and } H_{\ell\ j_f},$$

where $1 \leq e \neq f \leq |\widetilde{H}|$ and $j_e = \ell = i_f$. Then,

$$\Phi^{|\widetilde{H}|-e+1}(H_{i_e\ \ell}) = \Phi^{|\widetilde{H}|-e+1}(\Phi^{e-1}(H_{1\ 2})) = \Phi^{|\widetilde{H}|}(H_{1\ 2}) = H_{1\ 2}$$

and so

$$(\phi^d)^{|\widetilde{H}|-e+1}(i_e) = 1 \text{ and } (\phi^d)^{|\widetilde{H}|-e+1}(\ell) = 2$$

and, by Lemma 4.2(iii),

$$(\phi^{d\pm})^{|\widetilde{H}|-e+1}(i_e) = \pm 1 \text{ and } (\phi^{d\pm})^{|\widetilde{H}|-e+1}(\ell) = \pm 2 \tag{5.9}$$

Put

$$\Psi = \Phi^{|\widetilde{H}|-e+f}$$

Clearly, Ψ is a label automorphism which has a corresponding pointwise map, as in Lemma 3.12(ii),

$$\psi(a_i) = a_{i_0+ji},$$

for some $i_0, j \in \{1, \ldots, n\}$, and directed distance map $\psi^{d\pm}$. Then,

$$\begin{array}{rcll} \psi^{d\pm}(1) & = & (\phi^{|\widetilde{H}|-e+f})^{d\pm}(1) & \\ & = & (\phi^{d\pm})^{|\widetilde{H}|-e+f}(1) & \text{(by Lemma 4.2(ii))} \\ & = & (\phi^{d\pm})^{|\widetilde{H}|-e+1}((\phi^{d\pm})^{f-1}(1)) & \\ & = & \pm(\phi^{d\pm})^{|\widetilde{H}|-e+1}((\phi^{d})^{f-1}(1)) & \text{(by Lemma 4.2(iii))} \\ & = & \pm(\phi^{d\pm})^{|\widetilde{H}|-e+1}(\ell) & \text{(as } \Phi^{f-1}(H_{1\ 2}) = H_{\ell\ j_f}) \\ & = & \pm 2 & \text{(by (5.9))} \end{array}$$

It follows from this, putting $\phi = \psi$ and $k = 1$ in Lemma 4.2(i) and Lemma 4.3(i), that

$$\psi(a_i) = a_{i_0+2i} \text{ or } \psi(a_i) = a_{i_0-2i}$$

Therefore, the Hamilton cycles

$$H_{1\ 2}, \Psi(H_{1\ 2}), \ldots, \Psi^{|\widetilde{H}|-1}(H_{1\ 2}), \Psi^{|\widetilde{H}|}(H_{1\ 2}) \tag{5.10}$$

yield a corresponding sequence of directed labels for majority edges starting at +1:

$$+1, \pm 2, \ldots, (\pm 2)^{|\widetilde{H}|-1}, (\pm 2)^{|\widetilde{H}|} \tag{5.11}$$

Now,

$$\Psi^{|\widetilde{H}|} = (\Phi^{(|\widetilde{H}|-e+f)})^{|\widetilde{H}|} = (\Phi^{|\widetilde{H}|})^{|\widetilde{H}|-e+f}$$

is the identity mapping as $\Phi^{|\widetilde{H}|}$ is the identity mapping. In particular, the directed label for majority edges of $\Psi^{|\widetilde{H}|}(H_{1\ 2})$ equals $+1$. From (5.10) and (5.11) this means that

$$2^{|\widetilde{H}|} = 1 \text{ or } (-2)^{|\widetilde{H}|} = 1$$

But, $2^{|\widetilde{H}|} \neq 1$ as 2 is a primitive root and $|\widetilde{H}| \leq \varphi(n)/2 < \varphi(n)$. If $(-2)^{|\widetilde{H}|} = 1$ then, as $\varphi(n)/2$ is even, $|\widetilde{H}| \neq \varphi(n)/2$ else

$$1 = (-2)^{|\widetilde{H}|} = (-2)^{\varphi(n)/2} = 2^{\varphi(n)/2}$$

which cannot be the case as 2 is a primitive root. Thus, $|\widetilde{H}| = \varphi(n)/3$. But, then,

$$1 = ((-2)^{|\widetilde{H}|})^2 = 2^{2|\widetilde{H}|} = 2^{2\varphi(n)/3}$$

which is a contradiction as 2 is a primitive root and $2\varphi(n)/3 < \varphi(n)$. □

We show that the conditions of Theorem 5.3 are satisfied if $n = 5^k$ and hence that Theorem 5.3 holds for those n.

Lemma 5.4. *If $n = 5^k$ where $k \geq 1$, then $\varphi(n)/2$ is even, and 2 is a primitive root modulo n.*

Proof. We have that $\varphi(5^k) = 5^{k-1}(5-1)$ and so $\varphi(5^k)/2$ is even. Also, if j is a primitive root modulo a prime number p then j is a primitive root modulo all powers of p if j^{p-1} is not of the form $p^2K + 1$ (see [17]). As 2 is a primitive root of 5 and $2^{5-1} = 16$ is not of the form $25K + 1$, it follows that 2 is a primitive root of 5^k for all $k \geq 1$. □

Corollary 5.5. *If $n = 5^k$ where $k \geq 1$, then $|\widetilde{H}| \leq \varphi(n)/4$ for any strongly symmetric collection of disjoint Hamilton cycles.*

Proof. Follows from Theorem 5.3 and Lemma 5.4. □

5.2 Achieving the upper bound

Corollary 5.5 shows that there cannot be more than the bound of $\varphi(n)/4$ Hamilton cycles in any strongly symmetric collection if $n = 5^k$. In fact, the bound can be achieved. To show this, we will obtain the required edge-disjoint Hamilton cycles by scrutinizing the coprime constituents of the label automorphism Φ_n given by the pointwise map

$$\phi_n(a_i) = a_{2i} \quad (1 \leq i \leq n)$$

In [19] the inverse of this map is used to obtain $2\varphi(n)/9$ bounds on the numbers of disjoint Hamilton cycles for all odd n. Let us consider the case of $n = 25$. The only coprime directed constituent with $\varphi(n)$ elements is

$$(1\ 2\ 4\ 8\ 16\ 7\ 14\ 3\ 6\ 12\ 24\ 23\ 21\ 17\ 9\ 18\ 11\ 22\ 19\ 13)$$

which can be written, using negative numbers of arithmetic modulo n, as

$$(1\ \ 2\ \ 4\ \ 8 - 9\ \ 7 - 11\ \ 3\ \ 6\ \ 12 - 1 - 2 - 4 - 8\ \ 9 - 7\ \ 11 - 3 - 6 - 12) \tag{5.12}$$

The (undirected) constituent has half the number $(\varphi(n)/2)$ of elements:

$$(1\ 2\ 4\ 8\ 9\ 7\ 11\ 3\ 6\ 12)$$

Now, consider the pointwise map

$$\psi_{25}(a_i) = \phi_{25}^2(a_i) = a_{4i} \quad (1 \leq i \leq n)$$

i.e.

$$\psi_{25}(a_i) = a_{f^2(i)} \ \text{ where } \ \phi_{25}(a_i) = a_{f(i)}$$

It generates the Hamilton cycles

$$H_{1\ 2}, \Psi_{25}(H_{1\ 2}), \Psi_{25}^2(H_{1\ 2}), \Psi_{25}^3(H_{1\ 2}), \Psi_{25}^4(H_{1\ 2}),$$

by the corresponding label automorphism Ψ_{25}, which, by (5.12), give Hamilton cycles with majority and minority edges as follows:

$$H_{1\,2}, H_{4\,8}, H_{9\,7}, H_{11\,3}, H_{6\,12} \tag{5.13}$$

Applying Ψ_{25} to $H_{6\,12}$, i.e. calculating $\Psi_{25}^5 = \Psi_{25}^{\varphi(n)/4}(H_{1\ 2})$, yields a Hamilton cycle with majority edge labels equal to 1 and minority edge labels equal to 2. But, ψ_{25}^5 is not the identity mapping as $(\psi_{25}^{d\pm})^{\varphi(n)/4}(1) = -1$. Consider instead the mapping

$$\theta_{25}(a_i) = a_{-4i}$$

i.e.

$$\theta_{25}(a_i) = a_{-f^2(i)} \ \text{ where } \ \phi_{25}(a_i) = a_{f(i)}$$

Choosing a_i, a_j such that $\delta^{\pm}(a_i, a_j) = \ell$, where ℓ is a given directed label, we have that

$$\begin{array}{rcll} \theta_{25}^{d\pm}(\ell) & = & \theta_{25}^{d\pm}(\delta^{\pm}(a_i, a_j)) & \\ & = & \delta^{\pm}(\theta_{25}(a_i), \theta_{25}(a_j)) & \text{(by Lemma 4.2(i))} \\ & = & \delta^{\pm}(a_{-4i}, a_{-4j}) & \\ & = & -4(i-j) & \\ & = & -\delta^{\pm}(\phi_{25}^2(a_i), \phi_{25}^2(a_j)) & \\ & = & -(\phi_{25}^2)^{d\pm}(\delta^{\pm}(a_i, a_j)) & \text{(by Lemma 4.2(ii))} \\ & = & -(\phi_{25}^2)^{d\pm}(\ell) & \end{array}$$

i.e., by Lemma 4.2(ii),

$$\theta_{25}^{d\pm}(\ell) = -(\phi_{25}^2)^{d\pm}(\ell) = -(\phi_{25}^{d\pm})^2(\ell) \tag{5.14}$$

Also,

$$\begin{array}{rcll} (\theta_{25}^{d\pm})^{\varphi(n)/4}(1) & = & (\theta_{25}^{d\pm})^5(1) & \\ & = & (-(\phi_{25}^2)^{d\pm})^5(1) & \text{(by (5.14))} \\ & = & (-1)^5((\phi_{25}^2)^5)^{d\pm}(1) & \text{(by Lemma 4.2(ii))} \\ & = & -(\phi_{25}^{10})^{d\pm}(1) & \\ & = & -(\phi_{25}^{d\pm})^{10}(1) & \text{(by Lemma 4.2(ii))} \\ & = & -(-1) & \text{(from (5.12))} \\ & = & +1 & \end{array}$$

Thus, the Hamilton cycles

$$H_{1\ 2}, \Theta_{25}(H_{1\ 2}), \Theta_{25}^2(H_{1\ 2}), \Theta_{25}^3(H_{1\ 2}), \Theta_{25}^4(H_{1\ 2}),$$

where Θ_{25} is the corresponding label automorphism have, by (5.12) and (5.14), directed majority and minority edges as follows:

$$H_{1\ 2}, H_{-4\ -8}, H_{-9\ 7}, H_{11\ -3}, H_{6\ 12} \tag{5.15}$$

These have the same undirected edge labels as in (5.13), but θ_{25} differs from ψ_{25} in that $(\psi_{25}^{d\pm})^{\varphi(n)/4}(1) = -1$ whereas $(\theta_{25}^{d\pm})^{\varphi(n)/4}(1) = +1$. Thus, $\theta_{25}^{\varphi(n)/4}$ is the identity mapping by Lemma 4.5, and so (5.15) is a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles.

For $n = 25$, we have used the fact that 2 is a primitive root which generates $\varphi(n)/2$ directed labels, starting at $+1$ and finishing at -1, and that $\varphi(n)/2$ is even and $\varphi(n)/4$ is odd, i.e. $\varphi(n)/2$ is of the form $4K + 2$. This has allowed $-\phi_{25}^2$ to be applied an odd number $(\varphi(n)/4)$ of times thereby producing an odd number of pairs of labels of successive edge-disjoint Hamilton cycles, returning to label 1 and not -1. We shall show that these properties hold for $n = 5^k$ for all k. In fact, we give a more general theorem (Theorem 5.7 below) which also gives conditions where $\varphi(n)/4$ strongly symmetric disjoint Hamilton cycles can be obtained when 2 is a primitive lambda-root of n.

Lemma 5.6. *Let n be odd and let the pointwise maps ϕ_n and θ_n be defined by*

$$\phi_n(a_i) = a_{f(i)} \text{ and } \theta_n(a_i) = a_{-f^2(i)} \quad (1 \le i \le n),$$

where $f(i) = i_0 + ji$ for some $i_0, j \in \{1, \ldots, n\}$. Then, if $\phi_n^{d\pm}$ and $\theta_n^{d\pm}$ are the corresponding directed distance maps, we have that, for all directed labels ℓ,

$$\theta_n^{d\pm}(\ell) = -(\phi_n^{d\pm})^2(\ell)$$

Proof. Choosing a_i and a_j to be such that $\delta^{\pm}(a_i, a_j) = \ell$, where ℓ is a directed label, then we have that

$$\begin{aligned}
\theta_n^{d\pm}(\ell) &= \theta_n^{d\pm}(\delta^{\pm}(a_i, a_j)) \\
&= \delta^{\pm}(\theta_n(a_i), \theta_n(a_j)) && \text{(by Lemma 4.2(i))} \\
&= \delta^{\pm}(a_{-f^2(i)}, a_{-f^2(j)}) \\
&= -(f^2(i) - f^2(j)) \\
&= -\delta^{\pm}(\phi_n^2(a_i), \phi_n^2(a_j)) \\
&= -(\phi_n^2)^{d\pm}(\delta^{\pm}(a_i, a_j)) && \text{(by Lemma 4.2(ii))} \\
&= -(\phi_n^2)^{d\pm}(\ell) \\
&= -(\phi_n^{d\pm})^2(\ell) && \text{(by Lemma 4.2(ii))}
\end{aligned}$$

□

Theorem 5.7. *Let n be odd. Then, St_n has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles if either of the following conditions holds:*

(i) $\varphi(n)/2$ is even, $\lambda(n) = \varphi(n)/2$, 2 is a primitive lambda-root of n, and -1 is not a power of 2 (modulo n),

(ii) $\varphi(n)/2$ is of the form $4K + 2$ and 2 is a primitive root of n.

Proof. To prove (i), let the pointwise map ϕ_n be defined by

$$\phi_n(a_i) = a_{2i} \quad (1 \le i \le n)$$

Then, repeated application of the corresponding directed distance map $\phi_n^{d\pm}$ generates $\varphi(n)/2$ directed edge labels

$$1, 2, \ldots, 2^{\varphi(n)/2-1}$$

By Lemma 4.6(i), as $2^{\varphi(n)/2} = 2^{\lambda(n)} = 1$ and -1 is not a power of 2, these are distinct as undirected edge labels. Thus, the sequence of Hamilton cycles

$$H_{1\ 2}, \Phi_n^2(H_{1\ 2}), \ldots, (\Phi_n^2)^{\varphi(n)/4-1}(H_{1\ 2}), \tag{5.16}$$

where Φ_n is the label automorphism corresponding to ϕ_n, which have directed edge labels as follows:

$$H_{1\ 2}, H_{4\ 8}, \ldots, H_{2^{\varphi(n)/2-2}\ 2^{\varphi(n)/2-1}}$$

are edge-disjoint. If $(\phi_n^2)^{d\pm}$ is the directed distance map of the pointwise map ϕ_n^2 then, as $2^{\varphi(n)/2} = 1$, it follows by Lemma 4.2(ii) that

$$1 = 2^{\varphi(n)/2} = (\phi_n^{d\pm})^{\varphi(n)/2}(1) = ((\phi_n^2)^{d\pm}))^{\varphi(n)/4}(1) \tag{5.17}$$

Also,

$$(\phi_n^2)^{\varphi(n)/4}(a_n) = \phi_n^{\varphi(n)/2}(a_n) = a_n, \tag{5.18}$$

as $\phi_n(a_n) = a_n$. Thus, by (5.17), (5.18) and Lemma 4.5 (with ϕ equal to ϕ_n^2 and k equal to $\varphi(n)/4$),

$$(\phi_n^2)^{\varphi(n)/4}(a_i) = a_i \ \text{ for all } a_i \in \{a_1, ..., a_n\},$$

i.e. $(\phi_n^2)^{\varphi(n)/4}$ is the identity mapping, and so the Hamilton cycles in (5.16) are strongly symmetric.

For (ii), let the pointwise maps ϕ_n and θ_n be defined by:

$$\phi_n(a_i) = a_{f(i)} \text{ and } \ \theta_n(a_i) = a_{-f^2(i)}, \text{ where } f(i) = 2i.$$

Consider the Hamilton cycles

$$H_{1\ 2}, \Theta_n(H_{1\ 2}), \ldots, \Theta_n^{\varphi(n)/4-1}(H_{1\ 2}), \tag{5.19}$$

where Θ_n is the label automorphism corresponding to θ_n. These have directed edge labels as follows:

$$H_{1\ 2}, H_{\theta_n^{d\pm}(1)\ \theta_n^{d\pm}(2)}, \ldots, H_{(\theta_n^{d\pm})^{\varphi(n)/4-1}(1)\ (\theta_n^{d\pm})^{\varphi(n)/4-1}(2)} \tag{5.20}$$

Now, as $\phi_n(a_i) = a_{2i}$, we have that, by Lemma 4.3,

$$(\phi_n^{d\pm})^k(1) = 2^k \text{ and } (\phi_n^{d\pm})^k(2) = 2^{k+1} \text{ for all } k \geq 0. \tag{5.21}$$

Then, the sequence of directed labels in (5.20)

$$1,\ 2,\ \theta_n^{d\pm}(1),\ \theta_n^{d\pm}(2),\ \ldots,\ (\theta_n^{d\pm})^{\varphi(n)/4-1}(1),\ (\theta_n^{d\pm})^{\varphi(n)/4-1}(2)$$

which, by Lemma 5.6, equals

$$1,\ 2,\ -(\phi_n^{d\pm})^2(1),\ -(\phi_n^{d\pm})^2(2),\ \ldots,$$

$$\ldots\ (-1)^{\varphi(n)/4-1}((\phi_n^{d\pm})^2)^{\varphi(n)/4-1}(1), (-1)^{\varphi(n)/4-1}((\phi_n^{d\pm})^2)^{\varphi(n)/4-1}(2),$$

by (5.21) equals

$$1,\ 2,\ -2^2,\ -2^3,\ \ldots,\ (-1)^{\varphi(n)/4-1}2^{\varphi(n)/2-2},\ (-1)^{\varphi(n)/4-1}2^{\varphi(n)/2-1}. \tag{5.22}$$

As 2 is a primitive root of n, by Lemma 4.9, $2^{\varphi(n)/2} = -1$ and -1 does not occur in (5.22). By Lemma 4.6(ii), (5.22) corresponds to a sequence of distinct undirected edge labels. Hence, the Hamilton cycles in (5.19) are edge-disjoint.

It remains to prove that $\Theta_n^{\varphi(n)/4}$ is the identity mapping. We have that

$$\begin{aligned}
(\theta_n^{d\pm})^{\varphi(n)/4}(1) &= (-1)^{\varphi(n)/4}((\phi_n^{d\pm})^2)^{\varphi(n)/4}(1) && \text{(by Lemma 5.6)}\\
&= (-1)^{\varphi(n)/4}(\phi_n^{d\pm})^{\varphi(n)/2}(1) &&\\
&= (-1)^{\varphi(n)/4}2^{\varphi(n)/2} && \text{(by (5.21))}\\
&= -2^{\varphi(n)/2} && \text{(as } \varphi(n)/4 \text{ is odd)}\\
&= 1 && \text{(as } 2^{\varphi(n)/2} = -1)
\end{aligned}$$

As $\theta_n(a_n) = a_{-4n}$, $\theta_n(a_n) = a_n$ and so $\theta_n^{\varphi(n)/4}(a_n) = a_n$. It follows, by Lemma 4.5, that $\theta_n^{\varphi(n)/4}$, and therefore $\Theta_n^{\varphi(n)/4}$, is the identity mapping. □

Theorem 5.7(i) is satisfied by n equal to the following composite odd multiples of 3 less than 100:

$$15,\ 21,\ 39,\ 45,\ 57,\ 69,\ 75,\ 87,\ 99.$$

In the cases of $n = 51$ or $n = 93$, 2 is not a primitive lambda-root and, for $n = 63$, $\lambda(n) = \varphi(n)/6$. If $n = 33$, $\varphi(n)/2$ is even, $\lambda(n) = \varphi(n)/2$ and 2 is a primitive lambda-root, but -1 is a power of 2. The composite odd multiples of 5 less than 100 that satisfy Theorem 5.7(i) for n are:

$$15,\ 35,\ 45,\ 55,\ 75,\ 95.$$

For $n = 65$ or 85, $\lambda(n)$ does not equal $\varphi(n)/2$. In the cases of n equal to a power of certain primes, we can obtain a more general result using Theorem 5.7(ii).

Theorem 5.8. *Let $n = p^k$, where p is a prime number of the form $8K + 5$ and $k \geq 1$, be such that 2 is a primitive root of n. Then, St_n has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles. Moreover, the $\varphi(n)/4$ bound cannot be improved.*

Proof. We have that $\varphi(n) = \varphi(p^k) = p^{k-1}(p-1)$. So, $\varphi(n)/2 = p^{k-1}(p-1)/2$. As p is of the form $8K + 5$, $(p-1)/2$ is of the form $4K + 2$. Since p^{k-1} is an odd integer, $\varphi(n)/2 = p^{k-1}(p-1)/2$ is also of the form $4K + 2$. Then, as 2 is a primitive root modulo n, it follows by Theorem 5.7(ii), that St_n has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles. Furthermore, n satisfies the conditions of Theorem 5.3 and so the $\varphi(n)/4$ bound is optimal. □

Corollary 5.9. *Let $n = 5^k$, where $k \geq 1$. Then, St_n has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles and the $\varphi(n)/4$ bound cannot be improved.*

Proof. The conditions of Theorem 5.8 are satisfied for $n = 5^k$ $(k \geq 1)$ as 2 is a primitive root modulo n by Lemma 5.4. □

5.3 Failure to achieve the upper bound

The case of $n = 221$ is interesting because it does not satisfy the conditions of Theorem 5.7(i) or (ii) yet has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles. Consider the following two coprime constituents for the pointwise map ϕ_{221} for St_{221} defined by $\phi_{221}(a_i) = a_{6i}$ $(1 \leq i \leq 221)$:

(1 6 36 5 30 41 25 71 16 96 87 80 38 7 42 31 35 11 66 46 55 109 9 54 103
45 49 73 4 24 77 20 101 57 100 63 64 58 94 99 69 28 53 97 81 44 43 37)

(2 12 72 10 60 82 50 79 32 29 47 61 76 14 84 62 70 22 89 92 110 3 18 108 15
90 98 75 8 48 67 40 19 107 21 95 93 105 33 23 83 56 106 27 59 88 86 74)

The corresponding label automorphism Φ_{221} generates the Hamilton cycles

$$H_{1\ 2}, \Phi_{221}(H_{1\ 2}), \Phi_{221}^2(H_{1\ 2}), \ldots, \Phi_{221}^{47}(H_{1\ 2}) \tag{5.23}$$

whose edge labels are, respectively, as shown below:

$$H_{1\,2}, H_{6\,12}, H_{36\,72}, H_{5\,10}, H_{30\,60}, H_{41\,82}, H_{25\,50}, H_{71\,79}, H_{16\,32},$$

$$H_{96\,29}, H_{87\,47}, H_{80\,61}, H_{38\,76}, H_{7\,14}, H_{42\,84}, H_{31\,62}, H_{35\,70}, H_{11\,22},$$

$$H_{66\,89}, H_{46\,92}, H_{55\,110}, H_{109\,3}, H_{9\,18}, H_{54\,108}, H_{103\,15}, H_{45\,90}, H_{49\,98},$$

$$H_{73\,75}, H_{4\,8}, H_{24\,48}, H_{77\,67}, H_{20\,40}, H_{101\,19}, H_{57\,107}, H_{100\,21}, H_{63\,95},$$

$$H_{64\,93}, H_{58\,105}, H_{94\,33}, H_{99\,23}, H_{69\,83}, H_{28\,56}, H_{53\,106}, H_{97\,27}, H_{81\,59},$$

$$H_{44\,88}, H_{43\,86}, H_{37\,74}.$$

Moreover, Φ_{221}^{48} is the identity mapping. Also, $\varphi(221) = 192$ and $\lambda(221) = 48$. Thus, (5.23) is a strongly symmetric collection of $\varphi(221)/4 = 48$ disjoint Hamilton cycles. So far, in order to generate disjoint Hamilton cycles from $H_{1\,2}$ for odd n, we have sought pointwise maps ϕ_n which have produced coprime constituents in which 1 and 2 are present and adjacent to each other. The case of $n = 221$ has achieved the $\varphi(n)/4$ bound in a different way with 1 and 2 not occurring in the same coprime constituent. Given the case of $n = 221$, we may ask whether the $\varphi(n)/4$ bound can be achieved for strongly symmetric collections for all odd n, albeit using different methods to that in Theorem 5.7? The answer to this question is negative. The example we give is a partial converse to Corollary 5.9. We show below in Theorem 5.10 that if n is a power of 5 multiplied by the power of another prime, then there are infinitely many cases for which there does not exist a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles for St_n .

Theorem 5.10. *Let $n = 5^k p^m$ where $k > 1$, $m \geq 1$, and p is a prime number of the form $10K + 1$. Then, there does not exist a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles for St_n .*

Proof. As 5^k and p^m are coprime we have that

$$\begin{aligned} \varphi(5^k p^m) &= \varphi(5^k)\varphi(p^m) \\ &= (5^k(5-1)/5)(p^m(p-1)/p) \\ &= (5^{k-1}4)(p^{m-1}(p-1)) \end{aligned}$$

Since p is of the form $10K + 1$, $p - 1$ is of the form $10K$. Therefore,

$$\begin{aligned} \lambda(n) &= lcm(\lambda(5^k), \lambda(p^m)) && \text{(by Lemma 4.13(i))} \\ &= lcm(\phi(5^k), \phi(p^m)) && \text{(as } 5^k \text{ and } p^m \text{ have primitive roots)} \\ &= lcm(5^{k-1}4, p^{m-1}(p-1)) \\ &\leq (5^{k-1}4)(p^{m-1}(p-1))/10 \\ &= \varphi(n)/10 \end{aligned}$$

The result follows by Theorem 4.12 as $\lambda(n) < \varphi(n)/4$. □

Chapter 6

Conclusions

In Chapter 2, we introduced St_n as a star graph of $n!$ vertices and we have sought to obtain edge-disjoint Hamilton cycles as automorphic images of a known Hamilton cycle in St_n. This method provides a way of constructing edge-disjoint Hamilton cycles, as the known starting Hamilton cycle given in [11] of edge lengths 1 or 2 can itself be constructed from the construction of a special kind of Hamilton cycle, called a 'doubly adjacent Gray code', in path graphs [18]. It is unknown as to whether there are 2 edge-disjoint Hamilton cycles in any St_n for $n > 5$, which partition the edges of lengths 1 and 2. If that was the case, automorphisms by our method would only require $\lfloor n/4 \rfloor$ lower bounds to produce $\lfloor n/2 \rfloor$ edge-disjoint Hamilton cycles, i.e. a Hamilton decomposition of St_n. In this sense, the $\lfloor n/4 \rfloor$ bound is a kind of optimum bound that a method of generating edge-disjoint Hamilton cycles by automorphism can achieve. Our $\lfloor n/5 \rfloor$ lower bound for n equal to powers of certain primes, has come close to this ideal. However, although the Hamilton decomposition of St_5 in [25] gives an automorphism which is defined, as here, by means of a bijection of the a_is, and which, furthermore, maps lengths equal to 1 to lengths equal to 2 and vice-versa, such an automorphism does not exist for $n > 5$. Thus, if two edge-disjoint Hamilton cycles of edges lengths 1 and 2 do exist, this can only be proved by some other method. A number of open problems remain. Aside from the question mentioned above, of whether edge-disjoint Hamilton cycles of edge lengths 1 or 2 exist, the ultimate open problem of this chapter is as follows:

Open Problem 6.1. *Let St_n be a star graph. For which integers n is St_n Hamilton decomposable?*

Thus far, this has only been demonstrated for St_5 and there are no results proving non-existence for any n. In the meantime, we can ask if there are infinitely many Hamilton decomposable star graphs; this may be possible to prove for some prime n.

In Chapter 3 we have investigated symmetric collections of disjoint Hamilton cycles for labelled versions St_n. In that chapter, we have defined symmetric collections of disjoint Hamilton cycles for labelled St_n to be those for which, given a Hamilton cycle in the collection, there is an automorphism mapping labels consistently such that the chosen Hamilton cycle is the image of the base 2-labelled Hamilton cycle. We have shown that there are at most $\varphi(n)/2$ symmetric disjoint Hamilton cycles and this bound is sharp for all even n. So, if n is not a prime number, St_n is not symmetrically Hamilton decomposable. Our result in Section 3.5 of that chapter revealed that, the spanning subgraph $H'_{2\ 1}$ of St_n, comprising the edges of labels 1 and 2 that are not in $H_{1\ 2}$, is not a Hamilton cycle if n is even. Whilst the $\varphi(n)/2$ upper bound on the number of Hamilton cycles in a symmetric collection also holds for St_n if n is odd, it is not clear that this bound can be achieved for any odd n other than n equal to 5 [25]. Moreover, in that chapter, we have studied the labels for the majority of the edges and the labels for the minority of the edges. In the case of even n, the number of Hamilton cycles in a symmetric collection $\widetilde{H}$ is limited to $\varphi(n)/2$ because every majority edge label in $\widetilde{H}$ has to be coprime to n as the majority edge label 1 of the base Hamilton cycle $H_{1\ 2}$ is coprime to n. However, in the case of odd n, as both the majority and minority edge labels of $H_{1\ 2}$, i.e. 1 and 2, are coprime to n, both the majority and minority edge labels of Hamilton cycles in symmetric collections have to be coprime to n . For this reason, the least upper bound for symmetric collections for odd n may be $\varphi(n)/4$. This bound is nearly achieved by a $2\varphi(n)/9$ bound for all odd n other than $n = 127$ in Chapter 2. Consequently, the following problem has not been completely solved in Chapter 3:

Open Problem 6.2. *For all odd $n \neq 127$, does St_n has a symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles?*

We have given the bounds of the number of edge-disjoint Hamilton cycles based on graph automorphisms which produce edges with different labels incident at each vertex in the image of the base Hamilton cycle given in [11]. But, there may exist another structure for the base Hamilton cycle which is different from the structure of the cycle with edge lengths 1 or 2, that does not have majority or minority edges. Further work could investigate properties of such new Hamilton cycles as base Hamilton cycles without any majority or minority edges and similar or different automorphisms, to determine whether they could be used to establish or refute the existence of Hamiltonian decompositions.

In Chapters 4, we have investigated whether there are any cases where this optimal $\varphi(n)/2$ symmetric bound can be achieved by strongly symmetric collections for even n. In that chapter, we introduced directed labels and directed labelled star graphs and defined a collection of disjoint Hamilton cycles to be strongly symmetric if a single automorphism could generate all the Hamilton cycles from the base Hamilton cycle. We were not able to determine the exact optimum bound of strongly symmetric disjoint Hamilton cycles for star graphs St_n for all even n in that chapter. But, to find the largest strongly symmetric collection we have used primitive lambda-roots of n stemming from Carmichael's function $\lambda(n)$, and we have considered two different settings: that $n = 2^k$ in Section 4.4, and $n = 2p^k$ for prime p of the form $4K + 3$ in Section 4.5. We have shown that, for all such n, St_n has a strongly symmetric collection of $\varphi(n)/2$ edge-disjoint Hamilton cycles. However, not all even n have strongly symmetric collections at the $\varphi(n)/2$ bound. We have shown that if n is the product of any power of 2 greater than 2 and the power of any other prime, then there does not exist a strongly symmetric collection of $\varphi(n)/2$ disjoint Hamilton cycles for St_n.

In Chapter 5, we have continued our study of strongly symmetric properties of edge-disjoint Hamilton cycles in star graphs St_n where n is an odd integer. Theorem 5.7, gives conditions where $\varphi(n)/4$ strongly symmetric disjoint Hamilton cycles can be obtained when 2 is a primitive lambda-root of odd n. We have also discussed some interesting cases that does not satisfy the conditions of Theorem 5.7 yet has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles. Furthermore, as a result of Theorem 5.8, For $n = 5^k$ where $k \geq 1$, St_n has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles and the $\varphi(n)/4$ bound cannot be improved. We were unable to give optimum bounds for symmetric collections of disjoint Hamilton cycles for the case of odd n, but the $\varphi(n)/4$ bound is shown to be optimal and achievable for strongly symmetric collections for infinitely many odd n in that chapter. As a final result of this thesis, we have considered the question of whether St_n has a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles for all odd n? Theorem 5.10, has provided a negative answer to this problem for all integers n that are powers of 5 greater than 1 multiplied by the power of another prime. So, there are infinitely many cases for which there does not exist a strongly symmetric collection of $\varphi(n)/4$ disjoint Hamilton cycles for St_n. We now state an open problem arising from Chapters 4 and 5:

Open Problem 6.3. *Let St_n be a star graph. What are the optimum bounds for strongly symmetric collections of edge-disjoint Hamilton cycles that can be achieved for all even n or for all odd n?*

It would be interesting to investigate some other structures for labelling the edges of Hamilton cycles and introduce some other kind of labelled star graphs and label automorphisms. If we can device more efficient methods to obtain Hamilton cycles which are edge-disjoint, it may be possible to improve the existing bounds on symmetric and strongly symmetric collections of edge-disjoint Hamilton cycles for star graphs St_n given in this thesis.

To summarize, we give a table of results listing the bounds of the number of edge-disjoint Hamilton cycles which we have achieved in this thesis:

<table>
<tr><th>n</th><th>Symmetric Collection</th><th colspan="2">Strongly Symmetric Collection</th></tr>
<tr><td rowspan="3">Even</td><td rowspan="3">$= \varphi(n)/2$</td><td>2^k, $k \geq 3$
$(\lambda(n) = \varphi(n)/2)$</td><td>$= \varphi(n)/2$</td></tr>
<tr><td>$\lambda(n) = \varphi(n)$, $\varphi(n)/2$ is odd
(e.g. $n = 2p^k, p = 4K + 3$)</td><td>$= \varphi(n)/2$</td></tr>
<tr><td>$\lambda(n) < \varphi(n)/2$
(e.g. $n = 2^k p^r$, $k \geq 3$, $p > 2$)</td><td>$< \varphi(n)/2$</td></tr>
<tr><td rowspan="4">Odd</td><td rowspan="4">$= \varphi(n)/5$

($2\varphi(n)/9$, if $n \neq 127$)</td><td>$\varphi(n)/2$ is even, 2 is PR
(e.g. $n = 5^k, k \geq 1$)</td><td>$\leq \varphi(n)/4$</td></tr>
<tr><td>$\varphi(n)/2$ is even, $\lambda(n) = \varphi(n)/2$
2 is PLR, -1 is not a power of 2</td><td>$= \varphi(n)/4$</td></tr>
<tr><td>$\varphi(n)/2 = 4K + 2$, 2 is PR
(e.g. $n = 5^k, k \geq 1$)</td><td>$= \varphi(n)/4$</td></tr>
<tr><td>$n = 5^r p^m$, $r > 1$, $p = 10K + 1$</td><td>$< \varphi(n)/4$</td></tr>
</table>

- PR is primitive root, and PLR is primitive lambda root.
- p is a prime integer.

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