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## THE STUDY OF SOME NUMERICAL METHODS FOR

## SOLVINg PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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A Doctoral Thesis<br>Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of the Loughborough University of Technology<br>January, 1983.

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# Kenang-kenangan anak-anak-ku: 

Mohomad Ghazaly,
Fatimah,
Fatihah dan,
Mariam,
dan jugamu Abidah.

## ABSTRACT

The thesis commences with a description and classification of partial differential equations and the related matrix and eigenvalue theory. In most all cases the study of parabolic equations leads to initial boundary value problems and it is to this problem that the thesis is mainly concerned with. The basic (finite difference) methods to solve a (parabolic) partial differential equation are presented in the second chapter which is then followed by particular types of parabolic equations such as diffusion-convection, fourth order and non-1inear problems in the third chapter. An introduction to the finite element technique is also included as an alternative to the finite difference method of solution. The advantages and disadvantages of some different strategies in terms of stability and truncation error are also considered.

In Chapter Four the general derivation of a two time-level finite difference approximation to the simple heat conduction equation is derived. A new class of methods called the Group Explicit (GE) method is established which improves the stability of the previous explicit method. Comparison between the two methods in this class and the previous methods is also given. The method is also used in solving the two-space dimensional parabolic equation.

The derivation of a general two-time level finite difference approximation and the general idea of the Group Explicit method are extended to the diffusion-convection equation in Chapter Five. Some other explicit algorithms for solving this problem are also considered.

In the sixth chapter the Group Explicit procedure is applied to solve a fourth-order parabolic equation on two interlocking nets.

The concept of the GE method is also extendable to a non-linear partial
differential equation. Consideration of this extension to a particular problem can be found in Chapter Seven.

In Chapter Eight, some work on the finite element method for solving the heat-conduction and diffusion-convection equation is presented. Comparison of the results from this method with the finite-difference methods is given. The formulation and solution of this problem as a boundary value problem by the boundary value technique is also considered.

A special method for solving diffusion-convection equation is presented in Chapter Nine as well as an extension of the Group Explicit method to a hyperbolic partial differential equation is given. The thesis concludes with recommendations for further work.

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## Chapter One

INTRODUCTION AND MATHEMATICAL PRELIMINARIES

### 1.1 INTRODUCTION

The majority of the problems of physics and engineering fall naturally into one of three physical categories: equilibrium problems, eigenvalue problems and propagation problems. Equilibrium problems are problems of steady state in which the equilibrium configurations in a certain domain to be determined. Eigenvalue problems may be thought of as extension of equilibrium problems wherein critical values of certain parameters are to be determined in addition to the corresponding steady-state configurations. Propagation problems are initial value problems that have an unsteady state or transient nature, (W.F. Ames, 1977, p.3-4). Normally, in most cases, the dependent variable to any of these problems is expressed in terms of several independent variables. Such problems inherently give rise to the need for partial derivatives in the description of their behaviour. The study of differential equations arising from these problems constitutes the field of Partial Differential Equations.

Mathematically a partial differential equation (henceforth abbreviated as p.d.e.) for a dependent variable $u(x, y, \ldots)$ is a relation of the form

$$
\begin{equation*}
F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^{2} u}{\partial x^{2}} \frac{\partial^{2} u}{\partial x \partial y}, \ldots\right)=0 \tag{1.1.1}
\end{equation*}
$$

where $F$ is a given function of the independent variables $x, y, \ldots$ and the "unknown" function $u$ and of a finite number of its partial derivatives. We call $u$ as a solution of (1.1.1) if after substitution of $u(x, y, \ldots)$ and its partial derivatives (1.1.1) is satisfied identically in $x, y, \ldots$ in some region $\Omega$ in the space of these independent variables. The independent variables $x, y, \ldots$ are real (unless if stated otherwise) and $u$ and the derivatives of $u$ occurring in (1.1.1) are continuous functions of $x, y, \ldots$ in the real domain $\Omega$.

As in the theory of Ordinary Differential Equations (henceforth abbreviated as o.d.e.) a p.d.e. (1.1.1) is said to be of order $n$ if the order of the highest partial derivatives involved is n. Equation (1.1.1) is said to be Zinear if $F$ is linear in the unknown function and its derivatives and quasi-Zinear if $F$ is linear in at least the highest order derivatives, (F. John, 1975, p.1).

For example, the equation

$$
\begin{equation*}
\sqrt{x} \frac{\partial u}{\partial x}+u \frac{\partial u}{\partial y}=-u^{2} \tag{1.1.2}
\end{equation*}
$$

as a first order quasi-1inear p.d.e., and the equation

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}-32 u=0 \tag{1.1.3}
\end{equation*}
$$

is a second order linear p.d.e. Meanwhile the equation

$$
\begin{equation*}
\left(\frac{\partial^{2} u}{\partial x^{2}}\right)\left(\frac{\partial^{2} u}{\partial y^{2}}\right)-\left(\frac{\partial^{2} u}{\partial x \partial y}\right)^{2}=0 \tag{1.14}
\end{equation*}
$$

is the example of a second order non-linear. In equation (1.1.2)-(1.1.4) above, $x$ and $y$ are the independent variables and $u=u(x, y)$ is the dependent variable whose form is to be found.

A linear p.d.e. is said to be homogeneous if each term contains either the dependent variable or one of its derivatives. For example equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}-k \frac{\partial^{2} u}{\partial x^{2}}=0 \quad \text { (Heat Equation) } \tag{1.1.5}
\end{equation*}
$$

is homogeneous, whereas

$$
\begin{equation*}
\frac{\partial u}{\partial t}-a \frac{\partial^{2} u}{\partial x^{2}}=f(x, t), \quad a>0 \tag{1.1.6}
\end{equation*}
$$

where $f(x, t)$ is a given function, is an inhomogeneous equation.
The problem of finding the solution to the p.d.e. is a very difficult problem as the general method of solution is not available except for certain special types of linear or quasi-linear equations. Furthermore, the general solution of the linear p.d.e. which contains arbitrary functions
is of little use, since it has to be made to satisfy other conditions called boundary conditions which arise from the physical problem itself (see Section 1.3).

Similar to o.d.e., if $u_{1}, u_{2}, u_{3}, \ldots, u_{n}$ are $n$ different solutions of a Zinear homogeneous p.d.e, in some given domain then

$$
\begin{equation*}
u=c_{1} u_{1}+c_{2} u_{2}+\ldots+c_{n} u_{n} \tag{1.1.7}
\end{equation*}
$$

is also a solution in the same domain with $c_{1}, c_{2}, \ldots, c_{n}$ as arbitrary constants.

### 1.2 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

As the p.d.e. arise from different categories of physical phenomenon (e.g. steady viscous flow, resonance in electric circuits, propagation of heat,...), this suggests that the governing equations are also quite different in nature. Normally, they are classified in terms of their mathematical form such as elliptic, hyperbolic and parabolic equation, or in terms of the type of problems to which they apply i.e. the heat equation, the wave equation, ... .

The most general second order linear p.d.e. in two independent variables, is given by,

$$
\begin{gather*}
L U \equiv A(x, y) \frac{\partial^{2} u}{\partial x^{2}}+2 B(x, y) \frac{\partial^{2} u}{\partial x \partial y}+C(x, y) \frac{\partial^{2} u}{\partial y^{2}}+ \\
E\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0 . \tag{1.2.1}
\end{gather*}
$$

It is called elliptic, hyperbolic and parabolic according to the determinant

$$
\left|\begin{array}{ll}
\mathrm{B} & \mathrm{~A}  \tag{1.2.2}\\
\mathrm{C} & \mathrm{~B}
\end{array}\right|
$$

is negative, positive or zero. This classification depends in general on the region of the ( $\mathrm{x}, \mathrm{y}$ )-plane under consideration. Thus it is possible for a p.d.e. to change its classification within the different regions of the domain for which the problem is defined. The differential equation

$$
\begin{equation*}
y \frac{\partial^{2} u}{\partial x^{2}}+x \frac{\partial^{2} u}{\partial x \partial y}+y \frac{\partial^{2} u}{\partial y}=F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{1.2.3}
\end{equation*}
$$

for instance, is elliptic for $|x|<2|y|$, hyperbolic for $|x|>2|y|$ and parabolic along $|x|=2|y|$. This type of equation is said to be mixed type.

For a linear p.d.e. in more than two independent variables, for systems, and for non-linear p.d.e., a similar but complicated classification can be carried out. In the case of three independent variables, the terms elliptic, parabolic and hyperbolic are replaced by their three dimensional analogous such as ellipsoidal, etc.

The elliptic class are equilibrium problems and are usually described in terms of a closed region having boundary conditions prescribed at every point on the region's boundary. These are called the boundary value problems. The parabolic and hyperbolic problems are of propagation type and normally have a prescribed boundary condition on some part of the boundaries and an initial condition along the other part. It can also have open-ended regions into which the solution propagates. In mathematical parlance such problems are known as initial (boundary) value problems.

Another important aspect of the classification of the p.d.e. into hyperbolic, parabolic and elliptic is due to the characteristic equation. Here we can see that at every point of the $x-y$ plane there are two directions in which the integration of the p.d.e. reduces to the integration of an equation involving total differential only. In other words, the integration of an equation in certain directions is not complicated by the presence of partial derivatives in other directions (G.D. Smith, 1978, p.161).

Let the derivatives in equation (1.2.1) be denoted by

$$
\frac{\partial u}{\partial x}=p ; \quad \frac{\partial u}{\partial y}=q ; \quad \frac{\partial^{2} u}{\partial x^{2}}=r ; \quad \frac{\partial^{2} u}{\partial x \partial y}=s \text { and } \frac{\partial^{2} u}{\partial y^{2}}=t .
$$

Let $C$ be curve in the $x-y$ plane on which the values of $u$ and its derivatives satisfy equation (1.2.1). ( $C$ is not a curve on which initial values of $u, p$ and $q$ are given). Therefore, the differentials of $p$ and $q$ in the directions tangential to C are given by

$$
\begin{equation*}
d p=\frac{\partial p}{\partial x} d x+\frac{\partial p}{\partial y} d y=r d x+s d y \tag{1.2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
d q=\frac{\partial q}{\partial x} d x+\frac{\partial q}{\partial y} d y=s d x+t d y \tag{1.2.5}
\end{equation*}
$$

where the p.d.e. (1.2.1) is written as

$$
\begin{equation*}
\mathrm{Ar}+2 \mathrm{Bs}+\mathrm{Ct}+\mathrm{E}=0 \tag{1.2.6}
\end{equation*}
$$

Elimination of $r$ and $t$ in (1.2.6) using (1.2.4) and (1.2.5) gives:

$$
A \frac{d}{d x}(d p-s d y)+2 B s+C-\frac{d}{d y}(d q-s d x)+E=0
$$

i.e.

$$
\begin{equation*}
s\left\{A\left(\frac{d y}{d x}\right)^{2}-B\left(\frac{d y}{d x}\right)+C\right\}-\left\{A \frac{d p}{d x} \cdot \frac{d p}{d y}+C \frac{d q}{d x}+E \frac{d y}{d x}\right\}=0 . \tag{1.2.7}
\end{equation*}
$$

Now choose $\frac{d y}{d x}$, the tangent to $C$ at a point $P(x, y)$ to satisfy

$$
\begin{equation*}
A\left(\frac{d y}{d x}\right)^{2}-2 B\left(\frac{d y}{d x}\right)+c=0 . \tag{1.2.8}
\end{equation*}
$$

Therefore (1.2.7) leads to

$$
\begin{equation*}
A \frac{d p}{d x} \cdot \frac{d q}{d y}+C \frac{d p}{d x}+E \frac{d y}{d x}=0 \tag{1.2.9}
\end{equation*}
$$

which gives the relationship between the total differential $d p$ and $d q$ with respect to x and y .

This shows that at every point $P(x, y)$ of the solution domain there are two directions, given by the roots of equation (1.2.8), along which there is a relationship given by equation (1.2.9). The directions given by the roots of equation (1.2.8) are called the characteristic directions and the p.d.e. is said to be hyperbolic, parabolic or elliptic according to similar determinant requirement as in (1.2.2).

### 1.3 BOUNDARY CONDITIONS

As mentioned earlier in the chapter, the solution of the p.d.e. has to be made to satisfy the boundary conditions which arise from the problem formulation. There are four main types of such conditions which arise frequently in the description of physical phenomena, these are:

1. The First Boundary-Value Problem (the Dirichlet Problem), where the solution $u$ has to satisfy the given values

$$
\begin{equation*}
\left.u\right|_{s}=\phi \tag{1.3.1}
\end{equation*}
$$

on the boundary s. If $\phi \equiv 0$ the problem is called Homogeneous Dirichlet problem.
2. The Second Boundary-Vatue Problem (the Neumann Problem), where the solution $u$ has to satisfy the normal derivatives

$$
\begin{equation*}
\left.\frac{\partial u}{\partial v}\right|_{s}=\psi \tag{1.3.2}
\end{equation*}
$$

on the boundary of that region.
3. The Third Boundary-Value Problem (Mixed or Robin's Problem), where the solution $u$ has to satisfy a combination of $u$ and its derivatives namely

$$
\begin{equation*}
\left[\frac{\partial u}{\partial v}+h u\right]_{s}=\psi \tag{1.3.3}
\end{equation*}
$$

on the boundary s.
4. The Fourth Boundary-Value Problem (Periodic Boundary Problem). In this case we seek the solution such that it satisfies the periodicity conditions, for example,

$$
\begin{equation*}
\left.u\right|_{x}=\left.u\right|_{x+\ell},\left.\frac{\partial u}{\partial v}\right|_{x}=\left.\frac{\partial u}{\partial v}\right|_{x+\ell} \tag{1.3.4}
\end{equation*}
$$

where $\ell$ is called the period.

The physical meaning of the first three boundary-value problems can be illustrated by the problem of steady-state temperature distribution.

In the Dirichlet problem, the temperature is given on the boundary of
a solid. In the Neumann problem the loss or gain of heat through the boundary is given (it is proportional to $\partial u / \partial v$ ). In this problem in order to keep a steady-state distribution of temperature, the net flow of thermal energy passing through the boundary of a solid must be equal to zero, i.e.,

$$
\begin{equation*}
\int_{\mathbf{s}} \psi \mathrm{ds}=0 \tag{1.3.5}
\end{equation*}
$$

The third-boundary problem deals with the heat exchange with the surrounding medium the temperature of which is $\psi / h$, where $h$ is the coefficient of thermal conductivity divided by the specific heat, (S.G. Mikh1in, 1967, p.67).

### 1.4 PROPERLY POSED PROBLEMS

This is an important concept as for problems which are not properly posed cannot, in general, be attacked successfully with numerical methods. Definition 1.4.1

An initial value problem is called properly posed (or well posed) if it satisfies the following conditions:
(i) Existence: The set of initial values for which the problem has a solution is dense in the set of all initial values.
(ii) Uniqueness: For each initial value there exists at most one solution.
(iii) Continuous dependence on the initial values: The solution satisfies a

Lipschitz condition with respect to initial values for which the problem
is solvable.
(Theodor Meis and U1rich Marcowitz, 1981, p.2).
For boundary value problems, it is properly posed if there exists one and only one solution to the problem that satisfies the boundary condition and if small changes in the given function that occur in the boundary conditions cause only small changes in the solution (in other words, if the solution depends continuously on the boundary data). This last requirement is necessary if the theoretical results obtained by solving the boundaryvalue problem are to be used in practical applications where the boundary conditions are known only with whatever degree of accuracy may be provided by the measuring devices involved. In the case of a properly posed problem, admissible error in the determination of the boundary conditions do not invalidate the results found but lead only to insignificant quantitative deviations in the theoretical solution from the experimental results.

To demonstrate the properly posedness condition, consider the Laplace's equation in two-dimensions,

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=0 \tag{1.4.2}
\end{equation*}
$$

in the semi-strip $y>0,-\pi / 2 \leqslant x \leqslant \pi / 2$ under the conditions
with

$$
\begin{align*}
& u\left(-\frac{\pi}{2}, y\right)=u\left(\frac{\pi}{2}, y\right)=0 \\
& u(x, 0)=0 \\
& \frac{\partial u}{\partial y}(x, 0)=\phi(x)  \tag{1.4.3}\\
& \phi\left(-\frac{\pi}{2}\right)=\phi\left(\frac{\pi}{2}\right)=0
\end{align*}
$$

If we set $\phi(x)=0$, the only solution of this problem is $u(x, y) \equiv 0$. On the other hand, if we set

$$
\begin{equation*}
\phi(x)=e^{-\sqrt{2 n+1}} \cos (2 n+1) x \tag{1.4.4}
\end{equation*}
$$

the unique solution will be

$$
\begin{equation*}
u=\frac{1}{2 n+1} e^{-\sqrt{2 n+1}} \cos (2 n+1) x \sinh (2 n+1) y \tag{1.4.5}
\end{equation*}
$$

It is easy to show that the function $\phi$ and its derivatives for sufficiently large $n$ differ by an arbitrary slight amount from 0 . Yet for any non-zero $y$, the function $u$ has the form of a cosine function of arbitrarily large amplitude provided $n$ is large. Consequently, for sufficiently large $n$, this function differs by an arbitrarily great amount from the zero solution, (S.G. Mikhlin, 1967, p.19).

### 1.5 METHODS OF SOLUTION

Advancement in high-speed computing machines has greatly influenced the numerical methods for the solution of p.d.e.'s. One measure of the growth is the upsurge of new methods for solving these problems. However these methods can be classified into two different main techniques, namely finite difference and finite element methods for solving p.d.e.'s.

Finite difference methods are still far and away the most widely used and understood for problems in p.d.e.s. It is the only one that stands out as being universally applicable to both linear and non-linear problems. The main subclass of finite difference methods are methods of lines and methods of nets.

The finite element method which initially arose in structural mechanics has expanded into other areas due to promotion by Zienkiewicz (1971) (see references therein), Oden (1972) and conferences such as the Maryland Symposium of 1972 (A.K. Aziz, 1972) and the International Conference on Computational Methods in Non-linear Mechanics of 1974 (Oden, J.T., 1975).

This thesis deals mainly with finite difference methods where applied to solve parabolic partial differential equations. However, in order to suit the author's profession and future commitments, some brief work on finite element methods is also included.

### 1.6 BASIC MATRIX ALGEBRA

In the numerical solution of p.d.e.'s by the finite difference method or finite element method, the differential system is replaced by matrix system. In this section some useful properties of a matrix are outlined. Notations 1.6.1

A square matrix of order $n$
$a_{i j}$ real number which is the element in the $i^{\text {th }}$ row and $j^{\text {th }}$ column of the matrix $A$.
$A^{-1} \quad$ inverse of $A$
$A^{T} \quad$ transpose of $A$
$|A|$ determinant of $A$
I unit matrix of order $n$.
0 null matrix.
$\rho$ (A) spectral radius of $A$
x column vector with element $x_{i}, i=1,2, \ldots, n$
$\underline{x}^{T}$ row vector with element $x_{j}, j=1,2, \ldots, n$
$\overline{\mathrm{x}} \quad$ complex conjugate of x
$||A||$ norm of $A$
$||\underline{x}||$ norm of vector $x$
$\pi \quad$ permutation matrix which has entries of zeros and one only, with one non-zero entry in each row and column.

Definitions 1.6.1
The matrix $A$ is said to be
(i) non-singuzar if $|A| \neq 0$
(ii) diagonal if its only non-zero elements lie on the diagonal
(iii) symmetric if $A=A^{T}$ i.e. $a_{i j}=a_{j i}, i, j=1,2, \ldots, n$
(iv) orthogonal if $A^{-1}=A^{T}$.
(v) nuZZ if $a_{i j}=0$ for all $i$ and $j, i, j=1,2, \ldots, n$
(vi) diagonally dominant if $\left|a_{i i}\right| \geqslant \sum_{j=1}^{n}\left|a_{i j}\right|$ for all $i$
$\begin{array}{ll}\text { (vii) tridiagonal if } \\ \text { (viii) } \quad \text { block diagonal if }\end{array} a_{i j}=0$ for $\left\lvert\, \begin{aligned} & \mathrm{j} \neq \mathrm{j} \mid>1\end{aligned}\right.$

where each $B_{k}(k=1,2, \ldots, s)$ is a square matrix of order $k_{1}, k_{2}, \ldots, k_{s}$ with $k_{1}+k_{2}+\ldots+k_{s}=n$.
(ix) upper triangular if $\mathrm{a}_{\mathrm{i} j}=0$ for $\mathrm{i}>\mathrm{j}$
(x) Zower triongular if $a_{i j}=0$ for $j>i$
(xi) sparse if many of its elements are zero.

For the matrix A whose elements $a_{i j}$ which are not necessarily real numbers we denote $A^{H}$ as the conjugate transpose of $A$. A is called Hermition if $A^{H}=A$, i.e. if $\bar{a}_{j, i}=a_{i, j}$, for all $i$ and $j, i, j=1,2, \ldots, n$. The definition of a Hermitian matrix implies that the diagonal elements of the matrix are real. A real symmetric matrix is always Hermitian, but a Hermitian matrix is symmetric only if it is real.

If $A$ is real and $\underline{x}$ is complex, then $A$ is positive definite if

$$
(\underline{x}, \mathrm{Ax})>0 \quad \text { for all } \underline{x} \neq \underline{0} .
$$

(Note that the inner product, $(\underline{x}, \underline{y})$ of two complex vectors is $\sum_{i=1}^{n} x_{i} \vec{y}_{i}$ where $\bar{y}_{i}$ is the complex conjugate of $y_{i}$ ). A is non-negative or semi-positive definite if ( $\underline{x}, A \underline{x}$ ) $\geqslant 0$ for all $\underline{x} \neq \underline{0}$ with equality for at least one $\underline{x} \neq \underline{0}$.
$A$ is a band matrix of bandwidth $w=p+q+1$ if. $a_{i j}=0$ for $j>i+p$ or $i>j+q$. If $p=q=1$, then $A$ is tridiagonal and a pentadiagonal matrix can be obtained when $p=q=2$.

Two matrices are called commutative if $A B=B A$. They then possess the same set of eigenvectors.

## Theorem 1.1

A real matrix is positive (non-negative) definite if and only if it is symmetric and all its eigenvalues are positive (non-negative, with at least one eigenvalue equal to zero).

This theorem (without proof) is a very important one and sometimes it is used as a definition of positive (non-negative) definite.

The positive definite matrix can be written as $A=G J G^{-1}$ where $J$ is a positive diagonal matrix and is called Jordan canonical form of $A$. The matrix $G$ can be taken to be an orthogonal matrix (Young, 1971, p.16). If $J^{\frac{1}{2}}$ denotes the diagonal matrix whose elements are the positive square roots of the elements of $J$, then $A^{\frac{1}{2}}=G J^{\frac{1}{2}} G^{-1}$ is positive definite by Theorem 1.1. (It can be clearly seen that $\left.A=\left(A^{\frac{1}{2}}\right)^{2}=\left(G J^{\frac{1}{2}} G^{-1}\right)\left(G J^{\frac{1}{2}} G^{-1}\right)=G J G^{-1}\right)$.

## Theorem 1.2

A real symmetric matrix $A$ of order $n$ is positive (non-negative) definite if and only if it can be written in the form $A=P{ }^{T} P$ where $P$ is some nonsingular (singular) matrix of the same order.

## Proof:

(i) Assume that $A=P^{T} P$ with $P$ is some non-singular matrix, i.e. $|P| \neq 0$.

Then for any vector $x \neq 0$

$$
\begin{aligned}
\underline{x}^{T} \underline{x} & =\underline{x}^{T} P^{T} P \underline{x} \\
& =(P \underline{x})^{T}(P \underline{x})>0
\end{aligned}
$$

$\Rightarrow$ A is positive definite.
(ii) Let $A$ be positive definite (and real). Since $A=A^{\frac{1}{2}} A^{\frac{1}{2}}$ and $A^{\frac{1}{2}}$ is symmetric therefore $A=\left(A^{\frac{1}{2}}\right)^{T} A^{\frac{1}{2}}$. As $A^{\frac{1}{2}}$ is also positive definite $\left|A^{\frac{1}{2}}\right| \neq 0$. Thus, putting $P=A^{\frac{1}{2}}$ gives the required condition.

The proof for the case of A non-negative definite, follows in a similar fashion.

### 1.7 VECTOR AND MATRIX NORMS

For the purpose of analysing the errors in the later chapters, the approximate methods are usually associated with some vectors and matrices of which their magnitudes are measureable as non-negative scalars. Such measuring concept is called a norm.

## Definition 1.7.1

Let the vector $x$ be given by $\underline{x}^{T}=\left[x_{1}, x_{2}, \ldots, x_{n}\right]$, the following scalars are defined as the 1,2 and $\infty$ norm of a vector $x$ :

$$
\begin{align*}
& \| \underline{x}| |_{1}=\left|x_{i}\right|+\left|x_{2}\right|+\ldots+\left|x_{n}\right|  \tag{1.7.1a}\\
& \| \underline{x}| |_{2}=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{2}\right)^{\frac{1}{2}}  \tag{1.7.1b}\\
& \| \underline{x}| |_{\infty}=\sup _{1 \leqslant i \leqslant n}\left|x_{i}\right| \tag{1.7.1c}
\end{align*}
$$

$\left(\|x\|_{2}\right.$ is often called the length of $x$ ). In general $L_{p}$-norms are given by

$$
\begin{equation*}
\|x\|_{p}=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p} \quad, \quad 1 \leqslant p \leqslant \infty \tag{1.7.1d}
\end{equation*}
$$

Since normally matrices and vectors appear simultaneously, it is convenient to introduce the norm of a matrix in such a way that it is compatible with a given vector norm.

## Definition 1.7.2

A matrix norm is said to be compatible with a vector norm $\|\underline{x}\|$ if

$$
\|A \underline{x}\| \leqslant\|A| | \mid \underline{x}\|
$$

for all non-zero $x$.
To construct the matrix norm compatible with the vector norm of (1.7.1a)(1.7.1d), it is necessary that

$$
\begin{equation*}
\|A\|=\operatorname{sem}_{\substack{x \\ \underline{x} \\ \text { that }}}^{\|A \underline{x}\|} \| \tag{1.7.3}
\end{equation*}
$$

(G. Dahlquist, 1974, p.175),
which is equivalent to

$$
\begin{equation*}
\|\mathrm{A}\|=\sup _{\| \underline{\underline{x}}| |=1}\|\mathrm{Ax}\| . \tag{1.7.4}
\end{equation*}
$$

Definition 1.7.3
A matrix norm which is defined by (1.7.4) said to be subordinate to the corresponding vector norm.

Definition 1.7.4
Let $A$ be a matrix of order $n$ with eigenvalues $\lambda_{i}, 1 \leqslant i \leqslant n$, then spectral radius $\rho(A)$ is given by

$$
\rho(A)=\max _{1 \leqslant i \leqslant n}\left|\lambda_{i}\right| .
$$

The matrix norm subordinate to $\|\underline{x}\|_{p}$ is denoted by $\|A\|_{p}$ and these norms satisfy the relations:

$$
\begin{align*}
& \|A\|_{1}=\max _{1 \leqslant j \leqslant n} \sum_{i=1}^{n}\left|a_{i j}\right| \\
& \|A\|_{2}=\sqrt{\rho\left(A^{T} A\right)} \\
& \|A\|_{\infty}=\max _{1 \leqslant i \leqslant n} \sum_{j=1}^{n}\left|a_{i j}\right|  \tag{1.7.8}\\
& \text { (maximum absolute column sum) absolute row sum) } \tag{1.7.9}
\end{align*}
$$

For the derivations of (1.7.7)-(1.7.9) see for example K.E. Atkinson, 1978, p. 418.

Theorem 1.3
If $\underline{x}$ and $y$ are vectors then,
(i) $\quad\|\underline{x}\|>0$ for all $\underline{x} \neq \underline{0}$
(ii) $\quad||\alpha \underline{x}||=|\alpha|| | \underline{x}| |$ for any scalar $\alpha$
(iii) $\quad||\underline{x}+\underline{y}|| \leqslant||\underline{x}||+||\underline{y}||$

The proof is obvious from the Definition 1.7.1.

Theorem 1.4
If $A$ and $B$ are two matrices of order $n$, then
(i) $\|A\|>0$ if $A \neq 0$
(ii) $||k A||=|k|| | A| |$ for any scalar $k$
(iii) $\| A+B| | \leqslant||A||+||B||$ and
(iv) $\|A B||\leqslant||A||\|B\|$.

For the proof see Varga, 1962, p.9.

Theorem 1.5
If $A$ is a matrix of order $n$, then

$$
\begin{equation*}
\|A\| \geqslant \rho(A) \tag{1.7.10}
\end{equation*}
$$

Proof:
If $\lambda$ is any eigenvalue of $A$ and $\underline{x}$ is an eigenvector associated with the eigenvalue $\lambda$, then $A x=\lambda x$.

Thus, from Theorems 1.3 and 1.4

$$
|\lambda| \cdot||\underline{x}||=\|\lambda \underline{x}| |=\| \underline{A x}| | \leqslant||A|| \| \underline{x}| |
$$

from which we conclude that $\|A\| \geqslant|\lambda|$ for all eigenvalues of $A$, which proves (1.7.10).

Theorem 1.6
For any real symmetric matrix $A$ of order $n,\|A\|_{2}=\rho(A)$.

Proof:
Since A is symmetric,

$$
\|A\|_{2}^{2}=\rho\left(A^{T} A\right)=\rho\left(A^{2}\right)=\rho^{2}(A)
$$

and hence the result follows.

### 1.8 EIGENVALUES AND EIGENVECTORS OF A MATRIX

## Definition 1.8.1

If $A$ is a square matrix of order $n$ and if $x$ is a non-zero vector such that $A \underline{x}=\lambda \underline{x}$, where $\lambda$ is some number, then $\underline{x}$ is said to be an eigenvector of A with corresponding eigenvalue $\lambda$.

## Theorem 1.7

If $A$ is a square matrix of order $n$, any eigenvalue $\lambda$ satisfies the $n{ }^{\text {th }}$ degree polynomial equation $|A-\lambda I|=0$; this equation is known as the characteristic equation of $A$.

Proof:
We seek a scalar $\lambda$ and non-zero vector $\underline{x}$ such that $A \underline{x}=\lambda \underline{x}$ or ( $A-\lambda I$ ) $\underline{x}=0$. Since this is a system of $n$ simultaneous homogeneous equations in the $n$ unknowns, $x_{1}, x_{2}, \ldots, x_{n}$ (not all are zero), therefore $A-\lambda I$ must be singular, or in other words $|A-\lambda I|=0$.

## Theorem 1.8 (Gerschgorin's first theorem)

The largest of the moduli of the eigenvalues of the square matrix $A$ cannot exceed the largest sum of the moduli of the elements along any row or any column.

## Proof:

Let $\lambda_{i}$ be an eigenvalue of $A$ and $x_{i}$ is the corresponding eigenvector with components $v_{1}, v_{2}, \ldots, v_{n}$. Then the equation ${\underset{X}{x}}^{i}=\lambda \underline{x}_{\mathrm{i}}$ is in detail given by

$$
\begin{align*}
& a_{1,1} v_{1}+a_{1,2} v_{2}+\ldots+a_{1, n} v_{n}=\lambda_{i} v_{1} \\
& a_{2,1} v_{1}+a_{2,2} v_{2}+\ldots+a_{2, n} v_{n}=\lambda_{i} v_{2} \\
& \vdots  \tag{1.8.2}\\
& \vdots \\
& a_{s, 1} v_{1}+a_{s, 2} v_{2}+\ldots+a_{s, n} v_{n}=\lambda_{i} v_{s} \\
& \vdots \\
& \vdots \\
& a_{n, 1} v_{n}+a_{n, 2} v_{2}+\ldots+a_{n, n} v_{n}=\lambda_{i} v_{n}
\end{align*}
$$

Let $v_{s}$ be the largest in modulus of $v_{1}, v_{2}, \ldots, v_{n}$. Select the $s{ }^{\text {th }}$ equation and divide by $\mathrm{v}_{\mathrm{s}}$ giving

$$
\lambda_{i}=a_{s, 1}\left(\frac{v_{1}}{v_{s}}\right)+a_{s, 2}\left(\frac{v_{2}}{v_{s}}\right)+\ldots+a_{s, n}\left(\frac{v_{n}}{v_{s}}\right)
$$

Since $\left|\frac{v_{v}}{v_{s}}\right| \leqslant 1, i=1,2, \ldots, n$, therefore

$$
\left|\lambda_{i}\right| \leqslant\left|a_{s, 1}\right|+\left|a_{s, 2}\right|+\ldots+\left|a_{s, n}\right|
$$

In particular this holds for $\left|\lambda_{i}\right|=\max \left|\lambda_{s}\right|, s=1,2, \ldots, n$.
As the eigenvalues of the transpose of $A$ are the same as those of $A$, the proof is also true for columns. (G.D. Smith, 1978, p.87).

Theorem 1.9 (Gerschgorin's circle theorem)
Let $A$ have $n$ eigenvalues $\lambda_{i}, i=1,2, \ldots, n$. Then each $\lambda_{i}$ lies in the union of the $n$ circles,

$$
\begin{equation*}
\left|\lambda-a_{i, i}\right| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n}\left|a_{i, j}\right| \tag{1.8.3}
\end{equation*}
$$

Proof:
By the previous proof

$$
\lambda_{i}=a_{s, 1}\left(\frac{v_{1}}{v_{s}}\right)+a_{s, 2}\left(\frac{v_{2}}{v_{s}}\right)+\ldots+a_{s, s}+\ldots+a_{s, n}\left(\frac{v_{n}}{v_{s}}\right)
$$

Therefore,

$$
\begin{aligned}
&\left|\lambda_{i}-a_{s, s}\right|= \left\lvert\, a_{s, 1}\left(\frac{v_{1}}{v_{s}}\right)\right. \\
&+a_{s, 2}\left(\frac{v_{2}}{v_{s}}\right)+\ldots+a_{s, s-1}\left(\frac{v_{s-1}}{v_{s}}\right)+a_{s, s+1}\left(\frac{v_{s+1}}{v_{s}}\right) \\
& \left.+\ldots+a_{s, n}\left(\frac{v_{n}}{v_{s}}\right) \right\rvert\, \\
& \leqslant\left|a_{s, 1}\right|+\left|a_{s, 2}\right|+\ldots+\left|a_{s, s-1}\right|+\left|a_{s, s+1}\right|+\ldots+\left|a_{s, n}\right|
\end{aligned}
$$

As $\lambda_{i}$ is any eigenvalue, therefore,

$$
\begin{equation*}
\left|\lambda-a_{i, i}\right| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n}\left|a_{i, j}\right| \tag{1.8.3}
\end{equation*}
$$

Corollary 1: If $A$ is a square matrix of order $n$ and

$$
\begin{equation*}
v=\max _{1 \leqslant i \leqslant n} \sum_{j=1}^{n}\left|a_{i, j}\right| \tag{1.8.4}
\end{equation*}
$$

then $\rho(A) \leqslant \nu$.

Corollary 2: If $A$ is a square matrix of order $n$ and

$$
\begin{equation*}
v^{\prime}=\max _{1 \leqslant j \leqslant n} \sum_{i=1}^{n}\left|a_{1, j}\right| \tag{1.8.5}
\end{equation*}
$$

then $\rho(A) \leqslant \nu^{\prime}$.
These corollaries are the immediate result of theorem (1.9).

Theorem 1.10
If $A$ is a symmetric, diagonally dominant matrix with positive diagonal elements, it is positive definite.

Proof:
Since $A$ is symmetric, the eigenvalues of $A$ are real. As $A$ is diagonally dominant, the application of Gerschgorin's Theorem shows that the eigenvalues are all positive. Therefore according to Theorem 1.1 , A is positive definite.

## Theorem 1.11

Let $\lambda$ be an eigenvalue of $A$ with eigenvector $x$. Then,
(1) $\alpha \lambda$ is an eigenvalue of $A$ with eigenvector $x$
(2) $\lambda-\mu$ is an eigenvalue of $A-\mu I$ with eigenvector $x$
(3) if $A$ is non-singular, then $\lambda \neq 0$ and $\lambda^{-1}$ is an eigenvalue of $A^{-1}$ with
eigenvector x. (G.W. Stewart, 1973, p.266).

### 1.9 CONVERGENCE OF SEQUENCES OF MATRICES

## Definition 1.9.1

The powers of a sequence involving $A$ of order $n$ is convergent to zero if the sequence of matrices $A, A^{2}, A^{3}, \ldots$, converges to the null matrix 0 .

Theorem 1.12

$$
\begin{equation*}
\operatorname{Lim}_{r \rightarrow \infty} A^{r}=0 \text { if } \quad \| A| |<1 \tag{1.9.2}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
\left\|A^{r}\right\| & =\left\|A A^{r-1}\right\| \leqslant\|A\|\left\|A^{r-1}\right\| \\
& \leqslant\|A\|^{2}\left\|A^{r-2}\right\| \\
& \vdots \\
& \leqslant\|A\|^{r}
\end{aligned}
$$

and so the result follows which is only a sufficient condition and not necessary. The following theorem states the necessary and sufficient condition.

## Theorem 1.13

$$
\operatorname{Lim}_{r \rightarrow \infty} A^{r}=0 \text { if and only if }\left|\lambda_{i}\right|<1
$$

for all eigenvalues $\lambda_{i}(i=1,2, \ldots, n)$ of $A$.
Proof:
Consider the Jordan canonical form of A. A Jordan submatrix of A is of the form

where $\lambda_{i}$ is an eigenvalue of $A$. If this matrix is raised to the power $r$, then the result tends to the null matrix as $r \rightarrow \infty$ if and only if $\left|\lambda_{i}\right|<1$.

Theorem 1.14
Let

$$
F(z)=c_{0}+c_{1} z+\ldots+c_{k} z^{k}+\ldots
$$

be a power series with radius of convergence $R$ (i.e. it is absolutely convergent for $|z|<R)$. Then if $A$ is a square matrix of order $n$ with eigenvalues $\lambda_{i}(1 \leqslant i \leqslant n)$ the matrix series

$$
F(A)=c_{0} I+c_{1} A+c_{2} A^{2}+\ldots+c_{k} A^{k}+\ldots
$$

will be convergent provided $\left|\lambda_{i}\right|<R$ for all i.

### 1.10 EIGENVALUES OF SOME COMMON MATRICES

The eigenvalues of the $(n \times n)$ matrix

where $b$ and $c$ are both real and $a$ is real or complex, are given by

$$
\begin{equation*}
\lambda_{i}=a+2 \sqrt{b c} \quad \cos \left(\frac{i \pi}{n+1}\right), i=1,2, \ldots, n \tag{1.10.2}
\end{equation*}
$$

(G.D. Smith, 1978, p.113).

If $A$ is $a\left({ }^{\times} \times n\right)$ cyclic tridiagonal matrix, i.e.

$$
a=\left[\begin{array}{llllll}
a & b & & & c  \tag{1.10.3}\\
c & a & b & & 0 & \\
& \ddots & \ddots & \ddots & \\
& 0 & & c & a & b \\
b & & & & c & a
\end{array}\right]
$$

then the eigenvalues are given by

$$
\begin{equation*}
\lambda_{i}=a+2 \sqrt{b c} \cos \left(\frac{2 i \pi}{n}\right), i=0,1, \ldots, n-1 \tag{1.10.4}
\end{equation*}
$$

(Benson, A. 1968).

1. 11 ALGORITHMS FOR THE NUMERICAL SOLUTION OF SOME SPECIAL SYSTEMS OF EQUATIONS

Consider the system of equations

$$
\begin{equation*}
A u=f, \tag{1.11.1}
\end{equation*}
$$

where

$$
A=\left[\begin{array}{ccccc}
b_{1} & c_{1} & & &  \tag{1.11.2}\\
a_{2} & b_{2} & c_{2} & 0 & \\
& & \ddots & & \\
& 0 & & a_{n-1} & b_{n-1} \\
& & c_{n-1} \\
& & & a_{n} & b_{n}
\end{array}\right]
$$

and $f=\left[f_{1}, f_{2}, \ldots, f_{n}\right]^{T}$, the algorithm developed by Thomas (1949) which is an algorithmic form of Gaussian Elimination without pivoting is as follows First compute,

$$
\begin{equation*}
\beta_{i}=b_{i}-\frac{a_{i} c_{i-1}}{\beta_{i-1}} \text { with } \beta_{1}=b_{1}, i=2,3, \ldots, n \tag{1.11.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{i}=\frac{f_{i}^{-a_{i} \gamma_{i-1}}}{\beta_{i}} \quad \text { with } \gamma_{1}^{\gamma}=\frac{f_{1}}{b_{1}}, i=2,3, \ldots, n \tag{1.11.4}
\end{equation*}
$$

The components $u_{i}$ of the solution vector $u$ are then given recursively by

$$
\begin{equation*}
u_{n}=\gamma_{n} \text { and } u_{i}=\gamma_{i}-\frac{c_{i} u_{i+1}}{\beta_{i}} \tag{1.11.5}
\end{equation*}
$$

(D.U. Von Rosenberg, 1969, p.113).

Another algorithm which is frequently used is the system (1.11.1) where

$$
A=\left[\begin{array}{lllll}
a_{1} & b_{1} & & & c_{1}  \tag{1.11.6}\\
c_{2} & a_{2} & b_{2} & 0 & \\
& & 0 & \ddots & c_{n-1} \\
& & a_{n-1} & b_{n-1} \\
b_{n} & & & & c_{n} \\
& & & a_{n}
\end{array}\right]
$$

The solution can be obtained in a similar algorithmic manner. Let,

$$
g_{1}=\frac{b_{1}}{a_{1}} \quad, \quad g_{i}=\frac{b_{i}}{a_{i}-c_{i} g_{i-1}}
$$

$$
\begin{aligned}
h_{1} & =\frac{c_{1}}{a_{1}} ; \quad h_{i}=\frac{c_{i} h_{i-1}}{a_{i}-c_{i} g_{i-1}} \\
k_{1} & =\frac{f_{1}}{a_{1}} ; \quad k_{i}=\frac{f_{i}+c_{i} h_{i-1}}{a_{i}-c_{i} g_{i-1}}, i=2,3, \ldots, n-1 \\
G_{1} & =b_{n} ; \quad G_{i}=g_{i-1} G_{i-1} \\
H_{1} & =c_{n} ; \quad H_{i}=H_{i-1}-G_{i-1} h_{i-1} \\
F_{1} & =f_{n} ; \quad F_{i}=F_{i-1}+G_{i-1} k_{i-1} ; i=2,3, \ldots, n-1 \\
g_{n} & =h_{n}=F_{n}=G_{n}=0 \\
H_{n} & =H_{n-1}-\left(G_{n-1}+c_{n}\right)\left(g_{n-1}+h_{n-1}\right) ; \\
\text { and } \quad k_{n} & =F_{n-1}+\left(G_{n-1}^{\left.+a_{n}\right)} k_{n-1} .\right.
\end{aligned}
$$

The components $u_{i}$ of the solution vector $\underline{u}$ are then given recursively by

$$
\begin{equation*}
u_{n}=\frac{k_{n}}{H_{n}} ; \quad u_{i}=k_{i}+g_{i} u_{i+1}+h_{i} u_{n}, i=n-1, n-2, \ldots, 1 . \tag{1.11.8}
\end{equation*}
$$

(Evans and Atkinson, 1973).

## Chapter Two

PARABOLIC EQUATIONS: THE FINITE DIFFERENCE METHOD

### 2.1 PARABOLIC EQUATIONS IN ONE SPACE VARIABLE

The problem of the flow of heat along a rod whose temperature depends only on the coordinate $x$ and on the time $t$ leads to the equation,

$$
\begin{equation*}
c p \frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right)+f_{1}(x, t) \tag{2.1.1}
\end{equation*}
$$

where $u$ denotes the temperature, which is the unknown function of the space variable x and time $\mathrm{t} . \mathrm{k}$ is the thermal conductivity of the rod, c is the specific heat, $p$ is the density and $f_{1}(x, t)$ is the strength of heat sources located in the rod. Equation (2.1.1) is an example of a parabolic equation.

A linear parabolic equation is of ten written as

$$
\begin{equation*}
\frac{\partial u}{\partial t}=f\left(t, x, \frac{\partial u}{\partial x}, \frac{\partial^{2} u}{\partial x^{2}}\right) . \tag{2.1.2}
\end{equation*}
$$

However, the new ideas which arise in the treatment of a p.d.e. can be most readily demonstrated on the simplest parabolic equation. This equation describes conduction in a uniform, insulated rod and is

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}} \tag{2.1.3}
\end{equation*}
$$

The domain of solution for a parabolic equation usually has one of the forms illustrated in Figure (2.1.1) below.

$(t, x) \in(0, \infty) \times(-\infty, \infty)$
(a)

(b)

(c)

FIGURE (2.1.1)

The case (a) is called the semi-infinite plane. This leads to a purely initial value (Cauchy) problem, with initial values given as

$$
\begin{equation*}
u(x, 0)=f(x) \text { for } t=0 \text { and }-\infty<x<\infty \tag{2.1.4}
\end{equation*}
$$

The case (b) is called the open-rectangle plane. This leads to an initial-boundary value problem with the initial condition

$$
u(x, 0)=f(x) \quad \text { for } t=0 \text { and }-\infty<x<\infty
$$

and boundary conditions consisting of
and

$$
\left.\begin{array}{l}
\alpha_{0}(x, t) u+\alpha_{1}(x, t) \frac{\partial u}{\partial x}=\alpha_{2}(x, t) \text { at } x=0 \text { and } t>0  \tag{2.1.5}\\
\beta_{0}(x, t) u+\beta_{1}(x, t) \frac{\partial u}{\partial x}=\beta_{2}(x, t) \text { at } x=0 \text { and } t>0
\end{array}\right\}
$$

where

$$
\left.\begin{array}{l}
\alpha_{0} \geqslant 0, \quad \alpha_{1} \leqslant 0, \quad \alpha_{0}-\alpha_{1}>0  \tag{2.1.6}\\
\beta_{0} \geqslant 0, \quad \beta_{1} \geqslant 0 \quad \text { and } \quad \beta_{0}-\beta_{1}>0
\end{array}\right\}
$$

The case (c) which is a quarter plane also leads to an initial-boundary value problem with the initial condition as in (2.1.4) and the boundary condition at $\mathrm{x}=0$ as in (2.1.5). (Mitchell, A.R. 1976, p.17).

For a specific example, (Noye, 1978, p.7) consider a thin insulated rod initially at a temperature of $0^{\circ} \mathrm{C}$ suddenly has the temperature at each end raised to $100^{\circ} \mathrm{C}$ and held at that value for all t>0 (Figure 2.1.2). Then the initial and boundary conditions are

$$
\left.\begin{array}{l}
u(x, 0)=0, \quad 0<x<1, t=0  \tag{2.1.7}\\
u(0, t)=100, \quad t>0, x=0 \\
u(1, t)=100, \quad t>0, \quad x=0
\end{array}\right\}
$$



FIGURE (2.1.2)

A difficulty occurs when the initial temperature at $x=0$ and $x=1$ namely $u(0,0)$ and $u(1,0)$ are to be prescribed. In practice, the temperature at $x=0$ and $x=1$ cannot be altered instantaneously, but we want
and

$$
\begin{aligned}
& u(0,0-)=0 \text { while } u(0,0+)=100 \\
& u(1,0-)=0 \text { while } u(1,0+)=100
\end{aligned}
$$

For obvious reasons which will be evident later, we need to define $u(0,0)$ and $u(1,0)$ to have the values of 100 .

Equation (2.1.3) with the initial and boundary conditions as defined in (2.1.7) may be solved analytically by the method of separation of variables (Kreyszig, 1972, p.430), giving the solution

$$
\begin{equation*}
u(x, t)=100-\sum_{n=1}^{\infty} \frac{400}{(2 n-1) \pi} \sin \{(2 n-1) \pi x\} \exp \left\{-\alpha(2 n-1)^{2} \pi^{2} t\right\} \tag{2.1.8}
\end{equation*}
$$

for the value of temperature $u$ at any point in the solution region $0 \leqslant x \leqslant 1$, $t \geqslant 0$. For a given $x$ and $t$, this requires the summation of series (2.1.8). This is done by adding more and more terms of the infinite series until the successive partial sum remains unchanged with the required accuracy.

The method of separation of variable is only applicable for the solution of linear p.d.e.'s with constant coefficients and with certain types of boundary conditions. The finite difference method is one alternative method of solving the p.d.e. and is not restricted by such criteria.

### 2.2 FINITE DIFFERENCE GRID

To discuss this, let us assume that the solution region in $x-t$ space is an open-rectangle plane $(0,1) \times(0, \infty)$ which is covered by a rectangular grid (sometimes called a mesh or net), with grid spacing $\Delta x, \Delta t$ in the $x, t$ directions respectively. The values of $\Delta x$ and $\Delta t$ are assumed uniform throughout the region even though they are not necessarily so.

The grid points ( $x, t$ ) are given by

$$
\begin{equation*}
x=x_{i}=i \Delta x, \quad i=0,1, \ldots, m \tag{2.2.1}
\end{equation*}
$$

where $m=1 / \Delta x$ and

$$
\begin{equation*}
t=t_{j}=j \Delta t, j=0,1, \ldots, \tag{2.2.2}
\end{equation*}
$$

which lie on lines parallel to both axes as shown below in Figure (2.2.1).


The points with coordinates $\left(x_{i}, t_{j}\right)$ are called grid points and are denoted by (i,j). Methods will be developed which will approximately determine the values of $u$ at the interior grid points using those boundary and initial points (i.e. for $i=0$ or $i=1$ or $j=0$ ). The following notations will be used
for values of $u$ and $i$ ds derivatives at the grid point (i,j),

$$
\begin{align*}
& u_{i, j}=u(i \Delta x, j \Delta t) \\
& \left.\frac{\partial^{r} u}{\partial t^{r}}\right|_{i, j}=\left.\frac{\partial^{r} u}{\partial t^{r}}\right|_{\substack{x=i \Delta x \\
t}}=j \Delta t  \tag{2.2.3}\\
& \left.\frac{\partial^{s} u}{\partial x^{s}}\right|_{i, j}=\left.\frac{\partial^{s} u}{\partial x^{s}}\right|_{\substack{x=i \Delta x \\
t}}=j \Delta t \leq
\end{align*}
$$

### 2.3 FINITE DIFFERENCE APPROXIMATIONS TO DERIVATIVES

The following theorems are fairly elementary but they are important tools which are used throughout the chapter. The proof can be found in most calculus books.

## Theorem 2.1 (Taylor's Theorem)

Let $u(x)$ have ( $n+1$ ) continuous derivatives on $a \leqslant x \leqslant b$ for some $n \geqslant 0$, and let $x, x_{0} \in[a, b]$. Then,

$$
\begin{gather*}
u(x)=p_{n}(x)+R_{n+1}(x),  \tag{2.3.1}\\
p_{n}(x)=u\left(x_{0}\right)+\frac{\left(x-x_{0}\right)}{1!} \frac{d u\left(x_{0}\right)}{d x}+\ldots+\frac{\left(x-x_{0}\right)^{n}}{n!} \frac{d^{n} u\left(x_{0}\right)}{d x^{n}} \\
R_{n y 1}=\frac{1}{n!} \int_{x_{0}}^{x}(x-z) r^{n} \frac{d^{n+1} u}{d z^{n+1}} \cdot d z=\frac{\left(x-x_{0}\right)^{n+1}}{(n+1)!} u^{(n+1)}(\xi) \tag{2.3.2}
\end{gather*}
$$

for some $\xi$ between $x_{0}$ and $x$.

Theorem 2.2 (Tay1or's Theorem in two dimensions)
Let ( $x_{0}, t_{0}$ ) and ( $x_{0}+\xi, t_{0}+\eta$ ) be given points and assume $u(x, t)$ is ( $\left.n+1\right)$ times continuously differentiable for all ( $x, t$ ) in some neighbourhood of $L\left(x_{0}, t_{0}: x_{0}+\xi, t_{0}+n\right)$. Then

$$
\left.\begin{array}{rl}
u\left(x_{0}+\xi, t_{0}+n\right.
\end{array}\right)=u\left(x_{0}, t_{0}\right)+\left.\sum_{j=1}^{n} \frac{1}{j!}\left(\frac{h}{\xi} \frac{\partial}{\partial x}+\frac{\eta_{n}}{\eta t}\right)^{j} u(x, t)\right|_{\substack{x=x_{0} \\
t=t_{0}}}
$$

for some $0 \leqslant \theta \leqslant 1$. The point $\left(x_{0}+\theta \xi, t_{0}+\theta n\right)$ is an unknown point on the line $L\left(x_{0}, t_{0} ; x_{0}+\xi, t_{0}+\eta\right)$.

By using the above theorems, we can now write $u_{i, j \pm 1}$ and $u_{i \pm 1, j}$ as

$$
\begin{equation*}
u_{i, j \pm 1}=u_{i, j} \pm\left.\Delta t \frac{\partial u}{\partial t}\right|_{i, j}+\left.\frac{\Delta t^{2}}{2!} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j \pm \theta} \tag{2.3.5}
\end{equation*}
$$

$0 \leqslant \theta_{1} \leqslant 1$, and

$$
\begin{equation*}
u_{i \pm 1, j}=u_{i, j} \pm\left.\Delta x \frac{\partial u}{\partial x}\right|_{i, j}+\left.\frac{\Delta x^{2}}{2!} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i \pm \theta, j}, \tag{2.3.6}
\end{equation*}
$$

$0 \leqslant \theta_{2} \leqslant 1$ respectively.


FIGURE (2.3.1)

The value of $\partial u / \partial t$ at the grid point ( $i, j$ ) may be calculated by a number of different finite difference approximations. Figure (2.3.1) shows that, one of the approximations to the gradient at $P$ is given by the gradient of the chord $P Q$ if $P$ and $Q$ are closely located to each other, i.e. if $\Delta t$ is small enough. This is evident mathematically from equation (2.3.5), i.e.

$$
\begin{equation*}
\frac{u_{i, j+1}^{-u} i, j}{\Delta t}=\left.\frac{\partial u}{\partial t}\right|_{i, j}+\left.\frac{\Delta t}{2} \frac{\partial^{2} u^{2}}{\partial t^{2}}\right|_{i, j+\theta_{1}} . \tag{2.3.7}
\end{equation*}
$$

We may write

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{i, j}=\frac{u_{i, j+1}^{-u_{i, j}}}{\Delta t}+O(\Delta t) \text {, as } \Delta t \rightarrow 0 . \tag{2.3.8}
\end{equation*}
$$

Because the right hand side of this expression uses the value $u_{i, j+1}$ which is forward of $\mathbf{u}_{\mathbf{i}, \mathrm{j}}$ in time, it is called the forward difference approximation to the time derivative evaluated at the grid-point (i,j). Similarly, approximating the tangent of $u$ at $P$ by chord $R P$, i.e.

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{i, j}=\frac{u_{i, j}-u_{i, j-1}}{\Delta t}+O(\Delta t) \text {, as } t \rightarrow 0 \tag{2.3.9}
\end{equation*}
$$

leads to a backward difference approximation type formula.
If $u_{i, j+1}$ and $u_{i, j-1}$ are expanded about $u_{i, j}$ up to the third power of $\Delta t, i . e .$,

$$
u_{i, j \pm 1}=u_{i, j} \pm\left.\Delta t \frac{\partial u}{\partial t}\right|_{i, j}+\left.\frac{\Delta t^{2}}{2!} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j}+\left.\frac{\Delta t^{3}}{3!} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\theta}
$$

$0 \leqslant \theta_{1} \leqslant 1$, then

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{i, j}=\frac{u_{i, j+1} \mathbf{u}_{i, j-1}}{2 \Delta t}+o\left\{(\Delta t)^{2}\right\} \tag{2.3.10}
\end{equation*}
$$

and is called the central difference approximation. The error for the later type of approximation is much smaller than the first two types of approximations. Approximation (2.3.11) is equivalent to the slope of $R Q$ in Figure 2.3.1.

In a similar manner the finite difference approximation to the spatial derivatives of $u$ at the grid point ( $i, j$ ) may be obtained. The first order spatial derivative of the forward, backward and central difference approximations are respectively given by:

$$
\begin{align*}
& \left.\frac{\partial u}{\partial x}\right|_{i, j}=\frac{u_{i+1, j^{-u}} u_{, j}}{\Delta x}+O(\Delta x),  \tag{2.3.12}\\
& \left.\frac{\partial u}{\partial x}\right|_{i, j}=\frac{u_{i, j}-u_{i-1, j}}{\Delta x}+O(\Delta x), \tag{2.3.13}
\end{align*}
$$

and $\left.\quad \frac{\partial u}{\partial x}\right|_{i, j}=\frac{u_{i+1, j^{-u}}-1, j}{2 \Delta x}+0\left\{(\Delta x)^{2}\right\}$.
Finite difference approximation to the higher order derivatives can also be developed in a similar way. For example, from

$$
u_{i \pm 1, j}=u_{i, j} \pm\left.\Delta x \frac{\partial u}{\partial x}\right|_{i, j}+\left.\frac{\Delta x^{2}}{2!} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j} \pm\left.\frac{\Delta x^{3}}{3!} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j}+\left.\frac{\Delta x^{4}}{4!} \frac{\partial^{4} u}{\partial x^{4}}\right|_{i \pm \theta_{2}, j}
$$

$0 \leqslant \theta_{2} \leqslant 1$, we can obtain

$$
\left.\frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}=\frac{u_{i+1, j^{-2 u}}{ }_{i, j^{+u}}{ }_{i-1, j}}{(\Delta x)^{2}}+O\left\{(\Delta x)^{2}\right\}
$$

Having approximated the derivatives with the above finite difference approximations, we now use equations (2.3.8) and (2.3.16) respectively to approximate the left and right hand-side terms of the heat conduction equation (2.1.3) to give

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u_{i, j}}}{\Delta t}=\frac{u_{i+1, j^{-2 u_{i, j}}}{ }^{+u_{i-1, j}}}{(\Delta x)^{2}}+0\left\{(\Delta t)+(\Delta x)^{2}\right\} \tag{2.3.17}
\end{equation*}
$$

We may write (2.3.17) approximately and explicitly as

$$
\begin{equation*}
u_{i, j+1}=r u_{i-1, j}+(1-2 r) u_{i, j}+r u_{i+1, j} \tag{2,3.18}
\end{equation*}
$$

where $r=\Delta t /(\Delta x)^{2}$. Hence this method of approximation is called the explicit method. As this approximation is developed via the restrictions given in (2.3.16) and (2.3.17), therefore they are restrictions which apply to this approximation. In order to establish these restrictions, three important concepts will be considered later in this chapter. They are convergence, consistency and stability.

Without using the order notation the approximation (2.3.8) and (2.3.16) to (2.1.3) gives

$$
\begin{align*}
& u_{i, j+1}-r u_{i \oplus 1, j}-(1-2 r) u_{i, j}-r u_{i+1, j} \\
& \quad=\left.\Delta t\left(\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial x^{2}}\right)\right|_{i, j}+\left.\frac{1}{2}(\Delta t)^{2}\left(\frac{\partial^{2} u}{\partial t^{2}}-\frac{1}{6 r} \frac{\partial^{4} u}{\partial x^{4}}\right)\right|_{i, j}+\ldots
\end{align*}
$$

From equations (2.1.3), (2.3.18), (2.3.19) and if we define

$$
e_{i, j}=U_{i, j}-u_{i, j}
$$

as the difference between the exact solutions of the p.d.e. and the difference equation at grid point (i,j), the result

$$
\begin{equation*}
e_{i, j+1}=(1-2 r) e_{i, j}+r\left(e_{i+1, j}+e_{i-1, j}\right)+\frac{1}{2} \Delta t^{2}\left(\frac{\partial^{2} u}{\partial t^{2}}-\frac{1}{6 r} \frac{\partial^{4} u}{\partial x^{4}}\right){ }_{i, j}+\ldots \tag{2,3.20}
\end{equation*}
$$

is obtained. The quantity

$$
\begin{equation*}
\frac{1}{2}(\Delta t)^{2}\left(\frac{\partial^{2} u}{\partial t^{2}}-\frac{1}{6 r} \frac{\partial^{4} u}{\partial x^{4}}\right)_{i, j}+\ldots \tag{2.3.21}
\end{equation*}
$$

is defined as the local truncation error (henceforth abbreviated as L.T.E.) of formula (2.3.18) and the principal part of the L.T.E. is

$$
\begin{equation*}
\frac{1}{2}(\Delta t)^{2}\left(\frac{\partial^{2} u}{\partial t^{2}}-\frac{1}{6 r} \frac{\partial}{}_{4} x^{4}\right)_{i, j} \tag{2.3.22}
\end{equation*}
$$

This concept of the L.T.E. is necessary when we want to consider the three mentioned topics later in the chapter.

### 2.4 THE FINITE DIFFERENCE METHOD

Having introduced the method of approximation the solution of the p.d.e. by finite differences, we can now summarise as follows.

1) Discretise the region to form the grid points. At the grid points the approximate solution to the problem will be found.
2) The dependent variable and its derivatives are approximated by the finite difference approximation. This approximation will lead to either a single explicit equation or a system of linear difference equations. Non-linear problems normally give non-linear systems of equations which cannot be solved directly.
3) Wherever necessary and/or available, the values of the dependent variable are initialised. These initial values may be exact as to a real situation for a transient problem, or they may be an approximation (perhaps a rough guess) for a steady state solution.
4) The computational cycle begins. The approximation in (2) is used to calculate approximate values of the dependent variables at all grid points necessary in the solution region. Normally for the evolution problem, we calculate the solution from one time-level to the next time-level. For steady-state solutions, the rough guesses give the improved solutions which will then be used as input for the next iteration. The process is continued until a predetermined time is reached or until the dependent variable at each space position $\mathbf{i} \Delta \mathbf{x}$ remain unchanged when computed at successive time levels. In the latter case a steady state solution is assumed to have been reached.

### 2.5 CONVERGENCE

By solving (2.3.18) as an approximation to (2.1.3), the following questions may arise:

1) Does the approximate solution approach the exact solution of the differential equation when $\Delta x$ and $\Delta t$ tend to zero?
2) Is the difference scheme stable? In other words, what is the behaviour of the rounding errors when transmitted forward in time. Are they magnified or diminished during transmission.

## Definition 2.5.1

Let $U_{i, j}$ and $u_{i, j}$ represent the exact solution to the p.d.e, and its corresponding finite difference equation at the grid point (i,j) respectively. The value $\left\|U_{i, j}-u_{i, j}\right\|$, when $\|$.$\| is a suitable norm, is called the$ discretisation error.

From Section 2.3, as $u_{i, j}$ is found from the equation (2.3.18) using the approximation at the ( $\mathrm{j}-1$ ) th time level and with the L.T.E. at the ( $\mathrm{j}-1$ ) th time level, therefore the L.T.E. at the point ( $\mathrm{i}, \mathrm{j}-1$ ) is a measure of the (local) discretisation error at the point ( $i, j$ ) when the finitedifference scheme is applied once only to the exact solution values of the p.d.e., all arithmetic being exact, i.e. without rounding errors.

Definition 2.5.2
A difference scheme is said to be convergent if the discretization error tends to zero as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$.

Let $E_{j}$ denote the maximum value of $e_{i, j}$ along the $j^{\text {th }}$ time-level. If $r \leqslant \frac{1}{2}$, thus (2.3.20) gives

$$
\begin{equation*}
E_{j+1} \leqslant E_{j}+A\left\{(\Delta t)^{2}+\Delta t(\Delta x)^{2}\right\} \tag{2.5.3}
\end{equation*}
$$

where $A$ is the maximum modulus of the expression which is included in the
$0\left\{(\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right\}$. Therefore,

$$
\begin{aligned}
& E_{1} \leqslant E_{0}+A\left\{(\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right\}=A\left\{(\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right\} \\
& E_{2} \leqslant E_{0}+2 A\left\{(\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right\}=2 A\left\{(\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right\}
\end{aligned}
$$

$$
E_{M} \leqslant \operatorname{MA}\left\{(\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right\}
$$

as $E_{0}=\underset{i}{\max }\left|e_{i, 0}\right|=0$. Thus, it follows that

$$
\begin{equation*}
\lim _{\substack{\Delta x \rightarrow 0 \\ \Delta t \rightarrow 0}} E_{M} \leq \lim _{\substack{\Delta x \rightarrow 0 \\ \Delta t \rightarrow 0}} \operatorname{MA}\left\{(\Delta t)^{2}+(\Delta x)(\Delta x)^{2}\right\}=0 \tag{2.5.4}
\end{equation*}
$$

As this error term tends to zero as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$, therefore, under the assumption that $r \leqslant \frac{1}{2}$, the finite difference approximation (2.3.18) converges to the solution of the p.d.e. (2.1.3). However, $0 \leqslant r \leqslant \frac{1}{2}$ is a rather severe necessary condition for convergence.

### 2.6 CONS ISTENCY

The problem of consistency is the problem of finding the condition for which a discrete problem is an approximation of the corresponding continuous problem. To discuss this, let us define a more general initialboundary value problem.

Suppose we are required to find the solution of

$$
\begin{equation*}
L(U) \equiv 0 \tag{2.6.1}
\end{equation*}
$$

in a region $R$ of the $(x, t)$ space where $R=[0,1] \times[t, T]$ with the initial condition,

$$
\begin{equation*}
U(x, 0)=f(x), \text { for } t=0 \tag{2.6.2}
\end{equation*}
$$

and boundary condition,

$$
\begin{align*}
& U(0, t)=\phi_{1}(t), \quad x=0, \quad t>0  \tag{2.6.3}\\
& U(1, t)=\phi_{2}(t), \quad x=1, \quad t>0
\end{align*}
$$

where $L$ is any operator. Let $L_{\Delta x, \Delta t}$ denote an approximation to the operator L on the grid points as in Figure (2.2.1).

Definition 2.6.4
The approximation $L_{\Delta x, \Delta t}(u)$ is said to be consistent to the initialboundary value problem (2.6.1)-(2.6.3) if

$$
\begin{equation*}
\left\|L_{\Delta x, \Delta t}(\mathrm{U})-\mathrm{L}(\mathrm{U})\right\| \rightarrow 0 \tag{2.6.5}
\end{equation*}
$$

as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$, where $\|$.$\| is a suitable norm. The numerical value of$ (2.6.5) is called the error of approximation.

As an example, we again consider the approximation (2.3.17), i.e.

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u_{i, j}}}{\Delta t}=\frac{u_{i+1, j^{-2 u_{i, j}}}+u_{i-1, j}}{(\Delta x)^{2}} \tag{2.3.17}
\end{equation*}
$$

that is for the case of $L=\frac{\partial}{\partial t}-\frac{\partial^{2}}{\partial x^{2}}$ (this $L$ is assumed throughout the section). We want to see how closely equation (2.3.18) corresponds to equation (2.6.1). Thus,

$$
\begin{gathered}
\left\|L_{\Delta x, \Delta t}(u)-L(U)\right\|=\| \frac{u_{i, j+1}-u_{i, j}}{\Delta t}-\frac{u_{i+1}-2 u_{i, j}+u_{i-1, j}}{(\Delta x)^{2}}- \\
\left(\frac{\partial U}{\partial t}-\frac{\partial^{2} U}{\partial x^{2}}\right) \|
\end{gathered}
$$

By the substitution of the truncated Taylor series expansion about the grid point ( $i, j$ ) for each $u$ term in the right-hand side of the above equation we have

$$
\begin{equation*}
\left\|L_{\Delta x, \Delta t}(u)-L(U)\right\|=\left\|\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}-\frac{\Delta x^{2}}{6} \frac{\partial^{4} u}{\partial x^{4}}\right\| \rightarrow 0 \tag{2.6.6}
\end{equation*}
$$

as $\Delta x \rightarrow 0$ and $\Delta t=0\left\{(\Delta x)^{2}\right\} \rightarrow 0$, provided $\frac{\partial^{2} u}{\partial t^{2}}$ and $\frac{\partial^{4} u}{\partial x^{4}}$ are bounded at every point of the region. Therefore, the approximation (2.3.17) is consistent to the initial-boundary value problem (2.6.1). From equation (2.3.22), we can see that a certain difference approximation to a parabolic equation is consistent if

$$
\frac{\text { principal part of L.T.E. }}{\Delta t} \rightarrow 0 \text { as } \Delta x, \Delta t \rightarrow 0 .
$$

An example of an inconsistent difference approximation to (2.6.1) is given by the scheme

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u_{i, j-1}}}{2 \Delta t}-\frac{\left.u_{i+1, j^{-2\left\{\theta u_{i, j+1}\right.}}+(1-\theta) u_{i, j-1}\right\}+u_{i-1, j}}{(\Delta x)^{2}}=0 \tag{2.6.7}
\end{equation*}
$$

By Taylor series expansion, we get

$$
\begin{align*}
\left\|L_{\Delta x, \Delta t}(u)-L(U)\right\| \leqslant & \| \frac{\Delta t^{2}}{6} \frac{\partial^{3} u}{\partial t^{3}}-\frac{\Delta x^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+(2 \theta-1) \frac{2 \Delta t}{\Delta x^{2}} \frac{\partial u}{\partial t}+ \\
& \left(\frac{\Delta t}{\Delta x}\right)^{2} \frac{\partial^{2} u}{\partial t^{2}}| |+\left|\left|O\left(\frac{\Delta t^{3}}{\Delta x^{2}}, \Delta x^{4}, \Delta t^{4}\right)\right|\right| \tag{2.6.8}
\end{align*}
$$

For $\theta \neq \frac{1}{2}$, the $\|.\| \rightarrow \infty$ as $\Delta t, \Delta x \rightarrow 0$. For $\theta=\frac{1}{2}$, the norm (2.6.8) tends to

$$
\left\|\left(\frac{\Delta t}{\Delta x}\right)^{2} \frac{\partial^{2} u}{\partial t^{2}}\right\|
$$

as $\Delta x, \Delta t \rightarrow 0$. Therefore if $\frac{\Delta t}{\Delta x}=\alpha$, the scheme is not consistent with (2.6.1), but with the hyperbolic equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}-\alpha \alpha^{2} \frac{\partial^{2} u}{\partial t^{2}} \tag{2.6.9}
\end{equation*}
$$

However if $\Delta t \rightarrow 0$ faster than $\Delta x\left(e . g . \Delta t=0\left\{(\Delta x)^{2}\right\}\right)$, then (2.6.7) is consistent with the equation (2.6.1). The scheme (2.6.7) with $\theta=\frac{1}{2}$ is called the Du-Fort and Frankel scheme (1953). It was devised to overcome the well-known unfortunate formula due to Richardson, i.e.

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u_{i, j-1}}}{2 \Delta t}-\frac{u_{i-1, j^{-2 u_{i, j}}}{ }^{+u_{i+1, j}}}{(\Delta x)^{2}}=0 \tag{2.6.10}
\end{equation*}
$$

The reason for its ill-repute will become clear in the next section.

### 2.7 STABILITY

Another important feature of a finite difference equation is its stability, a property concerned with the behaviour of errors produced in the finite difference solution due to errors introduced in a previously calculated solution. If it were possible to carry out all numerical operations to an infinite number of decimal places, the exact solution to the difference equation would be found. In actual computation, however, each calculation made by the computer is carried out to a finite number of significant figures which introduce a 'round-off' error at every step, so that the solution actually found to equation (2.3.18) for example is not $u_{i, j}$ but instead $u_{i, j}^{*} u_{i, j}^{*}$ is called the numerical solution of the finite difference equation in contrast with its exact solution $u_{i, j}$.

Suppose in finding the solution to

$$
\begin{equation*}
L(U)=f \tag{2.7.1}
\end{equation*}
$$

which is associated with some boundary conditions

$$
\begin{equation*}
\ell(U)=\varnothing \tag{2.7.2}
\end{equation*}
$$

in some region $R$, we apply the approximate solution,

$$
\begin{align*}
& \mathrm{L}_{\Delta \mathrm{x}, \Delta \mathrm{t}}(\mathrm{u})=\mathrm{f}_{\Delta \mathrm{x}, \Delta \mathrm{t}} \\
& { }^{{ }_{\Delta x, \Delta t}}(\mathrm{u})=\phi_{\Delta \mathrm{x}, \Delta \mathrm{t}} \tag{2.7.3}
\end{align*}
$$

where the subscripts $\Delta x, \Delta t$ denote that the approximation is defined at the grid points in the ( $x, t$ ) space with grid-spacing $\Delta x$ and $\Delta t$. But due to the errors as mentioned above, instead of finding $u_{i, j}$ we get $u_{i, j}$, , where

$$
\begin{align*}
& \mathrm{L}_{\Delta x, \Delta t}\left(u^{*}\right)=f_{\Delta x, \Delta t}^{*}  \tag{2.7.4}\\
& \ell_{\Delta x, \Delta t}\left(u^{*}\right)=\phi_{\Delta x, \Delta t}^{*} .
\end{align*}
$$

From here we define that the difference scheme is stable if for a suitable norm,

$$
\begin{equation*}
\left\|\left\|_{\Delta x, \Delta t}-u_{\Delta x, \Delta t}^{*}\right\| \leqslant c\right\| f_{\Delta x, \Delta t}-f_{\Delta x, \Delta t}^{*} \| \tag{2.7.5}
\end{equation*}
$$

for some constant $C$ are for all values of $\Delta x$ and $\Delta t, 0<\Delta x<\Delta x_{0}$ and $0<\Delta t<\Delta t_{0}$ (i.e. when the mesh is refined). In other words, a difference scheme is stable, if small perturbations in the equations causes small perturbations in their solution uniformly for all small $\Delta x$ and $\Delta t$.

There are many methods of stability analysis, however, the most commonly used for examining this notion of stability are:

1) the matrix method
2) the Fourier Series method.

These methods will be described later in the chapter.
There is an important connection between the consistency of a stable finite difference scheme and the convergence of its solution to that of the p.d.e. it approximates.

Theorem 2.3 (Lax's Equivalence Theorem)
Given a properly posed initial-value problem and a finite-difference approximation that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence. (Lax and Richtmyer,1956).

The two restrictions which apply to this theorem should be carefully noted. Firstly, the initial-value problem must be well posed and secondly, the theorem only applies to a linear problem. This theorem is quite important since it is quite difficult to show the convergence of the solution of a finite difference equation to the solution of the p.d.e. that it approximates compared to the analysis of stability and consistency.

### 2.8 THE METHOD OF WEIGHTED AVERAGES

Crandall (1955) used a family of schemes called the method of a weighted average
as an approximation to the equation (2.1.3), where $0 \leqslant \theta \leqslant 1$. If $\theta=0$, the explicit scheme as discussed in Section 2.3 is obtained. Other values of $\theta$ give the implicit scheme i.e., for $\theta=1$ we get the classical implicit method due to $0^{\prime}$ Brien, Hyman and Kaplan (1950), i.e.,

$$
\begin{equation*}
-r u_{i+1, j+1}+(1+2 r) u_{i, j+1}-r u_{i-1, j+1}=u_{i, j} \tag{2.8.2}
\end{equation*}
$$

Meanwhile for $\theta=\frac{1}{2}$, (2.8.1) leads to the Crank-Nicolson method (1947) as

$$
\begin{equation*}
-\frac{r}{2} u_{i-1, j+1}+(1+r) u_{i, j+1}-\frac{r}{2} u_{i+1, j+1}=\frac{r}{2} u_{i-1, j}+(1-r) u_{i, j}+\frac{r}{2} u_{i+1, j} \tag{2.8.3}
\end{equation*}
$$

Application of (2.8.2) and (2.8.3) to every grid-point in the solution domain results in the system of linear equations,

$$
\begin{equation*}
\underline{\mathrm{A}}_{\mathrm{j}+1}=\mathrm{Bu}_{\mathrm{j}}+\underline{\mathrm{b}} . \tag{2.8.4}
\end{equation*}
$$

For the classical implicit formulae (2.8.2),

$$
A=\left[\begin{array}{ccccc}
1+2 r & -r & & &  \tag{2.8.5}\\
-r & 1+2 r & -r & 0 & \\
& & \ddots & \ddots & \ddots
\end{array}\right] \text {, }
$$

and $B$ is the identity matrix. While for the Crank-Nicolson formulae (2.8.3),
and

$$
\begin{align*}
& A=\frac{r}{2}\left(I+T_{n}\right), \\
& B=\frac{r}{2}\left(I-T_{n}\right) \tag{2.8.6}
\end{align*}
$$

where

$$
T_{n}=\left[\begin{array}{llllll}
2 & -1 & & &  \tag{2.8.7}\\
-1 & 2 & -1 & & 0 \\
& & \ddots & \ddots, & \\
& 0 & \ddots & \ddots & & \\
& & & \ddots & 2 & -1 \\
& & & & -1 & 2
\end{array}\right]
$$

$n$ is the order of the matrix which depends on the size of $\Delta x$. In both cases, $\underline{b}$ is the column vector whose elements associated with the boundary conditions.

The L.T.E. of (2.8.1) can be easily verified using Taylor's expansion and is of order $O\left(\Delta t+(\Delta x)^{2}\right)$, with its principal part as

$$
\begin{equation*}
\frac{\Delta x^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{\Delta t}{2} \cdot \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}-2(1-\theta) \frac{\partial^{2} u}{\partial x^{2}}\right]+(1-\theta) \frac{\Delta t^{2}}{2} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}} \tag{2.8.8}
\end{equation*}
$$

From the equation (2.8.8), it can be seen that the Crank-Nicolson formulae has a better L.T.E. compared to the others i.e. $O\left\{(\Delta x)^{2}+(\Delta t)^{2}\right\}$.

The accuracy of (2.8.1) can be further improved by suitably choosing $\theta$ and $r=\Delta t /(\Delta x)^{2}$ so that both terms in $\Delta t$ and $(\Delta x)^{2}$ disappear from the principle part of L.T.E. (2.8.8). Expression (2.8.8) may be rewritten as

$$
\begin{array}{ll} 
& \frac{\partial^{2} u}{\partial t^{2}}\left\{\frac{\Delta x^{2}}{12}+\frac{\Delta t}{2}-\Delta t(1-\theta)\right\}+(1-\theta) \frac{\Delta t^{2}}{2} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}} \\
\text { i.e. } & \frac{\partial^{2} u}{\partial t^{2}}\left\{\frac{\Delta x^{2}}{12}-\frac{\Delta t}{2}+\Delta t \theta\right\}+(1-\theta) \frac{\Delta t^{2}}{2} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}} \tag{2.8.9}
\end{array}
$$

the first part of which vanishes if $\theta$ is chosen to satisfy

$$
\begin{equation*}
\theta=\frac{1}{2}-\frac{1}{12 \mathrm{r}} \tag{2,8,10}
\end{equation*}
$$

This finite difference approximation then has the L.T.E. of $O\left\{(\Delta t)^{2}+(\Delta x)^{4}\right\}$.

### 2.9 STABILITY ANALYSIS BY THE MATRIX METHOD

This method of analysing the stability is applicable for initialboundary value problems as in Figure (2.1.2). In general, the finite difference approximation to a parabolic p.d.e. can be represented as a repetitive system of simultaneous linear equations, which in matrix form can be written as

$$
\begin{equation*}
A_{j} \underline{u}_{j+1}=B_{j} \underline{u}_{j}+\underline{b} \tag{2.9.1}
\end{equation*}
$$

For the linear parabolic equation with constant coefficient, $A_{j}$ and $B_{j}$ are constant matrices for $a l l \mathrm{j}$, where j denotes the evolutionary step. $\underline{b}$ is a column matrix which is associated with the boundary condition. For the $\operatorname{explicit}$ method, $A_{j}=I$ where $I$ is the identity matrix. In this analysis, we will only consider for the case where $A_{j}$ and $B_{j}$ are constant, i.e.,

$$
\begin{equation*}
{\underset{\mathrm{Au}}{j+1}}^{\mathrm{B}_{\mathrm{j}}}+\underline{\mathrm{b}} . \tag{2.9.2}
\end{equation*}
$$

Explicitly, (2.9.2) is written as

$$
\begin{equation*}
\underline{u}_{j+1}=A^{-1} \underline{B u}_{j}+\underline{\hat{b}}, \tag{2.9.3}
\end{equation*}
$$

where $\underline{\underline{b}}=A^{-1} \underline{b}$. Due to introduction of errors as discussed in Section (2.7), the u's that are actually calculated satisfy

$$
\begin{equation*}
\tilde{\underline{u}}_{j+1}=A^{-1} \underline{B}_{j}+\underline{\hat{b}} \tag{2.9.4}
\end{equation*}
$$

By defining the error vector

$$
\begin{equation*}
\underline{e}_{j+1}=\tilde{\underline{u}}_{j+1} \underline{u}_{j} \tag{2.9.5}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\underline{e}_{j+1}=A^{-1} \underline{B e}_{j} \tag{2.9.6}
\end{equation*}
$$

which leads to $\quad e_{j+1}=\left(A^{-1} B\right)^{j+1} e_{0}$.
As mentioned in Section (2.7), the method defined by system (2.9.2) will be stable provided the $\left|\underline{e}_{j+1}\right| \mid$ are bounded for all $j \geqslant 0$. Since

$$
\begin{equation*}
\left\|\underline{e}_{j+1}\right\| \leqslant\left\|\left(A^{-1} B\right)^{j+1}\right\|\left\|\underline{e}_{0}\right\| \tag{2.9.8}
\end{equation*}
$$

this occurs if and only if a constant $k$ (independent of grid spacing $\Delta x$ and $\Delta t$ )
can be found such that

$$
\begin{equation*}
\left\|\left(A^{-1} B\right)^{j+1}\right\| \leqslant k \tag{2.9.9}
\end{equation*}
$$

Now for any matrix $C=A^{-1} B$, the spectral radius (maximum modulus eigenvalue), $\rho(C)$, is related to the norm, $\|C\|$ by the inequality

$$
\rho^{j+1}(c) \leqslant\left\|c^{j+1}\right\| \leqslant\|c\|^{j+1}
$$

for all $j \geqslant 0$. Bearing these inequalities in mind, two simple criteria for regulating the error growth can be considered. They are:
(i) the spectral radius condition

$$
\begin{equation*}
\rho(C) \leqslant 1 \tag{2.9.11}
\end{equation*}
$$

which is necessary for stability and for $\rho(C)<1$ guarantees that the error vector ${\underset{j}{j}}^{\mathrm{f}} \boldsymbol{0}$ ( as $\mathrm{j} \rightarrow \infty$, but gives no indication of the magnitude of $\mathrm{e}_{\mathrm{j}}$ for finite j .
(ii) The norm condition

$$
\begin{equation*}
\| c| | \leq 1 \tag{2.9.12}
\end{equation*}
$$

which is sufficient for stability and guarantees an ever-diminishing error as $j$ increases. (Mitchell, A.R., 1980, p.41). If $C$ is a symmetric matrix, i.e. $\rho(C)=\|\left. C\right|_{2}$, then (2.9.11) is a necessary and sufficient condition for stability.

As an example of the matrix method for examining stability, we consider the scheme represented by the equation (2.3.18) namely,

$$
u_{i, j+1}=r u_{i-1, j}+(1-2 r) u_{i, j}+r u_{i+1, j}
$$

which in matrix form is written as

$$
\begin{equation*}
\underline{u}_{j+1}=\frac{A u_{j}}{}+\underline{b}, \tag{2.9.13}
\end{equation*}
$$

where
and b is associated with the boundary conditions.
From equation (1.10.2), the eigenvalues of this matrix are,

$$
\begin{equation*}
\lambda_{s}=1-4 r \sin ^{2} \frac{s \pi}{2 N}, \quad(s=1,2, \ldots, N-1), \tag{2.9.15}
\end{equation*}
$$

and from equation (2.9.11), it follows that the method is stable if

$$
\begin{equation*}
-1 \leqslant 1-4 r \sin ^{2} \frac{s \pi}{2 N} \leqslant 1 \quad,(s=1,2, \ldots, N-1), \tag{2.9.16}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
0<r \leqslant 1 / 2 . \tag{2.9.17}
\end{equation*}
$$

This analysis of stability can also be extended to the multi-level finite difference equations. An example of which is the scheme due to Du Fort and Frankel (1953), where the approximation to equation (2.1.3) is given by

$$
\begin{equation*}
(1+2 r) u_{i, j+1}=(1-2 r) u_{i, j-1}+2 r\left(u_{i-1, j}+u_{i+1, j}\right) \tag{2.9.18}
\end{equation*}
$$

For known boundary values and $N \Delta x=1$ these equations in matrix form are

$$
\begin{equation*}
\underline{u}_{j+1}=\frac{(1-2 r)}{(1+2 r)} \underline{I}_{j-1}+\frac{2 r}{(1+2 r)} \underline{A u}_{j}+\underline{b} \tag{2.9.19}
\end{equation*}
$$

where

$$
A=\left[\begin{array}{lllll}
0 & 1 & & 0  \tag{2.9.20}\\
1 & 0 & 1 & 0 & \\
1 & \ddots & \vdots & & \\
& 0 & & 1 & 0 \\
0 & & & 1 & 0
\end{array}\right]
$$

and $\underline{b}$ is associated with the boundary values.
If we define

$$
\underline{v}_{\mathrm{j}}=\left[\begin{array}{l}
\underline{u}_{j}  \tag{2.9.21}\\
\underline{u}_{j-1}
\end{array}\right]
$$

the equation (2.9.19) is given by

$$
\left[\begin{array}{l}
\underline{u}_{j+1}  \tag{2.9.22}\\
\underline{u}_{j}
\end{array}\right]=\left[\begin{array}{c:c}
\frac{2 r}{1+2 r} A & \frac{1-2 r}{1+2 r} \\
\hdashline I & \underline{0}
\end{array}\right]\left[\begin{array}{l}
\underline{u}_{j} \\
\underline{u}_{j-1}
\end{array}\right]+\left[\begin{array}{l}
\underline{b} \\
\underline{0}
\end{array}\right]
$$

Before we proceed the following theorem is useful:

## Theorem 2.4

If the matrix A can be written as

$$
A=\left[\begin{array}{cccc}
A_{1,1} & A_{1,2} \ldots \ldots & A_{1, m}  \tag{2.9.23}\\
A_{2,1} & A_{2,2} \ldots \ldots . & A_{2, m} \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \\
A_{m, 1} & A_{m, 2} \ldots \ldots & A_{m, m}
\end{array}\right]
$$

where each $A_{i, j}$ is an $n \times n$ matrix, and all the $A_{i, j}$ have a common set of $n$ linearly independent eigenvectors, then the eigenvalues of A are given by the eigenvalues of the matrices

$$
\left[\begin{array}{ccc}
\lambda_{1,1}^{(k)} & \lambda_{1,2}^{(k)} \ldots \ldots \lambda_{1, m}^{(k)}  \tag{2.9.24}\\
\lambda_{2,1}^{(k)} & \lambda_{2,2}^{(k)} \ldots \ldots \lambda_{2, m}^{(k)} \\
1 & 1 & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & 1 & \\
\lambda_{m, 1}^{(k)} & \lambda_{m, 2}^{(k)} \ldots \ldots & \lambda_{m, m}^{(k)}
\end{array}\right] \quad, k=1(1) n
$$

where $\lambda_{i, j}^{(k)}$ is the $k^{\text {th }}$ eigenvalue of $A_{i, j}$ corresponding to the $k^{\text {th }}$ eigenvector $\mathrm{v}_{\mathrm{k}}$ common to all the $\mathrm{A}_{\mathrm{i}, \mathrm{j}}$ 's (G.D. Smith, 1978, p.107).

From this theorem, therefore the eigenvalues $\lambda$ of the coefficient matrix in (2.9.22) are the eigenvalues of the matrix

$$
\left[\begin{array}{cc}
\frac{2 r}{1+2 r} \lambda_{k} & \frac{1-2 r}{1+2 r}  \tag{2.9.25}\\
1 & 0
\end{array}\right]
$$

where $\lambda_{k}$ is the $k^{\text {th }}$ eigenvalue of $A$. To find $\lambda$ we can directly evaluate giving $\quad \operatorname{det}\left[\begin{array}{cc}\frac{2 r}{1+2 r} \lambda_{k}-\lambda & \frac{1-2 r}{1+2 r} \\ 1 & -\lambda\end{array}\right]=0$

$$
\begin{equation*}
\lambda\left(\lambda-\frac{2 r}{1+2 r} \lambda_{k}\right)-\frac{1-2 r}{1+2 r}=0 \tag{2.9.27}
\end{equation*}
$$

where by (1.10.2)

$$
\begin{equation*}
\lambda_{k}=2 \cos \frac{k \pi}{N}, k=1,2, \ldots, N-1 \tag{2.9.28}
\end{equation*}
$$

Therefore the solution of the quadratic equation (2.9.27) is

$$
\begin{equation*}
\lambda=\frac{1}{1+2 r}\left\{2 r \cos \frac{k \pi}{N} \pm\left[1-4 r^{2} \sin ^{2} \frac{k \pi}{N}\right]^{\frac{1}{2}}\right\} \tag{2.9.29}
\end{equation*}
$$

For $r=\frac{1}{2}$, it is easily verified that $|\lambda| \leqslant 1$.
For $r>\frac{1}{2}$, i.e. $1-4 r^{2} \sin ^{2} \frac{k \pi}{N}<0$ then

$$
\begin{aligned}
|\lambda|^{2} & =\frac{1}{(1+2 r)^{2}}\left\{\left(2 r \cos \frac{k \pi}{N}\right)^{2}+4 r^{2} \sin ^{2} \frac{k \pi}{N}-1\right\} \\
& =\frac{4 r^{2}-1}{4 r^{2}+4 r+1}<1 \text { for all } r>0
\end{aligned}
$$

For $r<\frac{1}{2}$, i.e. $1-4 r^{2} \sin ^{2} \frac{k \pi}{N}>0$, then

$$
|\lambda|<\frac{2 r+1}{1+2 r}=1
$$

Therefore the equation is conditionally stable for all positive $r$.

### 2.10 STABILITY ANALYSIS BY THE FOURIER SERIES METHOD

This method of finding the stability criterion is to examine the propagation effect of a single row of errors, say along the line $t=0$. These initial errors are expressed in terms of a finite Fourier Series of the form $\sum_{n} e^{i n \pi x / L}$ where $i=\sqrt{-1}$ and $L$ is the interval throughout which the function is defined. To avoid confusion we need to change our usual notation $u_{i, j}$ to $u_{p, q}=u(p \Delta x, q \Delta t)$, say. In terms of this notation

$$
\begin{aligned}
A_{n} e^{i n \pi x / L} & =A_{n} e^{i n \pi p \Delta x / N \Delta x} \\
& =A_{n} e^{i p \beta_{n} \Delta x}
\end{aligned}
$$

where $\beta_{n}=n \pi / N \Delta x, N \Delta x=L$ and $A_{n}$ is the Fourier coefficient. Let $E_{p}$ denote the error at each pivotal point on $t=0,0<x<L$, therefore,

$$
\begin{equation*}
E_{p}=\sum_{n=0}^{N} A_{n} e^{i p \beta_{n} \Delta x}, p=0,1, \ldots, N \tag{2.10.1}
\end{equation*}
$$

Then, the $(N+1)$ equations are sufficient to determine the ( $N+1$ ) unknowns $A_{0}, A_{1}, \ldots, A_{n}$ uniquely, showing that an arbitrary distribution of initial errors can be expressed in this complex exponential form. However for the case of linear difference equations, it is enough to consider the contribution to the error due to a single term such as $e^{i \beta p \Delta x}$, as $A_{n}$ is a constant and can be neglected (G.D. Smith, 1978, p.93).

To investigate the propagation of this error as $t$ increases we need to find a solution of the finite-difference equation which reduces to $e^{i \beta p \Delta x}$ when $t=0$. Such a solution which is denoted by $E_{p, q}$ has the form,

$$
\begin{align*}
E_{p, q} & =e^{i \beta p \Delta x} \cdot e^{\alpha q \Delta t} \\
& =e^{i \beta p \Delta x} \xi^{q} \tag{2.10.2}
\end{align*}
$$

where $\xi=e^{\alpha \Delta t}$, and $\alpha$ in general, is a complex constant. This obviously reduces to $e^{i \beta p \Delta x}$ when $q=0$. It is also clear that the error will not
increase as $t$ increases provided

$$
\begin{equation*}
|\xi| \leq 1 . \tag{2.10.3}
\end{equation*}
$$

This criterion is necessary and sufficient for two time-level difference equations but is not always sufficient for three or more level equations although it is always necessary. (G.D. Smith, 1978, p.93).

To illustrate the method let us consider the method of weighted averages, (2.8.1), i.e.

$$
\begin{align*}
-r \theta u_{p-1, q+1}+(1+2 r \theta) u_{p, q+1}-r \theta u_{p+1, q+1} & =r(1-\theta) u_{p-1, q} \\
\{1-2(1-\theta) r\} u_{p, q} & +r(1-\theta) u_{p+1, q} \tag{2.10.4}
\end{align*}
$$

Substitution of (2.10.2) into (2.10.4) gives

$$
\begin{align*}
& -r \theta e^{i \beta(p-1) \Delta x_{\xi} q+1}+\{1+2 r \theta\} e^{i \beta p \Delta x_{\xi} q+1}-r \theta e^{i \beta(p+1) \Delta x_{\xi} q+1} \\
& \quad=r(1-\theta) e^{i \beta(p-1) \Delta x_{\xi} q}+\{1-2(1-\theta) r\} e^{i p \beta \Delta x q_{\xi}}+r(1-\theta) e^{i \beta(p+1) \Delta x_{\xi} q} \tag{2.10.5}
\end{align*}
$$

After the division by $e^{i \beta p \Delta x_{\xi}}{ }^{q}$ and with some manipulation we obtain the result

$$
\{-2 r \cos \beta \Delta x+(1+2 r \theta)\} \xi=r(1-\theta) 2 \cos \beta \Delta x+\{1-2(1-\theta) r\}
$$

which leads to

$$
\begin{equation*}
\xi=\frac{-4 r(1-\theta) \sin ^{2} \frac{\beta \Delta x}{2}+1}{4 r \theta \sin ^{2} \frac{\beta \Delta x}{2}+1} \tag{2.10.6}
\end{equation*}
$$

For stability we require $|\xi| \leqslant 1$ ( $\xi$ is called amplification factor). Therefore, the stability of the method of weighted average depends on $\theta$. For $\theta=1$, i.e. the classical implicit method,

$$
\xi=\frac{1}{1+4 r \sin ^{2} \frac{\beta \Delta x}{2}}
$$

which gives $|\xi| \leqslant 1$ for all $r>0$. For $\theta=0$, i.e., the explicit method,

$$
\xi=1-4 \operatorname{rsin}^{2} \frac{\beta \Delta x}{2},
$$

which guarantees stability if $r \leqslant \frac{1}{2}$. For $\theta=\frac{1}{2}$, i.e. .the Crank-Nicolson method,

$$
\xi=\frac{1-2 r \sin ^{2} \frac{\beta \Delta x}{2}}{1+2 r \sin ^{2} \frac{\beta \Delta x}{2}}
$$

which also implies unconditional stability for $r>0$.
It is important to notice that, although this method is unconditionally stable for $\theta>\frac{1}{2}$, for any $r>0$, we still have to choose the step length $\Delta x$ and $\Delta t$ small enough to obtain a reasonable accuracy, i.e. to make the truncation error for the finite-difference method negligible. The application of a large time-step not only disturbs the accuracy, for some schemes but also causes a jump in the solution which is called the noise effect (Danaee, A., 1980, p.44).

For the example of unconditional instability, we consider the scheme due to Richardson as given by the equation (2.6.10), i.e.,

$$
\begin{equation*}
u_{p, q+1}=2 r u_{p-1, q}-4 r u_{p, q}+2 r u_{p+1, q}+u_{p, q-1} \tag{2.10.7}
\end{equation*}
$$

Substitution of (2.10.2) into (2.10.7) gives

$$
\begin{gather*}
e^{i \beta p \Delta x_{\xi} q+1}=2 r e^{i \beta(p-1) \Delta x_{\xi} q}-4 r e^{i \beta p \Delta x_{\xi} q}+2 r e^{i \beta(p+1) \Delta x_{\xi} q}+ \\
e^{i \beta p \Delta x_{\xi} q-1} \tag{2.10.8}
\end{gather*}
$$

Upon division by $e^{i \beta p \Delta x_{\xi} q}$ and after some manipulation the final result is

$$
\begin{equation*}
\xi=-8 r \sin ^{2} \frac{\beta \Delta x}{2}+\xi^{-1} \tag{2.10.9}
\end{equation*}
$$

Multiplication by $\xi$ and solving the resulting quadratic equation in $\xi$ we find

$$
\begin{equation*}
\xi=-4 r \sin ^{2} \frac{\beta \Delta x}{2} \pm\left[1+8 r^{2} \sin ^{2} \frac{\beta \Delta x}{2}+O\left(r^{4}\right)\right] \tag{2.10.10}
\end{equation*}
$$

If we select the negative $s i g n$ we find that,

$$
\xi=-1-4 r \sin ^{2} \frac{\beta \Delta x}{2}\left(1+2 r \sin ^{2} \frac{\beta \Delta x}{2}\right)-O\left(r^{4}\right),
$$

or

$$
\begin{equation*}
|\xi|>1+4 r \sin ^{2} \frac{\beta \Delta x}{2} \tag{2.10.11}
\end{equation*}
$$

Consequently, for all $r>0, \quad|\xi|>1$ and the finite difference approximation (2.10.7) is always unstable. This method of analyzing stability effectively ignores the boundary conditions, and insofar as these may affect the stability criterion, the matrix method of analysis as in the previous section is more preferable (Ames, 1977, p.47).

### 2.11 STABILITY CRITERIA FOR PROBLEM WITH DERIVATIVE BOUNDARY CONDITIONS

To investigate the stability of this type, we assume that the problem
(2.1.3) is associated with the boundary conditions

$$
\begin{align*}
& \frac{\partial u}{\partial x}=h_{1}\left(u-v_{1}\right) \text { at } x=0, \quad t \geqslant 0 \\
& \frac{\partial u}{\partial x}=-h_{2}\left(u-v_{2}\right) \text { at } x=1, t \geqslant 0 \tag{2.11.1}
\end{align*}
$$

where $\mathrm{h}_{1}, \mathrm{~h}_{2}, \mathrm{v}_{1}, \mathrm{v}_{2}$ are constants with $\mathrm{h}_{1} \geqslant 0, \mathrm{~h}_{2} \geqslant 0$.
When the boundary conditions are approximated by the central difference equations

$$
\begin{align*}
& \frac{u_{1, j^{-u}}-1, j}{2 \Delta x}=h_{1}\left(u_{0, j}-v_{1}\right) \\
& \frac{u_{N+1, j^{-u}}{ }^{N}-1, j}{2 \Delta x}=-h_{2}\left(u_{N, j}-v_{2}\right) \tag{2.11.2}
\end{align*}
$$

with $N \Delta x=1$, and the differential equation by the Crank-Nicolson scheme (2.8.3), i.e.
$-\frac{r}{2} u_{i-1, j+1}+(1+r) u_{i, j+1}-\frac{r}{2} u_{i+1, j+1}=\frac{r}{2} u_{i-1, j}+(1-r) u_{i, j}+\frac{r}{2} u_{i+1, j}$,
elimination of $u_{-1, j}, u_{N+1, j}$ leads to the equation

$$
\begin{equation*}
\mathrm{Au}_{j+1}=\underline{B u}_{j}+\underline{b}_{j} \tag{2.11.3}
\end{equation*}
$$

where $\underline{u}_{j}$ denotes the column vector $\left[u_{0, j}, u_{1, j}, \ldots, u_{N, j}\right]^{T}$,
and

$$
\begin{align*}
& \mathrm{A}=\mathrm{I}-\frac{1}{2} \mathrm{rQ},  \tag{2.11.4}\\
& \mathrm{~B}=\mathrm{I}+\frac{1}{2} \mathrm{r} \mathrm{Q}, \tag{2.11.5}
\end{align*}
$$

where $Q$ is a matrix of order ( $N+1$ ) given by

and

$$
\mathrm{b}_{\mathrm{j}}^{\mathrm{T}}=\left[2 \mathrm{rv}_{1} \Delta \mathrm{xh}_{1}, 0, \ldots, 2 \mathrm{rv}_{2} \Delta \mathrm{xh}_{2}\right] .
$$

For the analysis of stability, we write (2.11.3) as

$$
\begin{equation*}
u_{j+1}=A^{-1} \underline{B u}_{j}+A^{-1} \underline{b}_{j}, \tag{2.11.7}
\end{equation*}
$$

i.e. $\quad \underline{u}_{j+1}=\left(I-\frac{1}{2} r Q\right)^{-1}\left(I+\frac{1}{2} r Q\right)_{\underline{u}}+\underline{b}_{j}$,
where $\hat{\underline{b}}_{j}=\left(I-\frac{1}{2} r Q\right)^{-1} \underline{b}_{j}$ and it is assumed that $\operatorname{det}\left(I-\frac{1}{2} r Q\right) \neq 0$.
$Q$ is not a symmetric matrix. Therefore we introduce the diagonal matrix,

$$
D=\left[\begin{array}{cccc}
\sqrt{2} & & &  \tag{2.11.8}\\
& \ddots & & 0 \\
& \ddots & \\
0 & & \ddots & \\
& & & \sqrt{2}
\end{array}\right]
$$

of order ( $N+1$ ) such that $Q$ is similar to the symmetric matrix,

$$
\begin{equation*}
\tilde{Q}=D^{-1} Q D . \tag{2.11.9}
\end{equation*}
$$

Then, $\left.\widetilde{\left(A^{-1} B\right.}\right)=D^{-1}\left(I-\frac{1}{2} r Q\right)^{-1}\left(I+\frac{1}{2} r Q\right) D$

$$
\begin{align*}
& =\left[D^{-1}\left(I-\frac{1}{2} r Q\right)^{-1} D\right]\left[D^{-1}\left(I+\frac{1}{2} r Q\right) D\right] \\
& =\left[D^{-1}\left(I-\frac{1}{2} r Q\right) D\right]^{-1}\left[D^{-1}\left(I+\frac{1}{2} r Q\right) D\right] \\
& =\left[I-\frac{1}{2} r \tilde{Q}\right]\left[I+\frac{1}{2} r \tilde{Q}\right] . \tag{2.11.10}
\end{align*}
$$

However the matrices $\left[I-\frac{1}{2} r \widetilde{Q}\right]$ and $\left[I+\frac{1}{2} r \widetilde{Q}\right]$ are symmetric and commute, and so ( $A^{-1} B$ ) is symmetric. Therefore $A^{-1} B$ is similar to the symmetric matrix $\widetilde{A}^{-1} B$ ) and

$$
\begin{equation*}
\rho\left(A^{-1} B\right) \equiv \rho\left(\widetilde{A^{-1}} B\right) \leqslant 1, \tag{2.11.11}
\end{equation*}
$$

is a necessary and sufficient condition for stability, where $\rho$ denotes the spectral radius.

The eigenvalues $\mu_{j}(j=0,1, \ldots, N)$ of $A^{-1} B$ are given

$$
\begin{equation*}
\mu_{j}=\frac{1+\frac{1}{2} r \lambda}{1-\frac{1}{2} r \lambda}{ }_{j} \tag{2.11.12}
\end{equation*}
$$

where $\lambda_{j}, j=0,1, \ldots, N$ are the eigenvalues of the matrix $Q$. Since $\mu$ and $\rho$ are related by

$$
\rho\left(A^{-1} B\right)=\max _{j}\left|\mu_{j}\right|
$$

the condition for stability (2.11.11) together with (2.11.12) gives

$$
\lambda_{j} \leqslant 0 \text { for all } j
$$

Using Theorem (1.9) of Gerschgorin's Circle Theorem, it is easily seen that $\lambda_{j}$ lies on the negative line for all $j$. Hence the equations (2.11.3) are unconditionally stable.

### 2.12 EQUATIONS OF INCREASED ACCURACY - I

The principal criteria of any finite-difference scheme, for the numerical approximation of an equation, are as follows in decreasing order of importance :

1. Stability
2. Order of error of approximation
3. Simplicity (Saul'yev, 1964, p.83).

If the scheme is unstable, it is impossible to use such a scheme. It is due to this reason, why we only consider the stable scheme in our discussion. Furthermore as stability is a necessary and sufficient condition for the consistent scheme of the well-posed problem, therefore it is logical for this stability condition to be on the top priority of any numerical scheme.

Convergence which depends on the order of error of approximation may be of various degrees. Slow convergence requires more computational work and is impractical whilst rapidly convergent processes need less arithmetical operations and hence are more desirable. Therefore, the rate and order of convergency is important. Also it is intuitively clear that the smaller the error of approximation, the smaller is the error in the solution. In fact, for most problems, it can be seen that the error in the solution has order which is similar to the order of the error in approximation. Hence, the order of the error of approximation is very important.

Let $L U=0$ denote a p.d.e. and let a corresponding finite difference approximation be denoted by $L_{\Delta x} u_{i, j}=0$. Then for a sufficiently smooth function $V(x, t)$ satisfying the equation $L V=0$, the expression

$$
\begin{equation*}
L_{\Delta x} V_{i, j}=0\left(\Delta x^{\alpha}\right) \quad(\alpha>0) \tag{2.12.1}
\end{equation*}
$$

denotes that the order of the error incurred by approximating the operator L by the operator $L_{\Delta x}$ at the grid point ( $i \Delta x, j \Delta t$ ) is $\alpha$, for the class of
functions satisfying the equation $\mathrm{LV}=0$ (Saul'yev, 1964, p.84). In accordance with this criterion, the classification of finite difference approximations is dependent upon the size $\alpha$ are as follows:

0 . $\alpha \leqslant 0$ : Range of divergence
I. $0<\alpha<2$ : Range of reduced accuracy
II. $\alpha=2$ : Range of optimal (standard) accuracy
III. $2<\alpha<6$ : Range of increased accuracy

IV: $\alpha=6$ : Range of extreme accuracy
We shall present examples of finite difference approximations illustrating each of the above-mentioned types as we have continued the discussion in this section except for the type II as few schemes such as explicit (2.3.18), implicit (2.8.2) and Crank-Nicolson (2.8.3) are already discussed in the previous sections. Also methods for attaining a higher order of accuracy $\alpha$ will also be considered.

The value of the third criterion (i.e. simplicity) is very clear, particularly for the application of electronic computers as this will reduce the number of operations and time.

### 2.12.1 Asymmetric Finite Difference Equations

Two asymmetric finite difference equations of order $O(\Delta x)$ approximating the equation (2.1.3) which have been introduced by Saul'yev (1964, p.31) are given by

$$
(1+r \alpha) u_{i, j+1}-r \alpha u_{i-1, j+1}=r(1-\alpha) u_{i-1, j}+[1+r(\alpha-2)] u_{i, j}+r u_{i+1, j},
$$

and

$$
\begin{equation*}
(1+r \alpha) u_{i, j+1}-r \alpha u_{i+1, j+1}=r u_{i-1, j}+[1+r(\alpha-2)] u_{i, j}+r(1-\alpha) u_{i+1, j}, \tag{2.12.3}
\end{equation*}
$$

where $0 \leqslant \alpha \leqslant 1$. These equations can be schematically described as in Fig. (2.12.1) in (a) and (b) respectively.


In particular, for $\alpha=0$ both formulae coincide and reduce to the classical explicit equation (2.3.18), i.e.,

$$
\begin{equation*}
u_{i, j+1}=r u_{i-1, j}+(1-2 r) u_{i, j}+r u_{i+1, j} \tag{2.12.5}
\end{equation*}
$$

For $\alpha=1$, equations (2.12.3) and (2.12.4) will reduce to

$$
\begin{equation*}
(1+r) u_{i, j+1}^{-r u_{i-1, j+1}}=(1-r) u_{i, j}+r u_{i+1, j} \tag{2.12.6}
\end{equation*}
$$

and

$$
\begin{equation*}
(1+r) u_{i, j+1}{ }^{r r} u_{i+1, j+1}=r u_{i-1, j}+(1-r) u_{i, j} \tag{2.12.7}
\end{equation*}
$$

The computational stencils are given by Fig. (2.12.2) (a) and (b) respectively.



FIGURE (2.12.2)
Due to the fact that (2.12.5)-(2.12.7) are derivable from (2.12.3) and (2.12.4), and in the context of the initial-boundary value problem, (2.12.6) and (2.12.7) are both implicity-explicit if the calculations are carried out in the positive direction and negative direction respectively, therefore the asymetric finite difference equations (2.12.3) and (2.12.4) may be considered as generalizations of the classical explicit equation. The use of these equations by themselves however for the numerical integration
of parabolic equations is inadvisable, in as much as they entail significant errors, viz. $O(\Delta x)$ instead of $O\left(\Delta x^{2}\right)$ as in the case of the classical explicit of (2.3.18) and implicit of (2.8.2) (Saul'yev, 1964, p.29). These formulae are examples of type $I$ of classification (2.12.2).

## Theorem 2.5

The finite difference approximation (2.12.3) and (2.12.4) are
absolutely stable when $\alpha=1$, i.e. for the case of equation (2.12.6) and (2.12.7).

## Proof

To investigate this, we will use the method of Fourier Series. Since the scheme is linear, the error function $E_{p, q}$ at point ( $p \Delta x, q \Delta t$ ) satisfies the same difference equations as $u_{p, q}$. As

$$
\begin{align*}
E_{p, q} & =e^{i \beta x_{e}^{\alpha t}}=e^{i \beta p \Delta x_{e} \alpha q \Delta t} \\
& =e^{i \beta p \Delta x_{\xi} q} \tag{2.12.8}
\end{align*}
$$

where $i=\sqrt{-1}, \xi=e^{\alpha \Delta t}$ and $\alpha$ in general is a complex constant. Substitution of (2.12.8) into (2.12.3) with notations ( $i, j$ ) are changed to ( $p, q$ ) gives,

$$
\begin{gather*}
-r \alpha e^{i \beta(p-1) \Delta x_{\xi} q+1}+(1+r \alpha) e^{i \beta p \Delta x_{\xi} q+1}=r(1-\alpha) e^{i \beta(p-1) \Delta x_{\xi} q} \\
+[1+r(\alpha-2)] e^{i \beta p \Delta x_{\xi} q}+r e^{i \beta(p+1) \Delta x_{\xi} q}, \tag{2.12.9}
\end{gather*}
$$

Division by $\mathrm{e}^{\mathrm{i} \beta \mathrm{p} \Delta \mathrm{x}_{\xi} \mathrm{q}}$ leads to

$$
\begin{equation*}
\xi=\frac{r(1-\alpha) e^{-i \beta \Delta x}+[1+r(\alpha-2)]+r e^{i \beta \Delta x}}{(1+r \alpha)-r \alpha e^{-i \beta \Delta x}} \tag{2.12.10}
\end{equation*}
$$

For $\alpha=1$,

$$
\xi=\frac{1-r\left(1-e^{i \beta \Delta x}\right)}{1+r\left(1-e^{-i \beta \Delta x}\right)}
$$

and upon simplification, ${ }^{1}$

$$
\xi=\frac{\left(1+2 r^{2} \cos \beta \Delta x-2 r^{2} \cos ^{2} \beta \Delta x\right)+i r \sin \beta \Delta x(2 r-2 r \cos \beta \Delta x)}{1+2 r(1-\cos \beta \Delta x)+2 r^{2}(1-\cos \beta \Delta x)}
$$

$\left.\therefore|\xi|=\frac{\left\{1+8 r^{2} \sin ^{2} \frac{\beta \Delta x}{2}-16 r^{2} \sin ^{4} \frac{\beta \Delta x}{2}+16 r^{4} \sin ^{4} \frac{\beta \Delta x}{2}\right\}^{\frac{1}{2}}}{\{1+4 r \sin }{ }^{2} \frac{\beta \Delta x}{2}+4 r^{2} \sin ^{2} \frac{\beta \Delta x}{2}\right\} \quad$
$\leqslant 1$ for all positive $r$.

For equation (2.12.4) or (2.12.7) in particular, precisely the same analysis is applicable. Therefore, equation (2.12.3) and (2.12.4) are absolutely stable for $\alpha=1$.

For other values of $\alpha$, it was mentioned by A.F. Filippov (Saul'yev, 1964, p.37) that in the case of Cauchy's problem, a necessary and sufficient condition for the stability of the equations (2.12.3) and (2.12.4) has the form that

$$
\begin{equation*}
r \leqslant \frac{1}{2(1-\alpha)} \tag{2.12.11}
\end{equation*}
$$

As has already been noted, equation (2.12.3) and (2.12.4) give such large errors that they are unsuitable for serious practical applications. However Saul'yev (1964, p.43) has proposed the use of these formula alternatively on even and odd time steps. The alternate use of these equations partially compensates for the asymmetry of the equations, and this influence on the components of the error, originally having order $O(\Delta x)$, substantially diminishes it. The method of alternate use of these equations is called the alternating method and may be written in the following form,

$$
u_{i, j+1}=\frac{1}{1+r \alpha}\left[r \alpha u_{i-1, j+1}+r(1-\alpha) u_{i-1, j}+[1+r(\alpha-2)] u_{i, j}+r u_{i+1, j}\right],
$$

and

$$
\begin{equation*}
u_{i, j+2}=\frac{1}{1+r \alpha}\left[r \alpha u_{i+1, j+1}+r u_{i-1, j}+[1+r(\alpha-2)] u_{i, j}+r(1-\alpha) u_{i+1, j}\right] \tag{2.12.12}
\end{equation*}
$$

where $j=0,2,4, \ldots$ and $i=1,2, \ldots, N-1$ for (2.12.12) and $i=N-1, N-2, \ldots, 2,1$ for (2.12.13). This method may still compete with the classical explicit and implicit methods, in contrast with the earlier asymmetric equations. To proceed further, we write equation (2.12.3) and (2.12.4) in the matrix form as
and

$$
\left.\begin{array}{l}
A \underline{u}_{j+1}=(A+C) \underline{u}_{j}+\underline{b}  \tag{2.12.14}\\
A^{T} \underline{u}_{j+1}=\left(A^{T}+C\right) \underline{u}_{j}+\underline{b}
\end{array}\right\}
$$

respectively, where
and $\underline{b}$ is the associated boundary conditions. If at every time step, the $u$ is taken as the average of $u$ found from both equations, i.e.,

$$
\begin{equation*}
u_{i, j+1}=\frac{\tilde{u}_{i, j+1}+\tilde{\widetilde{u}}_{i, j+1}}{2} \tag{2.12.16}
\end{equation*}
$$

where
and

$$
\left.\begin{array}{l}
A \tilde{u}_{j+1}=(A+C) \underline{u}_{j}+\underline{b}  \tag{2.12.17}\\
A^{T \tilde{z}} \underline{u}_{j+1}=\left(A^{T}+C\right) \underline{u}_{j}+\underline{b}
\end{array}\right\}
$$

this method of finding $u$ is called the 'method of mean arithmetic'. Thus the two equations (2.12.17) are equivalent to the following single equation,

$$
\begin{equation*}
\underline{u}_{j+1}=\left(I+\frac{A^{-1}+\left(A^{T}\right)^{-1}}{2} C\right) \underline{u}_{j}+\left(A^{-1}+\left(A^{T}\right)^{-1}\right) \underline{b} \tag{2.12.18}
\end{equation*}
$$

where

$$
\begin{aligned}
& \text { where } \begin{aligned}
& \frac{A^{-1}+\left(A^{T}\right)^{-1}}{2}=\frac{r}{2(1+\alpha r)}\left[\begin{array}{llll}
2 & a & a^{2} \ldots \ldots a^{N-2} \\
a & 2 & a \ldots \ldots \\
\vdots & 1 & 1 & \vdots \\
\vdots & 1 & & \vdots \\
1 & 1 & a^{N-3} \\
a^{N-2} & a^{N-3} & a^{N-4} \ldots \ldots
\end{array}\right] \\
& a=\frac{\alpha r}{1+\alpha r} .
\end{aligned}
\end{aligned}
$$

Theorem 2.6
Formulae (2.12.18) is stable for all r satisfying the inequality

$$
\begin{equation*}
r \leqslant \frac{1}{2(1-a)} \tag{2.12.19}
\end{equation*}
$$

(Saul'yev, 1964, p.53).
This method and the alternating method, has an error 'almost' $O\left(\Delta x^{2}\right)$.

This is an example of how the scheme of type $I$ according to classification (2.12.2) gradually moves to the type II. In the next discussion, it will be mentioned that with certain choices of $\alpha$ and a special treatment on (2.12.3) and (2.12.4), greater accuracy will be achieved.

### 2.12.2 Formulae with Choice of Parameters

If we denote $L$ as the operator to the parabolic equation, i.e.,

$$
\begin{equation*}
L=\frac{\partial}{\partial t}-\frac{\partial^{2}}{\partial x^{2}}, \tag{2.12.20}
\end{equation*}
$$

and if the combination of (2.12.3) and (2.12.4) is taken as the approximating equation to $L U=0$ and is denoted by $L_{\Delta x}{ }^{u}$, therefore

$$
\begin{gather*}
L_{\Delta x} u_{i, j} \equiv-r \alpha u_{i-1, j+1}+2(1+r \alpha) u_{i, j+1}-r \alpha u_{i+1, j+1}-r(2-\alpha) u_{i-1, j} \\
-2[1+r(\alpha-2)] u_{i, j}-r(2-\alpha) u_{i+1, j}=0 . \tag{2.12.21}
\end{gather*}
$$

This equation is implicit and generally belongs to the class of optimal accuracy, i.e. type II of (2.12.2), since it can be easily shown that the error is of order $O\left((\Delta x)^{2}\right)$ as it will reduce to equation (2.8.1) for $\alpha=2 \theta$. Thus, at the high cost of the loss of the explicit nature of the net equation, it has gone over from the first class of accuracy to the second. The following theorem will show how equation (2.12.21) may transform to a higher class of accuracy for certain choice of parameters.

## Theorem 2.7

If the solution $u$ of the operator (2.12.20) has derivatives up to eighth order which are bounded in absolute magnitude throughout $D$, then the following relations hold in D:

$$
L\left(u_{i, j}\right)-L_{\Delta x}\left(u_{i, j}\right)=\left\{\begin{array}{lr}
0\left(\Delta x^{2}\right) \text { if } \alpha \neq 1-\frac{1}{6 r}, & \alpha \geqslant 1-\frac{1}{2 r}  \tag{2.12.22}\\
0\left(\Delta x^{4}\right) \text { if } \alpha=1-\frac{1}{6 r}, & r \neq \frac{1}{2 \sqrt{5}} \\
0\left(\Delta x^{6}\right) \text { if } \alpha=1-\frac{1}{6 r}, & r=\frac{1}{2 \sqrt{5}}
\end{array}\right\}
$$

where $L\left(u_{i, j}\right)$ and $L_{\Delta x}\left(u_{i, j}\right)$ are the differential and difference expressions respectively.

## Proof:

First of all, we will investigate the stability of equation (2.12.21) by the matrix method. Hence we write (2.12.21) in the following form, $\left.\begin{array}{l}A_{1} \underline{u}_{j+1}=A_{2} \underline{u}_{j}+\underline{b} \\ \text { i.e. } \quad \underline{u}_{j+1}=A_{1}^{-1} A_{2} \underline{u}_{j}+A^{-1} \underline{b}\end{array}\right\}$
where

$$
\begin{equation*}
A_{1}=\alpha C-2 I, \quad A_{2}=-2 I+(\alpha-2) C, \tag{2.12.23}
\end{equation*}
$$

$C$ is the tridiagonal matrix as given in (2.12.15) and $I$ is the identity matrix. As the eigenvalues of $C$ are

$$
\begin{equation*}
\lambda_{s}(C)=-4 r \sin ^{2} \frac{s \pi}{2 N}, s=1,2, \ldots, N-1 \tag{2.12.24}
\end{equation*}
$$

therefore

$$
\begin{align*}
& \lambda_{s}\left(A_{1}\right)=-4 \alpha r \sin ^{2} \frac{s \pi}{2 N}-2  \tag{2.12.25}\\
& \lambda_{s}\left(A_{2}\right)=-2-4(\alpha-2) r \sin ^{2} \frac{s \pi}{2 N}, s=1,2, \ldots, N-1
\end{align*}
$$

and

For the stability of (2.12.21), it is sufficient that

$$
\begin{equation*}
\left|\frac{\lambda_{s}\left(A_{2}\right)}{\lambda_{s}\left(A_{1}\right)}\right|=\left|\frac{2+4(\alpha-2) r \sin ^{2} \frac{s \pi}{2 N}}{2+4 \alpha r \sin ^{2} \frac{s \pi}{2 N}}\right| \leqslant 1, \tag{2.12.26}
\end{equation*}
$$

or $-2-4 \alpha \sin ^{2} \frac{s \pi}{2 N} \leqslant 2+4(\alpha-2) r \sin ^{2} \frac{s \pi}{2 N} \leqslant 2+4 \alpha r \sin ^{2} \frac{s \pi}{2 N}$.
The right-hand side inequality is always fulfilled and for the left hand side we obtain the result

$$
\text { i.e. } \quad r \leqslant \frac{4+8 r(\alpha-1) \sin ^{2} \frac{s \pi}{2 N} \geqslant 0,}{2(1-\alpha)} \quad,
$$

which is the criterion for stability of (2.12.21). Thus for any $\alpha$, there is a specific value of $r$ which satisfies the stability condition.

For the accuracy of (2.12.21), Taylor series expansions are applied to replace the $u$ 's involved in (2.12.21) in the neighbourhood of (i $\Delta x,(j+1) \Delta t)$ in

$$
\begin{align*}
L_{\Delta x}\left(u_{i, j+1}\right)= & L\left(u_{i, j+1}\right)-\frac{\Delta x^{2}}{12}(6 r-6 \alpha r-1) \frac{\partial^{2} u_{i, j+1}}{\partial t^{2}}- \\
& \frac{\Delta x^{4}}{360}\left(120 r^{2}+1-30 r+15 \alpha r-90 \alpha r^{2}\right) \frac{\partial^{3} u_{i, j+1}}{\partial t^{3}}+0\left(\Delta x^{6}\right) \tag{2.12.29}
\end{align*}
$$

This formulae in general, i.e. for $\alpha \neq 1-\frac{1}{6 r}$, the first condition of (2.12.22) holds. For $\alpha=1-\frac{1}{6 r}$, the coefficient of $\Delta x^{2}$ in (2.12.29) vanishes, and hence the second condition of (2.12.22) holds. Finally, if for such a value of $\alpha$ we can choose $r$ such that the following equation holds

$$
120 r^{2}+1-30 r+15 \alpha r-90 \alpha r^{2}=0
$$

i.e., $r=\frac{1}{2 \sqrt{5}}$, then the coefficient of $(\Delta x)^{4}$ in (2.12.29) also vanishes, and the third condition (2.12.22) holds. Thus, an accuracy of $0\left(\Delta x^{6}\right)$ can be obtained. Therefore, with a tighter choice of parameters, it can be shown that, the scheme of type II according to classification (2.12.2), moves to the type IV, i.e. range of extreme accuracy.

## Remarks on Theorem 2.7

1. The condition (2.12.28) for the stability of the implicit equation coincides with the condition (2.12.19) for the stability of the explicit equation (2.12.18).
2. For $\alpha=0, r=1 / 6$, an explicit equation of the form

$$
u_{i, j+1}=\frac{1}{6}\left(u_{i-1, j}+u_{i+1, j}\right)+\frac{2}{3} u_{i, j}
$$

is obtainable with an accuracy of $O\left(\Delta x^{4}\right)$. This scheme was considered by Milne (1953, p.134) and D. Yu Panow (1955, p.125).
3. For $\alpha=1-\frac{1}{6 r}$, the equation coincides with the Douglas method having an accuracy of $O\left(\Delta x^{4}\right)$ if $r \neq \frac{1}{2 \sqrt{5}}$ and $O\left(\Delta x^{6}\right)$ if $r=\frac{1}{2 \sqrt{5}}$.
4. The identity $\frac{\partial^{s} u}{\partial t^{s}}=\frac{\partial^{s+3} u}{\partial^{s-\beta} t \partial^{2 \beta} x}, \quad s=1,2, \ldots$ has been applied to Theorem (2.7) whilst it cannot be extended directly to the cases of variable coefficient and multidimensional equations.

Up to this stage we can see that, higher order accuracy can be achieved if:
(i) the direction in space of how the solution is evaluated for every time level is alternate.
(ii) the mean arithmetic of the solutions in both directions is taken.
(iii) some certain choices in parameters $r$ and $\alpha$ of the equation (2.12.21) are taken.

Some other methods for attaining higher order of accuracy will be discussed in the next section.

### 2.13 EQUATIONS OF INCREASED ACCURACY - II

### 2.13.1 Introduction of Additional Nodes

One method for attaining higher order of accuracy is based on the introduction of additional nodes, beyond the minimal number, into the approximation for the derivatives appearing in the given equation $L(u) \equiv 0$. Using the formula,

$$
\begin{equation*}
\left.\frac{\partial^{2} u}{\partial x^{2}}\right|_{\substack{x=i \Delta x \\ t=j \Delta t}}=\frac{1}{(\Delta x)^{2}}\left(\delta^{2} u_{i, j}-\frac{1}{12} \delta^{4} u_{i, k}+\frac{1}{90} \delta^{6} u_{i, k}-\ldots\right) \tag{2.13.1}
\end{equation*}
$$

where $\delta^{2 k} u_{i, j}(k=1,2, \ldots)$ is the standard central-difference of even order, defined by the formulae:

$$
\left.\begin{array}{l}
\delta^{2} u_{i}=u_{i-1}-2 u_{i}+u_{i+1}  \tag{2.13.2}\\
\delta^{4} u_{i}=u_{i-2}-4 u_{i-1}+6 u_{i}-4 u_{i+1}+u_{i+2} \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{array}\right\}
$$

the approximation to (2.1.3) is given by,

$$
\begin{gather*}
\frac{u_{i, j+1}-u_{i, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left(\delta^{2} u_{i, j}-\frac{1}{12} \delta^{4} u_{i, j}+\frac{1}{90} \delta^{6} u_{i, j}-\ldots+\frac{2(-1)^{\ell-1}[(\ell-1)!]^{2}}{(2 \ell)!}\right. \\
\delta^{\left.2 \ell u_{i, j}\right),}(2.13 .3) \tag{2.13.3}
\end{gather*}
$$

with an error term of $O\left(\Delta t+(\Delta x)^{2 \ell}\right)$. For $l=1$, (2.13.3) reduces to the classical explicit approximation (2.3.18). Here it should be noted that for $\ell>1$ at nodes near the boundary nodes $x=0$ and $x=1$ (more precisely, at the nodes $(i \Delta x, j \Delta t), i=1,2, \ldots, j-1, i=n-1, n-2, \ldots, n-\ell+1 ; j=0,1, \ldots, r-1)$, complications arise. However few strategies, like the introduction of fictituous nodes or the use of asymmetric formulae, are able to overcome this problem. Even though the use of (2.13.3) increases the order of accuracy it worsens the stability restriction in the case of the explicit scheme, i.e., for stability it is now required,

$$
\begin{equation*}
\Delta t \leqslant \frac{(\Delta x)^{2}}{2\left(1+\frac{1}{3}+\frac{8}{45}+\ldots+\frac{2^{2 \ell-1}[(\ell-1)!]^{2}}{(2 \ell)!}\right.} \tag{2.13.4}
\end{equation*}
$$

(Saul'yev, 1964, p.89).
In the implicit case, one obtains

$$
\begin{align*}
u_{i, j+1} & -\frac{\Delta t}{(\Delta x)^{2}}\left(\delta^{2}-\frac{1}{12} \delta^{4}+\frac{1}{90} \delta^{6}-\ldots+\frac{2(-1)^{\ell-1}[(\ell-1)!]^{2}}{(2 \ell)!} \delta^{2 \ell}\right) u_{i, j+1} \\
& =u_{i, j} \tag{2.13.5}
\end{align*}
$$

which retains the unconditional stability of the method.

### 2.13.2 Alternating Method

In Section (2.12.1) we introduced the alternating method used on the asymmetry formulae. Now from the classical implicit formulae,

$$
\begin{equation*}
\frac{u_{i, j+1}-u_{i, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left(u_{i-1, j+1}-2 u_{i, j+1}+u_{i+1, j+1}\right) \tag{2.13.6}
\end{equation*}
$$

and classical exnlicit formulae

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u_{i, j}}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left(u_{i-1, j}-2 u_{i, j}+u_{i+1, j}\right) \tag{2.13.7}
\end{equation*}
$$

a similar alternating strategy can be applied. If we use equations (2.13.6) for even $j$, say, and equations (2.13.7) for odd $j$, then the step $\Delta t$ for even $j$ may be taken comparatively large, since equation (2.13.6) is stable for any $\Delta t$ and $\Delta x$. For odd $j$ the step $\Delta t$ should satisfy the condition $\Delta t \leqslant(\Delta x)^{2} / 2$. However, these combinations can be shown by the following theorem to prove otherwise.

## Theorem 2.8

The method

$$
\begin{align*}
& \frac{u_{i, 2 j+1^{-u}} i, 2 j}{\Delta t}=\frac{u_{i-1,2 j+1}-2 u_{i, 2 j+1} u_{i \neq 1,2 j+1}}{(\Delta x)^{2}}  \tag{2.13.8}\\
& \frac{u_{i, 2 j+2^{-u}}}{\Delta t}=\frac{u_{i-1,2 j+1}}{(\Delta x)^{2}} \tag{2.13.9}
\end{align*}
$$

is absolutely stable, if the step $\Delta t$ is constant or changes after an even number of steps, (Saul'yev, 1964, p.23).

Proof
To prove this, we write both equations in matrix form, i.e.,

$$
\begin{equation*}
\mathrm{Au}_{2 j+1}=\underline{\mathrm{u}}_{2 j} \tag{2.13.10}
\end{equation*}
$$

representing (2.13.8) and

$$
\begin{equation*}
\underline{\mathrm{u}}_{2 j+2}=\underline{\mathrm{Bu}}_{2 j+1} \tag{2.13.11}
\end{equation*}
$$

representing (2.13.9) where

$$
\begin{equation*}
\mathrm{A}=\mathrm{I}-\mathrm{rT} \text { and } \mathrm{B}=\mathrm{I}+\mathrm{rT} \tag{2.13.12}
\end{equation*}
$$

with T as given earlier in equation (2.8.7).
For stability, it is necessary and sufficient for the amplification matrix of the single equation

$$
\begin{equation*}
\underline{u}_{2 j+2}=\left(B A^{-1}\right) \underline{u}_{2 j}, \tag{2.13.13}
\end{equation*}
$$

satisfy $\quad\left|\mid B A^{-1} \| \leqslant 1\right.$.
As the eigenvalues of $T$, i.e. $\lambda(T)$ are given by

$$
\begin{align*}
\lambda(T) & =-2+2 \cos \frac{s \pi}{N}, s=1,2, \ldots, N-1, \\
& =-4 \sin ^{2} \frac{s \pi}{2 N}, \tag{2.13.15}
\end{align*}
$$

therefore

$$
\begin{equation*}
\lambda(A)=1+4 r \sin ^{2} \frac{s \pi}{2 N} \tag{2.13.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda(B)=1-4 r \sin ^{2} \frac{s \pi}{2 N} \tag{2.13.17}
\end{equation*}
$$

The stability condition (2.13.14) will be fulfilled if

$$
\begin{equation*}
\frac{1-4 r \sin ^{2} \frac{s \pi}{2 N}}{1+4 r \sin ^{2} \frac{s \pi}{2 N}} \leqslant 1 \tag{2.13.18}
\end{equation*}
$$

which is true for all $\mathrm{r}>0$.
Alternatively, we can also apply (2.13.9) and (2.13.8) in the reverse order and the Theorem (2.8) still holds.

Instead of alternating the implicit and explicit formulae according to each line, we can also alternate them according to each node. In this case
since the implicit equation is applied explicitly, they are essentially explicit. This Implicitly-Explicit method was proposed by Saul'yev (1964, p.66), Gordon (1965) and it was called Hopscotch later by Gourlay, A.R., (1970). The simplest form of this method is given by

$$
\begin{equation*}
u_{2 i, j+1}=r u_{2 i-1, j}+(1-2 r) u_{2 i, j}+r u_{2 i+1, j}, 1 \leqslant i \leqslant\left[\frac{n}{2}\right] \tag{2.13.19}
\end{equation*}
$$

and

$$
u_{2 i-1, j+1}=\frac{1}{(1+2 r)}\left\{u_{2 i-1, j}+r u_{2 i, j+1}+r u_{2 i-2, j+1}\right\}, 1 \leqslant i \leqslant\left[\frac{n+1}{2}\right](2.13 .20)
$$

### 2.13.3 Multi-Level Difference Schemes

In the construction of difference schemes, higher order accuracy can be achieved if one uses more time levels than the minimum number required. For example, an improvement over the classical implicit scheme for the heat equation (2.1.3),

$$
\begin{equation*}
\frac{u_{i, j+1} 1^{-u_{i, j}}}{\Delta t}=\frac{u_{i-1, j+1}-2 u_{i, j+1}+u_{i+1, j+1}}{(\Delta x)^{2}}+0(\Delta t)+0\left\{(\Delta x)^{2}\right\} \tag{2.13.21}
\end{equation*}
$$

is provided by the three-level equation

$$
\begin{align*}
& \frac{3}{2} \frac{u_{i, j+1}-u_{i, j}}{\Delta t}=\frac{1}{2} \frac{u_{i, j}-u_{i, j-1}}{\Delta t}=\frac{u_{i+1, j+1}-2 u_{i, j+1}+U_{i-1, j+1}}{(\Delta x)^{2}}+ \\
& \text { i.e. }  \tag{2.13.22}\\
& \frac{\frac{3}{2} u_{i, j+1}-2 u_{i, j}+\frac{1}{2} u_{i, j-1}}{\Delta t}=\frac{u_{i+1, j+1}-2 u_{i, j+1}+u_{i-1, j+1}}{(\Delta x)^{2}}+ \\
& O\left\{(\Delta t)^{2}\right\}+0\left\{(\Delta x)^{2}\right\} . \tag{2.13.23}
\end{align*}
$$

(Ritchmyer, R.B. and Morton, K.W., 1967, p.86).
Another example, where $r=\Delta t /(\Delta x)^{2}$ must have a specific value, is given by Mikeladze as,

$$
\begin{equation*}
u_{i, j+1}=\frac{1}{40}\left[2 u_{i, j-1}+32 u_{i, j}+3\left(u_{i-1, j-1}+u_{i+1, j-1}\right)\right], \tag{2.13.23}
\end{equation*}
$$

for $r=1 / 16$, and by Yushkov as

$$
\begin{equation*}
10 u_{i, j+1}=3\left(u_{i-1, j}+u_{i+1, j}\right)+2\left(u_{i, j}+u_{i, j-1}\right), \tag{2.13.24}
\end{equation*}
$$

for $r=1 / 4$ and,

$$
\begin{equation*}
72 u_{i, j+1}=25\left(u_{i-1, j}+u_{i+1, j}\right)+4 u_{i, j}+18 u_{i, j-1} \tag{2.13.25}
\end{equation*}
$$

for $r=1 / 36$. Equations (2.13.23)-(2.13.25) have errors of order $O\left((\Delta t)^{2}+\right.$ $(\Delta x)^{4}$ ) respectively (Saul'yev, 1964, p.95). As we can see one needs initial data on two time levels (say $t$ and $t+\Delta t$ ) to obtain the solution at $t+2 \Delta t$, which is greatly restrictive. The extra initial information can be taken from a simple two level scheme at the start of the procedure.

The general three level explicit difference scheme for (2.1.3) will involve seven points and may be written as

$$
\begin{equation*}
\left(1+\tau_{1}\right) u_{i, j+1}=\left[1+2 \tau_{1}+r\left(1-\gamma_{1}\right) \delta_{x}^{2}\right]_{u_{i, j}}-\left(\tau_{1}-r \gamma_{1} \delta_{x}^{2}\right) u_{i, j-1} \tag{2.13.26}
\end{equation*}
$$

where $\tau_{1}$ and $\gamma_{1}$ are arbitrary parameters (Jain, M.K., 1979, p.214). The truncation error of (2.13.26) is of order,
(i) $\left(\Delta t+(\Delta x)^{2}\right)$ if $\gamma_{1}$ and $\tau_{1}$ are arbitrary
(ii) $\left((\Delta t)^{2}+(\Delta x)^{2}\right)$ if $\tau_{1}+\gamma_{1}+\frac{1}{2}=0$ and either $\gamma_{1}$ and $\tau_{1}$ is arbitrary,
(iii) $\left((\Delta x)^{4}\right)$ if $\tau_{1}+\gamma_{1}+\frac{1}{2}-\frac{1}{12 r}=0$ and either $\gamma_{1}$ or $\tau_{1}$ is arbitrary.

The necessary and sufficient condition for (2.13.26) to be stable are (Jain, M.K., 1979, p.215).
(i) $1+2 \tau_{1}-2\left(1-2 \gamma_{1}\right) r \geqslant 0$
(ii) $1-4 \gamma_{1} r \geqslant 0$

The conditions (2.13.28) are shown in Figure (2.13.1).


We find that for
(i) $\gamma_{1} \leqslant 0$, the conditions (2.13.28) are satisfied if

$$
0<r \leqslant \frac{1+2 \tau_{1}}{2\left(1-2 \gamma_{1}\right)} \quad \text { and } \quad I+2 \tau_{1}>0
$$

(ii) $\quad \gamma_{1}<\frac{1}{2}$, the stability condition is obtained as $0<r<r$ min, where $1+2 \tau_{1}>0$ and

$$
\begin{equation*}
r_{\min }=\min \left[\frac{1}{4 \gamma_{1}}, \frac{1+2 \tau_{1}}{2\left(1-2 \gamma_{1}\right)}\right] \tag{2.13.29}
\end{equation*}
$$

For $\gamma_{1}=0$ and $\tau_{1}=-\frac{1}{2}+r$, scheme (2.13.26) will give the Du-Fort-Frankel formula

$$
\begin{equation*}
\left(\nabla_{t}-\frac{1}{2} \nabla_{t}^{2}+r \nabla_{t}^{2}\right) u_{i, j+1}=r \delta_{x}^{2} u_{i, j} \tag{2.13.30}
\end{equation*}
$$

For $\gamma_{1}=0$ and $\tau_{1}=-\frac{1}{2}$ the unstable Richardson formula is obtained. If we prescribe $\gamma_{1}$ and $\tau_{1}$ to satisfy (2.13.27) (iii), such as $\gamma_{1}=0, \tau_{1}=-\frac{1}{2}+\frac{1}{12 r}$, (2.13.26) may be written as,

$$
\begin{equation*}
\frac{1}{2}\left(1+\frac{1}{6 r}\right) u_{i, j+1}=r\left(u_{i-1, j}+u_{i+1, j}\right)-\left(2 r-\frac{1}{6 r}\right) u_{i, j}+\frac{1}{2}\left(1-\frac{1}{6 r}\right) u_{i, j-1} \tag{2.13.31}
\end{equation*}
$$

and alternatively $\tau_{1}=0$ and $\gamma_{1}=\frac{1}{2}+\frac{1}{12 r}$ gives the formulae

$$
\begin{array}{r}
u_{i, j+1}=\left(\frac{7}{6}-3 r\right) u_{i, j}+\frac{1}{2}\left(3 r-\frac{1}{6}\right)\left(u_{i-1, j}+u_{i+1, j}\right) \\
-\frac{1}{2}\left(r-\frac{1}{6}\right)\left(u_{i-1, j-1}-2 u_{i, j-1}+u_{i+1, j-1}\right) \tag{2.13.32}
\end{array}
$$

which is stable for $0<r \leqslant 1 / 3$. The truncation error becomes $O\left((\Delta x)^{6}\right)$, if $\mathrm{r}=\frac{1}{10}$ (Jain, M.K., 1979, p.218).

The general three level implicit formulae based upon nine points for approximating (2.1.3) is given by

$$
\begin{align*}
& {\left[\left(1+\tau_{1}\right)+\left\{\sigma\left(1+\tau_{1}\right)-r\left(1-\gamma_{1}+\gamma_{2}\right)\right\} \delta_{x}^{2}\right] u_{i, j+1}=} \\
& \quad\left[\left(1+2 \tau_{1}\right)+\left\{\sigma\left(1+2 \tau_{1}\right)+r\left(\gamma_{1}-2 \gamma_{2}\right)\right\} \delta_{x}^{2}\right] u_{i, j}-\left[\tau_{1}+\left(\sigma \tau_{1}-r \gamma_{2}\right) \delta_{x}^{2}\right] u_{i, j-1} \tag{2.13.33}
\end{align*}
$$

where $\tau_{1}, \gamma_{1}, \gamma_{2}$ and $\sigma$ are arbitrary (Jain, M.K., 1979, p.223).
The truncation error for this class of method is given by

$$
\begin{align*}
& T_{i, j}=\Delta t\left[\left(\sigma-\frac{1}{12}\right)+r\left(\tau_{1}+\gamma_{1}-\frac{1}{2}\right)\right](\Delta x)^{2} \frac{\partial^{4} u_{i, j}}{\partial x^{4}}+ \\
& \Delta t\left[\left(\frac{1}{2} \gamma_{1}-\gamma_{2}-\frac{1}{3}\right) r^{2}\right.+\left(\sigma\left(\tau_{1}+\frac{1}{2}\right)+\frac{1}{12}\left(\gamma_{1}-1\right)\right) r \\
&\left.+\frac{1}{12}\left(\sigma-\frac{1}{30}\right)\right)(\Delta x)^{4} \frac{\partial^{6} u_{i, j}}{\partial x^{6}}+\ldots \tag{2.13.34}
\end{align*}
$$

and the necessary and sufficient conditions for stability of (2.13.33) are

$$
\begin{gather*}
\left(\tau_{1}+\frac{1}{2}\right)\left(1-4 \sigma \sin ^{2} \frac{\beta \Delta x}{2}\right)+r \sin ^{2} \frac{\beta \Delta x}{2}\left(1-2 \gamma_{1}+4 \gamma_{2}\right)>0 \\
\frac{1}{4}\left(1-4 \sigma \sin ^{2} \frac{\beta \Delta x}{2}\right)+r \sin ^{2} \frac{\beta \Delta x}{2}\left(1-\gamma_{1}\right)>0 \tag{2.13.35}
\end{gather*}
$$

where $\sigma<\frac{1}{4}$ is taken (Jain, M.K., 1979, p.224).
A full discussion of the multi-level scheme is given by Jain, M.K., (1979, p.214-228) or Ritchmyer and Morton (1967, p.168-179) and Saul'yev (1964, p.95). Interested readers are referred to the quoted references.

Here we conclude that although it may appear in general that the multi-level formulae have an advantage over two-level schemes as far as the accuracy is concerned, they also have other disadvantages in a particular sense (e.g. stability for Richardson, consistency for Du-FortFrankel, restriction on $r$ as (2.13.23)-(2.13.25), etc.).

### 2.14 PARABOLIC EQUATIONS IN SEVERAL SPACE DIMENSIONS

We now consider finite-difference methods of the solution of the equation

$$
\begin{equation*}
\frac{\partial U}{\partial t}=L(U) \tag{2.14.1}
\end{equation*}
$$

where

$$
\begin{align*}
& L=\sum_{i=1}^{n} \frac{\partial}{\partial x_{i}}\left(a_{i}\left(x_{1}, x_{2}, \ldots, x_{n}, t\right) \frac{\partial}{\partial x_{i}}\right)+b_{i}\left(x_{1}, x_{2}, \ldots, x_{n}, t\right) \frac{\partial}{\partial x_{i}} \\
&-c\left(x_{1}, x_{2}, \ldots, x_{n}, t\right) \tag{2.14.2}
\end{align*}
$$

with $a_{1}, a_{2}, \ldots, a_{n}$ strictly positive and $c$ non-negative (Mitchell, A.R., 1980, p.53). There are two categories of finite-difference methods for parabolic equations with several space variables. Firstly, the generalization of standard methods which are presented for one dimensional problems and secondly, splitting methods which have no single space variable analog.

For simplicity, we consider the two dimensional heat flow equation in the rectangular region bounded by $R \times[0<t \leqslant T]$ where $R$ is a closed connected region in the $x-y$ plane, with continuous boundary $\partial R$ given by $R=[0 \leqslant x, y \leqslant 1]$, namely,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}} \tag{2.14.3}
\end{equation*}
$$

Appropriate initial and boundary data are provided as

$$
\begin{align*}
& u(x, y, 0)=f(x, y), \quad t=0, \\
& u(x, y, t)=g(x, y, t), \quad(x, y, t) \in \partial R \times[0<t \leqslant T] \tag{2.14.4}
\end{align*}
$$

where $f$ and $g$ are given for prescribed values of ( $x, y, t$ ).
In the same manner as the one dimensional case, the region $R$ is covered by a rectilinear grid with sides, parallel to the axes, with $\Delta x$ and $\Delta y$ the grid spacings in the direction $x$ and $y$ and $\Delta t$ in the time direction respectively. Unless otherwise stated; the space spacing $\Delta x=\Delta y$, throughout the discussion. The grid points ( $x, y, t$ ) are denoted by ( $i, j, k$ ) where $x=i \Delta x, y=j \Delta y$, and $t=k \Delta t$. The point $i=0, j=0, n=0$ is the origin
(Figure 2.14.1). The exact and approximate values at the grid point ( $i, j, k$ ) are denoted by $U_{i, j, k}$ and $u_{i, j, k}$ respectively.


FIGURE (2.14.1)

### 2.14.1 Explicit Methods

To extend the standard explicit methods to a second space dimension, we write equation (2.14.3) as

$$
\begin{equation*}
\frac{\partial u}{\partial t}=L u \tag{2.14.5}
\end{equation*}
$$

where

$$
\mathrm{L} \equiv \frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}} \equiv \mathrm{D}_{1}^{2}+\mathrm{D}_{2}^{2}
$$

From Taylor's expansion, it is possible to write $u(i \Delta x, j \Delta y,(k+1) \Delta t)$ in terms of $u(i \Delta x, j \Delta y, k \Delta t)$, i.e.

$$
\begin{align*}
u(i \Delta x, j \Delta y,(k+1) \Delta t) & =u_{i, j, k+1}=\left(1+\Delta t \frac{\partial}{\partial t}+\frac{\Delta t^{2}}{2} \frac{\partial^{2} u}{\partial t^{2}}+\ldots\right) u_{i, j, k} \\
& =\exp \left(\Delta t \frac{\partial}{\partial t}\right) u_{i, j, k} \\
& =\exp (\Delta t L) u_{i, j, k} \\
& =\exp \left(\Delta t D_{1}^{2}\right) \exp \left(\Delta t D_{2}^{2}\right) u_{i, j, k} \tag{2.14.6}
\end{align*}
$$

provided $\mathrm{D}_{1}^{2}$ and $\mathrm{D}_{2}^{2}$ commute, where

$$
\begin{equation*}
\mathrm{D}_{1}^{2}=\frac{1}{(\Delta \mathrm{x})^{2}}\left(\delta_{\mathrm{x}}^{2}-\frac{1}{12} \delta_{\mathrm{x}}^{4}+\frac{1}{90} \delta_{\mathrm{x}}^{6} \ldots\right) \tag{2.14.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{D}_{2}^{2}=\frac{1}{(\Delta \mathrm{y})^{2}}\left(\delta_{\mathrm{y}}^{2}-\frac{1}{12} \delta_{\mathrm{y}}^{4}+\frac{1}{90} \delta_{\mathrm{y}}^{6} \ldots\right) \tag{2.14.8}
\end{equation*}
$$

(Mitche11, A.R. and Griffiths, D.F., 1980, p.55). E1imination of $D_{1}^{2}$ and $D_{2}^{2}$ leads to the formula

$$
\begin{equation*}
u_{i, j, k+1}=\left[1+r \delta_{x}^{2}+\frac{1}{2} r\left(r-\frac{1}{6}\right) \delta_{x}^{4} \ldots\right]\left[1+r \delta_{y}^{2}+\frac{1}{2} r\left(r-\frac{1}{6}\right) \delta_{y}^{4} \cdots\right] u_{i, j, k}, \tag{2.14.9}
\end{equation*}
$$

where $r=\Delta t /(\Delta s)^{2}$ with $\Delta x=\Delta y=\Delta s$. Various explicit formulae can be obtained from equation (2.14.9) of which one is the well-known equation

$$
\begin{equation*}
u_{i, j, k+1}=\left[1+r\left(\delta_{x}^{2}+\delta_{y}^{2}\right)\right] u_{i, j, k}, \tag{2.14.10}
\end{equation*}
$$

which is the standard explicit formulae involving five points at the time level $t=k \Delta t$.

Equation (2.14.10) can be written as

$$
\begin{equation*}
u_{i, j, k+1}=(1-4 r) u_{i, j, k}+r\left(u_{i-1, j, k}+u_{i+1, j, k}+u_{i, j-1, k}+u_{i, j+1, k}\right) \tag{2.14.11}
\end{equation*}
$$

If we define the difference between the exact solution of differential and difference equations at the mesh point (i $\Delta x, j \Delta y, k \Delta t$ ) as

$$
\begin{equation*}
e_{i, j, k}=U_{i, j, k} u_{i, j, k} \tag{2.14.12}
\end{equation*}
$$

then from (2.14.11) we obtain

$$
\begin{align*}
e_{i, j, k+1}= & (1-4 r) e_{i, j, k}+r\left(e_{i-1, j, k}+e_{i+1, j, k}+e_{i, j-1, k}+e_{i, j+1, k}\right) \\
& +\frac{\Delta s^{2}}{2}\left(\frac{\partial^{2} u}{\partial t^{2}}\right)_{i, j, k}-\Delta t \frac{\Delta s^{2}}{12}\left(\frac{\partial^{4} u}{\partial x^{4}}+\frac{\partial^{4} u}{\partial y^{4}}\right)_{i, j, k}+\ldots(2.1 \tag{2.14.13}
\end{align*}
$$

This equation shows that the L.T.E. of formula (2.14.11) is $O\left((\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right)$ (Danaee A., 1980, p.64) or with order of accuracy of $O\left(\Delta t+(\Delta x)^{2}\right)$. Using the Fourier Series method of stability analysis, the stability condition is obtained as $0<r \leqslant 1 / 4$.

Another simple explicit formula which widens the stability condition up to $r \leqslant 1 / 2$ is

$$
\begin{equation*}
u_{i, j, k+1}=\left(1+r \delta_{x}^{2}\right)\left(1+r \delta_{y}^{2}\right) u_{i, j, k} \tag{2.14.14}
\end{equation*}
$$

which has the L.T.E. of $O\left((\Delta t)^{2}+(\Delta t)(\Delta x)^{2}\right)$ or order of accuracy of $0\left(\Delta t+(\Delta x)^{2}\right)$. For $r=1 / 6$, the formulae has order of accuracy of $O\left((\Delta t)^{2}+(\Delta x)^{4}\right)$ i.e.,

$$
\begin{equation*}
u_{i, j, k+1}=\left(1+\frac{1}{6} \delta_{x}^{2}\right)\left(1+\frac{1}{6} \delta_{y}^{2}\right) u_{i, j, k}, \tag{2.14.15}
\end{equation*}
$$

but it is of limited use since the step forward in time is so small, i.e. $\Delta t=1 / 6 \Delta x^{2}$. In order to obtain the unconditionally stable explicit difference schemes, Larkin (1963), introduced a class of explicit formulae of four basic different types, namely,

$$
\begin{align*}
\left(1+w+c^{2}\right) v_{i, j, k+1}= & v_{i-1, j, k+1}+v_{i+1, j, k}-\left(1-w+c^{2}\right) v_{i, j, k} \\
& +c^{2}\left[v_{i, j-1, k+1}+v_{i, j+1, k}\right] \tag{2.14.16}
\end{align*}
$$

 $+c^{2}\left[z_{i, j+1, k+1}{ }^{+z}{ }_{i, j-1, k}\right]$,

$$
\begin{align*}
\left(1+w+c^{2}\right) v_{i, j, k+1}= & v_{i-1, j, k+1}+v_{i+1, j, k}-\left(1-w+c^{2}\right) v_{i, j, k}  \tag{2.14.17}\\
& +c^{2}\left[v_{i, j-1, k}+v_{i, j+1, k+1}\right] \tag{2.14.18}
\end{align*}
$$

and

$$
\begin{align*}
\left(1+w+c^{2}\right) z_{i, j, k+1}= & z_{i-1, j, k}+z_{i+1, j, k+1}-\left(1-w+c^{2}\right) z_{i, j, k} \\
& +c^{2}\left[z_{i, j-1, k+1}^{+z} i, j+1, k\right. \tag{2.14.19}
\end{align*}
$$

where $c=\Delta x / \Delta y$ and $w=(\Delta x)^{2} / \Delta t=1 / r$. Dependent variables $v$ and $z$ are given diagrammatically by Fig. (2.14.2) (a), (b), (c) and (d).


For (2.14.16)
(a)



For (2.14.19)
(d)
(arrows show the marching direction)

Besides the individual schemes, combination of these schemes also offers good approximation such as
(i) the use of (2.14.16) and (2.14.17) alternately or (2.14.18) alternately with (2.14.19).
(ii) the use of (2.14.16) and (2.14.17) at each time level and average the results as

$$
\begin{equation*}
u_{i, j+1}=\frac{v_{i, j+1}+z_{i, j+1}}{2} . \tag{2.14.20}
\end{equation*}
$$

Then a similar stratety is applied to (2.14.18) and (2.14.19). The truncation error for equations (2.14.16) and (2.14.18) is given by

$$
\begin{equation*}
\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} v}{\partial x \partial t}+\frac{\Delta t}{2}\left(\frac{\partial^{2} v}{\partial t^{2}}-\frac{\partial^{3} v}{\partial t \partial x^{2}}\right)+\frac{(\Delta t)^{2}}{2 \Delta x} \frac{\partial^{3} v}{\partial x \partial t^{2}}+\ldots, \tag{2.14.21}
\end{equation*}
$$

whereas equations (2.14.17) and (2.14.19) have truncation errors of

$$
\begin{equation*}
-\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} z}{\partial x \partial t}+\frac{\Delta t}{2}\left(\frac{\partial^{2} z}{\partial t^{2}}-\frac{\partial^{3} z}{\partial t \partial z^{2}}\right)-\frac{(\Delta t)^{2}}{2 \Delta x} \frac{\partial^{3} z}{\partial x \partial t^{2}}+\ldots \tag{2.14.22}
\end{equation*}
$$

Consistency of the schemes with the equation (2.14.3) requires that $\Delta t / \Delta x \rightarrow 0$ as $\Delta x \rightarrow 0$. By diminishing $\Delta x$ and $\Delta t$ such that $w$ is constant, the leading term of equation (2.14.21) and (2.14.22) would be of first degree in $\Delta x$ with all other terms being of higher order. Because the leading term of (2.14.21) and (2.14.22) have opposite signs, averaging (2.14.20) will effectively produce a truncation error of second degree. The alternating use of the equations (2.14.16) and (2.14.17) or (2.14.18) and (2.14.19) is termed as Alternating Direction Explicit method.

Another explicit scheme, i.e. the Du-Fort-Frankel scheme for two space dimensions which can be written as

$$
\begin{equation*}
\left(\nabla_{t}^{-\frac{1}{2}} \nabla_{t}^{2}\right) u_{i, j, k+1}=r\left(\delta_{x}^{2}+\delta_{y}^{2}-2 \delta_{t}^{2}\right) u_{i, j, k}, \tag{2.14.23}
\end{equation*}
$$

is an unconditionally stable scheme also. The truncation error is given by (Jain, M.K., 1979, p.249).

$$
\begin{equation*}
\frac{1}{\Delta t} T_{i, j, k}=O\left((\Delta t)^{2}+(\Delta s)^{2}+\left(\frac{\Delta t}{\Delta s}\right)^{2}\right) \tag{2.14.24}
\end{equation*}
$$

where $\Delta x=\Delta y=\Delta s$, which tends to zero if $\Delta t / \Delta x \rightarrow 0$ as $\Delta t \rightarrow 0, \Delta s \rightarrow 0$. In order to apply this formulae, $u_{i, j, 1}$ are normally taken from an accurate two level difference scheme such as (2.14.15).

### 2.14.2 Implicit Difference Schemes

The general implicit difference scheme for two-dimensions as given by Jain, M.K. (1979, p.250) is

$$
\begin{align*}
& {\left[1+\left[\sigma-\left(1-\gamma_{1}\right) r\right\} \delta_{x}^{2}\right]\left[1+\left\{\sigma-\left(1-\gamma_{1}\right) r\right\} \delta_{y}^{2}\right] u_{i, j, k+1}=} \\
& \quad\left[1+\left(\sigma+r \gamma_{1}\right) \delta_{x}^{2}\right]\left[1+\left(\sigma+r \gamma_{1}\right) \delta_{y}^{2} I u_{i, j, k}+\left(1-2 \gamma_{1}\right) r^{2} \delta_{x}^{2} \delta_{y}^{2} u_{i, j, k}\right. \tag{2.14.25}
\end{align*}
$$

for arbitrary $\gamma_{1}$ and $\sigma$. The stability condition will be satisfied if

$$
\begin{equation*}
\sigma<\frac{1}{4} \text { and } \gamma_{1} \leqslant \frac{1}{2} \tag{2.14.26}
\end{equation*}
$$

and the truncation error is given by

$$
\begin{aligned}
\frac{1}{(\Delta t)} T_{i, j, k}= & {\left[\left(\sigma-\frac{1}{12}\right)(\Delta s)^{2}+\left(\gamma_{1}-\frac{1}{2}\right) \Delta t\right]\left(\frac{\partial^{4} u}{\partial x^{4}} i, j, k+\frac{\partial^{4} u}{\partial y^{4}} i, j, k\right)-} \\
& \left(1-2 \gamma_{1}\right) \Delta t{\frac{\partial^{4} u}{\partial x^{2} \partial y^{2}}}^{i}, j, k+0\left((\Delta t)^{2}+(\Delta s)^{4}\right) \ldots(2.14 .27)
\end{aligned}
$$

From (2.14.25) and (2.14.27), we find that the values:
(i) $\gamma_{1}=0$ and $\sigma=0$ give an implicit method of order $\left(\Delta t+(\Delta s)^{2}\right)$ namely,

$$
\begin{equation*}
\left(1-r \delta_{x}^{2}\right)\left(1-r \delta_{y}^{2}\right) u_{i, j, k+1}=u_{i, j, k}, \tag{2.14.28}
\end{equation*}
$$

and by neglecting the term $r^{2} \delta_{x}^{2} \delta_{y}^{2} u_{i, j, k+1}$ it gives the classical fully implicit scheme (Fig. 2.14.3) (a)

$$
\begin{align*}
& u_{i, j, k+1}+r\left(-u_{i-1, j, k+1}+2 u_{i, j, k+1}^{-u_{i+1}, j, k+1}\right) \\
& \quad+r\left(-u_{i, j-1, k+1}+2 u_{i, j, k+1}^{\left.-u_{i, j+1, k+1}\right)=u_{i, j, k}}\right. \tag{2.14.29}
\end{align*}
$$

for $i, j=1,2, \ldots, N$ and $N \Delta x=N \Delta y=1$. The compact form of (2.14.29) is

$$
\begin{equation*}
(I+r A){\underset{u}{u}+1}=u_{k}, k \geqslant 0 \tag{2.14.30}
\end{equation*}
$$

where $A$ is $a N^{2} \times N^{2}$ block tridiagonal matrix given by,
with

$$
\begin{align*}
& A=\left[\begin{array}{ccccc}
D & -I_{N} & & & \\
-I_{N} & D^{D} & -I_{N} & & 0 \\
\ddots & \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \\
0
\end{array}\right], D=4 I_{N}-\left(L_{N}+L_{N}^{T}\right)  \tag{2.14.31}\\
& 0 \\
& 0 \\
& { }^{-}-\mathrm{I}_{\mathrm{N}} \\
& L_{N}=\left[\begin{array}{lllll}
0 & & & & \\
1 & 0 & & & 0 \\
\ddots & \ddots & \ddots & & \\
0 & \ddots & \ddots & \\
0 & & & 0 & \\
& & & 1 & 0
\end{array}\right] \tag{2,14.32}
\end{align*}
$$

with

Hence at each time step, we are required to solve a very large system of linear equations, which in general is the drawback of the implicit method.
(ii) $\quad \gamma_{1}=\frac{1}{2}, \sigma=0$, gives the Crank-Nicolson method (Fig. (2.14.3) (b))

$$
\left(1-\frac{1}{2} r \delta_{x}^{2}\right)\left(1-\frac{1}{2} r \delta_{y}^{2}\right) u_{i, j, k+1}=\left(1+\frac{1}{2} r \delta_{x}^{2}\right)\left(1+\frac{1}{2} r \delta_{y}^{2}\right) u_{i, j, k},(2.14 .33)
$$

of order $\left((\Delta t)^{2}+(\Delta s)^{2}\right)$.
(iii) $\gamma_{1}=\frac{1}{2}$ and $\sigma=\frac{1}{12}$, gives the Mitchel1-Fairweather formula (Fig.(2.14.3)(b)

$$
\begin{align*}
& {\left[1+\left(\frac{1}{12}-\frac{1}{2} r\right) \delta_{x}^{2}\right]\left[1+\left(\frac{1}{12}-\frac{1}{2} r\right) \delta_{y}^{2}\right] u_{i, j}, k+1} \\
& \quad=\left[1+\left(\frac{1}{12}+\frac{1}{2} r\right) \delta_{x}^{2}\right]\left[1+\left(\frac{1}{12}+\frac{1}{2} r\right) \delta_{y}^{2}\right] u_{i, j, k} \tag{2.14.34}
\end{align*}
$$

of order $\left((\Delta t)^{2}+(\Delta s)^{4}\right)$. (Jain, M.K., 1979, p.251).
Even though the implicit methods are mostly unconditionally stable, however, due to the large system of equations to be solved, they are computationally expensive. We will discuss below alternative ways to manipulate the implicit methods in order to introduce computationally inexpensive explicit type schemes.

### 2.14.3 Splitting Methods

In the numerical treatment of parabolic differential equations, splitting is referred to as a method of breaking down a process into a series of simple processes. Amongst these are the Alternating Direction Implicit (A.D.I.), the Locally One Dimensional (L.O.D.) and the Hopscotch methods.

The A.D.I. method which was first introduced by Peaceman and Rachford (1955) is a two-step process involving the solution of tridiagonal sets of equations along lines parallel to the $x$ and $y$ axes at the first and second steps respectively.



For equations (2.14.33) and (2.14.34) (b)

For the Peaceman-Rachford ADI method, in the first step in advancing from $t_{k}$ to $t_{k}+\Delta t / 2$, implicit differences are used for $\partial^{2} u / \partial x^{2}$ and explicit differences are used for $\partial^{2} u / \partial y^{2}$. In the second step in advancing from $t_{k}+\Delta t / 2$ to $t_{k+1}$, a reverse procedure is used. Accordingly, the difference approximation to (2.14.3) can be expressed as

$$
\begin{equation*}
\frac{u_{i, j, k+\frac{1}{2}}{ }^{-u} u_{i, j, k}}{\Delta t / 2}=\frac{1}{(\Delta x)^{2}} \delta_{x}^{2} u_{i, j, k+\frac{1}{2}}+\frac{1}{(\Delta y)^{2}} \delta_{y}^{2} u_{i, j, k} \tag{i}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{u_{i, j, k+1}^{-u_{i, j, k+\frac{1}{2}}}}{\Delta t / 2}=\frac{1}{(\Delta x)^{2}} \delta_{x}^{2} u_{i, j, k+\frac{1}{2}}+\frac{1}{(\Delta y)^{2}} \delta_{y}^{2} u_{i, j, k+1} \tag{ii}
\end{equation*}
$$

which may also be written as (Fig. (2.14.4)(a))
(i)

$$
\begin{equation*}
\left(1-\frac{r}{2} \delta_{x}^{2}\right) u_{i, j, k+\frac{1}{2}}=\left(1+\frac{r}{2} \delta_{y}^{2}\right) u_{i, j, k} \tag{2.14.36}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1-\frac{r}{2} \delta_{y}^{2}\right) u_{i, j, k+1}=\left(1+\frac{r}{2} \delta_{x}^{2}\right) u_{i, j, k+\frac{1}{2}} \tag{ii}
\end{equation*}
$$

The equation (2.14.36) can also be obtained directly from the splitting of equation (2.14.33). This method is computationally feasible as it only requires the solution of a set of tridiagonal equations, and that each scheme, used on its own is conditionally stable. However, if they are used altermately, then the overall scheme is unconditionally stable. (Gourlay, A.R., 1977, p.761).

The higher order accuracy formulae of Mitchell-Fairweather (2.14.34) can be split into (Fig. (2.14.4)(a))
(i)

$$
\begin{equation*}
\left[1+\left(\frac{1}{12}-\frac{1}{2} r\right) \delta_{x}^{2}\right] u_{i, j, k+\frac{1}{2}}=\left[1+\left(\frac{1}{12}+\frac{1}{2} r\right) \delta_{y}^{2}\right] u_{i, j, k} \tag{2.14.37}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[1+\left(\frac{1}{12}-\frac{1}{2} r\right) \delta_{y}^{2}\right] u_{i, j, k+1}=\left[1+\left(\frac{1}{12}+\frac{1}{2} r\right) \delta_{x}^{2}\right] u_{i, j, k+\frac{1}{2}} \tag{ii}
\end{equation*}
$$

The implicit Crank-Nicolson formulae can be split in an alternative manner as suggested by D'Yakonov (1963) to give,
(i)

$$
\begin{equation*}
\left(1-\frac{1}{2} r \delta_{x}^{2}\right) u_{i, j, k+\frac{1}{2}}=\left(1+\frac{1}{2} r \delta_{x}^{2}\right)\left(1+\frac{1}{2} r \delta_{y}^{2}\right) u_{i, j, k} \tag{2.14.38}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1-\frac{1}{2} r \delta_{y}^{2}\right) u_{i, j, k+1}=u_{i, j, k+\frac{1}{2}} \tag{ii}
\end{equation*}
$$

and the corresponding higher-order accurate formulae to give

$$
\begin{equation*}
\left.\left[1+\left(\frac{1}{12}-\frac{1}{2} r\right) \delta_{x}^{2}\right] u_{i, j, k+\frac{1}{2}}=\left[1+\left(\frac{1}{12}+\frac{1}{2} r\right) \delta_{x}^{2}\right]\left[1+\left(\frac{1}{12}+\frac{1}{2} r\right) \delta_{y}^{2}\right)\right] u_{1, j, k} \tag{i}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[1+\left(\frac{1}{12}-\frac{1}{2} r\right) \delta_{y}^{2}\right] u_{i, j, k+1}=u_{i, j, k+\frac{1}{2}} \tag{ii}
\end{equation*}
$$

as in Fig.(2.14.3)(b).
The Peaceman-Rachford method (2.14.36) in matrix form can be written as (Danaee, A., 1980, p.66),
(i)

$$
\begin{equation*}
(I+r H)_{k+\frac{1}{2}}=(I-r V) u_{k}, \tag{2.14.40}
\end{equation*}
$$

and
(ii) $\quad(I+r V)_{-k+1}=(I-r H) u_{k+\frac{1}{2}}$,
where $\quad \begin{aligned} H & =2 I-\left(L+L^{T}\right), \\ V & =2 I-\left(B+B^{T}\right),\end{aligned}$

$$
L=\left[\begin{array}{lllll}
\mathrm{L}_{\mathrm{N}} & & & & \\
& \mathrm{~L}_{\mathrm{N}} & & 0 \\
& & \ddots & \\
& 0 & \ddots & \\
& & & \mathrm{~L}_{\mathrm{N}}
\end{array}\right] \text { and } B=\left[\begin{array}{lllll}
0 & & & \\
\mathrm{I}_{\mathrm{N}} & 0 & & & 0 \\
& \mathrm{I}_{\mathrm{N}} & \ddots & \\
& & \ddots & \ddots & \\
0 & & \ddots & \mathrm{I}_{\mathrm{N}} & 0
\end{array}\right]
$$

with $\mathrm{L}_{\mathrm{N}}$ given in (2.14.32). Douglas and Rachford (1956) in Fig.(2.14.4)(c)
proposed alternative to the Peaceman-Rachford method as follows,
(i) $\quad\left(1-r \delta_{x}^{2}\right) u_{i, j, k+\frac{1}{2}}=\left(1+r \delta_{y}^{2}\right) u_{i, j, k}$,
and
(ii) $\quad\left(1-r \delta_{y}^{2}\right) u_{i, j, k+1}=u_{i, j, k+\frac{1}{2}}-r \delta_{y}^{2} u_{i, j, k}$,.
which in the matrix form can be written as,
(i)

$$
\begin{equation*}
(I+r H){\underset{u}{k+\frac{1}{2}}}=(I-r V)_{k} \text {, } \tag{2.14.44}
\end{equation*}
$$

and
(ii) $\quad(\mathrm{I}+\mathrm{rV})_{u_{k+1}}=\underline{u}_{k+\frac{1}{2}}+r \underline{u}_{k}$,
respectively (Danaee, A., 1980, p.67). This method is also unconditionally stable and computationally feasible involving the solution of tridiagonal systems of difference equations along horizontal lines for $u_{k+\frac{1}{2}}$ and then along vertical lines for $u_{k+1}$.

It is important to note that $\underline{u}_{k+\frac{1}{2}}$ has no physical significance and it is only the first estimate or intermediate value, (some refer to this as $u_{k+1}^{*}$ ). It is not necessarily an approximation to the solution at any value of time. As a result, particularly with the high accuracy methods, the boundary values at the intermediate level must be obtained, if possible, in terms of the boundary values at $t=k \Delta t$ and $t=(k+1) \Delta t$. The methods of coping with this problem are given in Mitche11 A.R. and Griffiths D.F. (1980, p.62-63).

If we consider the equation (2.14.3) as the pair of equations

$$
\begin{equation*}
\frac{1}{2} \frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}} \tag{2.14.45}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{2} \frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial y^{2}}, \tag{2.14.46}
\end{equation*}
$$

then the simplest explicit formulae can be obtained as

$$
\begin{equation*}
u_{i, j, k+\frac{1}{2}}=\left(1+r \delta_{y}^{2}\right) u_{i, j, k}, \tag{2.14.47}
\end{equation*}
$$

and $\quad u_{i, j, k+1}=\left(1+r \delta_{x}^{2}\right) u_{i, j, k+\frac{1}{2}}$,
with the order of accuracy given by $O\left(\Delta t+(\Delta s)^{2}\right)$. The strategy of splitting a two-dimensional problem into a one-dimensional problem is called Locally One Dimensional (L.O.D.) method, and has been developed extensively by D'Yakonov, Ye. G. (1963) and Samarskii (1964). The equations (2.14.47) and
and (2.14.48) form the explicit scheme of L.O.D. and is described in Fig. (2.14.5). It is important to note that the elimination of $u_{i, j, k+\frac{1}{2}}$ from both equations leads to

$$
u_{i, j, k+1}=\left(1+r \delta_{x}^{2}\right)\left(1+r \delta_{y}^{2}\right) u_{i, j, k}
$$

which is the explicit equation (2.14.14).
Gourlay A.R. and Mitchell A.R. (1969) have exploited some interesting connections between the Peaceman-Rachford and L.O.D. methods.

Another class of splitting method was introduced by Saul'yev (1964, p.23), Gordon (1965) and marketed by Gourlay (1970) is called the Hopscotch method. The simplest form of Hopscotch method was mentioned in equation (2.13.8) and (2.13.9). To briefly discuss this class of method, we restrict ourselves to the linear parabolic equation of two dimension, i.e.,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=L(u)+g(x, y, t) \tag{2.14.49}
\end{equation*}
$$

where $L$ is the second-order linear, elliptic differential operator in the space variables $x$ and $y$.

A general splitting formula for the equation (2.14.49) is given by (Danaee, A., 1980, p.105)

$$
\begin{align*}
u_{i, j, k+1}-\Delta t & \left(\theta_{i, j, k+1} L_{\Delta x}^{(1)}+\eta_{i, j, k+1} L_{\Delta x}^{(2)}\right) u_{i, j, k+1} \\
= & u_{i, j, k}+\Delta t\left(\theta_{i, j, k}^{L}{ }_{\Delta x}^{(1)}+\eta_{i, j, k} L_{\Delta x}^{(2)}\right) u_{i, j, k} \\
& +\Delta t\left(\theta_{i, j, k+1} g_{i, j, k+1}^{(1)}+\eta_{i, j, k+1} g_{i, j, k+1}^{(2)}\right) \\
& +\Delta t\left(\theta_{i, j, k} g_{i, j, k}^{(1)}+\eta_{i, j, k} g_{i, j, k}^{(2)}\right), \tag{2.14.50}
\end{align*}
$$

with the restrictions,

$$
\left.\begin{array}{c}
\theta_{i, j, k+1}+\theta_{i, j, k}=1  \tag{2.14.51}\\
\eta_{i, j, k+1}+\eta_{i, j, k}=1
\end{array}\right\}
$$

where

$$
g_{i, j, k}^{(1)}+g_{i, j, k}^{(2)}=g_{i, j, k},
$$

and

$$
L_{\Delta x}^{(1)}+L_{\Delta x}^{(2)}=L_{\Delta x}
$$



For equation (2.14.36) and (2.14.37)


FIGURE (2.14.4)


For equation (2.14.38) and (2.14.39)
(b)


The odd-even Hopscotch method can be obtained by defining

$$
\theta_{i, j, k}=\eta_{i, j, k}=\left\{\begin{array}{l}
1, \text { if } i+j+k \text { even } \\
0, \text { if } i+j+k \text { odd }
\end{array}\right.
$$

which from (2.14.50) gives the explicit formulae

$$
\begin{equation*}
u_{i, j, k+1}=u_{i, j, k}+\Delta t\left(L_{\Delta x}^{(1)}+L_{\Delta x}^{(2)}\right) u_{i, j, k}+\Delta t\left(g_{i, j, k}^{(1)}+g_{i, j, k}^{(2)}\right) \tag{2.14.52}
\end{equation*}
$$

alternately with the implicit formulae

$$
\begin{equation*}
u_{i, j, k+1}-\Delta t\left(L_{\Delta x}^{(1)}+g_{i, j, k+1}^{(1)}\right)-\Delta t\left(L_{\Delta x}^{(2)}+g_{i, j, k+1}^{(2)}\right)=u_{i, j, k} \tag{2.14.53}
\end{equation*}
$$

Similar to the odd-even Hopscotch, the line-Hopscotch method in the $y$ direction can be obtained if we define,

$$
\theta_{i, j, k}=\eta_{i, j, k}= \begin{cases}1, & \text { if } i+k \text { odd }  \tag{2.14.54}\\ 0, & \text { if } i+k \text { even }\end{cases}
$$

and in the $x$-direction if we define,

$$
\theta_{i, j, k}=\eta_{i, j, k}= \begin{cases}1, & \text { if } j+k \text { odd }  \tag{2.14.55}\\ 0, & \text { if } j+k \text { even }\end{cases}
$$

The general splitting formulae, can also be used to derive the standard formulae like Crank-Nicolson equation if we take $\theta_{i, j, k}=\eta_{i, j, k}=$ $\theta_{i, j, k+1}=\eta_{i, j, k+1}=\frac{1}{2}$ for all $i, j$ and $k$. If $\theta_{i, j, k+1}=\eta_{i, j, k+1}=0$ and $\theta_{i, j, k}=\eta_{i, j, k}=1$ for all $i, j, k$ we obtain the $\operatorname{explicit}$ scheme. If

$$
\begin{aligned}
& \theta_{i, j, k}=\frac{1}{2}\left(1+(-1)^{k}\right)=\left\{\begin{array}{l}
1, \text { if } k \text { is even } \\
0, \text { if } k \text { is odd }
\end{array}\right. \\
& \eta_{i, j, k}=1-\theta_{i, j, k} \text { for all } i, j, k,
\end{aligned}
$$

we get the Peaceman-Rachford method with a time step of $2 \Delta t$ (McGuire, R.G., 1970, p.7).

Since the work completed on this class of methods is so extensive, the interested reader is recommended to read Gourlay (1970), Gourlay and McGuire (1971), Gane and Gourlay (1977), Gourlay and McKee (1977), Grieg and Morris (1976) and Danaee, A, (1980) to obtain a more detailed knowledge.

## Chapter Three

PARABOLIC EQUATIONS: FURTHER TOPICS

### 3.1 THE DIFFUSION-CONVECTION EQUATION

Parabolic equations containing first order spatial derivatives are called diffusion-convection equations because of the physical processes they describe. They arise frequently in the mathematical description of flow processes. Typically, a suspended material (effluent, sediment, etc) is carried along (convected) by a flow of fluid while at the same time its concentration is being attenuated (diffused) within the flow. These effects have the form of parabolic p.d.e, and thus the obvious approach when seeking numerical solutions is to utilize a method specially developed for that type of equation. When the problem under consideration is one dimensional in space, the methods discussed in the previous chapter should immediately come to mind.

Numerically, the more interesting problems arise when the first order (convection) spatial derivatives are large in relation to the second order (diffusion) ones. This is due to the fact that the methods for parabolic equations are normally subject to oscillation unless a very fine spatial grid is utilised. Several solution techniques have been developed to cope with this problem and they will be discussed later in this section.

A typical equation in its one-dimensional form is

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}-v \frac{\partial u}{\partial x} \tag{3.1.1}
\end{equation*}
$$

where $u$ describes the concentration of a suspension convected with velocity V and diffusing according to the diffusion coefficient D . Another source of the diffusion-convection equation lies in the NavierStokes equations, particularised for the case of incompressible flow. In that event,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{v^{2} u}{\partial x^{2}}-v \frac{\partial u}{\partial x} \tag{3.1.2}
\end{equation*}
$$

where $u=-\partial^{2} \psi / \partial x^{2}, V=-\partial \psi / \partial x$ with $v$ is the coefficient of kinematic viscosity and $\psi$ is a stream function (Gladwell and Wait, 1979, p.195). For the past two decades, scientists and mathematicians of various fields have attempted a numerical solution of equations of this kind. Some main contributions will be discussed in the later sections. The finite difference approximation for equations (3.1.1) or (3.1.2) using the CrankNicolson method are presented below for reference. Using the approximations

$$
\begin{align*}
&\left.\frac{\partial u}{\partial t}\right]_{i, j+\frac{1}{2}}=\frac{u_{i, j+1}-u_{i, j}}{\Delta t}+0\left(\Delta t^{2}\right)  \tag{3.1.3}\\
&\left.\frac{\partial u}{\partial x}\right]_{i, j+\frac{1}{2}}=\frac{1}{2}\left[\frac{u_{i+1, j+1}-u_{i-1, j+1}}{2 \Delta x}+\frac{u_{i+1, j}-u_{i-1, j}}{2 \Delta x}\right]+0\left(\Delta x^{2}\right)  \tag{3.1.4}\\
& \begin{aligned}
\left.\frac{\partial^{2} u^{2}}{\partial x^{2}}\right]_{i, j+\frac{1}{2}} & =\frac{1}{2}\left[\frac{u_{i+1, j^{-2 u_{i, j+1}}}+u_{i-1, j+1}}{(\Delta x)^{2}}+\frac{u_{i+1, j^{-2 u_{i, j}}}+u_{i-1, j}}{(\Delta x)^{2}}\right] \\
& +0\left(\Delta x^{2}\right)
\end{aligned}
\end{align*}
$$

Equation (3.1.1) yields the system of tridiagonal equations

$$
\begin{align*}
-r \alpha_{1} u_{i-1, j+1} & +(1+r D) u_{i, j+1}-r \alpha_{2} u_{i+1, j+1} \\
& =r \alpha_{1} u_{i-1, j}+(1-r D) u_{i, j}+r \alpha_{2} u_{i+1, j} \tag{3.1.6}
\end{align*}
$$

with $\alpha_{1}=\left(\frac{D}{2}+\frac{\Delta x V}{4}\right), \alpha_{2}=\left(\frac{D}{2}-\frac{\Delta x V}{4}\right)$ and $r=\Delta t /(\Delta x)^{2}$.
This scheme will be named as Crank-Nicolson with centred difference (CNCD). The approximations are second order in both space and time as regards truncation error.

Peaceman-Rachford (1962), who were among the early contributors suggested the scheme which is simplified to

$$
\begin{align*}
\frac{u_{i, j+1}-u_{i, j}}{\Delta t}= & \frac{D}{2(\Delta x)^{2}}\left[\left(u_{i-1, j^{-2}} u_{i, j}+u_{i+1, j}\right)+\left(u_{i-1, j+1}-2 u_{i, j+1}+\right.\right. \\
& \left.\left.u_{i+1, j+1}\right)\right]-\frac{V}{2 \Delta x}\left[u_{i+\frac{1}{2}, j^{*}} j^{*}+\frac{1}{1}, j+1-u_{i-\frac{1}{2}, j}^{*}-u_{i-\frac{1}{2}, j+1}^{*}\right] \tag{3.1.8}
\end{align*}
$$

where $u^{*}$ can be chosen either as,
or

$$
\begin{align*}
& u_{i+\frac{1}{2}}^{*}=\frac{1}{2}\left(u_{i+1}+u_{i}\right)  \tag{3.1.9}\\
& u_{i+\frac{1}{2}}^{*}=u_{i} . \tag{3.1.10}
\end{align*}
$$

The first choice (equation (3.1.9)) results in CNCD scheme (3.1.6) above and the second choice (equation (3.1.10)) gives a backward-indistance scheme. The latter is then termed by Spalding (1972) as upwinding. The backward-in-distance equation, unlike the CNCD equation, is first order in the space increment $\Delta x$ as regards the truncation error. Therefore it needs a prohibitively large number of distance increments to obtain an accurate solution. On the other hand, the backward-in-distance equation, is capable of eliminating overshooting (or oscillation in this case) for reasonably small increments in space. The combination of both choices in a particular manner, in a way, improved the accuracy, reduces overshooting and the required number of space intervals (Peaceman and Rachford, 1962).

A rather general finite-difference approximation to equation (3.1.1) was proposed by Stone and Brian (1963) which is given by

$$
\begin{align*}
& \frac{1}{\Delta t}\left[g\left(u_{i, j+1} u_{i, j}\right)+\frac{\theta}{2}\left(u_{i-1, j+1}-u_{i-1, j}\right)+m\left(u_{i+1, j+1}^{-u_{i+1, j}}\right)\right] \\
& =D\left[\Delta_{x}^{2} \frac{\left(u_{i, j}+u_{i, j+1}\right)}{2}\right]-\frac{v}{\Delta x}\left[a\left(u_{i+1, j}-u_{i, j}\right)+\frac{\varepsilon}{2}\left(u_{i, j}-u_{i-1, j}\right)\right. \\
& +c\left(u_{i+1, j+1}^{\left.\left.-u_{i, j+1}\right)+d\left(u_{i, j+1}^{-u_{i-1, j+1}}\right)\right] . ~}\right. \tag{3.1.11}
\end{align*}
$$

The coefficients $a, \varepsilon / 2, c$ and $d$ are the weighting coefficients chosen to match the finite-difference analogue of the convective term in equation (3.1.1). The coefficients $g, \theta / 2$ and $m$ are similarly the weighting coefficients used to match the finite-difference analogue of the time derivative. Consequently, these coefficients are subject to the restrictions,

$$
\left.\begin{array}{l}
a+\frac{\varepsilon}{2}+c+d=1  \tag{3.1.12}\\
g+\frac{\theta}{2}+m=1
\end{array}\right\}
$$

If the initial condition is represented by the sum of a number of
sinusoidal variations of different frequencies, or by a Fourier series in the variable $x$, and if the boundary conditions are appropriate, the super-position principle yields the solution of equation (3.1.1) as

$$
\begin{equation*}
u(x, t)=\sum_{\omega=1}^{\infty} A_{\omega} e^{-\omega^{2} \pi^{2} D t} \sin \omega \pi(x-V t) \tag{3.1.13}
\end{equation*}
$$

In this method, the analysis of the accuracy of the rate of decay of the sine wave harmonics which comprise the solution (3.1.13), and its relation with the weighting coefficients (3.1.12) is considered. Also, the accuracy of the convective propagation of the appropriate sine-wave harmonics is considered too. As the analysis is very complicated the author feels it is not necessary to elaborate on it here and the attention of the interested readers is directed to the original literature.

Price, Varga and Warren (1966) introduced rather specialised weighting factors for the first space derivative analogue:

while using similar analogues as the $C N C D$ method for the other derivatives. They found that this difference analoguegreatly reduces the tendency for spurious oscillations in the solution, permitting much larger spatial grids to be used and an appreciably more efficient solution is obtained. However, the introduction of an additional unknown value, $u_{i-2, j+1}$ leads to a system of four unknown variables which required the unusual algorithm for the solution of pentadiagonal systems. This more complex solution algorithm somewhat offsets the advantages gained with the coarser grid. It is also worth noting that the approximation (3.1.14) cannot be applied at the first unknown grid point as a value of $u$ exterior to the boundary would be involved. A simple two point backward formula (of first order) is normally used instead.

### 3.2 THE STABILITY CONDITION FOR SOME FINITE DIFFERENCE TECHNIQUES IN

## SOLVING THE DIFFUSION-CONVECTION EQUATION

In the solution of the diffusion-convection equation (3.1.1), the numerical stability condition is often more stricter than ordinary diffusion equations as the convection velocity $V$ and diffusion constant $D$ both affect the stability. For example, an explicit finite difference representation

$$
\begin{equation*}
u_{i, j+1}=u_{i, j}+\frac{D \Delta t}{(\Delta x)^{2}}\left(u_{i-1, j}-2 u_{i, j}+u_{i+1, j}\right)-\frac{v \Delta t}{2 \Delta x}\left(u_{i+1, j}-u_{i-1, j}\right), \tag{3.2.1}
\end{equation*}
$$

using the Von Neumann's method of analysis of stability as in Section (2.10), gives the complex amplification factor

$$
\begin{equation*}
\overline{\xi \bar{\xi}}=|\xi|^{2}=\left[1+\frac{2 D \Delta t}{(\Delta x)^{2}}(\cos \beta \Delta x-1)\right]^{2}+\left(\frac{V \Delta t}{\Delta t}\right)^{2}\left(1-\cos ^{2} \beta \Delta x\right) \tag{3.2.2}
\end{equation*}
$$

which requires the conditions,

$$
\begin{equation*}
\frac{V \Delta t}{\Delta x} \leqslant 1 \text { and } \frac{D \Delta t}{(\Delta x)^{2}} \leqslant \frac{1}{2} \tag{3.2.3}
\end{equation*}
$$

for stability. (S. Biringen, 1981). The restrictions (3.2.3) are tighter than restriction (2.9.17) as the two conditions have to be fulfilled. Conditions (3.2.3), according to Fromm (1964) are classified as

$$
\begin{equation*}
R_{c}=\mathrm{V} \frac{\Delta \mathrm{x}}{\mathrm{D}} \leqslant 2 \quad \text { (convective stability) } \tag{3.2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{d}=\mathrm{D} \frac{\Delta \mathrm{t}}{(\Delta \mathrm{x})^{2}} \leqslant \frac{1}{2} \quad \text { (diffusive stability) } \tag{3.2.5}
\end{equation*}
$$

where $R_{c}$ is the cell Reynolds number.
On the other hand, the fully-implicit scheme

$$
\begin{align*}
u_{i, j+1}^{-u_{i, j}} & +\frac{V}{2} \frac{\Delta t}{\Delta x}\left(u_{i+1, j+1}-u_{i-1, j+1}\right) \\
& =\frac{D \Delta t}{(\Delta x)^{2}}\left(u_{i+1, j+1}-2 u_{i, j+1}+u_{i-1, j+1}\right) \tag{3.2.6}
\end{align*}
$$

gives the complex amplification factor

$$
\begin{equation*}
\xi \bar{\xi}=|\xi|^{2}=\frac{1}{\left[1+4 d^{2} \sin ^{2}\left(\frac{\beta \Delta x}{2}\right)\right]^{2}+c^{2} \sin ^{2}(\beta \Delta x)} \tag{3.2.7}
\end{equation*}
$$

where $d=D \Delta t /(\Delta x)^{2}$ and $c=V \Delta t / \Delta x$. From equation (3.2.7), it is clear that the stability condition, which requires that $|\xi|^{2} \leqslant 1$, is unconditionally satisfied. In practice, this unconditionally stable implicit scheme produces instabilities when the convection velocity $V$ (or $\mathrm{R}_{\mathrm{c}}$ ) is large. The 'jump' in the convection velocity $V$ is observed to have a significant effect on the stability of the scheme (Biringen, $S$; 1981).

Siemieniuch and Gladwell (1978) have shown the 'inability' of certain stability techniques to control error growth in calculations arising from diffusion-convection problems such as

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}-\lambda \frac{\partial u}{\partial x} \quad, \lambda>0 \tag{3.2.8}
\end{equation*}
$$

in the domain $0<x<1, t>0$ with initial and boundary conditions
and

$$
\left.\begin{array}{ll}
u(x, 0)=0, & 0<x<1  \tag{3.2.9}\\
u(0, t)=1, & \frac{\partial u}{\partial x}(1, t)=0
\end{array}\right\}
$$

respectively.
In their concluding remarks they said "we have no explanation for the discrepancy between the theoretically derived stability region and the computed region". Morton (1980) explains: "Unfortunately most of the analysis was based on the so-called matrix method, and an associated concept of stability, which is misleading in both theory and practice for such problems". He also mentions that the erroneous stability limit obtained with the matrix method is often hidden by the 'Gerschgorin rescue act'. In order to verify this statement it is necessary to inspect some of his analysis of stability in greater detail.

To commence, we write the approximation schemes to equation (3.2.8) as follows:

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u}{ }_{i, j}}{\Delta t} \approx \frac{\delta^{2} u_{i, j}}{(\Delta x)^{2}}-\lambda \frac{\Delta_{0} u_{i, j}}{(\Delta x)} \text { (central) } \tag{3,2,10}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{u_{i, j+1}{ }^{-u_{i, j}}}{\Delta t}=\frac{\delta^{2} u_{i, j}}{(\Delta x)^{2}}-\lambda \frac{\Delta_{-} u_{i, j}}{(\Delta x)} \quad \text { (upwind) } \tag{3.2.11}
\end{equation*}
$$

$i=1,2, \ldots, M$ where $M \Delta x=1$ and the standard notation $\delta^{2} u_{i, j}=u_{i-1, j}-2 u_{i, j}+u_{i+1, j}$ $\left.\Delta_{0} u_{i, j}{ }^{\frac{1}{2}\left(u_{i+1, j}\right.}{ }^{-u}{ }_{i-1, j}\right), \Delta_{-} u_{i, j}=u_{i, j}{ }^{-u}{ }_{i-1, j}$. The initial and boundary conditions (3.2.9) are now written as
and

$$
\left.\begin{array}{ll}
u_{i, 0}=0, & 0<i<M  \tag{3.2.12}\\
u_{0, j}=1, & u_{i+1, j}=u_{i-1, j}
\end{array}\right\}
$$

Approximations (3.2.10) and (3.2.11) together with the boundary condition (3.2.12) in matrix form may be written as

$$
\begin{equation*}
\underline{u}_{j+1}=E \underline{u}_{j}+r s, \tag{3.2.13}
\end{equation*}
$$

where in case of (3.2.10),

$$
E=\left[\begin{array}{lrrrr}
1-2 r & r(1-L) & &  \tag{3.2.14}\\
r(1+L) & 1-2 r & r(1-L) & 0 \\
\ddots & \ddots & \ddots & \\
0 & \ddots & \ddots & \ddots & \\
0 & & \ddots(1+L) & 1-2 r & r(1-L) \\
& & & & 2 r \\
& & 1-2 r
\end{array}\right]_{(M \times M)}
$$

with $L=\frac{\lambda \Delta x}{2}, \underline{s}^{T}=[1+L, 0, \ldots, 0]$ and in the case of (3.2.11),

$$
E=\left[\begin{array}{cccc}
\begin{array}{cccc}
1-r\left(2+L_{1}\right) & r & & \\
r\left(1+L_{1}\right) & 1-r\left(2+L_{1}\right) & r & 0 \\
& \ddots, & \ddots, ~ & \ddots
\end{array}  \tag{3.2.15}\\
0 & \ddots r\left(1+L_{1}\right) & 1-r\left(2+L_{1}\right) & \ddots r \\
& & & 2 r+L_{1} \\
& & 1-r\left(2+L_{1}\right)
\end{array}\right]_{(M \times M)}
$$

with $L_{1}=\lambda \Delta x, \underline{s}^{T}=\left[1+L_{1}, 0,0, \ldots, 0\right], \underline{u}_{j+1}^{T}=\left[u_{1, j+1}, u_{2, j+1}, \ldots, u_{M-1, j+1}\right.$,
$\left.u_{M, j+1}\right]$. The analysis of stability by the matrix method (Section 2. ) requires

$$
\begin{equation*}
\left\|\mathrm{E}^{\mathrm{n}}\right\| \leq \mathrm{K} \text { for all } \mathrm{n} \tag{3.2.16}
\end{equation*}
$$

and any choice of matrix norm. As $E$ is an oscillation matrix (Price et al, 1966) with distinct eigenvalues $n_{j}$ and possesses non-singular matrices
of eigenvectors $S$ such that

$$
S^{-1} E^{n} S=\operatorname{diag}\left\{n_{i}^{n}\right\}
$$

i.e. $\quad\left|\left|E^{n}\right|\right| \leqslant||s|| \cdot| | s^{-1}| | \cdot \max _{1 \leqslant i \leqslant M}\left|n_{i}\right|^{n}$.

Thus, for stability, it is necessary and sufficient to calculate the eigenvalues of $E$ and ensure that

$$
\begin{equation*}
\max _{1 \leqslant i \leqslant M}\left|n_{i}\right| \leqslant 1 \tag{3.2.17}
\end{equation*}
$$

Using (3.2.17) as the basis for stability, and with the help of Gerschgorin's bound, Siemieniuch and Gladwell (1978) were able to derive the following conditions

$$
0<r \leqslant\left\{\begin{array}{l}
{\left[1+\sqrt{ }\left(1-L^{2}\right)\right]^{-1} \text { for } L<1}  \tag{3,2,18}\\
1 / L^{2} \text { for } L \geqslant 1
\end{array}\right.
$$

for approximation (3.2.10), and

$$
\begin{equation*}
0<r \leqslant \frac{1}{\left[2+L_{1}+2 \sqrt{\left(1+L_{1}\right)}\right]}, L_{1} \geqslant 0, \tag{3.2.19}
\end{equation*}
$$

for approximation (3.2.11). These conditions are too large and particularly misleading according to Morton (1980). In explaining this, he used the Fourier series method of analysis which requires the amplification factor $\xi(\beta)$ to satisfy

$$
\begin{equation*}
|\xi(B)| \leqslant 1+K \Delta t, \tag{3.2,20}
\end{equation*}
$$

where $K$ is a constant and independent of $\beta, \Delta t$ and $\Delta x$. In the case of equations (3.2.10) and (3.2.11)
and

$$
\begin{align*}
& \xi(\beta)=1-4 \operatorname{rin}^{2} \frac{1}{2} \beta \Delta x-i(\lambda \Delta t / \Delta x) \sin \beta \Delta x  \tag{3.2.21}\\
& \xi(\beta)=1-4 \operatorname{rin}^{2} \frac{1}{2} \beta \Delta x-(\lambda \Delta t / \Delta x)\left(1-e^{-i \beta \Delta x}\right) \tag{3.2.22}
\end{align*}
$$

respectively. Therefore for (3.2.10)

$$
\begin{align*}
|\xi(B)| & =\left(1-4 r \sin ^{2} \frac{1}{2} \beta \Delta x\right)^{2}+\lambda^{2} r \Delta t \sin ^{2} \beta \Delta x  \tag{3.2.23}\\
& \leqslant 1+O(\Delta t)
\end{align*}
$$

in which the first term requires $r \leqslant \frac{1}{2}$ when $\beta \Delta x=\pi$, and if this is satisfied, the second term is $O(\Delta t)$. The upwinding scheme (3.2.11) is a little more
complicated and gives,

$$
\begin{align*}
|\xi(\beta)| & =\left[1-(4 r+2 \lambda \Delta t / \Delta x) \sin ^{2} \frac{1}{2} \beta \Delta x\right]^{2}+\lambda^{2} r \Delta t \sin ^{2} \beta \Delta x \\
& \leqslant 1+0(\Delta t) \tag{3.2.24}
\end{align*}
$$

in which $r<\frac{1}{2}$ is needed for stability.
These conditions are regarded as the minimal asymptotic stability conditions which are required since they are necessary for convergence in the presence of arbitrary perturbations which are induced by rounding errors. We may still have considerable growth of errors for finite $\Delta x$ and $\Delta t$, because of the term $K \Delta t$ in (3.2.20). Therefore, it is usually necessary to impose a more severe condition by requiring

$$
\begin{equation*}
|\xi(\beta)| \leqslant 1 \tag{3.2.25}
\end{equation*}
$$

which is called 'practical stability' by Richtmyer and Morton (1967, p.146). Analysis of $(3.2 .23)$ and (3.2.24) for the two schemes will give the condition

$$
\begin{equation*}
(\lambda \Delta t / \Delta x)^{2} \leqslant 2 r \leqslant 1 \tag{3.2.26}
\end{equation*}
$$

for central scheme (3.2.10) and

$$
\begin{equation*}
r \leqslant 1 /(2+\lambda \Delta x) \tag{3.2.27}
\end{equation*}
$$

for upwinding scheme (3.2.11), (Morton, 1980). It can be noted that condition (3.2.26) is similar to (3.2.3).

Griffith et al (1980) explain this phenomena in a different manner, i.e. the confusion lies as no indication is obtainable about the magnitude of $E^{N}$ in (3.2.16) for finite $N$. In other words, when (3.2.17) is satisfied, it is possible for the error vector to grow considerably with $N$ before it eventually diminishes or is bounded.

### 3.3 SPECIAL EXPLICIT/IMPLICIT FINITE-DIFFERENCE TECHNIQUES

From equation (3.1.1), we observe that when $V$ is sufficiently large the differential equation is not dissimilar to the hyperbolic equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=-v \frac{\partial u}{\partial x} \tag{3.3.1}
\end{equation*}
$$

In this technique (Siemieninch and G1adwell, 1978) the convective term is treated by hyperbolic equation techniques. This builds a stronger dependence on $V$ into the difference replacements of (3.1.1), and is achieved by weighting the relative dependence of the diffusive and convective terms in the differential equation at the two-levels of the time discretization. A low order fully-implicit scheme to equation (3.1.1) with upwinding is derived

$$
\begin{align*}
u_{i, j+1} u_{i, j}= & r D\left\{u_{i+1, j+1}-2 u_{i, j+1}+u_{i-1, j+1}\right\} \\
& -\frac{v \Delta t}{\Delta x}\left\{u_{i, j+1}-u_{i-1, j+1}\right\}+\Delta t \tau_{i} \tag{3.3.2}
\end{align*}
$$

where

$$
\begin{equation*}
\tau_{i}=-\left.\frac{1}{2} v \Delta x \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}-\left.\frac{1}{2} \Delta t \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j} \tag{3.3.3}
\end{equation*}
$$

We observe that the presence of the second order spatial derivative term in the truncation error may cause problems of numerical diffusion for large $V \Delta x$. It is suggested that to eliminate this term from the truncation error by a suitable discretization, i.e., approximating the term $\left.\frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}$ (3.3.3) by central difference approximations at level $j$ and substituting in (3.3.2) will give rise to

$$
\begin{align*}
u_{i, j+1}-u_{i, j}= & r D\left\{u_{i+1, j+1}-2 u_{i, j+1}+u_{i-1, j+1}\right\}-\frac{V \Delta t}{\Delta x}\left\{u_{i, j+1}-u_{i-1, j+1}\right. \\
& \left.+\frac{1}{2}\left(u_{i+1, j}-2 u_{i, j}+u_{i-1, j}\right)\right\}+\Delta t \sigma_{i} \tag{3.3.4}
\end{align*}
$$

where

$$
\sigma_{i} \simeq \tau_{i}+\left.\frac{1}{2} v \Delta x \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}
$$

Therefore, the scheme (3.4.4) is an implicit scheme which is second order spatially accurate.

Another special technique (Siemieniuch and Gladwell, 1978) is to consider the explicit type of approximation as in (3.2.11) but with a $\theta$ weighted type of upwinding for the first spatial derivative in (3.1.1) at two time levels, i.e. to use the approximation

$$
\frac{\partial u}{\partial x} \simeq \frac{\theta}{\Delta x}\left\{u_{i, j+1}^{-u_{i-1, j+1}}\right\}+\frac{(1-\theta)}{\Delta x}\left\{u_{i, j}-u_{i-1, j}\right\}
$$

where $0 \leqslant \theta \leqslant 1$. Therefore, the approximation (3.2.11) will change to

$$
\begin{gather*}
(1+r \theta \beta) u_{i, j+1}-r \theta B u_{i-1, j+1}=r D u_{i+1, j}+\{1-r(2 D+(1-\theta) \beta)\} u_{i, j}+ \\
r\{D+(1-\theta) \beta\} u_{i-1, j}+\Delta t \tau_{i}, \tag{3.3.5}
\end{gather*}
$$

where $\beta=V \Delta x$ and

$$
\begin{align*}
\tau_{i} \simeq & -\frac{1}{2} \Delta t\left\{-\left.\frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j}-\left.2 v \theta \frac{\partial^{2} u}{\partial x \partial t}\right|_{i, j}+v \theta-\left.\frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j}\right\} \\
& \left.\frac{1}{2} v \Delta x \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j} \tag{3.3.6}
\end{align*}
$$

Scheme (3.3.5), for stability, requires,

$$
\begin{equation*}
0<r \leqslant \frac{2+\beta}{4+4 \beta(1-\theta)+\beta^{2}(1-2 \theta)} \tag{3.3.7}
\end{equation*}
$$

for $0 \leqslant \theta \leqslant \frac{1}{2}$, whilst for $\frac{1}{2}<\theta \leqslant 1$ and $0 \leqslant \beta \leqslant \frac{2}{2 \theta-1}$ requires

$$
\begin{equation*}
0<r \leqslant \frac{2+\beta}{4+4 \beta(1-\theta)-\beta^{2}(2 \theta-1)} \tag{3.3.8}
\end{equation*}
$$

and for $\beta \geqslant \frac{2}{2 \theta-1}, r$ is unrestricted.
Although the scheme (3.3.5) has greatly improved stability properties over the scheme (3.2.11), it again introduces 'numerical diffusion', which is quite significant for large $V$, in the truncation error term (3.3.6). To eliminate this effect a similar approach as before is used, i.e. rewriting equation (3.3.5) as

$$
\begin{align*}
& (1+r \theta \beta) u_{i, j+1}-r \theta \beta u_{i-1, j+1}=r D u_{i+1, j}+\left\{1-r(2 D \dot{+}(1-\theta) \beta\} u_{i, j}+\right. \\
& +r\{D+(1-\theta) \beta\} u_{i-1, j}-\left.\frac{1}{2} \Delta t \Delta x v \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}+\Delta t \tau_{i} \tag{3.3.9}
\end{align*}
$$

and approximating the second derivatives by central difference to give

$$
\begin{gather*}
(1+r \theta \beta) u_{i, j+1}^{-r \theta \beta u_{i-1, j+1}=r\left\{\left(D-\frac{1}{2} \beta\right) u_{i+1, j}+\left[D+\left(\frac{1}{2}-\theta\right) \beta\right] u_{i-1, j}\right\}} \\
+\{1-r(2 D-\beta \theta)\}_{u_{i, j}}+\Delta t \rho_{i} . \tag{3.3.10}
\end{gather*}
$$

The effect of the cross-derivative in (3.3.6) can also be avoided by approximating $\quad \frac{\partial^{2} u}{\partial x^{\partial} t} \simeq\left\{u_{i, j+1}+u_{i-1, j} \mathbf{u}_{i-1, j+1} \mathbf{u}_{i, j}\right\} / \Delta x \Delta t$
to yield,

$$
\begin{equation*}
u_{i, j+1}=\{1-r(2 D+B)\} u_{i, j}+r\left\{D u_{i+1, j}+(D+\beta) u_{i-1, j}\right\}+\Delta t n_{i} . \tag{3.3.12}
\end{equation*}
$$

$\tau_{i}$ and $\eta_{i}$ in (3.3.9) and (3.3.12) respectively are the remainder terms.
From here we see that the equations (3.3.5), (3.3.10) and (3.3.12) are all explicit. The equation (3.3.5) is a generalization of the explicit low-order upwinding scheme, whereas the equations (3.3.10) and (3.3.12) are special schemes which arise after the significant terms in the truncation error are taken into account.

### 3.4 THE LAX-WENDROFF TYPE SCHEME

This scheme emanated from the hyperbolic type of P.D. equation which is represented by

$$
\begin{equation*}
\frac{\partial u}{\partial t}+\frac{\partial F(u)}{\partial x}=0, \tag{3.4.1}
\end{equation*}
$$

where $F$ is a function of $u$.
The Lax-Wendroff approach started from a Taylor series in $t$ (Richtmyer and Morton, 1967), viz.

$$
u_{i, j+1}=u_{i, j}+\left.\Delta t \frac{\partial u}{\partial t}\right|_{i, j}+\left.\frac{\Delta t^{2}}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j}+\ldots
$$

The $t$ derivatives indicated are replaced by $x$-derivatives by means of the equation (3.4.1) and the further equation

$$
\frac{\partial^{2} u}{\partial t^{2}}=-\frac{\partial}{\partial t} \frac{\partial F(u)}{\partial x}=-\frac{\partial}{\partial x} \frac{\partial F}{\partial t}=-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u} \frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u} \frac{\partial F}{\partial x}\right)
$$

Therefore, if

$$
\begin{equation*}
\frac{\partial u}{\partial t}=-v \frac{\partial u}{\partial x} \tag{3.4.2}
\end{equation*}
$$

is approximated by

$$
\begin{equation*}
u_{i, j+1}=u_{i, j}-\frac{v}{2} \frac{\Delta t}{\Delta x}\left(u_{i+1, j}-u_{i-1, j}\right)+\frac{v^{2}}{2}\left(\frac{\Delta t}{\Delta x}\right)^{2}\left(u_{i+1, j}-2 u_{i, j}+u_{i-1, j}\right) \tag{3.4.3}
\end{equation*}
$$

and the remainder second derivative terms are approximated by the ordinary central difference approximation at level $j$, the approximation to equation (3.1.1) can now be written as

$$
\begin{equation*}
u_{i, j+1}=\left\{1-2 r D-(\beta r)^{2}\right\}_{i, j}+r\left\{\left(D+\frac{1}{2} \beta+\frac{1}{2} \beta^{2} r\right) u_{i-1, j}+\left(D-\frac{1}{2} \beta+\frac{1}{2} \beta^{2} r\right) u_{i+1, j}\right\}+\tau_{i} \tag{3.4.4}
\end{equation*}
$$

where $\tau_{i}=-\left.\frac{\beta^{2} r}{2} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}-\left.\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j}-\left.\frac{\Delta t^{2}}{6} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j}-\left.\frac{\beta \Delta x}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j}$

$$
\begin{equation*}
+\left.\left(D+\frac{\beta^{2} r}{2}\right) \frac{\Delta x^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}\right|_{i, j} \tag{3.4.5}
\end{equation*}
$$

The Fourier series analysis of stability shows that the scheme (3.4.3)
will be stable if (Siemieniuch and Gladwell, 1976),

$$
\begin{equation*}
0<r \leqslant \frac{1}{\left(1+\sqrt{\left.1+\beta^{2}\right)}\right.} \tag{3.4.6}
\end{equation*}
$$

with $\beta=V \Delta x$.

### 3.5 THE DIFFUSION-CONVECTION EQUATION IN TWO SPACE-DIMENSION

The diffusion-convection equation in two space dimension is given by

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}\right)-v_{1} \frac{\partial u}{\partial x}-v_{2} \frac{\partial u}{\partial y}, v_{1}, v_{2}>0 . \tag{3.5.1}
\end{equation*}
$$

In this discussion it is assumed that $\Delta x=\Delta y=\Delta x$.
One of the possible schemes is to use the explicit type with the central difference approximation for the spatial derivatives, namely,

$$
\begin{align*}
u_{i, j, k+1}= & u_{i, j, k}-\frac{1}{2} c_{x}\left(u_{i+1, j, k}-u_{i-1, j, k}\right)-\frac{1}{2} c_{y}\left(u_{i, j+1, k}-u_{i, j-1, k}\right) \\
& +s\left(u_{i+1, j, k}-2 u_{i, j, k}+u_{i-1, j, k}\right) \\
& +s\left(u_{i, j+1, k}-2 u_{i, j, k}+u_{i, j-1, k}\right), \tag{3.5.2}
\end{align*}
$$

The stability of equation (3.5.2) may be determined by the application of the Fourier series method of analysis, i.e. the error factor is defined
as

$$
\begin{equation*}
E_{i, j, k}=e^{\sqrt{-1}\left(i \beta_{x}+j \beta_{y}\right) \Delta s_{\xi} k} \tag{3.5.3}
\end{equation*}
$$

which when after substitution in (3.5.2) will give

$$
\begin{equation*}
\xi=1-2 s\left\{2-\cos \beta_{x} \Delta x-\cos \beta_{y} \Delta x\right\}-\sqrt{-1}\left(C_{x} \sin \beta_{x} \Delta s+C_{y} \sin \beta_{y} \Delta s\right) \tag{3.5.4}
\end{equation*}
$$

The value of $|\xi|$ is therefore given by the equation

$$
\begin{align*}
|\xi|^{2}= & 1-4 s\left\{2-\cos \beta_{x} \Delta s-\cos \beta_{y} \Delta s\right\}+4 s^{2}\left\{2-\cos \beta_{x} \Delta s-\cos \beta_{y} \Delta s\right\} \\
& +C_{x}^{2} \sin ^{2} \beta_{x} \Delta s+2 C_{x} C_{y} \sin \beta_{x} \Delta s \sin \beta_{y} \Delta s+C_{y}^{2} \sin ^{2} \beta_{y} \Delta s . \tag{3.5.5}
\end{align*}
$$

If we let $C_{x} \leqslant 2 \mathrm{~s}, \mathrm{C}_{\mathrm{y}} \leqslant 2 \mathrm{~s}$, (3.5.5) becomes

$$
\begin{align*}
&|\xi|^{2} \leqslant 1-4 s\left\{2-\cos \beta_{x} \Delta s-\cos \beta_{y} \Delta s\right\}+4 s^{2}\left\{2-\cos \beta_{x} \Delta s-\cos \beta_{y} \Delta s\right\}^{2} \\
&+4 s^{2}\left\{\sin ^{2} \beta_{x} \Delta s+2 \sin \beta_{x} \Delta s \sin \beta_{y} \Delta s+\sin ^{2} \beta_{y} \Delta s\right\} \\
& \leqslant 1-4 s(1-4 s)\left(2-\cos \beta_{x} \Delta s-\cos \beta_{y} \Delta s\right) \tag{3.5.6}
\end{align*}
$$

From (3.5.6) it is clear that $|\xi| \leqslant 1$ so long as

$$
\begin{equation*}
s \leq \frac{1}{4} \tag{3.5.6}
\end{equation*}
$$

If $C_{x}=C_{y}=C$ then we require,

$$
\begin{equation*}
C \leqslant 2 s \leqslant \frac{1}{2} \tag{3.5.6}
\end{equation*}
$$

Similar to the one-dimensional case, it is also usual to use upwinding difference approximations to the spatial derivatives in the convective terms to give

$$
\begin{align*}
u_{i, j, k+1}= & u_{i, j, k}-C_{x}\left(u_{i, j, k}-u_{i-1, j, k}\right)-C_{y}\left(u_{i, j, k}-u_{i, j-1, k}\right) \\
& +s\left(u_{i+1, j, k}-2 u_{i, j, k}+u_{i-1, j}\right) \\
& +s\left(u_{i, j+1, k^{-2 u_{i}}}\right)  \tag{3.5.7}\\
& \left.=k^{+u_{i, j}-1, k}\right)
\end{align*}
$$

A similar approach of analysis of stability will give the condition,

$$
\begin{equation*}
4 s+C_{x}+C_{y} \leqslant 1 \quad \text { for stability } \tag{3.5.8}
\end{equation*}
$$

If $C_{x}=C_{y}=C$ then,

$$
\begin{equation*}
(2 s+C) \leqslant \frac{1}{2} \tag{3.5.9}
\end{equation*}
$$

Conditions (3.5.6) and (3.5.9) if compared to the conditions for the onedimensional case i.e. (3.2.26) and (3.2.27) respectively, can be seen to be twice as restrictive as in the one-dimensional case.

A Taylor series expansion of each term of equation (3.5.7) about the ( $i, j, k$ ) grid point will show that the method is consistent to the equation (3.5.1) with the truncation error term

$$
\alpha_{x}^{\prime} \frac{\partial^{2} u}{\partial x^{2}}+\alpha_{y}^{\prime} \frac{\partial^{2} u}{\partial y^{2}}+0\left(\Delta t+(\Delta x)^{2}\right)
$$

where

$$
\left.\begin{array}{l}
\alpha_{x}^{\prime}=\frac{1}{2} v_{1} \Delta s\left(1-C_{x}\right)  \tag{3.5.11}\\
\alpha_{y}^{\prime}=\frac{1}{2} \nu_{2} \Delta s\left(1-C_{y}\right)
\end{array}\right\}
$$

As in the one-dimensional case, the accuracy of this method is strongly influenced by the false (numerical) diffusion which is introduced by $\alpha_{x}^{\prime}$ and $\alpha_{y}^{\prime}$. Unless $\alpha_{x}^{\prime} \ll \alpha$ and $\alpha_{y}^{\prime} \ll \alpha$, the presence of this artificial diffusion may produce an approximate solution $u$ which differs greatly from the true solution (B.J. Noye, 1978, p.69).

On the other hand, either fully or Crank-Nicolson-type implicit methods
are also possible. The Crank-Nicolson type is given by

$$
\begin{equation*}
u_{i, j, k+1} u_{i, j, k}=\Delta t\left(p_{i, j, k+1}{ }^{+p_{i, j}, k}\right) \tag{3.5.12}
\end{equation*}
$$

where

$$
\begin{aligned}
p_{i, j, k}= & -\frac{1}{2} v_{1} \frac{u_{i+1, j, k^{-u_{i-1, j, k}}}^{2 \Delta x}-\frac{1}{2} v_{2} \frac{u_{i, j+1, k^{-u_{i, j-1, k}}}^{2 \Delta x}}{}}{}+\frac{1}{2} \alpha \frac{u_{i+1, j, k^{-2 u_{i, j, k}}+u_{i-1, j, k}}^{(\Delta x)^{2}}}{(\Delta y)^{2}} \\
& +\frac{1}{2} \alpha \frac{u_{i, j+1, k^{-2 u_{i, j, k}}{ }^{+u_{i, j-1, k}}}^{2}}{}
\end{aligned}
$$

with the stability guaranteed if

$$
\begin{equation*}
2(1+2 s)>\left(s+\frac{1}{2} C_{y}\right)+\left(s+\frac{1}{2} C_{x}\right)+\left|s-\frac{1}{2} C_{x}\right|+\left|s-\frac{1}{2} C_{y}\right| . \tag{3.5.13}
\end{equation*}
$$

(B.J. Noye, 1978, p.75).

As the implicit type method involves the solution of a system of equations which is usually large and sparse, the iterative methods of solution are recommended.

### 3.6 THE FOURTH-ORDER PARABOLIC EQUATION

The governing equation of the vibration of a thin beam which is clamped at its ends and set into vibration, is well known to be a fourthorder parabolic equation whose non-dimensional form is given by

$$
\begin{equation*}
\frac{\partial^{2} y}{\partial t^{2}}+\frac{\partial^{4} y}{\partial x^{4}}=0, \quad 0 \leqslant x \leqslant 1, t>0 \tag{3.6.1}
\end{equation*}
$$

where $y=y(x, t)$ denotes the displacement from the equilibrium position at a distance $x$ along the beam from one end at time $t$. The equation (3.6.1) is usually associated with the initial conditions
and

$$
\left.\begin{array}{l}
y(x, 0)=g_{0}(x)  \tag{3.6.2}\\
\frac{\partial y}{\partial t}(x, 0)=g_{1}(x)
\end{array}\right\} \quad \text { for } 0 \leqslant x \leqslant 1
$$

and with boundary conditions at the end points $x=0$ and 1 of the form,
and

$$
\begin{align*}
& y(0, t)=f_{0}(t), \quad y(1, t)=f_{1}(t) \\
& \frac{\partial^{2} y}{\partial x^{2}}(0, t)=p_{0}(t), \frac{\partial^{2} y}{\partial x^{2}}(1, t)=p_{1}(t) . \tag{3.6.3}
\end{align*}
$$

Various finite-difference replacements have been applied to solve the equations (3.6.1)-(3.6.3). Among them are:
1 .

$$
\begin{equation*}
\frac{\delta_{t}^{2} y_{i, j}}{(\Delta t)^{2}}+\frac{\delta_{x}^{4} y_{i, j}}{(\Delta x)^{4}}=0 \tag{3.6.4}
\end{equation*}
$$

due to Collatz (Saulyev, 1964), Fig. (3.6.1a),
2. $y_{i, j+1}=y_{i, j-1}+\frac{1}{8}\left(y_{i-3, j}+y_{i+3, j}\right)-\frac{9}{8}\left(y_{i-1, j}+y_{i+1, j}\right), \frac{\Delta t}{\Delta x}=\frac{\sqrt{3}}{2}$
due to Nishimura (1954), Fig. (3.6.1b),
3. $\frac{\delta_{t}^{2} y_{i, j}}{(\Delta t)^{2}}+\frac{\delta_{x}^{4} y_{i, j+1}}{2(\Delta x)^{4}}+\frac{\delta_{x}^{4} y_{i, j-1}}{2(\Delta x)^{4}}=0$
due to Crandall (1954), Fig. (3.6.1c),
4. $\quad \frac{\delta_{t}^{2} y_{i, j}}{(\Delta t)^{2}}+\frac{\delta_{x}^{4} y_{i, j+1}}{4(\Delta x)^{4}}+\frac{\delta_{x}^{4} y_{i, j}}{2(\Delta x)^{4}}+\frac{\delta_{x}^{4} y_{i, j-1}}{4(\Delta x)^{4}}=0$
due to Conte, Royster (1956), Fig. (3.6.1d),
5. $\quad y_{i-1, j+1}-\left(2+\frac{1}{r^{2}}\right) y_{i, j+1}+y_{i+1, j+1}$

$$
\begin{align*}
& =y_{i-2, j}^{-2 y_{i-1, j}}+2\left(1-\frac{1}{r^{2}}\right) y_{i, j}-2 y_{i+1, j}+y_{i+2, j} \\
& -y_{i-1, j-1}+\left(2+\frac{1}{2}\right) y_{i, j-1} y_{i+1, j-1} \tag{3.6.8}
\end{align*}
$$

due to Conte (1957), Fig. (3.6.1e).
6. $\left(1+4 r^{2}\right) y_{i, j+2}-8 r^{2}\left(y_{i-1, j+1}+y_{i+1, j+1}\right)+4 r^{2}\left(y_{i-2, j}+y_{i+2, j}\right)+2\left(8 r^{2}-1\right) y_{i, j}$

$$
\begin{equation*}
-8 r^{2}\left(y_{i-1, j-1}+y_{i+1, j-1}\right)+\left(1+4 r^{2}\right) y_{i, j-2}=0 \tag{3.6.9}
\end{equation*}
$$

due to Albrecht (Saul'yev, 1964), Fig. (3.6.1f).
The equations (3.6.4), (3.6.5) and (3.6.9) even though explicit suffer from severe restrictions on stability (i.e. $r \leqslant \frac{1}{2}$ ), applicable only for one particular $r$ and needs five time-levels instead of the standard three timelevel respectively. The implicit equations (3.6.6) and (3.6.7) are absolutely stable, but the calculation of each layer requires the solution of a system of algebraic equations with a quindiagonal matrix. The implicit equation (3.6.8) is very attractive as it involves only the system of tridiagonal equations.

A quite different approach (Richtmyer, 1967, p.271) is by introducing two new variables $u$ and $v$ such that

$$
\begin{equation*}
u=\frac{\partial y}{\partial t} \quad, \quad v=\frac{\partial^{2} y}{\partial x^{2}} \tag{3.6.10}
\end{equation*}
$$

The equation (3.6.1) can now be rewritten as a system of two second order parabolic equations:

$$
\left.\begin{array}{l}
\frac{\partial u}{\partial t}=-\frac{\partial^{2} v}{\partial x^{2}}  \tag{3.6.11}\\
\frac{\partial v}{\partial t}=\frac{\partial^{u}}{\partial x^{2}}
\end{array}\right\}
$$

or as a second order system
where

$$
\begin{aligned}
& \frac{\partial \underline{\omega}}{\partial t}=A \frac{2}{\partial \underline{\omega}} \\
& \partial x^{2} \\
& \underline{\omega}=\left[\begin{array}{l}
u \\
v
\end{array}\right] \quad \text { and } \quad A=\left[\begin{array}{lr}
0 & -1 \\
1 & 0
\end{array}\right] .
\end{aligned}
$$


jth level

(b)

(c)

(d)

(e)

(f)

FIGURE (3.6.1)

Now the system (3.6.11) may be solved by different techniques including the Du Fort-Frankel method (Evans, D.J., 1965) and the explicitimplicit methods (Fairweather and Gourlay, 1966) and an hopscotch approach (Danaee, A., 1980).

### 3.7 NON-LINEAR EQUATIONS

Until now, we have considered only linear parabolic equations, but many of the mathematical formulations of natural processes are non-linear. For example the equation of heat conduction in a bar is given by

$$
\begin{equation*}
\mathrm{pc} \frac{\partial \mathrm{u}}{\partial \mathrm{t}}=\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{k} \frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right) \tag{3.7.1}
\end{equation*}
$$

where $p, c$ and $k$ may depend on $u$ (Albasiny, 1956). Also, a boundary layer problem concerning the flow, near a semi-infinite flat plate, of a stream of incompressible viscous liquid with negative pressure gradient is given by a non-linear equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\sqrt{\left[(1-t)^{2}-u\right]} \frac{\partial^{2} u}{\partial x^{2}}, \tag{3.7.2}
\end{equation*}
$$

(Mitchell and Thomson, 1958).
A general definition of a non-linear parabolic p.d.e. is given by Friedman (1964, p.78) who states that

$$
\begin{equation*}
F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^{2} u}{\partial x^{2}}\right)=0 \tag{3.7.3}
\end{equation*}
$$

is a non-linear parabolic equation in some domain $D$ with respect to a solution $u(x, t)$ if
and

$$
b(x, t) \equiv \frac{\partial F}{\partial(\partial u / \partial t)} \text { is strictly negative in } D
$$

$$
c(x, t) \equiv \frac{\partial F}{\partial\left(\partial^{2} u / \partial x^{2}\right)} \text { is strictly positive in } D
$$

the argument of $F$ being

$$
\left(x, t, u(x, t), \frac{\partial u(x, t)}{\partial x}, \frac{\partial u(x, t)}{\partial t}, \frac{\partial^{2} u(x, t)}{\partial x^{2}}\right)
$$

Based on this definition, the general non-linear parabolic equation can also be written either in the form

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}=f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \tag{3.7.4}
\end{equation*}
$$

where $\frac{\partial f}{\partial(\partial u / \partial t)} \geqslant b>0$ for real $b$, or in the form,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=g\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial^{2} u}{\partial x^{2}}\right) \tag{3.7.5}
\end{equation*}
$$

where $\frac{\partial g}{\partial\left(\partial^{2} u / \partial x^{2}\right)} \geqslant a>0$ for real $a$.
Condition (3.7.4) and (3.7.5) are important as they ensure that each representation is properly posed, i.e. physically stable.

## Finite Difference Approximations

There is a certain amount of work which has been devoted to the solution of parabolic non-linear equations by the method of finite difference approximation. This is due to the fact that many of the methods and proofs, based on linear equations with constant coefficient carry over directly to non-linear equations. However, in this case both the numerical process and the analysis of stability and convergence becomes more complicated (Ames, 1977, p.73). In practice, usually a very specialised form of non-linear equation is considered rather than the more general form (3.7.3). For example Richtmyer and Morton (1967, p.201) considered the non-linear problem of the form

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u^{n}}{\partial \mathbf{x}^{2}} \tag{3.7.6}
\end{equation*}
$$

with $n=5$ and Douglas (1956) considered the quasi-linear parabolic equation

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}=F(x, t, u) \frac{\partial u}{\partial t}+G(x, t, u), F \geqslant a>0 . \tag{3.7.7}
\end{equation*}
$$

In a way, this specialised approach probably relaxes the complications or difficulties which may arise in the analysis of convergence and stability.

For non-linear problems, stability depends not only on the form of the finite difference system but also generally upon the solution being obtained. The system may be stable for some values of $t$ and not for others. In practice, in the case of conditionally stable difference approximations
it is necessary to monitor the stability by checking the stability condition and to alter $\Delta t$ in order to restore the stability.

Here we will investigate briefly some explicit and implicit methods and the convergence and stability requirements for them. The general $\theta$-weighted formula corresponding to (3.7.5) is

$$
\begin{align*}
u_{i, j+1}= & u_{i, j}+\Delta t . g\left(i \Delta x,[j+(1-\theta)] \Delta t,\left[\theta u_{i, j}+(1-\theta) u_{i, j+1}\right]\right. \\
& \theta\left(\frac{\left\{u_{i+1, j}-u_{i-1, j}\right\}}{2 \Delta x}, \frac{\delta_{x}^{2} u_{i, j}}{(\Delta x)^{2}}\right), \\
& \left.(1-\theta)\left(\frac{\left\{u_{i+1, j+1}-u_{i-1, j+1}\right\}}{2 \Delta x}, \frac{\delta_{x}^{2} u_{i, j+1}}{(\Delta x)^{2}}\right)\right) \tag{3.7.8}
\end{align*}
$$

For $\theta=1$ equation (3.7.8) becomes an explicit scheme, whilst for $\theta=0$ it becomes fully implicit and $\theta=\frac{1}{2}$ is the Crank-Nicolson scheme.

Alternatively, for equation (3.7.4), the $\theta$-formulae is

$$
\begin{align*}
& \frac{1}{(\Delta x)^{2}}\left[\theta \Delta_{x}^{2} u_{i, j+1}+(1-\theta) \Delta_{x}^{2} u_{i, j}\right]=f\left(i \Delta x,\left[j+(1-\theta) \Delta t, \theta u_{i, j}+(1-\theta) u_{i, j+1}\right.\right. \\
&  \tag{3.7.9}\\
& \left.\left[\theta\left(\frac{u_{i+1, j}-u_{i-1, j}}{2 \Delta x}\right)+(1-\theta)\left(\frac{u_{i+1, j+1}-u_{i-1, j+1}}{2 \Delta x}\right)\right], \frac{1}{\Delta t}\left[u_{i, j+1}-u_{i, j}\right]\right) .
\end{align*}
$$

It is proved in Ames (1977, p.74) that equation (3.7.8) is convergent and stable for $\theta=1$ if
and

$$
\left.\begin{array}{l}
\Delta x \leqslant 2 a / c  \tag{3.7.10}\\
0<r \leqslant \frac{1-c \Delta t}{2 c}
\end{array}\right\},
$$

where

$$
\frac{\partial g}{\partial\left(\partial^{2} u / \partial x^{2}\right)} \geqslant a>0
$$

and

$$
\left|\frac{\partial g}{\partial u}\right|+\left|\frac{\partial g}{\partial(\partial u / \partial x)}\right|+\frac{\partial g}{\partial\left(\partial^{2} u / \partial x^{2}\right)} \leqslant c .
$$

Rose (1956) demonstrates the convergence of equation (3.7.9) under the assumption that

$$
\begin{equation*}
r=\frac{\Delta t}{\Delta x^{2}} \leqslant \frac{b}{2 \theta}, \tag{3.7.11}
\end{equation*}
$$

where $\partial f / \partial(\partial u / \partial t) \geqslant b>0$. Both equations have an error term of $0\left(\Delta t+(\Delta x)^{2}\right)$.

The three time-level method of Du-Fort Frankel (1953) for (3.7.5)
is given by

$$
\begin{array}{r}
\frac{\delta_{t} u_{i, j}}{2 \Delta t}=g\left(i \Delta x, j \Delta t, \frac{\left\{u_{i, j+1}+u_{i, j-1}\right\}}{2}, \frac{\delta_{x} u_{i, j}}{2 \Delta x},\right. \\
\left.\frac{\left\{u_{i+1, j^{-u_{i, j-1}}{ }^{-u_{i, j+1}}+u_{i-1, j}}\right\}}{(\Delta x)^{2}}\right) \tag{3.7.12}
\end{array}
$$

Another three time-level scheme was given by Lees (1966) who considered the self-adjoint form

$$
\begin{equation*}
b(u) \frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a(u) \frac{\partial u}{\partial x}\right), \quad(a(u)>0, b(u)>0) \tag{3.7.13}
\end{equation*}
$$

and approximated by

$$
\begin{equation*}
b\left(u_{i, j}\right) \frac{\delta_{t}\left(u_{i, j}\right)}{2 \Delta t}=\frac{1}{(\Delta x)^{2}} \delta_{x}\left(a\left(u_{i, j}\right) \delta_{x}^{2}\right) u_{i, j} \tag{3.7.14}
\end{equation*}
$$

to give

$$
\begin{align*}
& b\left(u_{i, j}\right)\left(u_{i, j+1} u_{i, j-1}\right)=2 r\left[a\left(u_{i+\frac{1}{2}, j}\right)\left(u_{i+1, j}-u_{i, j}\right)-\right. \\
&\left.-a\left(u_{i-\frac{1}{2}, j}\right)\left(u_{i, j}-u_{i-1, j}\right)\right] \tag{3.7.15}
\end{align*}
$$

In order to avoid instability, $u_{i+1, j}, u_{i, j}$ and $u_{i-1, j}$ are replaced by

$$
\begin{aligned}
& \frac{1}{3}\left(u_{i+1, j+1}+u_{i+1, j}+u_{i+1, j-1}\right) \\
& \frac{1}{3}\left(u_{i, j+1}+u_{i, j}+u_{i, j-1}\right) \\
& \frac{1}{3}\left(u_{i-1, j+1}+u_{i-1, j}+u_{i-1, j-1}\right)
\end{aligned}
$$

respectively. Also, $a\left(u_{i+\frac{1}{2}, j}\right)$ and $a\left(u_{i-\frac{1}{2}, j}\right)$ are replaced by $a\left(\frac{u_{i+1, j}+u_{i, j}}{2}\right)$ and $a\left(\frac{u_{i, j}{ }^{+}{ }_{i-1, j}}{2}\right)$ respectively. Therefore equation (3.7.15) leads to a three time-level equation, i.e.,

$$
\begin{align*}
& b\left(u_{i, j}\right)\left(u_{i, j+1}-u_{i, j}\right)=\frac{2}{3} r\left[a _ { 1 } \left\{\left(u_{i+1, j+1}^{-u_{i, j+1}}\right)+\left(u_{i+1, j^{-u_{i, j}}}\right)\right.\right. \\
& \left.\quad+\left(u_{i+1, j-1}-u_{i, j-1}\right)\right\}-a_{2}\left\{\left(u_{i, j+1}-u_{i-1, j+1}\right)+\left(u_{i, j}-u_{i-1, j}\right)\right. \\
&  \tag{3.7.16}\\
& \left.\left.\quad+\left(u_{i, j-1}-u_{i-1, j-1}\right)\right\}\right]
\end{align*}
$$

where $a_{1}=a\left(\frac{u_{i+1, j^{+}} u_{i, j}}{2}\right) \quad$ and $\quad a_{2}=a\left(\frac{u_{i, j^{+}} u_{i-1, j}}{2}\right)$.

Equations (3.7.16) was proved to be convergent with the error term of order $\left((\Delta x)^{2}+(\Delta t)^{2}\right)$. Equation (3.7.6) is included in the class of equation (3.7.13) as it is written as

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(n u^{n-1} \frac{\partial u}{\partial x}\right) \tag{3.7.17}
\end{equation*}
$$

There are two main difficulties in regard to the above approximation schemes. We observe that if we use the explicit method for a problem of type (3.7.4) and (3.7.5), i.e. in the case $\theta=1$, they are quite easy to solve, but suffers from disadvantages on the time step chosen to maintain stability such as condition (3.7.11). This limitation can be avoided by using an implicit method, i.e. $\theta=\frac{1}{2}$ or $\theta=1$. But, in this case we normally end up with a non-linear system of equations to solve, and depending on $f$ or $g$ in (3.7.4) or (3.7.5) respectively, the algebraic problem of finding the solution may become difficult and normally one needs to use an iteration technique to evaluate the solution. However, in some cases a linearization technique is also possible.

One approach which is called the linearization of the difference scheme (Richtmyer, 1967, p.201) is normally used in the case of problems containing terms such as $u^{n}$ (such as equation (3.7.6)) where we approximate

$$
\begin{equation*}
\left(u_{i, j+1}\right)^{n} \simeq\left(u_{i, j}\right)^{n}+n\left(u_{i, j}\right)^{n-1} v_{i, j+1} \tag{3.7.18}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{i, j+1}=u_{i, j+1} u_{i, j} \tag{3.7.19}
\end{equation*}
$$

After this the remaining system is linear in $v_{i, j+1}$ and $u_{i, j+1}$ is easily obtained from (3.7.19).

To illustrate the non-linearity of the system of equations from the implicit method more clearly, equation (3.7.4) is written in quasilinear form as $\quad \frac{\partial^{2} u}{\partial x^{2}}+f(x, t, u) \frac{\partial u}{\partial x}+g(x, t, u)=p(x, t, u) \frac{\partial u}{\partial t}$.

When the Crank-Nicolson concept is applied, i.e. $\theta=\frac{1}{2}$ for (3.7.9), the finite difference algorithm assumes the form

$$
\begin{gather*}
\frac{1}{2(\Delta x)^{2}} \delta_{x}^{2}\left[u_{i, j+1}+u_{i, j}\right]+\frac{1}{4 \Delta x} f\left[i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{1}{2}\left(u_{i, j+1}+u_{i, j}\right)\right] \delta_{x}\left(u_{i, j+1}+u_{i, j}\right) \\
+g\left[i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{1}{2}\left(u_{i, j+1}+u_{i, j}\right)\right]= \\
p\left[i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{1}{2}\left(u_{i, j+1}+u_{i, j}\right)\right]  \tag{3.7.21}\\
\frac{\left(u_{i, j+1}-u_{i, j}\right)}{\Delta t}
\end{gather*}
$$

which after some simplification leads to the non-linear equations

$$
\begin{array}{r}
-\left(\frac{r}{2}-\frac{\Delta t}{4 \Delta x} f_{i, j+1}\right) u_{i-1, j+1}+\left(p_{i, j+1}+r\right) u_{i, j+1}-\left(\frac{r}{2}+\frac{\Delta t}{4 \Delta x} f_{i, j+1}\right) u_{i+1, j+1} \\
-\frac{\Delta t}{4 \Delta x}\left(u_{i+1, j}-u_{i-1, j}\right) f_{i, j+1}-u_{i, j} p_{i, j+1}-\Delta t g_{i, j+1}= \\
\frac{r}{2}\left(u_{i-1, j}-2 u_{i, j}+\right. \\
\left.+u_{i+1, j}\right),
\end{array}
$$

where, $\quad f_{i, j+1}=f\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{1}{2}\left(u_{i, j+1}+u_{i, j}\right)\right)$

$$
\begin{equation*}
g_{i, j+1}=g\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{1}{2}\left(u_{i, j+1}+u_{i, j}\right)\right) \tag{3.7.23}
\end{equation*}
$$

and

$$
p_{i, j+1}=p\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{1}{2}\left(u_{i, j+1}+u_{i, j}\right)\right), i=1,2, \ldots, N-1
$$

If we assume that the initial condition is given and the boundary conditions are homogeneous Dirichlet form, the non-linear equations (3.7.22) can be written in matrix form as,

$$
\left[\begin{array}{c}
u_{1, j+1} \\
u_{2, j+1} \\
\vdots \\
\vdots \\
\vdots \\
u_{N-2, j+1} \\
u_{N-1, j+1}
\end{array}\right]-\left[\begin{array}{cc}
\frac{\Delta t}{4 \Delta x}\left(u_{2, j}-u_{0, j}\right) f_{1, j+1}+u_{1, j} p_{1, j+1}+\Delta t g_{1, j+1} \\
\frac{\Delta t}{4 \Delta x}\left(u_{3, j}-u_{1, j}\right) f_{2, j+1}+u_{2, j} p_{2, j+1}+\Delta t g_{2, j+1} \\
1 & 1 \\
\vdots & \vdots \\
\frac{\Delta t}{4 \Delta x}\left(u_{N-1, j} u_{N-3, j}\right) f_{N-3, j+1}+u_{N-2, j} p_{N-2, j+1}+\Delta t g_{N-2, j+1} \\
\frac{\Delta t}{4 \Delta x}\left(u_{N, j}-u_{N-3, j}\right) f_{N-1, j+1}+u_{N-1, j} p_{N-1, j+1}+\Delta t g_{N-1, j+1}
\end{array}\right]=
$$

The system (3.7.24) is normally solved by an iterative method which will be discussed later in this chapter.

One special point we can observe from the equations (3.7.24) is that it can be made linear if the elements of the coefficient matrix in the lefthand side of the equations are independent of $u_{i, j+1}$. In doing so, we use $f_{j}, g_{j}$ and $p_{j}$ instead of $f_{j+1}, g_{j+1}$ and $p_{j+1}$ respectively, where,

$$
\begin{aligned}
& f_{i, j}=f\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, u_{i, j}\right) \\
& g_{i, j}=g\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, u_{i, j}\right) \\
& p_{i, j}=p\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, u_{i, j}\right)
\end{aligned}
$$

which results in a system of linear equations which can be more easily solved. The convergence proof remains valid under this modification, (Douglas, 1956).

An example of a non-linear problem in the class of can be approximated linearly as above is given by the we equation,

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}=\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x} \tag{3.7.25}
\end{equation*}
$$

which was derived as a mathematical model of turbulence, (Burger, 1949).
Another technique that is worthy of mention and extensively used in the numerical solution of ordinary differential equations (see Lambert, 1978 or Fox 1962) and extendible to the numerical solution of non-linear p.d.e.'s is the predictor-corrector method. As an example assume the linear version of (3.7.24) as the predictor formula and the non-linear one of (3.7.24) as the corrector with the solutions from the predictor becoming input to the corrector formula.

### 3.8 ITERATIVE SOLUTION TECHNIQUES

With the application of finite-difference methods or finite element methods to the solution of p.d.e's one might expect one of the following fundamental situations:

1. In most non-linear problems, either explicit or implicit schemes result in a non-linear or a system of non-linear equations to solve.
2. In the case of a linear p.d.e., an implicit finite-difference technique usually leads to a large sparse system of linear equations where highly efficient algorithms are provided by iterative techniques. We shall study both situations in this chapter. Basically in both situations some iterative methods are necessary. Any iterative method consists of three parts as follows:
(1) an initial estimate (guess) of the solution;
(2) a formula for updating the approximate solution;
(3) a 'fail-safe' procedure for stopping the updating process.

In this section, we will consider the first situation, i.e. iterative methods for determining the zeros of the equation,

$$
\begin{equation*}
\underline{f}(\underline{x})=0, \tag{3.8.1}
\end{equation*}
$$

where $\underline{f}$ and $\underline{x}$ are vectors of the same dimension $N$. For $N=1$, we have a single equation and for $N>1$, (3.8.1) a system of $N$ equations.

Functional Iteration: To discuss this method, we assume $N=1$. In this method, equation (3.8.1) is written as

$$
\begin{equation*}
x_{m+1}=q\left(x_{m}\right), \quad m=0,1, \ldots, \tag{3.8.2}
\end{equation*}
$$

where the equation $\quad x=g(x)$,
has the same solution as equation (3.8.1), that is $x=\alpha$ for instance. Normally, for any $f$, there are a few possible forms for $g$ to be defined
to form the iteration (3.8.2), but not all of those will converge to the root $\alpha$. The following theorem, i.e. the contraction mapping theorem is necessary in order to determine the function $g$ that will converge.

## Theorem 3.1

Let $g(x)$ satisfy the Lipschitz condition

$$
\begin{equation*}
\left|g(x)-g\left(x^{\prime}\right)\right| \leqslant \lambda\left|x-x^{\prime}\right| \tag{3.8.4}
\end{equation*}
$$

for all values $x, x^{\prime}$ in the closed interval $I=\left[x_{0}-p, x_{0}+p\right]$, where the Lipschitz constant $\lambda$, satisfies $0 \leqslant \lambda \leqslant 1$, and let the initial estimate $x_{0}$ be such that,

$$
\begin{equation*}
\left|x_{0}-g\left(x_{0}\right)\right| \leqslant(1-\lambda) f \tag{3.8.5}
\end{equation*}
$$

Then,
(i) all the iterates $x_{m}$, defined by (2.16.3), lie within the interval I, i.e.

$$
\begin{equation*}
x_{0}-p \leqslant x_{m} \leq x_{0}+p \tag{3.8.6}
\end{equation*}
$$

(ii) (Existence) the iterates $x_{i}$ converge to some point, say

$$
\left.\lim _{m \rightarrow \infty} x_{m}=\alpha \quad \text { (in fact }\left|x_{m}-\alpha\right| \leqslant \lambda^{m} p\right)
$$

which is a root of (3.8.3), and
(iii) (Uniqueness) $\alpha$ is the only root in $\left[x_{0}-p, x_{0}+p\right]$.

For proof see E. Is aacson and H.B. Keller, 1966, p. 86.

## Corollary

If $\left|g^{\prime}(x)\right| \leqslant \lambda<1$ for $\left|x-x_{0}\right| \leqslant p$ and (3.8.5) is satisfied, then the conclusion of Theorem 3.1 is valid.

## Proof

The mean value theorem implies $g\left(x_{1}\right)-g\left(x_{2}\right)=g^{\prime}(\xi)\left(x_{1}-x_{2}\right)$, where $\lambda$ may serve as the Lipschitz constant in (3.8.4).

## Convergence Criterion

## Definition 3.8.1

Let $x_{0}, x_{1}, \ldots$ be a sequence which converges to $\alpha$, and $\varepsilon_{m}=x_{m}-\alpha$.
If there exists a number $\sigma$ and a constant $\mathrm{c} \neq 0$ such that,

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \frac{\left|\varepsilon_{m+1}\right|}{\left|\varepsilon_{m}\right| \sigma}=c \tag{3.8.7}
\end{equation*}
$$

the $\sigma$ is called the order of convergence of the sequence and $c$ the asymptotic error constant. For $\sigma=1,2,3$ the convergence is said to be 1inear, quadratic and cubic respectively.

## Newton-Raphson Method

The method starts by expanding by Taylor's series $f(x)=0$ about $x_{m}$ to give $f(x)=f\left(x_{m}\right)+\left(x-x_{m}\right) f^{\prime}\left(x_{m}\right)+\frac{\left(x-x_{m}\right)^{2}}{2!} f^{\prime \prime}(\xi)$,
$\xi \in\left(x_{0}, x\right)$ by the mean value theorem. By neglecting the quadratic term, and rewriting the equation in iterative form, i.e. $x=x_{m+1}$, we have

$$
\begin{equation*}
f^{\prime}\left(x_{m}\right)\left(x_{m+1}-x_{m}\right)+f\left(x_{m}\right) \simeq f(x)=0, m=0,1, \ldots \tag{3.8.9}
\end{equation*}
$$

Thus, the Newton-Raphson method is defined by

$$
\begin{align*}
x_{m+1} & =x_{m}-\frac{f\left(x_{m}\right)}{f^{\prime}\left(x_{m}\right)} \\
\text { i.e. } \quad x_{m+1}=x_{m} & +\delta_{m}, m=0,1,2, \ldots \\
\delta_{m} & =-\frac{f\left(x_{m}\right)}{f^{\prime}\left(x_{m}\right)} . \tag{3.8.10}
\end{align*}
$$

Comparison with (3.8.2) gives the iteration function

$$
\begin{equation*}
g\left(x_{m}\right)=x_{m}-\frac{f\left(x_{m}\right)}{f^{\prime}\left(x_{m}\right)} \tag{3.8.11}
\end{equation*}
$$

For the convergence of the method, we define

$$
\varepsilon_{\mathrm{m}}=\mathrm{x}_{\mathrm{m}}-\mathrm{x}
$$

and from (3.8.8)


FIGURE (3.8.1)

$$
x-x_{m+1}=\frac{f\left(x_{m}\right)}{f^{\prime}\left(x_{m}\right)}+\left(x-x_{m}\right)=-\frac{1}{2} \frac{\left(x-x_{m}\right)^{2} f^{\prime \prime}(\xi)}{f^{\prime}\left(x_{m}\right)}
$$

Thus, we have

$$
\begin{equation*}
\varepsilon_{m+1}=\frac{1}{2} \varepsilon_{m}^{2} \frac{f^{\prime \prime}(\xi)}{f^{\prime}\left(x_{m}\right)} \tag{3.8.12}
\end{equation*}
$$

and as $\mathrm{X}_{\mathrm{m}} \rightarrow \mathrm{x}$,

$$
\frac{\varepsilon_{\mathrm{m}+1}}{\varepsilon_{\mathrm{m}}} \rightarrow \frac{1}{2} \frac{\mathrm{f}^{\prime \prime}(\mathrm{x})}{\mathrm{f}^{\prime}(\mathrm{x})}
$$

Since $\varepsilon_{m+1}$ is approximately proportional to the square of $\varepsilon_{m}$, the NewtonRaphson method is said to be quadratically convergent or to be of second order (Atkinson, 1978, p.54).

However, for the convergence of the above-mentioned method, a good initial estimate must be provided. This is indicated by the following theorem.

Theorem 3.2
Suppose that $f^{\prime}(x) \neq 0$ and $f^{\prime \prime}(x)$ does not change sign in the interval
$[a, b]$, and that $f(a) . f(b)<0$. Then, if

$$
\left|\frac{f(a)}{f^{\prime}(a)}\right|<b-a \text { and }\left|\frac{f(b)}{f^{\prime}(b)}\right|<b-a
$$

the Newton-Raphson method converges from an arbitrary initial approximation $x_{0} \in[a, b]$.

This theorem is described by Figure (3.8.1). While the iteration procedure converges from any point $x_{0} \in[a, b]$ it may also diverge from some points $x_{0}^{\prime} \in\left[a_{1}, b_{1}\right]$. This is actually a serious drawback of the method.

For the system of non-linear equations, we consider the Taylor's equation in $N$ dimensions, i.e.,

$$
\begin{equation*}
\underline{f}(\underline{x})=\underline{f}\left(\underline{x}_{m}\right)+\underline{f}^{\prime}\left(\underline{x}_{m}\right)\left(\underline{x}-\underline{x}_{m}\right)+0\left(\left|\underline{x}-x_{-m}\right|^{2}\right), \tag{3.8.13}
\end{equation*}
$$

where $x_{-m}$ is the iterate vector, $f^{\prime}\left(x_{m}\right)$ is the Jacobian matrix with elements

$$
\begin{equation*}
f_{i, j}^{\prime}(\underline{x})=\frac{\partial f_{i}(\underline{x})}{\partial x_{j}} \quad, 1 \leqslant i, j \leqslant N \tag{3.8.14}
\end{equation*}
$$

Neglecting the remainder part of (3.8.13) leads to the Newton-Raphson method in N dimension, i.e.,
$\left.\begin{array}{ll}\text { where } \quad & x_{m+1}=\frac{x}{m}+\delta_{m} \\ \underline{f}^{\prime}\left(\underline{x}_{m}\right) \delta_{m}=\underline{f}\left(\underline{x}_{m}\right),\end{array}\right\}$
which is a linear system of equation for $x_{m+1}$ and if $f^{\prime}\left(x_{m}\right)$ is nonsingular, this system can be solved by a direct method. However if $\underline{f}^{\prime}\left(x_{-m}\right)$ is a large sparse matrix an indirect or iterative method is more suitable.

Inspite of the rapid convergence, each step of the solution of (3.8.15) requires the recalculation of $N^{2}$ entries of the ( $N \times N$ ) matrix $f^{\prime}({\underset{m}{m}})$ and also the solution of a set of $N$ linear equations. Therefore, the Newton-Raphson method is rather expensive from the viewpoint of the computational work. One may reduce the amount of work by only
evaluating $\underline{f}^{\prime}\left(\underline{x}_{\mathrm{m}}\right)$ occasionally and not at every step (e.g. every $5^{\text {th }}$ step).

## Algorithmic Considerations

In using the equation (3.8.15), the following procedure may be followed:

1. Calculate $\underline{f}\left(\underline{x}_{\mathrm{m}}\right)$ and $\underline{f}^{\prime}\left(\frac{x_{m}}{\mathrm{~m}_{\mathrm{m}}}\right)$,
2. Solve for $\delta_{m}$ from $\underline{f}^{\prime}\left(\underline{x}_{m}\right) \delta_{m}=\underline{f}\left(x_{m}\right)$
3. Evaluate $\frac{x_{m+1}}{}$ from $x_{m+1}=\frac{x_{m}}{\mathrm{x}_{\mathrm{m}}}+\delta_{\mathrm{m}}$
4. Calculate $\underline{f}\left(\underline{x}_{\mathrm{m}}^{\mathrm{m}} \mathrm{l}\right)$.

These cases can now arise:
(a) if $\left|\left|\underline{f}\left(\frac{\mathrm{x}}{\mathrm{m}+1}\right)\right|\right| \ll\left|\left|\underline{f}\left(\frac{\mathrm{x}}{\mathrm{m}}\right)\right|\right|$, we continue with the same Jacobian,
(b) if $\left\|\underline{f}\left(\underline{x}_{\mathrm{m}+1}\right)\right\|<\left\|\underline{f}\left(\underline{x}_{\mathrm{m}}\right)\right\|$ go to step 1
 found by solving the linear system (3.8.15), and $\lambda_{m}=\frac{1}{2^{m}}, m=0,1, \ldots$, until a reduction in $\left\|\underline{f}\left(\underline{x}_{\mathrm{m}+1}\right)\right\|$ is obtained (J. Walsh, 1976). This method which involves a parameter $\lambda_{\mathrm{m}}$ is called the "damped Newton method" (where $\lambda_{\mathrm{m}}$ can also be chosen to be $1 / 10^{\mathrm{m}}$ ).

As the calculation of the Jacobian matrix is either impossible or expensive for some non-linear systems, the functional iteration method which does not use the Jacobian matrix at all, or replaces the Jacobian matrix by some approximation to it is recommended. One such method is the secant method which will be discussed below.

## The Secant Method

The secant method can be derived from the Newton-Raphson method if we approximate $f^{\prime}\left(x_{m}\right)$ by the difference quotient

$$
\begin{equation*}
f^{\prime}\left(x_{m}\right)=\frac{f\left(x_{m}\right)-f\left(x_{m-1}\right)}{x_{m}-x_{m-1}} \tag{3.8.16}
\end{equation*}
$$

which then leads to the following method.
Given initial approximations $x_{0}$ and $x_{1}$, the sequence $x_{2}, x_{3}, \ldots$ is computed recursively, i.e.,
where

$$
\begin{align*}
& x_{m+1}=x_{m}+\delta_{m} \\
& \delta_{m}=-f\left(x_{m}\right)\left[\frac{x_{m}-x_{m}-1}{f\left(x_{m}\right)-f\left(x_{m-1}\right)}\right], f\left(x_{m}\right) \neq f\left(x_{m-1}\right) \tag{3.8.17}
\end{align*}
$$

In this case the iteration function is $g\left(x_{m}\right) \equiv_{m}+\delta_{m}$. As the values at two points, i.e. $x_{m-1}$ and $x_{m}$ are required in order to get $x_{m+1}$, this method is one of a type called two-point iteration formula.

The choice between the secant method and the Newton-Raphson method depends on the amount of work required to compute $f^{\prime}(x)$. Suppose the amount of work to compute $f^{\prime}(x)$ is $\theta$ times the amount of work to compute a value of $f(x)$, then if $\theta>0.44$, the use of the secant method is recommended otherwise use Newton-Raphson's method (Dahlquist, 1974, p.228).

For a system of non-linear equations, if we let

$$
\begin{equation*}
\underline{r}_{\mathrm{m}}=\underline{f}\left(\underline{\mathrm{x}}_{\mathrm{m}+1}\right)-\mathrm{f}\left(\underline{\mathrm{x}}_{\mathrm{m}}\right) \tag{3.8,19}
\end{equation*}
$$

then from (3.8.13) with the error term neglected, we obtain the system

$$
\begin{equation*}
\mathrm{B}_{\mathrm{m}+1} \frac{\delta}{\mathrm{~m}}=\Upsilon_{\mathrm{m}} \tag{3.8.20}
\end{equation*}
$$

where $B_{m}$ is the approximation to $\underline{f}^{\prime}\left(\underline{x}_{m}\right)$ and is given by

$$
\begin{equation*}
B_{m+1}=B_{m}-\frac{\left(B_{m} \delta-r_{m}\right) q_{m}^{T}}{q_{m} \mathcal{q}_{m}}, q_{m}^{T} \delta \neq 0 \tag{3.8.21}
\end{equation*}
$$

with $q_{m}$ chosen arbitrarily for example $q_{m}=\delta_{m}$. This method is also called the Barnes Secant method, (Wait, 1979, p.119).

### 3.9 ITERATIVE METHODS FOR LINEAR EQUATION

Iterative methods of solution are of importance because practical problems invariably lead to large sparse matrices which are highly structured and iterative methods can make good use of these properties to provide highly efficient algorithms.

The majority of iterative methods for linear systems are stationary Iinear iteration methods: that is they can be written as

$$
\begin{equation*}
\underline{x}^{(m+1)}=\underline{M x}^{(m)}+\underline{c}, m=0,1, \ldots \tag{3.9.1}
\end{equation*}
$$

where $M$ is a constant matrix and $\subseteq$ is a constant vector. For later purposes we define the matrices $D, L$ and $U$ such that the 1inear system: $B \underline{x}=g$, $B=\left\{b_{i, j}\right\}$ and the matrices

$$
\begin{align*}
& D=\operatorname{diag}\left(b_{i i}\right) \\
& L= \begin{cases}-b_{i j} & i>j \\
0 & i \leqslant j\end{cases} \tag{3.9.2}
\end{align*}
$$

and

$$
U=\left\{\begin{array}{ll}
-b_{i j} & i<j \\
0 & i \geqslant j
\end{array} \quad\right. \text { (strictly upper triangular) }
$$

is related by $\mathrm{B}=\mathrm{D}-\mathrm{L}-\mathrm{U}$.

Jacobi Method
The (point) Jacobi Method which is also known as the method of simultaneous displacements is the simplest of all iterative technqiues. This method can be written as
or

$$
\begin{align*}
& \underline{x x}^{(m+1)}=\frac{(L+U) x^{(m)}+\underline{g}}{x_{i}^{(m+1)}}=\frac{-\sum_{\substack{j \neq i \\
j=1}}^{N} b_{i, j} x_{j}+g_{i}}{b_{i, i}} \quad, i=1,2, \ldots, N . \tag{3.9.3}
\end{align*}
$$

Then by taking the limit of (3.9.4) i.e. $1 \mathrm{im} \underline{\mathrm{x}}^{(\mathrm{m}+1)}=\underline{\underline{x}}$, it can be shown that $u$ is a solution of the original equation (3.9.1). Inspite of its simplicity, it is seldom used as it is very slow to converge.

## Gauss-Seide1 Method

In the Jacobi Method, one does not use the new values $x_{i}^{(m+1)}$ until every component of the vector $x$ has been evaluated. In the Gauss Seidel method, the recently calculated values of $x_{r}^{(m+1)}, r=1,2, \ldots, i-1$, are used in evaluating $x_{i}^{(m+1)}$. This method can be written in matrix form as

$$
\begin{equation*}
(D-L) \underline{x}^{(m+1)}=U \underline{x}^{(m)}+\underline{g}, \tag{3.9.5}
\end{equation*}
$$

or

$$
b_{i, i} x_{i}^{(m+1)}=-\sum_{j=1}^{i-1} b_{i, j} x_{j}^{(m+1)}-\sum_{j=i+1}^{N} b_{i, j} x_{j}^{(m)}+g_{i}
$$

i.e.

$$
\begin{equation*}
x_{i}^{(m+1)}=\frac{-\sum_{j=1}^{i-1} b_{i, j} x_{j}^{(m+1)}-\sum_{j=i+1}^{N} b_{i, j} x_{j}^{(m)}+g_{i}}{b_{i, i}} \tag{3.9.6}
\end{equation*}
$$

Here, as the values of $x_{i}$ are successively updated and overwritten, i.e. only one approximation for each $x_{i}$ needs to be stored at a time, thus saving vital computer memory, therefore the method is also called a successive displacement method. In both cases, it is assumed that $b_{i, i} \neq 0$.

## Point Successive Over- or Under-Relaxation Method

This method is closely related to the point Gauss-Seidel iterative method. We define $\hat{\underline{x}}^{(m)}$ from (3.9.6) i.e.

$$
\begin{equation*}
b_{i, i} \hat{x}_{i}^{(m+1)}=-\sum_{j=1}^{i=1} b_{i, j} x_{j}^{(m+1)}-\sum_{j=i+1}^{N} b_{i, j} x_{j}^{(m)}+g_{i}, 1 \leqslant i \leqslant N \tag{3.9.7}
\end{equation*}
$$

The actual components $\mathrm{x}_{\mathrm{i}}^{(m+1)}$ of this method are then defined from

$$
\begin{align*}
x_{i}^{(m+1)} & =x_{i}^{(m)}+\omega\left\{\hat{x}_{i}^{(m+1)}-x_{i}^{(m)}\right\}  \tag{3.9.8}\\
& =(1-\omega) x_{i}^{(m)}+\omega \hat{x}_{i}^{(m+1)}
\end{align*}
$$

The quantity $\omega$ is called the relaxation factor with for the ranges $0 \leqslant \omega<2$ (<1) correspond to overrelaxation (underrelaxation), respectively. A comparison of (3.9.7) and (3.9.8) gives the system as a single equation, i.e.,

$$
\begin{gather*}
b_{i, i} x_{i}^{(m+1)}=b_{i, i} x_{i}^{(m)}+w\left\{-\sum_{j=1}^{i-1} b_{i, j} x_{j}^{(m+1)}-\sum_{j=i+1}^{N} b_{i, j} x_{j}^{(m)}+\right. \\
\left.g_{i}-b_{i, i} x_{i}^{(m)}\right\}, \tag{3.9.9}
\end{gather*}
$$

in which the auxiliary iterates $\hat{x}_{i}$ do not appear. In matrix notation, this can be written as

$$
\begin{equation*}
(D-\omega L) \underline{x}^{(m+1)}=\{(1-\omega) D+\omega U\} \underline{x}^{(m)}+\omega \underline{g}, \tag{3.9.10}
\end{equation*}
$$

and as ( $D-\omega L$ ) is non-singular for any choice of $\omega$, then

$$
\begin{align*}
\underline{x}^{(m+1)} & =\left\{I-\omega\left(D^{-1} L\right)\right\}^{-1}\left\{(1-\omega) I+\omega D^{-1} U\right\}_{\underline{x}}^{(m)}+\omega\left(I-\omega D^{-1} L\right)^{-1} D^{-1} \underline{g} \\
& =\left(I-\omega L_{1}\right)^{-1}\left\{(1-\omega) I+\omega U_{1}\right\} x^{(m)}+\omega\left(I-\omega L_{1}\right)^{-1} D^{-1} \underline{g}, \tag{3.9.11}
\end{align*}
$$

with $L_{1}=D^{-1} L$ and $U_{1}=D^{-1} U$.
In all the three methods above, the iteration matrix $M$ is given by,

$$
\begin{align*}
& M_{J}=D^{-1}(L+U)  \tag{3.9.12}\\
& M_{G S}=(D-L)^{-1} U \tag{3.9.13}
\end{align*}
$$

and $\quad M_{S O R}=\left(I-\omega L_{1}\right)^{-1}\left\{(1-\omega) I+\omega U_{1}\right\} \quad$ respectively.
If we define $\underline{e}^{(n)}$ as the error in the nth approximation to the exact solution $\underline{x}$, i.e.

$$
\begin{equation*}
\underline{e}^{(\mathrm{n})}={\underline{x}-\underline{x}^{(n)}}^{(n)} \tag{3.9.15}
\end{equation*}
$$

hence, it follows by the subtraction of equation (3.9.1) from equation

$$
\begin{equation*}
\underline{x}=M \underline{x}+c \tag{3.9.16}
\end{equation*}
$$

we obtain $\quad \underline{e}^{(n+1)}=M e^{(n)}$.
Therefore, using (3.9.17) successively we obtain

$$
\begin{equation*}
\underline{e}^{(n)}=M \underline{e}^{(n-1)}=M^{2} e^{(n-2)}=\ldots=M^{n} e^{(0)} . \tag{3.9.18}
\end{equation*}
$$

The sequence of iterative values $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}$, will converge to $\underline{x}$ as $n$ tends to infinity if

$$
\lim _{n \rightarrow \infty} \underline{e}^{(n)}=\underline{0}
$$

Thus, it follows from (3.9.18) that the iteration will converge if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty} M^{n}=\underline{0} \tag{3.9.19}
\end{equation*}
$$

Now let $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}$ denote the eigenvalues of $M$ with the corresponding linearly independent eigenvectors $\underline{v}_{s}, s=1(1) N$. Therefore, the components of the initial vector $e_{-}^{(0)}$ may be represented by

$$
\underline{e}^{(0)}=\sum_{s=1}^{N} c_{s} v_{s}
$$

and hence

$$
\begin{equation*}
\underline{e}^{(n)}=\sum_{s=1}^{N} c_{s} \lambda_{s}^{n} v_{s} \tag{3,9.20}
\end{equation*}
$$

From equation (3.9.20), we indicate that the iteration (3.9.1) is convergent if $\left|\lambda_{i}\right|<1, i=1,2, \ldots, N$. Therefore the following theorem now can be established.

Theorem 3.3
A necessary and sufficient condition for a stationary iterative method (3.9.1) to converge from an arbitrary initial approximation is that

$$
\begin{equation*}
\rho(M)=\max _{1 \leqslant i \leqslant N}\left|\lambda_{i}(M)\right|<1, \tag{3.9.21}
\end{equation*}
$$

where $\rho()$ denotes the spectral radius.
From (3.9.4) the matrix $M_{J}$ is defined to have the elements

$$
m_{i, j}=-\frac{\left|b_{i, j}\right|}{\left|b_{i, i}\right|}, i \neq j \text { and } b_{i, i} \neq 0
$$

Thus,

$$
\begin{equation*}
\left\|M_{J}\right\|_{\infty}=\max _{1 \leqslant i \leqslant N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{{ }^{-b} i_{, 2}, j}{b_{i, i}} \tag{3.9.22}
\end{equation*}
$$

Therefore from (3.9.22) if the matrix B of (3.9.2) is strictly diagonally dominant, the Jacobi method is convergent.

For the Gauss-Seidel, we apply the subordinate matrix norm, namely,

$$
\begin{equation*}
\left\|M_{G S}\right\| \|_{\infty}=\max _{\substack{x \neq 0}} \frac{\left\|M_{G S} x^{(\mathrm{m})}\right\|_{\infty}}{\left\|\underline{x}^{(\mathrm{m})}\right\|_{\infty}} \tag{3.9.23}
\end{equation*}
$$

From (3.9.6),

$$
\begin{align*}
& \left\|\underline{x}^{(m+1)}\right\|_{\infty} \leqslant\|\tilde{\tilde{L}}\|_{\infty}\left\|\underline{x}^{(m+1)}\right\|\left\|_{\infty}+\right\| \tilde{u}\left\|_{\infty}\right\| \underline{x}^{(m)}\| \|_{\infty}+\|\underline{\hat{g}}\|_{\infty} \\
& \quad=s_{i}\left\|\underline{x}^{(m+1)}\right\|+r_{i}\left\|\underline{x}^{(m)}\right\|+\hat{g}_{i}  \tag{3.9.24}\\
& \quad \leqslant s_{i}\left\|\underline{x}^{(m+1)}\right\|+r_{i}\left\|\underline{x}^{(m)}\right\|
\end{align*}
$$

where

$$
s_{i}=\sum_{j=1}^{i-1} \frac{\left|b_{i, j}\right|}{\left|b_{i, i}\right|}, r_{i}=\sum_{j=i+1}^{N} \frac{\left|b_{i, j}\right|}{\left|b_{i, i}\right|} \text { and } \hat{g}_{i}=\frac{\left|g_{i}\right|}{|b i, i|}
$$

Therefore we have,

$$
\left(1-s_{i}\right)\left\|\underline{x}^{(m+1)}\right\|_{\infty} \leqslant r_{i}\left\|\underline{x}^{(m)}\right\|_{\infty}
$$

i.e.

$$
\left\|\underline{\underline{x}}^{(m+1)}\right\|_{\infty} \leqslant \frac{r_{i}\left\|_{\underline{x}}^{(m)}\right\|_{\infty}}{\left(1-s_{i}\right)}
$$

which results in

$$
\begin{equation*}
\left\|M_{G S}\right\|=\max _{1 \leqslant i \leqslant N} \frac{\left\|\underline{x}^{(m+1)}\right\|}{\left\|\underline{x}^{(m)}\right\|}=\max _{1 \leqslant i \leqslant N} \frac{r_{i}}{\left(1-s_{i}\right)} \tag{3.9.25}
\end{equation*}
$$

From equation (3.9.25) it follows that the Gauss-Seidel iteration is also convergent when A is strictly diagonally dominant.

In the case of the S.O.R. method with the iteration matrix defined by (3.9.14), the following theorem is established.

Theorem 3.4
For the S.O.R. iteration matrix (3.9.14) we have

$$
\begin{equation*}
\rho\left(M_{S O R}\right) \geqslant|\omega-1| . \tag{3.9.26}
\end{equation*}
$$

So the method can only converge for $0<\omega<2$.
Proof
Since the determinant of a triangular matrix is the product of its diagonal elements and $\left(I-\omega L_{1}\right)^{-1}$ and $\left[(I-\omega) I+\omega U_{1}\right]$ are both triangular matrices, hence we obtain

$$
\begin{aligned}
\operatorname{det}\left(M_{S O R}\right) & =\operatorname{det}\left(I-\omega L_{1}\right)^{-1} \operatorname{det}\left[(1-\omega) I+\omega U_{1}\right] \\
& =(1-\omega)^{N} .
\end{aligned}
$$

On the other hand, if the eigenvalues of $M_{S O R}$ are denoted by $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}$, then $\operatorname{det}\left(M_{S O R}\right)=\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}$. Therefore, from both expressions for $\operatorname{det}\left(M_{S O R}\right)$ it follows that

$$
\max _{\mathbf{i}}\left|\lambda_{\mathbf{i}}\right| \geqslant|1-\omega|, 0<\omega<2
$$

## Theorem 3.5

Let $A$ be a symmetric matrix with positive diagonal elements. Then, the S.O.R. method converges if and only if $A$ is positive definite and $0<\omega<2$. For proof, see Young, D.M., 1971, p.113.

As it is often impractical to find the eigenvalues of $B$, and therefore Theorem 3.3 for convergence is difficult to apply. Instead one can apply the following property of the norm-inequality on (3.9.18) and obtain,

$$
\begin{equation*}
\left\|\underline{e}^{(n)}\right\| \leqslant\|M\|^{n}\left\|\underline{e}^{(0)}\right\| \tag{3.9.27}
\end{equation*}
$$

Then, the sufficient condition for convergence now becomes $||M||<l$, for some consistent matrix-norm. An estimate for the error in $x^{n}$ can be found by using the relation,

$$
\begin{equation*}
\underline{x}^{n}-\underline{x}=-M\left(\underline{x}^{n}-\underline{x}^{n-1}\right)+M\left(\underline{x}^{n}-\underline{x}\right) \tag{3.9.28}
\end{equation*}
$$

If the norm of $M$ is denoted by $\beta<1$, then (3.9.28) becomes

$$
\begin{equation*}
\left|\left|\underline{x}^{n}-\underline{x}\right|\right| \leqslant \frac{\beta}{1-\beta}| | \underline{x}^{n}-\underline{x}^{n-1}| | \tag{3.9.29}
\end{equation*}
$$

We note that (3.9.29) is a rough estimate for checking the error but one should also consider the effect of round-off errors in the iterative process.

The rate of convergence of the iterative method is measured by the asymptotic rate of convergence which is defined by

$$
\begin{equation*}
R=-\log _{10}(\rho(M)) \tag{3.9.30}
\end{equation*}
$$

## Some Conclusive Points

1. In the finite-difference and finite element approximations of partial differential equations, we frequently have positive matrices.
2. In practice $\omega$ usually lies between 1 and 2 .
3. The optimum $\omega$ denoted by $\omega_{\text {opt }}$ for the maximum rate of convergence is given by (Young, D.M., 1971, p.169)

$$
\omega_{\text {opt }}=\frac{2}{1+\sqrt{1-\mu^{2}}}
$$

where $\mu$ is the spectral radius of the Jacobi iteration matrix $D^{-1}(L+U)$ associated with matrix $B$.
4. The methods which we have discussed so far are point iterative methods. There are also some iteration techniques which correspond to the evaluation of a group of points simultaneously. These methods are called block iterative methods.

### 3.10 FINITE ELEMENT - AN INTRODUCTION

Since the work in this thesis is not mainly on this field of method, the description in this section will be brief and will concentrate only on those aspects that are connected with the author's work in Chapter 8. Firstly, we assume the given p.d.e. as

$$
\begin{equation*}
L(u)=0, \tag{3.10.1}
\end{equation*}
$$

where $L$ is some operator. Now a solution is sought within some domain D subject to given appropriate conditions on $S$ the boundary of $D$. An approximate solution, defined by

$$
\begin{equation*}
v=\sum_{j=1}^{n} N_{j}(x, t) \cdot \bar{v}_{j} \tag{3.10.2}
\end{equation*}
$$

is introduced where $\bar{v}_{j}$ are the nodal values of $v$ which are the unknowns and $N_{j}(x, t)$ are functions of $x$ and $t$ and are referred to as (global) shape functions and $n$ is the element number.

In a finite element representation, we divide the domain or region of interest into a series of sub-domains, sub-regions or elements. In the case of parabolic p.d.e.'s as the domain or region of interest is always rectangular (open-sided), our concentration is therefore on the rectangular element.

As in the construction of the method we are dealing with the individual element, local element shape functions are more of interest as they allow us to locally interpolate the nodal displacements at any point within an element. Thus, they have the property

$$
N_{i}=\left\{\begin{array}{l}
1 \text { at the } i^{\text {th }} \text { node }  \tag{3.10.3}\\
0 \text { at all other nodes }
\end{array}\right.
$$

In Fig. (3.10.1) the diagram shows the adoption of a natural coordinate system ( $\xi, \eta$ ) in order to define the element geometry.


FIGURE (3.10.1)

The elements have side $\xi= \pm 1$ and $\eta= \pm 1$. For an element of size 2 a by $2 b$

$$
\begin{align*}
& \xi=\frac{x^{x-x_{c}}}{a} \\
& \eta=\frac{t-t_{c}}{b}, \tag{3.10.4}
\end{align*}
$$

and
where $\left(x_{c}, y_{c}\right)$ are the coordinates at the centre of the element. Thus, we have

$$
\begin{equation*}
\frac{\mathrm{d} \xi}{\mathrm{dx}}=\frac{1}{\mathrm{a}} \quad ; \quad \frac{\mathrm{d} \eta}{\mathrm{dt}}=\frac{1}{\mathrm{~b}} \tag{3.10.5}
\end{equation*}
$$

and an elemental area of the rectangular element is given as

$$
\begin{equation*}
\mathrm{dxd} t=a b d \xi \mathrm{~d} \eta . \tag{3.10.6}
\end{equation*}
$$

Thus to integrate any function $f(x, t)$ over the element we transform to the natural coordinate system so that

$$
\begin{equation*}
\int_{\Omega} \int_{(e)} f(x, t) d x d t=\int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) a b d \xi d \eta \tag{3.10.7}
\end{equation*}
$$

where $\Omega^{(e)}$ denotes the integration over the whole of the element.
Note that for all two-dimensional rectangular elements, the shape function must satisfy the condition,

$$
\begin{equation*}
\sum_{i}^{(e)}(\xi, \eta)=1 \tag{3.10.8}
\end{equation*}
$$

Now we will consider the shape functions for the 4,8 and 9 elements.
(i) 4-node element

For the 4 -node element (Fig. 3.10.2), the shape functions are given as (Hinton and Owen, 1979, p.246),

$$
\begin{equation*}
N_{i}(\xi, \eta)=\frac{1}{4}\left(1+\xi \xi_{i}\right)\left(1+\eta \eta_{i}\right) \tag{3.10.9}
\end{equation*}
$$




## (ii) Serendipity 8-node element

For the serendipity 8 -node element (Fig. 3.10.3) the shape function have the form (Hinton and Owen, 1979, p.247),
(a) for corner nodes

$$
\begin{equation*}
N_{i}^{(e)}=\frac{1}{4}\left(1+\xi \xi_{i}\right)\left(1+\eta n_{i}\right)\left(\xi \xi_{i}+n \eta_{i}-1\right), i=1,3,5,7, \tag{3.10.10}
\end{equation*}
$$

(b) for mid-side nodes

$$
\begin{equation*}
N_{i}(e)=\frac{\xi_{i}^{2}}{2}\left(1+\xi \xi_{i}\right)\left(1-\eta^{2}\right)+\frac{n_{i}^{2}}{2}\left(1+\eta \eta_{i}\right)\left(1-\xi^{2}\right), i=2,4,6,8 \tag{3.10.11}
\end{equation*}
$$



| Local Node <br> Number | $\xi_{\mathbf{i}}$ | $\eta_{\mathbf{i}}$ |
| :---: | :---: | :---: |
| 1 | -1 | -1 |
| 2 | 0 | -1 |
| 3 | 1 | -1 |
| 4 | 1 | 0 |
| 5 | 1 | 1 |
| 6 | 0 | 1 |
| 7 | -1 | 1 |
| 8 | -1 | 0 |
| 9 | 0 | 0 |

FIGURE (3.10.3)
(iii) Lagrangian 9-node element

For the 9 -node Lagrangian element shown in Fig.(3.10.3), the shape function has the form (Hinton and Owen, 1979, p.249):
(a) for corner nodes

$$
\begin{equation*}
N_{i}^{(e)}=\frac{1}{4}\left(\xi^{2}+\xi \xi_{i}\right)\left(\eta^{2}+\eta n_{i}\right), i=1,3,5,7 \tag{3.10.12}
\end{equation*}
$$

(b) for midside nodes

$$
\begin{equation*}
N_{i}^{(e)}=\frac{1}{2} \eta_{i}^{2}\left(\eta-\eta \eta_{i}\right)\left(1-\xi^{2}\right)+\frac{1}{2} \xi_{i}^{2}\left(\xi^{2}-\xi \xi_{i}\right)\left(1-\eta^{2}\right), i=2,4,6,8 \tag{3.10.13}
\end{equation*}
$$

(c) for the central node

$$
\begin{equation*}
N_{i}^{(e)}=\left(1-\xi^{2}\right)\left(1-\eta^{2}\right) . \tag{3.10.14}
\end{equation*}
$$

Since the Cartesian coordinates were changed to natural (local) coordinates, therefore for all rectangular elements, if we wish to evaluate Cartesian shape function derivatives we must use the expression

$$
\begin{align*}
& \frac{\partial N_{i}}{\partial x}=\frac{\partial N_{i}}{\partial \xi} \frac{\partial \xi}{\partial x}=\frac{1}{a} \frac{\partial N_{i}}{\partial \xi}, \\
& \frac{\partial N_{i}}{\partial t}=\frac{\partial N_{i}}{\partial \eta} \frac{\partial \eta}{\partial t}=\frac{1}{b} \frac{\partial N_{i}}{\partial \eta}, \tag{3.10.15}
\end{align*}
$$

Having defined the shape functions as required in (3.10.2), the finite element method proceeds by substituting equation (3.10.2) into equation (3.10.1), which then results in a residual given by

$$
\begin{equation*}
r=L(v) . \tag{3.10.16}
\end{equation*}
$$

Application of the weighted residual integral gives

$$
\begin{equation*}
\int_{0} \int_{i} \omega_{i}(x, t) d x d y=0, i=1,2, \ldots, n \tag{3.10.17}
\end{equation*}
$$

where $\omega_{i}$ is the ith element weighting function.
In the normal procedure, by the use of Green's theorem, equations (3.10.6), (3.10.7) and (3.10.15), the equation (3.10.17) will give

$$
\begin{equation*}
\sum_{j=1}^{n} c_{i j} \bar{v}_{j}=b_{i}, i=1,2, \ldots, n \tag{3.10.18}
\end{equation*}
$$

which is the system of finite element equations (C.A.J. Fletcher, 1978).
The author, in Chapter 8, has established a noval approach in obtaining the system of finite element equations which closely parallels the finite difference approach which is a tridiagonal system of linear equations incorporating the six neighbourhood points from 2 time-levels. The application of Boundary Value Technique (Greenspan, D. (1967), (1974), Carasso, A. (1968) and Carasso, A. and Parter, S.V. (1970)) for solving these finite element equations is also carried out.

## Chapter Four

## A NEW GROUP EXPLICIT METHOD FOR

PARABOLIC EQUATIONS

### 4.1 INTRODUCTION

One of the strategies for the determination of a more accurate numerical solution to the exact solution of a problem without upgrading the order of the approximation is by using different numerical algorithms which give truncation errors of different signs. In obtaining these different algorithms, the author commenced with the derivation of a generalised approximation, so that the difference in sign of the errors can be clearly observed.

It is well known that generally, an explicit type method is the simplest type method. As simplicity is normally related to the computational cost, therefore it is worth trying to retain the explicit type of approximation.

Starting from different algorithms of opposite sign in truncation errors and having in mind explicitness implies simplicity, the author has developed this study which later resulted in a new class of Group Explicit methods. The derivation of the method is based on the simplest heatconduction problem

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}, 0 \leqslant x \leqslant 1, \quad t \geqslant 0, \tag{4.1.1}
\end{equation*}
$$

This equation is chosen for the sake of simplicity in the following discussion. In any case, the derivation can always be carried over to a. more general problem, i.e.

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}+g(x, t), \quad 0 \leqslant x \leqslant 1, t \geqslant 0 . \tag{4.1.2}
\end{equation*}
$$

In this chapter the method will be discussed in great detail, together with the analysis of stability and an estimate of the truncation errors.

A comparison of the accuracy of the schemes in this method with the standard known method is also included. The use of the method is also extended to the two-space dimensional problem.

### 4.2 A GENERALISED TWO TIME-LEVEL FINITE DIFFERENCE APPROXIMATION

We approximate equation (4.1.1) at the point (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ by the finite-difference approximation

$$
\begin{align*}
\frac{u_{i, j+1}{ }^{-u} i, j}{\Delta t}= & \frac{1}{(\Delta x)^{2}}\left\{\theta_{1} \delta_{x} u_{i+\frac{1}{2}, j+1}-\theta_{2} \delta_{x} u_{i-\frac{1}{2}, j+1}+\right. \\
& \left.+\theta_{1}^{\prime} \delta_{x}^{u}{ }_{i+\frac{1}{2}, j}-\theta_{2}^{\prime} \delta_{x}^{u}{ }_{i-\frac{1}{2}, j}\right\} \tag{4.2.1}
\end{align*}
$$

with the compulsory conditions on the parameters $\theta_{i}, \theta_{i}^{\prime}, i=1,2$ given by

$$
\begin{align*}
& 0 \leqslant \theta_{i}, \theta_{i}^{\prime} \leqslant 1, i=1,2  \tag{4.2.2}\\
& \sum_{i=1}^{2}\left(\theta_{i}+\theta_{i}^{\prime}\right)=2  \tag{4.2.3}\\
& -\theta_{1}+\theta_{2}-\theta_{1}^{\prime}+\theta_{2}^{\prime}=0 \tag{4.2.4}
\end{align*}
$$

and the optional conditions

$$
\begin{align*}
& -\theta_{1}+\theta_{2}+\theta_{1}^{\prime}-\theta_{2}^{\prime}=0  \tag{4.2.5}\\
& -\theta_{1}-\theta_{2}+\theta_{1}^{\prime}+\theta_{2}^{\prime}=0 \tag{4.2.6}
\end{align*}
$$

The need for the compulsory and optional conditions will be given later. However, in general, all well known two-time level formulae fulfill the compulsory conditions, while the optional conditions determine the order of accuracy and the consistency of the formulae.

Examples of how the known standard formulae are derived from (4.2.1)(4.2.6) are as follows:
(a) If all conditions (4.2.2)-(4.2.6) are fulfilled, $\theta_{1}=\theta_{2}=\theta_{1}^{\prime}=\theta_{2}^{\prime}=\frac{1}{2}$
(4.2.1) yields,

$$
\begin{aligned}
\frac{u_{i, j+1}{ }^{-u}}{\Delta t}= & \frac{1}{2(\Delta x)^{2}}\left\{\left(u_{i+1, j+1}-2 u_{i, j+1}+u_{i-1, j+1}\right)+\right. \\
& \left.+\left(u_{i+1, j^{-2}} u_{i, j}+u_{i-1, j}\right)\right\}
\end{aligned}
$$

i.e.

$$
\begin{equation*}
-r u_{i+1, j+1}+(2+2 r) u_{i, j+1}-r u_{i-1, j+1}=r u_{i+1, j}+(2-2 r) u_{i, j}+r u_{i-1, j} \tag{4.2.7}
\end{equation*}
$$

which is known as the Crank-Nicolson (1947) formulae and is denoted pictorially in Fig. (4.2.1). This formulae is known to be unconditionally stable for all $r>0$ and has a principal truncation error of $O\left((\Delta x)^{2}+(\Delta t)^{2}\right)$.


FIGURE (4.2.1)


FIGURE (4.2.2)


FIGURE (4.2.3)
b) If $\theta_{i}^{\prime}=0, \theta_{i}=1, i=1,2$, all conditions except (4.2.6) are fulfilled.

This leads to

$$
\begin{equation*}
\frac{u_{i, j+1}^{-u_{i, j}}}{\Delta t}=\frac{u_{i+1, j+1}^{-2 u_{i, j+1}+u_{i-1, j+1}}}{(\Delta x)^{2}} \tag{4.2.8}
\end{equation*}
$$

i.e. $\quad-r u_{i-1, j+1}+(1+2 r) u_{i, j+1}-r u_{i+1, j+1}=u_{i, j}$
which is known as the fully-implicit formulae. This scheme is also unconditionally stable for all $\mathrm{r}>0$. The principal truncation error is of $O\left(\Delta t+(\Delta x)^{2}\right)$ and the molecular diagram given in Fig. (4.2.2).
c) If $\theta_{i}=0, \theta_{i}^{\prime}=1, i=1,2$ again the condition (4.2.6) is not fulfilled. This gives the classical explicit formulae (Fig.(4.2.3),

$$
\begin{equation*}
u_{i, j+1}=r u_{i-1, j}+(1-2 r) u_{i, j}+4 u_{i+1, j} \tag{4.2.9}
\end{equation*}
$$

which is stable for $r \leqslant \frac{1}{2}$ and has a principal truncation error of $O\left(\Delta t+(\Delta x)^{2}\right)$, (Smith, 1969).
d) If $\theta_{i}=\alpha / 2, \theta_{i}^{\prime}=1-\alpha / 2, i=1,2$ where $\alpha$ is a free parameter, conditions (4.2.2)-(4.2.5) are fulfilled and condition (4.2.6) given by

$$
-\theta_{1}-\theta_{2}+\theta_{1}^{\prime}+\theta_{2}^{\prime}=-2 \alpha+2
$$

This will also be satisfied if $\alpha=1$ (as in the case of (a)), and will give the finite difference formula

$$
\begin{align*}
& -\alpha r\left(u_{i-1, j+1}+u_{i+1, j+1}\right)+2(1+\alpha r) u_{i, j+1} \\
& \quad=(2-\alpha) r\left(u_{i-1, j}+u_{i+1, j}\right)+2(1-2 r+\alpha r) u_{i, j} \tag{4.2.10}
\end{align*}
$$

which is due to Saul'yev (1964: 91, eq. 8.16). Diagrammatically this is given by Fig. (4.2.4). It can be easily seen that equations (4.2.7), (4.2.8) and (4.2.9) are special cases of equation (4.2.10). This formulae has the stability condition

$$
\begin{equation*}
r \leqslant \frac{1}{2(1-\alpha)} \tag{4.2.11}
\end{equation*}
$$

and the principal truncation error

$$
e[u]= \begin{cases}O\left(\Delta t+(\Delta x)^{2}\right) & \alpha \neq 1  \tag{4,2.12}\\ \left.o(\Delta t)^{2}+(\Delta x)^{2}\right) & \alpha=1\end{cases}
$$



## FIGURE (4.2.4)

e) If $\theta_{i}=\alpha-\frac{1}{6 r}, \theta_{i}^{\prime}=1-\left(\alpha-\frac{1}{6 r}\right), i=1,2$ where $\alpha$ is as given in (d), conditions (4.2.2)-(4.2.5) are fulfilled and the condition (4.2.6) given by

$$
-\theta_{1}-\theta_{2}+\theta_{1}^{\prime}+\theta_{2}^{\prime}=-4\left(\alpha-\frac{1}{6 r}\right)+2
$$

will also be satisfied if $\alpha=\frac{1}{2}+\frac{1}{6 r}$.
This will lead to the cubic spline approximation of Papamichael and Whiteman (1973) which is

$$
\begin{align*}
& (1-6 r \alpha)\left(u_{i-1, j+1}+u_{i+1, j+1}\right)+2(2+6 r \alpha) u_{i, j+1} \\
& \quad=\{1+6 r(1-\alpha)\}\left(u_{i-1, j}+u_{i+1, j}\right)+2\{2-6 r(1-\alpha)\} u_{i, j} \tag{4.2.13}
\end{align*}
$$

The scheme (4.2.13) is unconditionally stable for $\frac{1}{2} \leqslant \alpha \leqslant 1$ and stable when $r \leqslant \frac{1}{6(1-2 \alpha)}$ for $0 \leqslant \alpha \leqslant \frac{1}{2}$. The principal truncation error is $0\left(\Delta t+(\Delta x)^{2}\right)$ and the molecular diagram is in Fig. (4.2.5).


FIGURE (4.2.5)
f) If $\theta_{1}=\theta_{2}^{\prime}=1$ and $\theta_{2}=\theta_{1}^{\prime}=0$ we see that the condition (4.2.5) is not fulfilled. This gives the formulae due to Saul'yev (1964: 32, 3.10), Fig. (4.2.6), i.e.,

$$
\begin{equation*}
-r u_{i+1, j+1}+(1+r) u_{i, j+1}=r u_{i-1, j}+(1-r) u_{i, j} \tag{4.2.14}
\end{equation*}
$$



## FIGURE (4.2.6)

g) Another formulae due to Saul'yev (1964: 32, 3.9) is found when $\theta_{2}=\theta_{1}^{\prime}=1$ and $\theta_{1}=\theta_{2}^{\prime}=0$ and again the condition (4.2.5) is not fulfilled. It is

$$
\begin{equation*}
(1+r) u_{i, j+1}-r u_{i-1, j+1}=r u_{i+1, j}+(1-r) u_{i, j} \tag{4,2.15}
\end{equation*}
$$

as in Fig. (4.2.7)


FIGURE (4.2.7)

From the above examples, we can see that from equation (4.2.1) it is possible:

1) to establish the well-known standard explicit, implicit and Crank-Nicolson formulas (i.e. (4.2.7), (4.2.8) and (4.2.9)).
2) to obtain the weighted six points formula, i.e. (4.2.10)
3) to obtain assymetry formulae such as (4.2.14) and (4.2.15).
4) to establish any six point formula with truncation error of order less or equal to $\left\{(\Delta t)^{2}+(\Delta x)^{2}\right\}$, of which one of the examples is equation (4.2.13).

Therefore, it is obvious that the approximation (4.2.1) is a general twolevel six point finite difference approximation.

### 4.3 TRUNCATION ERRORS FOR THE GENERAL APPROXIMATION

The general two-time level six point finite difference approximation (4.2.1) can be written as

$$
\begin{align*}
& -r \theta_{1} u_{i+1, j+1}+\left[1+r\left(\theta_{1}+\theta_{2}\right)\right] u_{i, j+1}-r \theta_{2} u_{i-1, j+1} \\
& \quad=r \theta_{1}^{\prime} u_{i+1, j}+\left[1-r\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)\right] u_{i, j}+r \theta_{2}^{\prime} u_{i-1, j} \tag{4.3.1}
\end{align*}
$$

To estimate the truncation error for this approximation, we expand each term in a Taylor's series expansion about the point (i,j+1) to result in

$$
\begin{align*}
& \left(\frac{\partial u}{\partial t}-\frac{\gamma_{1}}{2} \frac{\partial^{2} u}{\partial x^{2}}\right)_{i, j+\frac{1}{2}}+\gamma_{2}\left(\frac{1}{\Delta x} \frac{\partial u}{\partial x}+\frac{\Delta x}{6} \frac{\partial^{3} u}{\partial x^{3}}+\frac{1}{8} \frac{\Delta t^{2}}{\Delta x} \frac{\partial^{3} u}{\partial x \partial t^{2}}\right)_{i, j+\frac{1}{2}} \\
& \quad-\gamma_{1}\left(\frac{(\Delta x)^{2}}{24} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{16} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right)_{i, j+\frac{1}{2}} \\
& \quad+\gamma_{3}\left(\frac{1}{2}\left(\frac{\Delta t}{\Delta x} \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{12} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{48} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right)_{i, j+\frac{1}{2}}\right. \\
& \quad+\left.\frac{\gamma_{4}}{4} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha}, \Delta t^{\alpha}\right)=0 \tag{4.3.2}
\end{align*}
$$

with $\alpha_{1}+\alpha_{2}=5$, and $\gamma_{1}, \gamma_{2}, \gamma_{3}$ and $\gamma_{4}$ are given by the left-hand sides of equations (4.2.3), (4.2.4), (4.2.5) and (4.2.6) respectively. Assuming the compulsory conditions are fulfilled (i.e. $\gamma_{1}=2$ and $\gamma_{2}=0$ ), (4.3.2) is now written as

$$
\begin{align*}
& \left(\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial x^{2}}\right)_{i, j+\frac{1}{2}}+\gamma_{3}\left(\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} u}{\partial x^{2} t}+\frac{1}{12} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{48} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right) \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \\
& +\left.\frac{\gamma_{4}}{4} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha}, \Delta t^{\alpha}\right)=0 \tag{4.3.3}
\end{align*}
$$

From this analysis it is clearly evident why it is necessary to define the
conditions (4.2.2)-(4.2.4) as compulsory and the conditions (4.2.5) and (4.2.6) as optional. If the compulsory conditions are not fulfilled, it is clear that the approximation (4.2.1) does not approximate equation (4.1.1). However if $\gamma_{3}$ and $\gamma_{4}$ do not vanish, scheme (4.2.1) can still approximate equation (4.1.1) provided some restrictions are put on either $\Delta t, \Delta x$ or their ratio.

Using equation (4.3.3) we can easily verify the following truncation errors for the equation (4.2.7) as

$$
\begin{equation*}
T_{4.2 .7}=-\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \quad+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u+\frac{1}{2}}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} t^{\alpha} t^{\alpha}\right) \tag{4.3.4}
\end{equation*}
$$

for equation (4.2.8),

$$
\begin{align*}
& T_{4.2 .8}=-\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}\right.\left.+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) i, j+\frac{1}{2} \\
&-\left.\frac{1}{2} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}  \tag{4.3.5}\\
&+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} o\left(\Delta x^{\alpha} t^{\alpha}\right)
\end{align*}
$$

for equation (4.2.9),

$$
\begin{gather*}
T_{4.2 .9}=-\left(\frac{(\Delta x)^{2}}{12}\right. \\
\left.\frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \quad+\left.\frac{1}{2} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}  \tag{4.3.6}\\
+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} \Delta^{\alpha} t^{2}\right)
\end{gather*}
$$

for equation (4.2.10),

$$
\left.\begin{array}{rl}
T_{4.2 .10}=-\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \\
i, j+\frac{1}{2} \tag{4.3.7}
\end{array}+\left.\frac{\Delta t}{2}(1-\alpha) \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}\right)
$$

and finally for equation (4.2.13),

$$
\left.\begin{array}{rl}
T_{4.2 .13}=-\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial^{2}}\right) \\
i, j+\frac{1}{2} \tag{4.3.8}
\end{array}+\left.\frac{\Delta t}{2}\left[1-2\left(\alpha-\frac{1}{6 r}\right)\right] \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}\right)
$$

For the asymmetric equations (4.2.14) and (4.2.15) the truncation errors are given by

$$
\begin{align*}
& T_{4.2 .14}=-\left(\frac{\Delta t}{\Delta x} \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \quad \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right) i, j+\frac{1}{2} \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right)_{i, j+\frac{1}{2}}+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} 1_{\Delta t}{ }^{\alpha}\right)^{\alpha} \tag{4.3.9}
\end{align*}
$$

and

$$
\begin{align*}
T_{4.2 .15}= & \left(\frac{\Delta t}{\Delta x} \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right)_{i, j+\frac{1}{2}} \\
& \left.-\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right)_{i, j+\frac{1}{2}}+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} 1_{\Delta t}^{\alpha}\right)^{\alpha}\right) \tag{4.3.10}
\end{align*}
$$

respectively. Therefore equation (4.3.3) gives the general expression for the truncation error of the approximation represented by (4.2.1).

### 4.4 STABILITY ANALYSIS FOR THE GENERALISED APPROXIMATION

To investigate the stability condition of equation (4.2.1) or (4.3.1), we use the method of Fourier series as discussed in Section 2.10. Substitution of the error function at any point (i,j), i.e.,

$$
\begin{aligned}
E_{i, j} & =e^{\sqrt{-1} \beta x} e^{\alpha t} \\
& =e^{\sqrt{-1} i \beta \Delta x} e^{\alpha j \Delta t} \\
& =e^{\sqrt{-1} i \beta \Delta x} \xi^{j}, \quad \xi=e^{\alpha \Delta t}, \quad \alpha \text { complex }
\end{aligned}
$$

into the original approximating equation (4.3.1), will result in

$$
\begin{align*}
\xi & =\frac{r \theta_{1}^{\prime} e^{\sqrt{-1} \beta \Delta x}+\left[1-r\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)\right]+r \theta_{2}^{\prime} e^{-\sqrt{-1} \beta \Delta x}}{-r \theta_{1} e^{\gamma-1 \beta \Delta x}+\left[1+r\left(\theta_{1}+\theta_{2}\right)\right]-r \theta_{2} e^{-\sqrt{-1 \beta \Delta x}}}  \tag{4.4.1}\\
& =\frac{1-r(1-\cos \beta \Delta x)\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)+\sqrt{-1} r\left(\theta_{1}^{\prime}-\theta_{2}^{\prime}\right) \sin \beta \Delta x}{1-r(\cos \beta \Delta x-1)\left(\theta_{1}+\theta_{2}\right)-\sqrt{-1} r\left(\theta_{1}-\theta_{2}\right) \sin \beta \Delta x}  \tag{4.4.2}\\
\text { i.e. } \quad|\xi|^{2} & =\frac{1+16 r^{2} \theta_{1}^{\prime} \theta_{2}^{\prime} s^{4}-4 r\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right) s^{2}+4 r^{2}\left(\theta_{1}^{\prime}-\theta_{2}^{\prime}\right)^{2} s^{2}}{1+16 r^{2} \theta_{1} \theta_{2} s^{4}+4 r\left(\theta_{1}+\theta_{2}\right) s^{2}+4 r^{2}\left(\theta_{1}-\theta_{2}\right)^{2} s^{2}} \tag{4.4.3}
\end{align*}
$$

where $s=\sin ^{2} \frac{\beta \Delta x}{2}$.
For stability, we need $|\xi| \leqslant 1$. The expression (4.4.3) can be used to verify the stability condition of all the schemes given in Section (4.2). For the equation (4.2.7), we have

$$
\begin{equation*}
|\xi|=\frac{1+4 r^{2} s^{4}-4 r s^{2}}{1+4 r^{2} s^{4}+4 r^{2}}=\frac{1+4 r s^{2}\left(\mathrm{rs}^{2}-1\right)}{1+4 r s^{2}\left(\mathrm{rs}^{2}+1\right)} \leqslant 1 \tag{4.4.4}
\end{equation*}
$$

for all $r>0$. Similarly for the equation (4.2.8)

$$
\begin{align*}
|\xi|= & \frac{1}{1+16 \mathrm{r}^{2} \mathrm{~s}^{4}+8 \mathrm{rs}^{2}}  \tag{4.4.5}\\
& \leqslant 1 \text { for all } r>0 .
\end{align*}
$$

Meanwhile the stability of (4.2.9) can be fulfilled if

$$
1+8 r s^{2}\left(2 r^{2}-1\right) \leqslant 1,
$$

i.e.,

$$
\begin{equation*}
r \leqslant \frac{1}{2 s^{2}} \leq \frac{1}{2} \tag{4.4.6}
\end{equation*}
$$

For the equation (4.2.10), (4.4.3) will be less than (or equal) to unity if

$$
\begin{equation*}
r \leqslant \frac{1}{2(1-\alpha) s^{2}} \leqslant \frac{1}{2(1-\alpha)} \tag{4.4.7}
\end{equation*}
$$

i.e. similar to the original condition derived by Saul'yev.

In the case of equation (4.2.13), the stability condition can be fulfilled if
i.e.

$$
\begin{equation*}
8 \operatorname{rs}^{2}\left[2 \operatorname{rs}^{2}\left\{1-2\left(\alpha-\frac{1}{6 r}\right)\right\}-1\right] \leqslant 0 \tag{4.4.8}
\end{equation*}
$$

For the equation (4.2.14) and (4.2.15), equation (4.4.3) is equal to

$$
\begin{equation*}
\frac{1-4 r s^{2}+4 r^{2} s^{2}}{1+4 r s^{2}+4 r^{2} s^{2}} \tag{4.4.9}
\end{equation*}
$$

which is less than (or equal to) unity for all $r>0$. Therefore both equations are unconditionally stable. Finally, the following theorems can therefore be established.

## Theorem 4.1

The finite difference approximation (4.2.1) is stable if (4.4.3) is less than (or equal to) unity for any choice of $\theta_{i}$ and $\theta_{i}^{\prime}$ which satisfy (4.2.2)-(4.2.4).

## Theorem 4.2

The finite difference approximation (4.2.14) and (4.2.15) are unconditionally stable for all $r>0$.

Theorem 4.1 gives the general stability condition for the general six point two-level finite difference approximation (4,2.1) and Theorem 4.2 is necessary for purposes which will be developed later.

### 4.5 A NEW GROUP EXPLICIT METHOD

Now consider any group of two points, i.e. $\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ and $\left((i+1) \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$, in which equation (4.2.14) and (4.2.15) are used simultaneously to calculate the values of $u$ at these points respectively. Therefore, at point $\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ the solution is approximated by

$$
\begin{equation*}
-r u_{i+1, j+1}+(1+r) u_{i, j+1}=r u_{i-1, j}+(1-r) u_{i, j} \tag{4.5.1}
\end{equation*}
$$

whilst at point $\left((i+1) \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ the solution is given by

$$
\begin{equation*}
-r u_{i, j+1}+(1+r) u_{i+1, j+1}=(1-r) u_{i+1, j}+r u_{i+2, j} \tag{4.5.2}
\end{equation*}
$$

If we now write equation (4.5.1) and (4.5.2) simultaneously in the matrix form,

$$
\left[\begin{array}{cc}
1+r & -r \\
-r & 1+r
\end{array}\right]\left[\begin{array}{l}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
1-r & 0 \\
0 & 1-r
\end{array}\right]\left[\begin{array}{l}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{l}
r u_{i-1, j} \\
r u_{i+2, j}
\end{array}\right] \text { (4.5.3) }
$$

whose $(2 \times 2)$ matrix of coefficients can easily be inverted so that the equation can be written in explicit form as

$$
\left[\begin{array}{c}
u_{i, j+1}  \tag{4.5.4}\\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{|A|}\left[\begin{array}{cc}
1+r & r \\
r & 1+r
\end{array}\right]\left\{\left[\begin{array}{cc}
1-r & 0 \\
0 & 1-r
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{c}
r u_{i-1, j} \\
r u_{i+2, j}
\end{array}\right]\right\}
$$

where $|A|=1+2 r$. This simplifies to

$$
\left[\begin{array}{c}
u_{i, j+1}  \tag{4.5.5}\\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{|A|}\left[\begin{array}{l}
r(1+r) u_{i-1, j}+\left(1-r^{2}\right) u_{i, j}+r(1-r) u_{i+1, j}+r^{2} u_{i+2, j} \\
r^{2} u_{i-1, j}+r(1-r) u_{i, j}+\left(1-r^{2}\right) u_{i+1, j}+r(1+r) u_{i+2, j}
\end{array}\right]
$$

which is diagramatically represented by Fig. (4.5.1).
For the ungrouped (single) points near the right and left boundaries we can use equation (4.2.14) and (4.2.15) respectively, i.e. for the right boundary

$$
\begin{equation*}
u_{m-1, j+1}=\frac{1}{(1+r)}\left\{r u_{m, j+1}+r u_{m-2, j}+(1-r) u_{m-1, j}\right\} \tag{4.5.6}
\end{equation*}
$$

and for the left boundary


FIGURE (4.5.1)


$$
\begin{equation*}
u_{1, j+1}=\frac{1}{1+r}\left\{r u_{0, j+1}+r u_{2, j}+(1-r) u_{1, j}\right\} \tag{4.5.7}
\end{equation*}
$$

Now we will consider a variety of schemes for this class of methods which are possible for solving the equation (4.1.1) for an odd or even number of intervals.

## Even Number of Intervals

Assume the line segment $0 \leqslant x \leqslant 1$ is divided into a number $m$ of equal intervals with $m$ even. Therefore, at every time level, the number of internal points is odd, i.e. (m-1). This results in one ungrouped point at either end as shown in Figure (4.5.2).
i) GER scheme: This scheme is obtained by the use of equation (4.5.5) for $\frac{1}{2}(m-2)$ times for the first ( $m-2$ ) points grouped 2 at a time and equation (4.5.6) for the ( $m-1$ ) th point at every time-level. This is called the Group Explicit with Right ungrouped point (GER) scheme.

In implicit matrix form, it is given by

where

$$
\underline{b}_{-1}^{T}=\left[r u_{0, j}, 0, \ldots, 0, r u_{m, j+1}\right],
$$

and consists of known boundary values, or

$$
\begin{equation*}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1}, \tag{4.5.9}
\end{equation*}
$$

where


with $\quad G^{(i)}=\left[\begin{array}{cc}1 & -1 \\ -1 & 1\end{array}\right], i=1,2, \ldots, \frac{1}{2}(m-2), j=0,1,2, \ldots$.
The scheme can be described diagrammatically in Figure (4.5.3).
ii) GEL scheme: The second scheme is obtained by the use of equation (4.5.7) for the lst point and $\frac{1}{2}(\mathrm{~m}-2)$ times of equation (4.5.5) for the rest of the points at every time-level. This is called Group Explicit with Left ungrouped point (GEL) scheme. This scheme is given by

$$
\begin{equation*}
\left(I+r G_{2}\right) \underline{u}_{j+1}=\left(I-r G_{1}\right) \underline{u}_{j}+\underline{b}_{2}, \tag{4.5.12}
\end{equation*}
$$

$\underline{b}_{2}^{\mathrm{T}}=\left[r u_{0, j+1}, 0, \ldots, 0, r u_{m, j}\right]$ and the 'brick' diagram as in Fig. (4.5.4).
iii) The (S)AGE scheme: Another variation is the coupled use of the first
scheme (4.5.9) and the second scheme (4.5.12) at every alternate time level. This scheme is called the (Single) Alternating Group Explicit ((S)AGE) and is given by

$$
\left.\begin{array}{l}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1}  \tag{4.5.13}\\
\left(I+r G_{2}\right) \underline{u}_{j+2}=\left(I-r G_{1}\right) \underline{u}_{j+1}+\underline{b}_{2}
\end{array}\right\}
$$

and the diagram is given by Figure (4.5.5).
(iv) The (D)AGE scheme: The final scheme developed is a periodic rotation of the (S)AGE two time-level as given in the previous scheme resulting in a four time-level step process with the second half cycle in reverse order. The scheme is called the (Double) Alternating Group Explicit ((D)AGE) method and is given by the formulae,

$$
\left.\begin{array}{l}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1}  \tag{4.5.14}\\
\left(I+r G_{2}\right) \underline{u}_{j+2}=\left(I-r G_{1}\right) \underline{u}_{j+1}+\underline{b}_{2} \\
\left(I+r G_{2}\right) \underline{u}_{j+3}=\left(I-r G_{1}\right) \underline{u}_{j+2}+\underline{b}_{2} \\
\left(I+r G_{1}\right) \underline{u}_{j+4}=\left(I-r G_{2}\right) \underline{u}_{j+3}+\underline{b}_{1}
\end{array}\right\}
$$

Diagrammatically it is represented by Figure (4.5.6).

## Odd Number of Intervals

In this case, mis assumed to be odd. Therefore at every time-level, the number of internal points is even, i.e. (m-1). This gives at every time-level either ( $\frac{m-1}{2}$ ) complete groups of two points or ( $\frac{m-3}{2}$ ) groups of two points and one ungrouped point, adjacent to each boundary.

The first scheme for this case is obtained by using at every time level, equation (4.5.7) at the left ungrouped point, ( $\frac{\mathrm{m}-3}{2}$ ) times equation (4.5.5) from the second point to (m-2) th point and equation (4.5.6) for the last (m-1) th point which is left ungrouped at the far-right of the line adjacent to the boundary. This scheme is written as,

$$
\begin{equation*}
\left(I+r \hat{G}_{1}\right) \underline{u}_{j+1}=\left(I-r \hat{G}_{2}\right) \underline{u}_{j}+\underline{b}_{3}, \tag{4.5.15}
\end{equation*}
$$

where

$$
\underline{b}_{3}^{T}=\left[r u_{0, j+1}, 0, \ldots, 0, r u_{m, j+1}\right]
$$



$$
\widehat{G}_{2}=\left[\begin{array}{llll}
G^{(1)} & & &  \tag{4.5.17}\\
& G^{(2)} & 0 & \\
& 0^{-} & \ddots & \\
& & & G^{\frac{1}{2}(m-3)} \\
& & G^{\frac{1}{2}(m-1)}
\end{array}\right]
$$

and $G^{(i)}, i=1,2, \ldots, \frac{1}{2}(m-1)$ are ( $2 \times 2$ ) matrices as previously defined.
This scheme is denoted as Group Explicit with both Ungrouped ends (GEU) and described by Figure (4.5.7).

The second scheme is obtained by using ( $\frac{m-1}{2}$ ) times equation (4.5.5) for the first to ( $\mathrm{m}-1$ ) th point (Figure 4.5.8) with the equation given by

$$
\begin{equation*}
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+1}=\left(I-r \hat{G}_{1}\right) \underline{u}_{j}+\underline{b}_{4}, \tag{4.5.18}
\end{equation*}
$$

$b_{4}^{T}=\left[r u_{0, j}, 0, \ldots, 0, r u_{m, j}\right]$, and the scheme is known as the Group Explicit Complete (GEC) method.

The alternating schemes corresponding to the schemes (4.5.13) and
(4.5.14) above are given by,

$$
\left.\begin{array}{l}
\left(I+r \hat{G}_{1}\right) \underline{u}_{j+1}=\left(I-r \hat{G}_{2}\right) \underline{u}_{j}+\underline{b}_{3}  \tag{4.5.19}\\
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+2}=\left(I-r \hat{G}_{1}\right) \underline{u}_{j+1}+\underline{b}_{4}
\end{array}\right\}
$$

for (S)AGE (Fig. 4.5.9) and

$$
\left.\begin{array}{l}
\left(I+r \hat{G}_{1}\right) \underline{u}_{j+1}=\left(I-r \hat{G}_{2}\right) \underline{u}_{j}+\underline{b}_{3} \\
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+2}=\left(I-r \hat{G}_{1}\right) \underline{u}_{j+1}+\underline{b}_{4} \\
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+3}=\left(I-r \hat{G}_{1}\right) \underline{u}_{j+2}+\underline{b}_{4} \\
\left(I+r \hat{G}_{1}\right) \underline{u}_{j+4}=\left(I-r \hat{G}_{2}\right) \underline{u}_{j+3}+\underline{b}_{3}
\end{array}\right\}
$$

for (D)AGE (Fig. 4.5.10).
 FIGURE (4.5.3)

G.E.L. (Group Explicit with Left ungrouped point) method

> FIGURE (4.5.4)

S.A.G.E. (Single Alternating Group Explicit) method

D.A.G.E. (Double Alternating Group Explicit) method

G.E.U. (Group Explicit with Ungrouped ends) method FIGURE (4.5.7)


FIGURE (4.5.8)

S.A.G.E. (Single Alternating Group Explicit) method

D.A.G.E. (Double Alternating Group Explicit) method FIGURE (4.5.10)

### 4.6 AGE METHODS - FRACTIONAL SPLITTING

Now if we imagine that the levels j and ( $\mathrm{j}+1$ ) of the time variable can be divided into a small number of fractional or 'artificial' sub-levels, then instead of using the (S)AGE and (D)AGE schemes for every two and four time levels respectively, we can use them as a complete cycle at every timelevel since we have intermediate 'results' which fall on the artifical levels Therefore a fractional splitting type of (S)AGE scheme equation (4.5.13) is given by

$$
\left.\begin{array}{l}
\left(I+\frac{r_{2}}{2} G_{1}\right) \underline{u}_{j+\frac{1}{2}}=\left(I-\frac{r_{2}}{2}\right)_{2} \underline{u}_{j}+\underline{b}_{1}  \tag{4.6.1}\\
\left(I+\frac{r}{2} G_{2}\right) \underline{u}_{j+1}=\left(I-\frac{r}{2} G_{1}\right) \underline{u}_{j+\frac{1}{2}}+\underline{b}_{2}
\end{array}\right\}
$$

Similarly the fractional splitting form of (D) AGE scheme equation (4.5.14) is given by

$$
\left.\begin{array}{l}
\left(I+\frac{r_{2}}{2} G_{1}\right) u_{j+\frac{1}{2}}=\left(I-\frac{r_{2}}{2} G_{2}\right) \underline{u}_{j}+\underline{b}_{1}  \tag{4.6.2}\\
\left(I+\frac{r_{2}}{2} G_{2}\right) \underline{u}_{j+1}=\left(I-\frac{r}{2} G_{1}\right) \underline{u}_{j+\frac{1}{2}}+\underline{b}_{2} \\
\left(I+\frac{r_{2}}{2} G_{2}\right) \underline{u}_{j+\frac{3}{2}}=\left(I-\frac{r_{2}}{2} G_{1}\right) \underline{u}_{j+1}+\underline{b}_{2} \\
\left(I+\frac{r_{2}}{2} G_{1}\right) \underline{u}_{j+2}=\left(I-\frac{r}{2} G_{2}\right) \underline{u}_{j+\frac{3}{2}}+\underline{b}_{1}
\end{array}\right\}
$$

where the complete splitting process takes place every two time-levels.
Alternatively if the levels $j$ and ( $\mathrm{j}+\mathrm{l}$ ) are divided into four sublevels, the fractional or quarter-step of the (D)AGE scheme is then given as

$$
\left.\begin{array}{l}
\left(I+\frac{r_{4}}{G_{1}}\right) \underline{u}_{j+\frac{1}{4}}=\left(I-\frac{r_{4}}{G_{2}}\right) \underline{u}_{j}+\underline{b}_{1}  \tag{4.6.3}\\
\left(I+\frac{r_{4}}{4} G_{2}\right) \underline{u}_{j+\frac{1}{2}}=\left(I-\frac{r_{G}}{4} G_{1}\right) \underline{u}_{j+\frac{1}{4}}+\underline{b}_{2} \\
\left(I+\frac{r_{G}}{4} G_{2}\right) \underline{u}_{j+\frac{3}{4}}=\left(I-\frac{r_{4}}{4} G_{1}\right) \underline{u}_{j+\frac{1}{2}}+\underline{b}_{2} \\
\left(I+\frac{r}{4} G_{1}\right) \underline{u}_{j+1}=\left(I-\frac{r_{1}}{4} G_{2}\right) \underline{u}_{j+\frac{3}{4}}+\underline{b}_{1}
\end{array}\right\}
$$

### 4.7 THE GROUP EXPLICIT METHOD FOR PERIODIC BOUNDARY CONDITIONS

All the schemes discussed in the previous section are applicable only for the Dirichlet boundary condition. In the case where equation (4.1.1) is associated with the periodic boundary condition as in (1.3.4), the matrix of coefficients involved will be different. If it is assumed that the number of intervals is even, then the number of unknown points is also even. The Group Explicit Complete (GEC) scheme is thus given by


or $\quad\left(I+r \hat{G}_{2}\right) \underline{u}_{j+1}=\left(I-r \hat{\hat{G}}_{1}\right) \underline{u}_{j}$,
where $\hat{G}_{2}$ is given by (4.5.17),

$$
\hat{\hat{G}}_{1}=\left[\begin{array}{ccccc}
1 & { }_{G}^{(1)} & & 0 &  \tag{4.7.3}\\
& & G^{(2)} & & \\
& 0 & \ddots & { }_{G}\left(\frac{m-2}{2}\right) & \\
-1 & & & & 1
\end{array}\right]
$$

and the $G^{(i)}, i=1,2, \ldots,\left(\frac{m-2}{2}\right)$, are as previously defined. Meanwhile the Group Explicit with both Ungrouped end points (GEU) is given by


Now from the Group Explicit Complete (GEC) and Group Explicit with both Ungrouped ends (GEU), the following fractional step schemes can be developed:
(i) Single Alternating Group Explicit ((S)AGE)i.e.

$$
\left.\begin{array}{rl}
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+1} & =\left(I-r \hat{\hat{G}}_{1}\right) \underline{u}_{j}  \tag{4.7.6}\\
\left(I+r \hat{\hat{G}}_{1}\right) \underline{u}_{j+2} & =\left(I-r \hat{G}_{2}\right) \underline{u}_{j+1}
\end{array}\right\}
$$

(ii) Double Alternating Group Explicit ((D)AGE) i.e.

$$
\left.\begin{array}{l}
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+1}=\left(I-r \hat{\hat{G}}_{1}\right) \underline{u}_{j}  \tag{4.7.7}\\
\left(I+r \hat{\hat{G}}_{1}\right) \underline{u}_{j+2}=\left(I-r \hat{G}_{2}\right) \underline{u}_{j+1} \\
\left(I+r \hat{\hat{G}}_{1}\right) \underline{u}_{j+3}=\left(I-r \hat{G}_{2}\right) \underline{u}_{j+2}- \\
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+4}=\left(I-r \hat{\hat{G}}_{1}\right) \underline{u}_{j+3}
\end{array}\right\}
$$

(iii) Half-step Single AGE, i.e.

$$
\left.\begin{array}{l}
\left(I+\frac{r}{2} \hat{G}_{2}\right) \underline{u}_{j+\frac{1}{2}}=\left(I-\frac{r}{2} \hat{\hat{G}}_{1}\right) \underline{u}_{j}  \tag{4.7.8}\\
\left(I+\frac{r}{2} \hat{\hat{G}}_{1}\right) \underline{u}_{j+1}=\left(I-\frac{r}{2} \hat{G}_{2}\right) \underline{u}_{j+\frac{1}{2}}
\end{array}\right\}
$$

(iv) Half-step Double AGE i.e.

$$
\left.\begin{array}{l}
\left(I+\frac{r}{4} \hat{G}_{2}\right) \underline{u}_{j+\frac{1}{2}}=\left(I-\frac{r}{4} \hat{\hat{G}}_{1}\right) \underline{u}_{j}  \tag{4.7.9}\\
\left(I+\frac{r}{4} \hat{\hat{G}_{1}}\right) \underline{u}_{j+1}=\left(I-\frac{r}{4} \hat{G}_{2}\right) \underline{u}_{j+\frac{1}{2}} \\
\left(I+\frac{r}{4} \hat{G}_{1}\right) \underline{u}_{j+\frac{3}{2}}=\left(I-\frac{r}{4} \hat{G}_{2}\right) \underline{u}_{j+1} \\
\left(I+\frac{r}{4} \hat{G}_{2}\right) \underline{u}_{j+2}=\left(I-\frac{r}{4} \hat{G}_{1}\right) \underline{u}_{j+\frac{3}{2}}
\end{array}\right\}
$$

(v) Quarter-step Double AGE i.e.

$$
\left.\begin{array}{l}
\left(I+\frac{r}{4} \hat{G}_{2}\right) \frac{u}{j+\frac{1}{4}}=\left(I-\frac{\left.r \hat{\hat{G}}_{I}\right) u_{j}}{4}=\left(I-\frac{r}{4} \hat{G}_{2}\right) u_{j+\frac{1}{4}}\right.  \tag{4.7.10}\\
\left(I+\frac{r}{4} \hat{\hat{G}}_{1}\right) u_{j+\frac{1}{2}}=\left(I-\frac{r}{4} \hat{G}_{2}\right) \frac{u}{j+\frac{1}{2}} \\
\left(I+\frac{r}{4} \hat{G}_{1}\right) \underline{u}_{j}+\frac{3}{4}=\left(I-\frac{r}{4} \hat{G}_{1}\right) u_{j}+\frac{3}{4}
\end{array}\right\}
$$

From the above schemes, it can be easily seen that the schemes are similar to the case of the odd number of intervals in section (4.5) except for the definition of $\hat{\mathrm{G}}_{1}$.

Now if it is assumed that the number of intervals is odd, then the number of unknowns is also odd, i.e. $u_{0, j}, u_{1, j}, \ldots, u_{m-1, j}$ for $j=1,2, \ldots$. Therefore the Group Explicit with right ungrouped point (GER) scheme is given by

$$
\begin{equation*}
\left(I+r \breve{G}_{1}\right) \underline{u}_{j+1}=\left(I-r \breve{G}_{2}\right) \underline{u}_{j} \tag{4.7.11}
\end{equation*}
$$

where

and

$$
\breve{G}_{2}=\left[\begin{array}{r:c:c:c}
1 & & & \ldots  \tag{4.7.13}\\
\hdashline & 1 & -1 & \ldots
\end{array}\right)
$$

While the scheme corresponding to the Group Explicit method with Left ungrouped point (GEL) is given by

$$
\begin{equation*}
\left(I+r \breve{G}_{2}\right) \underline{u}_{j+1}=\left(I-r \breve{G}_{1}\right) \underline{u}_{j} \tag{4.7.14}
\end{equation*}
$$

Finally from the two schemes (4.7.11) and (4.7.14) we can derive in an exactly similar manner the corresponding alternating scheme to equations (4.7.6)-(4.7.10).

### 4.8 THE ESTIMATE OF THE TRUNCATION ERRORS FOR THE GROUP EXPLICIT METHOD

To estimate the truncation errors for the schemes in the Group Explicit method, firstly, we need to estimate the truncation errors for the approximations (4.5.5), (4.5.6) and (4.5.7).

From (4.3.9) and (4.3.10), the estimate of the truncation errors for (4.5.6) and (4.5.7) are respectively given by

$$
\begin{align*}
T_{4.5 .6}= & -\left(\frac{\Delta t}{\Delta x} \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right) \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right)_{m-1, j+\frac{1}{2}} \\
& +\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{m-1, j+\frac{1}{2}}  \tag{4.8.1}\\
& 0\left(\Delta x^{\alpha} \Delta t^{\alpha}\right)
\end{align*}
$$

and

$$
\begin{aligned}
& T_{4.5 .7}=\left(\frac{\Delta t}{\Delta x} \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right)_{1, j+\frac{1}{2}} \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \quad+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{1, j+\frac{1}{2}} \\
& +\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} 1_{\Delta t}{ }^{\alpha}\right) .
\end{aligned}
$$

Expanding each term of both equations in (4.5.5) about ( $i \Delta x,\left(j+\frac{1}{2}\right) \Delta t$ ) and ( (i+1) $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ respectively using a Taylor's series expansion will give the following:

For the first equation of (4.5.5) it results in the expression,

$$
\begin{align*}
& \left(\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial x^{2}}\right)_{i, j+\frac{1}{2}}-\left.\frac{\Delta t}{\Delta x}\left(\frac{1}{1+2 r}\right) \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}}+\left.\Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}} \\
& -\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}-\left.\frac{(\Delta x)^{2}}{24} \frac{(r+8)}{(1+2 r)} \frac{\partial^{4} u}{\partial x^{4}}\right|_{i, j+\frac{1}{2}} \\
& +\left.\frac{\Delta t \Delta x}{2} \frac{r}{1+2 r} \frac{\partial^{4} u}{\partial x^{3} \partial t}\right|_{i, j+\frac{1}{2}}-\left.\frac{\Delta t^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} \Delta_{t}^{\alpha}\right)=0 \tag{4.8.3}
\end{align*}
$$

While for the second equation of (4.5.5) it results in the expression

$$
\begin{align*}
& \left(\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial x^{2}}\right)\left(i+1, j+\frac{1}{2}+\left.\frac{\Delta t}{\Delta x}\left(\frac{1}{1+2 r}\right) \frac{\partial^{3} u}{\partial x^{3}}\right|_{i+1, j+\frac{1}{2}}+\left.\Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i+1, j+\frac{1}{2}}\right. \\
& \quad-\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i+1, j+\frac{1}{2}}-\left.\frac{(\Delta x)^{2}}{24} \frac{(r+8)}{(1+2 r)} \frac{\partial^{4} u}{\partial x^{4}}\right|_{i+1, j+\frac{1}{2}} \\
& \quad-\left.\frac{\Delta t \Delta x}{2} \frac{r}{1+2 r} \frac{\partial^{4} u}{\partial^{3,4} \partial t}\right|_{i+1, j+\frac{1}{2}}-\left.\frac{\Delta t^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} l_{\Delta t^{2}}^{\alpha}\right)=0 \tag{4.8.4}
\end{align*}
$$

Having found the truncation error terms for equations (4.5.5),(4.5.6) and (4.5.7) which are the basic equations for all the schemes used in the class of Group Explicit methods, we can now proceed to examine the truncation error for the Group Explicit method.
(i) GER scheme: The truncation errors at a group of two points for any time level are respectively given by the error terms of equation (4.8.3) and (4.8.4) namely

$$
\begin{align*}
& \left\{-\frac{\Delta t}{\Delta x}\left(\frac{1}{1+2 r}\right) \frac{\partial^{3} u}{\partial x^{3}}+\Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}-\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}-\frac{(\Delta x)^{2}}{24} \frac{(r+8)}{(1+2 r)} \frac{\partial^{4} u}{\partial x^{4}}\right. \\
& \left.+\frac{\Delta t \Delta x}{2}\left(\frac{r}{1+2 r}\right) \frac{\partial^{4} u}{\partial x^{3} \partial t}-\frac{\Delta t^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\alpha} 1_{\Delta t}{ }^{\alpha}\right)\right\}_{i, j+\frac{1}{2}} \tag{4.8.5}
\end{align*}
$$

and

$$
\begin{align*}
& \left\{\frac{\Delta t}{\Delta x}\left(\frac{1}{1+2 r}\right) \frac{\partial^{3} u}{\partial x^{3}}+\Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}-\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}-\frac{(\Delta x)^{2}}{24} \frac{(r+8)}{(1+2 r)} \frac{\partial^{4} u}{\partial x^{4}}\right. \\
& \left.-\frac{\Delta t \Delta x}{2}\left(\frac{r}{1+2 r}\right) \frac{\partial^{4} u}{\partial x^{3} \partial t}-\frac{\Delta t^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}+\frac{1}{\Delta t} O\left(\Delta x^{\alpha} t^{\alpha} t^{\alpha}\right)\right\}_{i+1, j+\frac{1}{2}}^{i=1} \tag{4.8.6}
\end{align*}
$$

$$
i=1,3,5, \ldots, m-3
$$

and at the last ungrouped point the error term is given by equation (4.8.1).
(ii) GEL scheme: The truncation error at the first point from the lefthand side boundary is given by the equation (4.8.2) for any time level.

Similarly the truncation errors at the remaining groups of two points are given by equations (4.8.5) and (4.8.6) respectively for $i=2,4, \ldots, m-2$.
(iii) GEU Scheme: This scheme which occurs in the case of an odd number of intervals has the truncation errors at the first point given by the equation (4.8.2), at every group of two points from the second point up to (m-2)th point given by the equation (4.8.5) and (4.8.6) respectively and at the last point, i.e. the (m-1)th point given by the equation (4.8.1).
(iv) GEC Scheme: Since the number of points along each line form (m-1)/2 complete groups of two points each, then the truncation error for this scheme is given by the equation (4.8.5) and (4.8.6) respectively for $i=1,3, \ldots, m-4, m-2$, where $m$ is odd.
(v) (S)AGE Scheme: For an even number of intervals, the (S)AGE schemes constitute the alternating use of the GER and GEL schemes. If it is assumed that for the $j$ th and $(j+1)$ th time-levels the GER and GEL schemes are used, then the truncation error along the $j$ th and ( $j+1$ ) th time-level is given by the truncation errors of the GER and GEL schemes respectively. Due to the difference in signs of the truncation errors, these alternating errors will tend to cancel the effect of the $(\Delta t / \Delta x)$ and $\Delta t \Delta x$ terms at most internal points, thus leaving the truncation error to approximately $0\left(\Delta t+(\Delta x)^{2}\right)$, Fig. (4.8.1a) for each of the schems.
(vi) (D)AGE Scheme: The distribution of errors for this scheme can be clearly explained by the diagram (Fig. 4.8.1b)). This scheme with a similar reasoning tend to have the approximate error of $O\left(\Delta t+(\Delta x)^{2}\right)$ also. However the numerical experiments will show in a later section that this scheme produces more accurate results than the (S)AGE scheme.


FIGURE (4.8.1a)


The distribution of errors at every time-level

### 4.9 STABILITY OF THE GROUP EXPLICIT METHOD

All schemes in the class of the group explicit method above are formed independently by the combination of the equations (4.5.5), (4.5.6) and (4.5.7). Theorem 4.2 has already concluded that the schemes (4.5.6) and (4.5.7) are unconditionally stable for r>0. Now we try to establish the stability of the formulae (4.5.5) by the matrix method as outlined in Section (2.10).

Equation (4.5.5) can be written individually as,

$$
\begin{equation*}
u_{i, j+1}=\frac{1}{|A|}\left\{r(1+r) u_{i-1, j}+\left(1-r^{2}\right) u_{i, j}+r(1-r) u_{i+1, j}+r^{2} u_{i+2, j}\right\} \tag{4.9.1}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{i+1, j+1}=\frac{1}{|A|}\left\{r^{2} u_{i-1, j}+r(1-r) u_{i, j}+\left(1-r^{2}\right) u_{i+1, j}+r(1+r) u_{i+2, j}\right\} \tag{4.9.2}
\end{equation*}
$$

where $|A|=1+2 r$. These equations in matrix form are written as

$$
\begin{equation*}
\underline{u}_{j+1}=\frac{B u_{j}}{}+\underline{b}, \tag{4.9.1a}
\end{equation*}
$$

and

$$
\begin{equation*}
\underline{u}_{j+1}=B_{B_{j}^{T}}+\underline{b}, \tag{4.9.2a}
\end{equation*}
$$

where

For stability it is necessary that $||B|| \leqslant 1$.

Using the maximum norm it can be shown that the stability condition is achieved if $r \leqslant 1$.

Therefore equation (4.5.5) is conditionally stable for all $\mathrm{r} \leqslant 1$. It would also be expected that the schemes in the class of Group Explicit method, which are formed from the combination of (4.5.5), (4.5.6) and (4.5.7) should also be conditionally stable.

The formulae GER of equation (4.5.9), GEL of equation (4.5.12), GEU of equation (4.5.15) and GEC of equation (4.5.18) can be written explicitly as

$$
\begin{equation*}
\underline{u}_{j+1}=\underline{T u}_{j}+\underline{b}^{\prime} \tag{4.9.3}
\end{equation*}
$$

where the amplification matrices $T$ for each case are defined as follows:


and



Now the system (4.9.3) will be stable against growth of rounding errors if,

$$
\begin{aligned}
& \|\mathrm{T}\|_{\infty} \leqslant 1 \text {, where we define } \\
& \|T\|_{\infty}=\max _{1 \leqslant i \leqslant m-1}^{m-1}\left\{\sum_{j=1}^{m}\left|t_{i, j}\right|\right\} \\
& = \\
& \quad \text { maximum absolute row sum of the }(m-1) \times(m-1) \\
& \quad \text { matrix } T \text { with elements } t_{i, j}
\end{aligned}
$$

Thus, from (4.9.4)-((4.9.7) we have,

$$
\begin{aligned}
&\left\|T_{G E R}\right\|_{\infty}= \max \left\{\frac{1}{1+2 r}\left(\left|1-r^{2}\right|+|r(1-r)|+r^{2}\right), \frac{1}{1+2 r}\left(|r(1-r)|+\left|1-r^{2}\right|+|r(1+r)|\right),\right. \\
& \frac{1}{1+2 r}\left(|r(1+r)|+\left|1-r^{2}\right|+|r(1-r)|+r^{2}\right), \\
&\left.\frac{1}{1+r}(|r|+|1-r|)\right\} \leqslant 1 \\
& \text { for all } r \leqslant 1 .
\end{aligned}
$$

Similarly, we find that $\left\|T_{G E L}\right\|\left\|_{\infty},\right\| T_{G E U}\| \|_{\infty}$ and $\left\|T_{G E C}\right\|_{\infty}$ fulfill the condition when $r \leqslant 1$. Therefore all single time step group explicit methods of GER, GEL, GEU and GEC are conditionally stable for $r \leqslant 1$. This result can be easily confirmed from the numerical experiments.

Alternatively, if we preserve the implicit form of (4.5.9),(4.5.12) and (4.5.15), i.e.

$$
\begin{equation*}
\left(I+r G_{A}\right) \underline{u}_{j+1}=\left(I-r G_{B}\right) \underline{u}_{j}+\underline{b}, \tag{4.9.9}
\end{equation*}
$$

we can still reach a similar stability condition using the following lẹma of Kellogg (1964):

Lemma 4.1:
If $\rho>0$ and ( $B+B^{*}$ ) is non-negative (or positive) definite, then $(\rho \mathrm{I}+\mathrm{B})^{-1}$ is bounded and $\left\|(\rho \mathrm{I}+\mathrm{B})^{-1}\right\| \|_{2 \leqslant \rho}^{-1}$.

From equations (4.5.10), (4.5.11), (4.5.16) and (4.5.17), it is easily verified (Appendix 1) that ( $r G_{A}+r G_{A}^{*}$ ) is non-negative (or positive) definite, then $\left(I+r G_{A}\right)^{-1}$ is banded and $\left\|\left(I+r G_{A}\right)^{-1}\right\| \|_{2} \leqslant 1$.

For stability, from (4.9.8) we need

$$
\|T\|=\left\|\left(I+r G_{A}\right)^{-1}\left(I-r G_{B}\right)\right\| \leqslant 1 .
$$

Thus

$$
\begin{aligned}
\left\|\left(I+r G_{A}\right)^{-1}\left(I-r G_{B}\right)\right\|_{2} & \leqslant\left\|\left(I+r G_{A}\right)^{-1}\right\|_{2}\left\|\left(I-r G_{B}\right)\right\|_{2} \\
& \leqslant\left\|\left(I-r G_{B}\right)\right\|_{2} \\
& =\max \{|1-r|,|1-2 r|, 1\}
\end{aligned}
$$

It can be easily shown that the eigenvalues of matrix $G_{A}$ and $G_{B}$, irrespective of the schemes described, are $1,1-2 \mathrm{r}$ and $1-\mathrm{r}$ (the number of 1 and $1-2 \mathbf{r}$ is the multiplicity of group). Therefore

$$
\left\|\left(I+r G_{A}\right)^{-1}\left(I-r G_{B}\right)\right\|_{2} \leqslant 1
$$

for all $r \leqslant 1$, i.e. similar to that of the earlier condition. This leads to the following theorem.

## Theorem 4.3

The GER scheme of (4.5.9), GEL scheme of (4.5.12), GEU scheme of (4.5.15) and the GEC scheme of (4.5.18) are conditionally stable for all $\mathrm{r} \leqslant 1$.

We now consider the stability of the two-step process (Single)
Alternating Group Explicit as in the equations (4.5.13), (4.5.19) and (4.6.1). Generally, these equations are written as

$$
\left.\begin{array}{l}
\left(I+r \theta G_{A}\right) \underline{u}_{j+\theta}=\left(I-r \theta G_{B}\right) \underline{u}_{j}+\underline{b}  \tag{4.9.9}\\
\left(I+r \theta G_{B}\right) \underline{u}_{j+2 \theta}=\left(I-r \theta G_{A}\right) \underline{u}_{j+\theta}+\underline{b}
\end{array}\right\}
$$

where

$$
\theta=\left\{\begin{array}{l}
1, \text { for equations }(4.5 .13) \text { and (4.5.19) } \\
\frac{1}{2}, \text { for equation }(4.6 .1)
\end{array}\right.
$$

Eliminating $\underline{u}_{j+\theta}$ from (4.9.9) gives

$$
\begin{equation*}
\underline{u}_{j+2 \theta}=\underline{T u}_{j}+\underline{b}^{\prime} \tag{4.9.10}
\end{equation*}
$$

where $\underline{b}^{\prime}$ is independent of $\underline{u}$ 's and

$$
\begin{equation*}
T=\left(I+r \theta G_{B}\right)^{-1}\left(I-r \theta G_{A}\right)\left(I+r \theta G_{A}\right)^{-1}\left(I-r \theta G_{B}\right) \tag{4.9.11}
\end{equation*}
$$

In proving the stability condition we will state another lemma of Kellogg (1964) which we employ below.

Lemma 4.2
If $\rho>0$ and ( $B+B *$ ) is non-negative (or positive) definite, then $(\rho I-B)(\rho I+B)^{-1}$ is bounded and $\left\|(\rho I-B)(\rho I+B)^{-1}\right\|_{2} \leqslant 1$.

Now we define the matrix,

$$
\begin{equation*}
\widetilde{T}=\left(I+r \theta G_{B}\right) T\left(I+\infty \theta G_{B}\right)^{-1}, \tag{4.9.12}
\end{equation*}
$$

which is similar to $T$ and thus has the same eigenvalues as $T$. With (4.9.11) it follows that we can express $\widetilde{T}$ as

$$
\begin{equation*}
\widetilde{T}=\left\{\left(I-r \theta G_{A}\right)\left(I+r G_{A}\right)^{-1}\right\}\left\{\left(I-r \theta G_{B}\right)\left(I+r \theta G_{B}\right)^{-1}\right\} \tag{4.9.13}
\end{equation*}
$$

From our knowledge of matrix norms and spectral radii, it is evident that

$$
\begin{equation*}
\rho(T)=\rho(\widetilde{T})=\|\widetilde{T}\|_{2} \leqslant\left\|\left(\operatorname{Fr} \theta G_{A}\right)\left(I+r \theta G_{A}\right)^{-1}\right\|_{2}\left\|\left(I+r \theta G_{B}\right)\left(I-r \theta G_{B}\right)^{-1}\right\|_{2} \tag{4.9.14}
\end{equation*}
$$

Since $r \theta\left(G_{A}{ }^{+} G_{A}^{*}\right)$ is non-negative (or positive definite for all cases of (4.5.13), (4.5.19) and (4.6.1), and $r \theta\left(G_{B}+G_{B}^{*}\right)$ can also be easily shown then from (4.9.14) and Lerma 4.2 we get the unconditional stability condition for schemes governed by the equation (4.9.9). Hence we can state the following theorem.

## Theorem 4.4

The (Single) Alternating Group Explicit method (4.5.13), (4.5.19) and (4.6.1) for the solution of the heat conduction equation (4.1.1) on the region $R$ of the open rectangle is unconditionally stable for all $r>0$.

The analysis of stability for the four-step process (Double) Alternating Group Explicit method as in equation (4.5.14), (4.5.20), (4.6.2) and (4.6.4) almost resembles the previous one. In general, these equations are written as

$$
\left.\begin{array}{l}
\left(I+r \theta G_{A}\right) \underline{u}_{j+\theta}=\left(I-r \theta G_{B}\right) \underline{u}_{j}+\underline{b}  \tag{4.9.15}\\
\left(I+r \theta G_{B}\right) \underline{u}_{j+2 \theta}=\left(I-r \theta G_{A}\right) \underline{u}_{j+\theta}+\underline{b} \\
\left(I+r \theta G_{B}\right) \underline{u}_{j+3 \theta}=\left(I-r \theta G_{A}\right) \underline{u}_{j+2 \theta}+\underline{b} \\
\left(I+r \theta G_{A}\right) \underline{u}_{j+4 \theta}=\left(I-r \theta G_{B}\right) \underline{u}_{j+4 \theta}^{3}+\underline{b}
\end{array}\right\}
$$

where

$$
\theta=\left\{\begin{array}{l}
1, \text { for equations }(4.5 .14) \text { and }(4.5 .20) \\
\frac{1}{2}, \text { for equation }(4.6 .2) \\
\frac{1}{4}, \text { for equation (4.6.4) }
\end{array}\right.
$$

Eliminating $\underline{u}_{j+\theta},{ }_{j}{ }_{j+2 \theta}$ and $u_{j+3 \theta}$ from (4.9.15) gives

$$
\begin{equation*}
\underline{u}_{j+4 \theta}=T^{\prime} \underline{u}_{j}+\underline{b}^{\prime \prime} \tag{4.9.16}
\end{equation*}
$$

where $\underline{b}$ " is independent of $\underline{u}$ 's and where

$$
\begin{equation*}
T^{\prime}=\left(I+r G_{A}\right)^{-1}\left(I-r \theta G_{B}\right)\left(I+r \theta G_{B}\right)^{-1}\left(I-r \theta G_{A}\right)\left(I+r \theta G_{B}\right)^{-1}\left(I-r \theta G_{A}\right)\left(I+r \theta G_{A}\right)^{-1}\left(I-r \theta G_{B}\right) \tag{4.9.17}
\end{equation*}
$$

Let us define
and

$$
\left.\begin{array}{rl}
T_{1}^{\prime} & =\left(I+r \theta G_{A}\right)^{-1}\left(I r \theta G_{B}\right)\left(I+r \theta G_{B}\right)^{-1}\left(I-r \theta G_{A}\right)  \tag{4.9.18}\\
T_{2}^{\prime} & =\left(I+r \theta G_{B}\right)^{-1}\left(I-r \theta G_{A}\right)\left(I+r \theta G_{A}\right)^{-1}\left(I-r \theta G_{B}\right)
\end{array}\right\}
$$

$$
\begin{equation*}
\text { then } \quad T^{\prime}=T_{1}^{\prime} T_{2}^{\prime} \tag{4.9.19}
\end{equation*}
$$

If $\widetilde{T}_{1}^{\prime}$ and $\tilde{T}_{2}^{\prime}$ are defined such that
$\left.\begin{array}{rlrl} & \widetilde{T}_{1}^{\prime} & =\left(I+r \theta G_{A}\right) T_{1}^{\prime}\left(I+r \theta G_{A}\right)^{-1} \\ \text { and } \quad & \widetilde{T}_{2}^{\prime} & =\left(I+r \theta G_{B}\right) T_{2}^{\prime}\left(I+r \theta G_{B}\right)^{-1}\end{array}\right\}$
then $\widetilde{T}_{1}^{\prime}$ and $\tilde{T}_{2}^{\prime}$ are similar to $T_{1}^{\prime}$ and $T_{2}^{\prime}$ respectively. Therefore $T_{1}^{\prime}$ and $\mathrm{T}_{2}^{\prime}$ have the same eigenvalues (and spectral radius) as $\tilde{\mathrm{T}}_{1}^{\prime}$ and $\tilde{\mathrm{T}}_{2}^{\prime}$ respectively. For stability we need $\left|\left|T^{\prime}\right|\right| \leqslant 1$
i.e., $\quad\left\|T^{\prime}\right\|_{2}=\left\|T_{1}^{\prime} T_{2}^{\prime}\right\|_{2}$

$$
\leqslant\left\|T_{1}^{\prime}\right\|_{2}\left\|T_{2}^{\prime}\right\|_{2}
$$

$$
=\rho\left(T_{1}^{\prime}\right) \rho\left(T_{2}^{\prime}\right)
$$

$$
=\rho\left(T_{1}^{\prime}\right) \rho\left(T_{2}^{\prime}\right)
$$

$$
=\left\|\left(I-r^{\theta} G_{B}\right)\left(I+r^{\theta} G_{B}\right)^{-1}\left(I-r^{\theta} G_{A}\right)\left(I+r^{\theta} G_{A}\right)^{-1}\right\|
$$

$$
\times\left\|\left(I-r^{\theta} G_{A}\right)\left(I+r^{\theta} G_{A}\right)^{-1}\left(I-r^{\theta} \mathcal{G}_{B}\right)\left(I+r^{\theta} G_{B}\right)^{-1}\right\|
$$

$$
\begin{equation*}
\leqslant\left\|\left(I-r \theta_{A}\right)\left(I+r^{\theta} G_{A}\right)^{-1}\right\|^{2}\left\|\left(I-r^{\theta} G_{B}\right)\left(I+r^{\theta} G_{B}\right)^{-1}\right\|^{2} \tag{4.9.21}
\end{equation*}
$$

Since in all cases $r \theta\left(G_{A}+G_{A}^{*}\right)$ and $r \theta\left(G_{B}+G_{B}^{*}\right)$ are positive (or nonnegative) definite matrices, therefore from Lemma 4.2 and (4.9.21) we get

$$
\begin{equation*}
\left\|T^{\prime}\right\|_{2} \leqslant 1 \tag{4.9.22}
\end{equation*}
$$

which means the schemes governed by (4.9.15) are unconditionally stable for all r>0. Thus the theorem below follows immediately.

Theorem 4.5
The (Double) Alternating Group Explicit method for the solution of the heat conduction equation (4.1.1) on the region $R$ of the open rectangle is unconditionally stable for all $r>0$.

### 4.10 STABILITY OF THE GROUP EXPLICIT METHOD: PERIODIC CASE

This case has to be treated differently because its matrix of coefficients is quite different from the previous one. Equations (4.7.2) and (4.7.5) can be explicitly written as

$$
\begin{equation*}
\underline{u}_{\mathrm{j}+1}=\mathrm{T}_{3} \underline{\mathrm{u}}_{\mathrm{j}} \tag{4.10.1}
\end{equation*}
$$

where for (4.7.2)

$$
\begin{equation*}
T_{3}=\left(I+r \hat{G}_{2}\right)^{-1}\left(I-r \hat{\hat{G}}_{1}\right) \tag{4.10.2}
\end{equation*}
$$

and for (4.7.5)

$$
\begin{equation*}
T_{3}=\left(I+r \hat{\hat{G}}_{1}\right)^{-1}\left(I-r \hat{G}_{2}\right) \tag{4.10.3}
\end{equation*}
$$

From equation (4.7.1) and (4.7.4) we can easily get that in the case of (4.7.2) $\mathrm{T}_{3}$ is given by

and in the case of (4.7.5) $\mathrm{T}_{3}$ is given by

| $T_{3}=\frac{1}{1+2 r}$ | $\left[\begin{array}{llll}1-r^{2} & r(1+r) & \vdots & \\ & \\ \end{array}\right.$ |
| :---: | :---: |
|  |  |
|  | $\begin{array}{c:ll:l} r(1+r) & 1-r^{2} & r(1-r) & r^{2} \\ r^{2} & r(1-r) & 1-r^{2} & r(1+r) \end{array}$ |
|  |  |
|  | rr(1-r) $\mathrm{r}^{2} \mathrm{l}$ |

In both cases of (4.10.4) and (4.10.5), we can easily verify that $\left\|T_{3}\right\|_{\infty} \leqslant 1$
if $r \leqslant 1$. This gives the similar stability condition as the corresponding GER, GEL, GEU and GEC schemes as discussed in the earlier section.

For the two-step (equations (4.7.6) and (4.7.8) and four-step (equations (4.7.7), (4.7.9) and (4.7.10) processes, since $r\left(\hat{\hat{G}}_{1}+\hat{\hat{G}}_{1}\right)^{2}$ ) and $r\left(\hat{G}_{2}+\hat{G}_{\hat{2}}\right)$ are both positive (non-negative) definite (see Appendix 1) and since they can be written respectively as equation (4.9.9) and (4.9.15) with $b=\underline{0}$, the unconditional stability condition is similarly evident.

It is observed that the two-step process Single (AGE) and four-step process Double (AGE), either in fractional form or not, for both cases of boundary conditions are unconditionally stable. While the one step processes, i.e. the GER, GEL, GEU and GEC schemes, still retain a larger stability condition compared to the ordinary explicit method. However, this conditional stability, is only to be expected since these schemes are formed from the combination of the unconditionally stable formulae (4.5.6) and (4.5.7) and the conditionally stable formula (4.5.5).

At this stage it is important to mention that the unconditional stability condition does not mean that the time increment can be taken indefinitely large, particularly at small time levels if the solution is to satisfactorily approximate that of the equation (4.1.1). This behaviour is common to all unconditionally stable explicit or semi-explicit methods.

## PAGE

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### 4.11 NUMERICAL EXAMPLES AND COMPARATIVE RESULTS

As a simple example, the following problem was considered

$$
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}, \quad 0 \leqslant x \leqslant 1
$$

with the initial condition

$$
\begin{equation*}
u(x, 0)=4 x(1-x), 0 \leqslant x \leqslant 1 \tag{4.11.1}
\end{equation*}
$$

and the boundary conditions,

$$
u(0, t)=u(1, t)=0, \quad t \geqslant 0
$$

The exact solution (Saul'yev, 1964, p.34) is given by

$$
\begin{equation*}
U(x, t)=\frac{32}{\pi^{3}} \sum_{k=1,3,5, \ldots, k^{3}}^{\infty} e^{-k^{2} \pi^{2} t} \sin (k \pi x) \tag{4.11.2}
\end{equation*}
$$

Experiment 1:
In this experiment, accuracies of the various schemes in the class of Group Explicit methods are compared. The schemes are implemented using various values of $r$. At each point, the absolute error (A.E.)

$$
\begin{equation*}
\left|e_{i, j}\right|=\left|u_{i, j}-u_{i, j}\right| \tag{4.11.3}
\end{equation*}
$$

and the percentage error (P.E.)

$$
\begin{equation*}
\% \text { error }=\frac{\left|e_{i, j}\right|}{\left|U_{i, j}\right|} \times 100 \tag{4.11.4}
\end{equation*}
$$

are calculated.
Tables (4.11.1) and (4.11.2) give the comparison of the absolute errors and the percentage errors of GER (equation 4.5.9), GEL (equation 4.5.12), (S)AGE (equation 4.5.13) and (D)AGE (equation 4.5.14) for $r=0.1$ after the 10 th and 50th time-steps respectively. Tables (4.11.3) and (4.11.4) give a similar comparison for $r=0.5$ and tables (4.11.5) and (4.11.6) for $r=1.0$.

## Experiment 2:

In this experiment, the accuracy of the fractional type of equation
is studied. Tables (4.11.7)-(4.11.10) give the comparison of the absolute errors and the percentage errors of the schemes (4.6.1), (4.6.2) and (4.6.4) for $r=0.1, r=0.5, r=1.0$ and $r=2.0$ after the 50th time-step.

## Experiment 3:

In this experiment, a comparison of some of the GE schemes with standard methods like CN, explicit and DuFort-Frankel are made for values of $\mathrm{r}=0.1,0.5,1.0,1.5$. Tables (4.11.11)-(4.11.14) show the comparison (in terms of percentage errors) for each after the 50th time-step.

## Experiment 4:

The behaviour of the percentage errors of the GER, GEL and (D)AGE schemes for $\mathrm{r}=0.1, \mathrm{x}=0.1,0.5,0.9$ at a certain time-interval are observed. The behaviour of the solutions of GER, GEL and (D)AGE for certain values of $r$ is also observed. These observations are given graphically in Figs. (4.11.1)-(4.11.3).

## Experiment 5:

In this experiment, the interval $x=[0,1]$ is divided into an odd number of sub-intervals, i.e. m=11. The accuracy of GEC, (S)AGE and (D) AGE are then compared, some of which are given in Tables (4.11.15)(4.11.17) for $\mathrm{r}=0.121,0.605$ and 1.21 . In this experiment the problem (4.11.1) is considered to have different initial conditions

$$
u(x, 0)= \begin{cases}2 x, & 0 \leqslant x \leqslant 0.5  \tag{4.11.5}\\ 2(1-x), & 0.5 \leqslant x \leqslant 1.0\end{cases}
$$

and hence the exact solution is (Smith, G.D., 1978, p.15),

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty} \frac{8}{\pi^{2} n^{2}} \sin (n \pi x) \sin \frac{n \pi}{2} e^{-n^{2} \pi^{2} t} \tag{4.11.16}
\end{equation*}
$$

## Experiment 6:

In this experiment, the (D)AGE scheme is implemented on the heat conduction problem with derivative boundary conditions, i.e.,

$$
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}, \quad 0 \leqslant x \leqslant 1
$$

with the initial condition

$$
\begin{equation*}
u(x, 0)=\sin x+\cos x, x \geqslant 0 \tag{4.11.7}
\end{equation*}
$$

and the boundary conditions

$$
\begin{align*}
& \frac{\partial u}{\partial x}(0, t)=e^{-t}, \quad t \geqslant 0  \tag{4.11.8}\\
& \frac{\partial u}{\partial x}(1, t)=e^{-t}(\cos (1.0)-\sin (1.0)), t \geqslant 0
\end{align*}
$$

with the exact solution (Mitchell and Wait, 1977, p.179),

$$
\begin{equation*}
u(x, t)=e^{-t}(\sin x+\cos x), \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0 \tag{4.11.9}
\end{equation*}
$$

Table (4.11.18) give the results of this problem compared with the exact solution (4.11.9) in terms of absolute error, for $r=0.1,0.5,1.0$ and 1.5. The derivatives at the boundary $x=0$ and 1.0 are approximated by the forward and backward difference formulae respectively.

## Experiment 7:

For this experiment, the heat conduction problem with periodic boundary conditions is considered. The problem is

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial t^{2}}+10(1-x) x t \tag{4.11.10}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
u(x, 0)=x(1-x) \tag{4.11.11}
\end{equation*}
$$

and the boundary conditions

$$
u(0, t)=u(1, t)
$$

The exact solution to the problem is given by

$$
\begin{align*}
u(x, t)= & \frac{\left(1+5 t^{2}\right)}{6}-\frac{5}{8} \sum_{n=1}^{\infty} \frac{\cos 2 n \pi x}{(n \pi)^{6}}\left\{4 n^{2} \pi^{2} t-1+e^{-4 n^{2} \pi^{2} t}\right\} \\
& -\sum_{n=1}^{\infty} \frac{e^{-4 n^{2} \pi^{2} t}}{n^{2} \pi^{2}} \cos 2 n \pi x . \tag{4.11.12}
\end{align*}
$$

For some values of $t$ and $x$ a proper comparison with the exact solution cannot be made as the solution (4.11.12) converges very slowly. Table (4.11.19) give a comparison of the numerical and exact solutions for some specimen values of $r$.
$\mathrm{r}=0.1, \mathrm{t}=0.01$ (10th step)

| Scheme $\quad x$ |  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GER | A.E. | 0.0001 | 0.0012 | 0.0004 | 0.00015 | 0.00005 | 0.00015 | 0.0007 | 0.0012 | 0.0030 |
|  | P.E. | 0.039 | 0.214 | 0.047 | 0.017 | 0.005 | 0.017 | 0.090 | 0.214 | 0.990 |
| GEL | A.E. | 0.0030 | 0.0012 | -0.0007 | 0.00015 | 0.00005 | 0.00015 | 0.0004 | 0.0012 | 0.0001 |
|  | P.E. | 0.990 | 0.214 | 0.090 | 0.017 | 0.005 | 0.017 | 0.047 | 0.214 | 0.039 |
| S (AGE) | A.E. | 0.0016 | 0.0010 | 0.0005 | 0.0001 | 0.00004 | 0.0001 | 0.0005 | 0.0013 | 0.0013 |
|  | P.E. | 0.520 | 0.182 | 0.064 | 0.014 | 0.004 | 0.017 | 0.061 | 0.226 | 0.439 |
| D (AGE) | A.E. | 0.0015 | 0.0011 | 0.0005 | 0.0001 | 0.00004 | 0.0001 | 0.00051 | 0.0012 | 0.0014 |
|  | P.E. | 0.492 | 0.196 | 0.064 | 0.015 | 0.004 | 0.016 | 0.061 | 0.212 | 0.467 |
| Exact <br> Solution |  | 0.3023887 | 0.5645662 | 0.7606421 | 0.8800478 | 0.9200077 | 0.8800478 | 0.7606421 | 0.5645662 | 0.3023887 |

First line is the absolute error (A.E.) and second line is percentage error (P.E.)
TABLE (4.11.1)

| Scheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GER | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.00006 \\ & 0.032 \end{aligned}$ | $\begin{aligned} & 0.0016 \\ & 0.443 \end{aligned}$ | $\begin{aligned} & 0.0015 \\ & 0.300 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 0.383 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 0.373 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 0.383 \end{aligned}$ | $\begin{aligned} & 0.0026 \\ & 0.519 \end{aligned}$ | 0.0016 <br> 0.443 | 0.0019 <br> 0.970 |
| GEL | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0019 \\ & 0.970 \end{aligned}$ | 0.0016 <br> 0.443 | $\begin{aligned} & 0.0026 \\ & 0.519 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 0.383 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 0.373 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 0.383 \end{aligned}$ | $\begin{aligned} & 0.0015 \\ & 0.300 \end{aligned}$ | 0.0016 <br> 0.443 | $\begin{aligned} & 0.00006 \\ & 0.032 \end{aligned}$ |
| S (AGE) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0009 \\ & 0.507 \end{aligned}$ | $\begin{aligned} & 0.0014 \\ & 0.390 \end{aligned}$ | $\begin{aligned} & 0.0020 \\ & 0.384 \end{aligned}$ | $\begin{gathered} 0.0020 \\ 0.342 \end{gathered}$ | 0.0021 <br> 0.338 | $\begin{aligned} & 0.0021 \\ & 0.353 \end{aligned}$ | $\begin{aligned} & 0.0019 \\ & 0.365 \end{aligned}$ | 0.0016 <br> 0.435 | 0.0008 <br> 0.416 |
| D (AGE) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0009 \\ & 0.470 \end{aligned}$ | $\begin{aligned} & 0.0015 \\ & 0.405 \end{aligned}$ | $\begin{aligned} & 0.0019 \\ & 0.379 \end{aligned}$ | 0.0021 <br> 0.347 | 0.0021 <br> 0.339 | 0.0021 <br> 0.351 | $\begin{aligned} & 0.0019 \\ & 0.372 \end{aligned}$ | 0.0016 <br> 0.421 | 0.0009 <br> 0.449 |
| Exact Solution |  | 0.1950648 | 0.37 P 7705 | 0.5098716 | 0.5989617 | 0.6296137 | 0.5989617 | 0.5098716 | 0.3707705 | 0.1950648 |

First line is the absolute error (A.E.) and the second line is percentage error (P.E.)
$r=0.5, t=0.05$ ( 10 th step)

| Scheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GER | A.E. | 0.0040 | 0.0015 | 0.0009 | 0.0020 | 0.0020 | 0.0020 | 0.0047 | 0.0015 | 0.0057 |
|  | P.E. | 2.0644 | 0.4155. | 0.1792 | 0.3366 | 0.3236 | 0.3366 | 0.9139 | 0.4155 | 2.9406 |
| GEL | A.E. | 0.0057 | 0.0015 | 0.0047 | 0.0020 | 0.0020 | 0.0020 | 0.0009 | 0.0015 | 0.0040 |
|  | P.E. | 2.9406 | 0.4155 | 0.9139 | 0.3366 | 0.3236 | 0.3366 | 0.1792 | 0.4155 | 2.0644 |
| S (AGE) | A.E. | 0.0028 | 0.0033 | 0.0012 | 0.0041 | 0.0035 | 0.0024 | 0.0038 | 0.0010 | 0.0017 |
|  | P.E. | 1.4528 | 0.8985 | 0.2451 | 0.6885 | 0.5600 | 0.4089 | 0.7375 | 0.2602 | 0.8609 |
| D (AGE) |  | 0.0027 | 0.0029 | 0.0004 | 0.0018 | 0.0011 | 0.0003 | 0.0023 | 0.0019 | 0.0013 |
|  | P.E. | 1.3698 | 0.7936 | 0.0790 | 0.3012 | 0.1803 | 0.0553 | 0.4428 | 0.5064 | 0.6826 |
| Exact Solution |  | 0.1950648 | 0.3707705 | 0.5098716 | 0.5989617 | 0.6296137 | 0.5989617 | 0.5098716 | 0.3707705 | 0.1950648 |

First line is the absolute error (A.E.) and the second line is the percentage error (P.E.)

[^0]$r=0.5, t=0.25$ (50th step)

| Scheme |  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GER | A.E. | 0.0001 | 0.0011 | 0.0011 | 0.0017 | 0.0018 | 0.0017 | 0.0019 | 0.0011 | 0.0012 |
|  | P.E. | 0.4213 | 2.0783 | 1.4858 | 2.0783 | 2.0751 | 2.0783 | 2.6644 | 2.0783 | 4.5715 |
| GEL | A.E. | 0.0012 | 0.0011 | 0.0019 | 0.0017 | 0.0018 | 0.0017 | 0.0011 | 0.0011 | 0.0001 |
|  | P.E. | 4.5715 | 2.0783 | 2.6644 | 2.0783 | 2.0751 | 2.0783 | 1.4858 | 2.0783 | 0.4213 |
| $S$ (AGE) | A.E. | 0.0002 | 0.0016 | 0.0018 | 0.0025 | 0.0025 | 0.0023 | 0.0021 | 0.0010 | 0.0008 |
|  | P.E. | 0.743 | 3.073 | 2.474 | 2.962 | 2.863 | 2.714 | 3.029 | 2.020 | 3.098 |
| D (AGE) | A.E. | 0.0002 | 0.0007 | 0.0003 | 0.0007 | 0.0007 | 0.0005 | 0.0007 | 0.0003 | 0.0003 |
|  | P.E. | 0.8389 | 1.3155 | 0.4557 | 0.8690 | 0.7534 | 0.6235 | 1.0178 | 0.0526 | 1.2330 |
| Exact <br> Solution |  | 0.0270461 | 0.0514447 | 0.0708075 | 0.0832392 | 0.0875229 | 0.0832392 | 0.0708075 | 0.0514447 | 0.0270461 |

First line is the absolute error (A.E.) and the second line is the percentage error (P.E.)
$\mathrm{r}=1.0, \mathrm{t}=0.1$, (10th step)

| Scheme | - | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GER | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0050 \\ & 4.1651 \end{aligned}$ | $\begin{aligned} & 0.0017 \\ & 0.7682 \end{aligned}$ | $\begin{aligned} & 0.0015 \\ & 0.4826 \end{aligned}$ | $\begin{aligned} & 0.0026 \\ & 0.6996 \end{aligned}$ | $\begin{aligned} & 0.0027 \\ & 0.7056 \end{aligned}$ | $\begin{aligned} & 0.0026 \\ & 0.6996 \end{aligned}$ | $\begin{aligned} & 0.0059 \\ & 1.8945 \end{aligned}$ | $\begin{aligned} & 0.0017 \\ & 0.7682 \end{aligned}$ | $\begin{aligned} & 0.0066 \\ & 5.5781 \end{aligned}$ |
| GEL | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0066 \\ & 5.5781 \end{aligned}$ | $\begin{aligned} & 0.0017 \\ & 0.7682 \end{aligned}$ | $\begin{aligned} & 0.0059 \\ & 1.8945 \end{aligned}$ | $\begin{aligned} & 0.0026 \\ & 0.6996 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 0.0027 \\ & 0.7056 \end{aligned}\right.$ | $\begin{aligned} & 0.0026 \\ & 0.6996 \end{aligned}$ | $0.0015$ | $\begin{aligned} & 0.0017 \\ & 0.7682 \end{aligned}$ | $\begin{aligned} & 0.0050 \\ & 4.1651 \end{aligned}$ |
| (S) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 1.957 \end{aligned}$ | $\begin{aligned} & 0.0187 \\ & 8.260 \end{aligned}$ | $\begin{aligned} & 0.0175 \\ & 5.636 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 0.0284 \\ & 7.751 \end{aligned}\right.$ | $\begin{aligned} & 0.0281 \\ & 7.310 \end{aligned}$ | $\begin{aligned} & 0.0244 \\ & 6.675 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 0.0251 \\ & 8.050 \end{aligned}\right.$ | $\begin{aligned} & 0.0082 \\ & 3.641 \end{aligned}$ | $\begin{aligned} & 0.0099 \\ & 8.359 \end{aligned}$ |
| (D) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0053 \\ & 4.44 \end{aligned}$ | $\begin{aligned} & 0.0095 \\ & 4.21 \end{aligned}$ | $\begin{aligned} & 0.0022 \\ & 0.70 \end{aligned}$ | $\begin{aligned} & 0.0059 \\ & 1.61 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 0.0052 \\ & 1.35 \end{aligned}\right.$ | $\left\lvert\, \begin{aligned} & 0.0029 \\ & 0.78 \end{aligned}\right.$ | $\left\lvert\, \begin{aligned} & 0,0061 \\ & 1,97 \end{aligned}\right.$ | $\left\lvert\, \begin{aligned} & 0.0025 \\ & 1.12 \end{aligned}\right.$ | $\begin{aligned} & 0.0046 \\ & 3.83 \end{aligned}$ |
| EXACT SOLUTION |  | 0.1188685 | 0.2260983 | 0.3111923 | 0.3658234 | 0.3846475 | 0.3658234 | 0.3111923 | 0.2260983 | 0.1188685 |

First line is the absolute error (A.E.) and the second line is percentage error (P.E.)
$r=1.0, t=0.5$ (50th step)

| heme |  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\begin{aligned} & \text { GER } \\ & (4.5 .9) \end{aligned}$ | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 9 \times 10^{-6} \\ & 0.4185 \end{aligned}$ | $\begin{aligned} & 0.0002 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.0002 \\ & 3.4864 \end{aligned}$ | $\begin{aligned} & 0.0003 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.0003 \\ & 4.6930 \end{aligned}$ | $\begin{aligned} & 0.0003 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.0004 \\ & 5.9000 \end{aligned}$ | $\begin{aligned} & 0.0002 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.0002 \\ & 9.8045 \end{aligned}$ |
| $\begin{aligned} & \text { GEL } \\ & (4.5 .12) \end{aligned}$ | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.00022 \\ & 9.8045 \end{aligned}$ | $\begin{aligned} & 0.00021 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.00035 \\ & 5.9000 \end{aligned}$ | $\begin{aligned} & 0.00033 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.00035 \\ & 4.6930 \end{aligned}$ | $\begin{aligned} & 0.00033 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 0.00021 \\ & 3.4864 \end{aligned}$ | $\begin{aligned} & 0.00021 \\ & 4.7062 \end{aligned}$ | $\begin{aligned} & 9 \times 10^{-6} \\ & 0.4185 \end{aligned}$ |
| $\begin{aligned} & S(A G E) \\ & (4.5 .13) \end{aligned}$ | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.0006 \\ & 26.5 \end{aligned}$ | $\begin{aligned} & 0.0015 \\ & 33.9 \end{aligned}$ | $\begin{aligned} & 0.0019 \\ & 32.0 \end{aligned}$ | $\begin{aligned} & 0.0024 \\ & 33.5 \end{aligned}$ | $\begin{aligned} & 0.0025 \\ & 33.2 \end{aligned}$ | $\begin{aligned} & 0.0023 \\ & 32.7 \end{aligned}$ | $\begin{aligned} & 0.0020 \\ & 33.7 \end{aligned}$ | $\begin{aligned} & 0.0013 \\ & 30.5 \end{aligned}$ | $\begin{aligned} & 0.0008 \\ & 33.9 \end{aligned}$ |
| $\begin{aligned} & \mathrm{D}(\mathrm{AGE}) \\ & (4.5 .14) \end{aligned}$ | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.00004 \\ & 1.56 \end{aligned}$ | $\begin{aligned} & 0.00030 \\ & 6.80 \end{aligned}$ | $\begin{aligned} & 0.00020 \\ & 3.37 \end{aligned}$ | $\begin{aligned} & 0.00030 \\ & 4.29 \end{aligned}$ | $\begin{aligned} & 0.00030 \\ & 4.03 \end{aligned}$ | $\begin{aligned} & 0.00024 \\ & 3.45 \end{aligned}$ | $\begin{aligned} & 0.00028 \\ & 4.60 \end{aligned}$ | $\begin{aligned} & 0.00007 \\ & 1.57 \end{aligned}$ | $\begin{aligned} & 0.00015 \\ & 6.38 \\ & \hline \end{aligned}$ |
| EXACT SOLUTION |  | 0.0022936 | 0.0043628 | 0.00600483 | 0.0070591 | 0.0074224 | 0.0070591 | 0.00600483 | 0.0043628 | 0.0022936 |

First line is the absolute error (A.E.) and the second line is percentage error (P.E.)

TABLE (4.11.6)
$r=0.1, \Delta t=0.001, t=0.05$ (50th step)

| e | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (S)AGE A.E. HALF P.E. | $\begin{aligned} & 9.3466 \times 10^{-4} \\ & 0.479 \end{aligned}$ | $\begin{aligned} & 15.9499 \times 10^{-4} \\ & 0.430 \end{aligned}$ | $\begin{aligned} & 20.6200 \times 10^{-4} \\ & 0.404 \end{aligned}$ | $\begin{aligned} & 22.4217 \times 10^{-4} \\ & 0.374 \end{aligned}$ | $\begin{aligned} & 23.0245 \times 10^{-4} \\ & 0.366 \end{aligned}$ | $\begin{aligned} & 22.5814 \times 10^{-4} \\ & 0.377 \end{aligned}$ | $\begin{aligned} & 20.3814 \times 10^{-4} \\ & 0.400 \end{aligned}$ | $\begin{aligned} & 16.3671 \times 10^{-4} \\ & 0.441 \end{aligned}$ | $\begin{aligned} & 8.9063 \times 10^{-4} \\ & 0.457 \end{aligned}$ |
| $\begin{array}{ll} \text { (D)AGE } & \text { A.E. } \\ \text { HALF } & \text { P.E. } \\ \text { STEP } & \end{array}$ | $\begin{aligned} & 9.0948 \times 10^{-4} \\ & 0.466 \end{aligned}$ | $\begin{aligned} & 16.1951 \times 10^{-4} \\ & 0.437 \end{aligned}$ | $\begin{aligned} & 20.4765 \times 10^{-4} \\ & 0.402 \end{aligned}$ | $\begin{aligned} & 22.5184 \times 10^{-4} \\ & 0.376 \end{aligned}$ | $\begin{aligned} & 23.0290 \times 10^{-4} \\ & 0.365 \end{aligned}$ | $\begin{aligned} & 22.4928 \times 10^{-4} \\ & 0.376 \end{aligned}$ | $\begin{aligned} & 20.5305 \times 10^{-4} \\ & 0.403 \end{aligned}$ | $\begin{aligned} & 16.1243 \times 10^{-4} \\ & 0.435 \end{aligned}$ | $\begin{aligned} & 9.1585 \times 10^{-4} \\ & 0.470 \end{aligned}$ |
| (D)AGE A.E. २TR. P.E. | $\begin{aligned} & 0.1534 \times 10^{-4} \\ & 0.549 \end{aligned}$ | $\begin{aligned} & 16.3812 \times 10^{-4} \\ & 0.442 \end{aligned}$ | $\begin{aligned} & 20.8466 \times 10^{-4} \\ & 0.409 \end{aligned}$ | $\begin{aligned} & 22.9235 \times 10^{-4} \\ & 0.383 \end{aligned}$ | $\begin{aligned} & 23.4665 \times 10^{-4} \\ & 0.373 \end{aligned}$ | $\begin{aligned} & 22.9203 \times 10^{-4} \\ & 0.383 \end{aligned}$ | $\begin{aligned} & 20.8534 \times 10^{-4} \\ & 0.409 \end{aligned}$ | $\begin{aligned} & 16.3723 \times 10^{-4} \\ & 0.442 \end{aligned}$ | $\begin{aligned} & 9.614 \times 10^{-4} \\ & 0.470 \end{aligned}$ |
| EXACT <br> SOLUTION | 0.1950648 | 0.3707705 | 0.5098716 | 0.5989617 | 0.6296137 | 0.5989617 | 0.5098716 | 0.3707705 | 0.1950648 |

$r=0.5, \Delta t=0.005, t=0.25$ (50th step)

| Schem | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (S)AGE half STEP | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 3.6602 \times 10^{-4} \\ & 1.353 \end{aligned}$ | $\begin{aligned} & 3.9462 \times 10^{-4} \\ & 0.767 \end{aligned}$ | $\begin{aligned} & 6.4997 \times 10^{-4} \\ & 0.918 \end{aligned}$ | $\begin{aligned} & 6.6179 \times 10^{-4} \\ & 0.795 \end{aligned}$ | $\begin{aligned} & 7.1775 \times 10^{-4} \\ & 0.820 \end{aligned}$ | $\begin{aligned} & 7.1386 \times 10^{-4} \\ & 0.858 \end{aligned}$ | $\begin{aligned} & 5.5099 \times 10^{-4} \\ & 0.778 \end{aligned}$ | $\begin{aligned} & 5.3094 \times 10^{-4} \\ & 1.032 \end{aligned}$ | $\begin{aligned} & 2.0579 \times 10^{-4} \\ & 0.761 \end{aligned}$ |
| (D) AGE HALF STEP | A.E. | $\begin{aligned} & 2.7866 \times 10^{-4} \\ & 1.030 \end{aligned}$ | $\begin{aligned} & 6.0368 \times 10^{-4} \\ & 1.173 \end{aligned}$ | $\begin{aligned} & 6.8814 \times 10^{-4} \\ & 0.972 \end{aligned}$ | $\begin{aligned} & 8.6351 \times 10^{-4} \\ & 1.037 \end{aligned}$ | $\begin{aligned} & 8.8297 \times 10^{-4} \\ & 1.009 \end{aligned}$ | $\begin{aligned} & 8.2293 \times 10^{-4} \\ & 0.989 \end{aligned}$ | $\begin{aligned} & 7.6631 \times 10^{-4} \\ & 1.082 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 4.9133 \times 10^{-4} \\ & 0.955 \end{aligned}\right.$ | $\begin{aligned} & 3.7489 \times 10^{-4} \\ & 1.386 \end{aligned}$ |
| (D) AGE <br> QTR <br> STEP | A.E. | $\begin{aligned} & 4.8143 \times 10^{-4} \\ & 1.780 \end{aligned}$ | $\begin{aligned} & 9.1641 \times 10^{-4} \\ & 1.781 \end{aligned}$ | $\begin{aligned} & 12.3655 \times 10^{-4} \\ & 1.746 \end{aligned}$ | $\begin{aligned} & 14.5913 \times 10^{-4} \\ & 1.753 \end{aligned}$ | $\left\{\begin{array}{l} 15.2990 \times 10^{-4} \\ 1.748 \end{array}\right.$ | $\left\lvert\, \begin{aligned} & 14.5325 \times 10^{-4} \\ & 1.746 \end{aligned}\right.$ | $\begin{aligned} & 12.4778 \times 10^{-4} \\ & 1.762 \end{aligned}$ | $\begin{aligned} & 9.0079 \times 10^{-4} \\ & 1.751 \end{aligned}$ | $\begin{aligned} & 4.9474 \times 10^{-4} \\ & 1.829 \end{aligned}$ |
| EXACT SOLUTION |  | 0.02704606 | 0.05144467 | 0.07080751 | 0.08323922 | 0.08752290 | 0.08323922 | 0.07080751 | 0.05144467 | 0.02704606 |

TABLE (4.11.8)
$r=1.0, \Delta t=0.01, t=0.5$ (50th step)

| Scheme | ${ }^{\mathbf{x}}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (S) AGE <br> HALF <br> STEP | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 8.188 \times 10^{-5} \\ & 3.570 \end{aligned}$ | $\begin{aligned} & 25.452 \times 10^{-5} \\ & 5.834 \end{aligned}$ | $\begin{aligned} & 31.536 \times 10^{-5} \\ & 5.252 \end{aligned}$ | $\begin{aligned} & 40.421 \times 10^{-5} \\ & 5.726 \end{aligned}$ | $\begin{aligned} & 41.785 \times 10^{-5} \\ & 5.630 \end{aligned}$ | $\begin{aligned} & 38.718 \times 10^{-5} \\ & 5.485 \end{aligned}$ | $\begin{aligned} & 34.776 \times 10^{-5} \\ & 5.791 \end{aligned}$ | $\begin{aligned} & 20.991 \times 10^{-5} \\ & 4.811 \end{aligned}$ | $\begin{aligned} & 13.436 \times 10^{-5} \\ & 5.858 \end{aligned}$ |
| (D) AGE <br> HALF <br> STEP | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 4.417 \times 10^{-5} \\ & 1.926 \end{aligned}$ | $\begin{aligned} & 3.288 \times 10^{-5} \\ & 0.754 \end{aligned}$ | $\begin{aligned} & 1.0281 \times 10^{-4} \\ & 1.712 \end{aligned}$ | $\begin{aligned} & 9.322 \times 10^{-5} \\ & 1.321 \end{aligned}$ | $\begin{aligned} & 10.758 \times 10^{-5} \\ & 1.449 \end{aligned}$ | $\left\{\begin{array}{l} 11.025 \times 10^{-5} \\ 1.563 \end{array}\right.$ | $\begin{aligned} & 6.929 \times 10^{-5} \\ & 1.154 \end{aligned}$ | $\begin{aligned} & 8.759 \times 10^{-5} \\ & 2.008 \end{aligned}$ | $\begin{aligned} & 0.302 \times 10^{-5} \\ & 0.132 \end{aligned}$ |
| (D) AGE <br> QTR. <br> STEP | A.E. P.E. | $\begin{aligned} & 4.731 \times 10^{-5} \\ & 2.063 \end{aligned}$ | $\begin{aligned} & 9.630 \times 10^{-5} \\ & 2.207 \end{aligned}$ | $\begin{aligned} & 12.032 \times 10^{-5} \\ & 2.004 \end{aligned}$ | $\left\{\begin{array}{l} 14.612 \times 10^{-5} \\ 2.070 \end{array}\right.$ | $\begin{aligned} & 15.150 \times 10^{-5} \\ & 2.041 \end{aligned}$ | $\begin{aligned} & 14.264 \times 10^{-5} \\ & 2.021 \end{aligned}$ | $\begin{aligned} & 12.701 \times 10^{-5} \\ & 2.115 \end{aligned}$ | $\begin{aligned} & 8.668 \times 10^{-5} \\ & 1.987 \end{aligned}$ | $\begin{aligned} & 5.556 \times 10^{-5} \\ & 2.422 \end{aligned}$ |
| EXACT SOLUTION |  | $2.2936 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $7.4224 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $4.3628 \times 10^{-2}$ | $2.2936 \times 10^{-3}$ |

TABLE (4.11.9)
$r=2.0, \Delta t=0.02, t=1.0$ (50th step)

| cheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (S) AGE HALF <br> STEP | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 8.44 \times 10^{-6} \\ & 51.168 \end{aligned}$ | $\begin{aligned} & 1.759 \times 10^{-5} \\ & 56.066 \end{aligned}$ | $\begin{aligned} & 2.367 \times 10^{-5} \\ & 54.813 \end{aligned}$ | $\begin{aligned} & 2.835 \times 10^{-5} \\ & 55.835 \end{aligned}$ | $\begin{aligned} & 2.969 \times 10^{-5} \\ & 55.628 \end{aligned}$ | $\begin{aligned} & 2.808 \times 10^{-5} \\ & 55.316 \end{aligned}$ | $\begin{aligned} & 2.417 \times 10^{-5} \\ & 55.975 \end{aligned}$ | $\begin{aligned} & 1.690 \times 10^{-5} \\ & 53.860 \end{aligned}$ | $\begin{aligned} & 9.26 \times 10^{-6} \\ & 56.118 \end{aligned}$ |
| (D) AGE <br> HALF <br> STEP | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 1.57 \times 10^{-6} \\ & 9.530 \end{aligned}$ | $\begin{aligned} & 1.53 \times 10^{-6} \\ & 4.874 \end{aligned}$ | $\begin{aligned} & 3.37 \times 10^{-6} \\ & 7.808 \end{aligned}$ | $\begin{aligned} & 3.40 \times 10^{-6} \\ & 6.698 \end{aligned}$ | $\begin{aligned} & 3.87 \times 10^{-6} \\ & 7.255 \end{aligned}$ | $\begin{aligned} & 3.81 \times 10^{-6} \\ & 7.508 \end{aligned}$ | $\begin{aligned} & 2.86 \times 10^{-6} \\ & 6.614 \end{aligned}$ | $\begin{aligned} & 3.12 \times 10^{-6} \\ & 9.934 \end{aligned}$ | $\begin{aligned} & 0.31 \times 10^{-6} \\ & 1.856 \end{aligned}$ |
| (D) AGE <br> QTR <br> STEP | A.E. P.E. | $\begin{aligned} & 5.4 \times 10^{-7} \\ & 3.297 \end{aligned}$ | $\begin{aligned} & 6.7 \times 10^{-7} \\ & 2.141 \end{aligned}$ | $\begin{aligned} & 13.3 \times 10^{-7} \\ & 3.086 \end{aligned}$ | $\begin{aligned} & 13.7 \times 10^{-7} \\ & 2.700 \end{aligned}$ | $\begin{aligned} & 15.1 \times 10^{-7} \\ & 2.827 \end{aligned}$ | $\begin{aligned} & 14.9 \times 10^{-7} \\ & 2.939 \end{aligned}$ | $\begin{aligned} & 10.9 \times 10^{-7} \\ & 2.535 \end{aligned}$ | $\begin{aligned} & 10.6 \times 10^{-7} \\ & 2.277 \end{aligned}$ | $\begin{aligned} & 2.1 \times 10^{-7} \\ & 1.268 \end{aligned}$ |
| EXACT SOLUTION |  | $1.650 \times 10^{-5}$ | $3.138 \times 10^{-5}$ | $4.319 \times 10^{-5}$ | $5.077 \times 10^{-5}$ | $5.338 \times 10^{-5}$ | $5.077 \times 10^{-5}$ | $4.319 \times 10^{-5}$ | $3.138 \times 10^{-5}$ | $1.650 \times 10^{-5}$ |

TABLE (4.11.10)
$r=0.1, \Delta t=0.001, t=0.1$

| Scheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| CN | P.E. | 0.811 | 0.809 | 0.808 | 0.807 | 0.806 | 0.807 | 0.808 | 0.809 | 0.811 |
| Explicit | P.E. | 0.324 | 0.324 | 0.323 | 0.323 | 0.323 | 0.323 | 0.323 | 0.324 | 0.324 |
| DFF | P.E. | 0.716 | 0.710 | 0.714 | 0.708 | 0.712 | 0.708 | 0.714 | 0.710 | 0.716 |
| GER | P.E. | 0.317 | 0.809 | 0.691 | 0.807 | 0.806 | 0.807 | 0.924 | 0.809 | 1.304 |
| (D)AGE | P.E. | 0.759 | 0.754 | 0.733 | 0.734 | 0.731 | 0.731 | 0.741 | 0.738 | 0.784 |
| (D)AGE <br> QTR. STEP | P.E. | 0.809 | 0.806 | 0.804 | 0.803 | 0.802 | 0.803 | 0.804 | 0.806 | 0.809 |
| EXACT SOLUTION | 0.118869 | 0.226098 | 0.311192 | 0.365823 | 0.384647 | 0.365823 | 0.311192 | 0.226098 | 0.118869 |  |

$r=0.5, \Delta t=0.005, t=0.5$

| Scheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| CN | P.E. | 4.022 | 4.022 | 4.022 | 4.022 | 4.022 | 4.022 | 4.022 | 4.022 |  |
| Explicit | P.E. | 7.945 | 8.061 | 7.945 | 8.061 | 7.945 | 8.061 | 7.945 | 8.061 | 7.945 |
| DFF | P.E. | 7.945 | 8.061 | 7.945 | 8.061 | 7.945 | 8.061 | 7.945 | 8.061 | 7.945 |
| GER | P.E. | 1.716 | 4.270 | 3.664 | 4.270 | 4.266 | 4.270 | 4.868 | 4.270 | 6.816 |
| (D)AGE | P.E. | 1.926 | 0.754 | 1.712 | 1.321 | 1.449 | 1.563 | 1.154 | 2.008 | 0.312 |
| (D)AGE <br> QTR. STEP | P.E. | 3.566 | 3.568 | 3.532 | 3.539 | 3.534 | 3.531 | 3.548 | 3.537 | 3.616 |
| EXACT SOLUTION | 0.002294 | 0.004363 | 0.006005 | 0.007059 | 0.007422 | 0.007059 | 0.006005 | 0.004363 | 0.002294 |  |


| $x$ <br> Scheme | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CN | 7.577 | 7.577 | 7.577 | 7.577 | 7.577 | 7.577 | 7.577 | 7.577 | 7.577 |
| GER | 4.534 | 9.913 | 8.633 | 9.913 | 9.900 | 9.913 | 11.166 | 9.913 | 15.265 |
| (D) AGE | 9.530 | 4.874 | 7.808 | 6.670 | 7.255 | 7.508 | 6.614 | 9.934 | 1.856 |
| (D) AGE HALF | 3.297 | 2.141 | 3.086 | 2.700 | 2.827 | 2.939 | 2.535 | 3.377 | 1.268 |
| (D) AGE QTR. | 4.159 | 4.307 | 4.099 | 4.167 | 4.137 | 4.116 | 4.213 | 4.082 | 4.526 |
| (S) AGE HALF | 8.985 | 11.121 | 10.572 | 11.020 | 10.929 | 10.792 | 11.081 | 10.156 | 11.144 |
| EXACT SOLUTION | $1.650 \times 10^{-\frac{8}{4}}$ | $3.138 \times 10^{-4}$ | $4.319 \times 10^{-7}$ | $5.077 \times 10^{-4}$ | $5.338 \times 10^{-4}$ | $5.077 \times 10^{-4}$ | $4.319 \times 10^{-4}$ | $3.138 \times 10^{-x_{\text {a }}}$ | $1.650 \times 10^{-8}$ |

TABLE (4.11.13)
$r=1.5, \Delta t=0.015, t=0.75$

| Scheme | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CN | 4.855 | 4.855 | 4.855 | 4.855 | 4.855 | 4.855 | 4.855 | 4.855 | 4.855 |
| (S) AGE |  |  |  |  |  |  | 7 |  |  |
| (D) AGE HALF | 6.660 | 3.815 | 5.764 | 4.967 | 5.236 | 5.448 | 4.723 | 6.727 | 1.967 |
| (D)AGE QTR. | 0.2046 | 0.7439 | 0.2185 | 0.4198 | 0.3472 | 0.2868 | 0.5228 | 0.1089 | 1.2444 |
| (S) AGE HALF | 21.126 | 25.401 | 24.304 | 25.198 | 25.016 | 24.744 | 25.321 | 23.472 | 25.446 |
| EXACT SOLUTION | $0.194 \times 10^{-3}$ | $0.3700 \times 10^{-3}$ | $0.5092 \times 10^{-3}$ | $0.598 \times 10^{-3}$ | $0.6295 \times 10^{-3}$ | $0.5987 \times 10^{-3}$ | $0.5092 \times 10^{-3}$ | $0.3700 \times 10^{-3}$ | $0.1945 \times 10^{-3}$ |

TABLE (4.11.14)

## PERCENT. ERROR



$r=0.121, \Delta x=1 / 11, \Delta t=0.00 /, t=0.09$

|  |  | 1/11 | 2/11 | 3/11 | 4/11 | 5/11 | 6/11 | 7/11 | 8/11 | 9/11 | 10/11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GEC | A.E. | $\begin{aligned} & 1.17 \times 10^{-4} \\ & 0.125 \end{aligned}$ | $\begin{aligned} & 6.64 \times 10^{-4} \\ & 0.369 \end{aligned}$ | $\begin{aligned} & 6.19 \times 10^{-4} \\ & 0.246 \end{aligned}$ | $\begin{aligned} & 11.33 \times 10^{-4} \\ & 0.374 \end{aligned}$ | $\begin{aligned} & 11.62 \times 10^{-4} \\ & 0.352 \end{aligned}$ | $\begin{aligned} & 12.17 \times 10^{-4} \\ & 0.369 \end{aligned}$ | $\begin{aligned} & 12.97 \times 10^{-4} \\ & 0.428 \end{aligned}$ | $\begin{aligned} & 8.80 \times 10^{-4} \\ & 0.350 \end{aligned}$ | $\begin{aligned} & 10.05 \times 10^{-4} \\ & 0.559 \end{aligned}$ | $\begin{aligned} & 2.79 \times 10^{-4} \\ & 0.298 \end{aligned}$ |
| (S) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 3.49 \times 10^{-4} \\ & 0.372 \end{aligned}$ | $\begin{aligned} & 4.77 \times 10^{-4} \\ & 0.265 \end{aligned}$ | $\begin{aligned} & 7.69 \times 10^{-4} \\ & 0.306 \end{aligned}$ | $\begin{aligned} & 8.82 \times 10^{-4} \\ & 0.291 \end{aligned}$ | $\begin{aligned} & 10.33 \times 10^{-4} \\ & 0.314 \end{aligned}$ | $\begin{aligned} & 10.89 \times 10^{-4} \\ & 0.331 \end{aligned}$ | $\begin{aligned} & 1.0 .46 \times 10^{-4} \\ & 0.345 \end{aligned}$ | $\begin{aligned} & 10.31 \times 10^{-4} \\ & 0.410 \end{aligned}$ | $\begin{aligned} & 8.18 \times 10^{-4} \\ & 0.455 \end{aligned}$ | $\begin{aligned} & 7.45 \times 10^{-4} \\ & 0.797 \end{aligned}$ |
| (D) AGE | A.E. | $\begin{aligned} & 3.16 \times 10^{-4} \\ & 0.337 \end{aligned}$ | $\begin{aligned} & 5.11 \times 10^{-4} \\ & 0.284 \end{aligned}$ | $\begin{aligned} & 7.61 \times 10^{-4} \\ & 0.302 \end{aligned}$ | $\left\{\begin{array}{l} 9.11 \times 10^{-4} \\ 0.301 \end{array}\right.$ | $\begin{aligned} & 10.48 \times 10^{-4} \\ & 0.318 \end{aligned}$ | $\begin{aligned} & 11.03 \times 10^{-4} \\ & 0.335 \end{aligned}$ | $\begin{aligned} & 10.75 \times 10^{-4} \\ & 0.355 \end{aligned}$ | $\begin{aligned} & 10.23 \times 10^{-4} \\ & 0.407 \end{aligned}$ | $\begin{aligned} & 8.52 \times 10^{-4} \\ & 0.474 \end{aligned}$ | $\begin{aligned} & 7.12 \times 10^{-4} \\ & 0.762 \end{aligned}$ |
| EXACT <br> SOLUTION |  | 0.093909 | 0.180027 | 0.251668 | 0.302916 | 0.329600 | 0.329545 | 0.302752 | 0.251407 | 0.179686 | 0.093413 |

TABLE (4.11.15)
$r=0.605, \Delta x=1 / 11, \Delta t=0.005, t=0.45$

|  | x | 1/11 | 2/11 | 3/11 | 4/11 | 5/11 | 6/11 | 7/11 | 8/11 | 9/11 | 10/11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GEC | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 2.44 \times 10^{-5} \\ & 0.911 \end{aligned}$ | $\begin{aligned} & 8.09 \times 10^{-5} \\ & 1.574 \end{aligned}$ | $\begin{aligned} & 6.53 \times 10^{-5} \\ & 0.909 \end{aligned}$ | $\begin{aligned} & 13.00 \times 10^{-5} \\ & 1.503 \end{aligned}$ | $\begin{aligned} & 12.58 \times 10^{-5} \\ & 1.336 \end{aligned}$ | $\begin{aligned} & 12.75 \times 10^{-5} \\ & 1.355 \end{aligned}$ | $\begin{aligned} & 13.50 \times 10^{-5} \\ & 1.562 \end{aligned}$ | $\left\{\begin{array}{l} 7.32 \times 10^{-5} \\ 1.019 \end{array}\right.$ | $\left\{\begin{array}{l} 9.11 \times 10^{-5} \\ 1.775 \end{array}\right.$ | $\begin{aligned} & 1.29 \times 10^{-5} \\ & 0.483 \end{aligned}$ |
| (S) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 1.47 \times 10^{-4} \\ & 5.47 \end{aligned}$ | $\begin{aligned} & 4.22 \times 10^{-4} \\ & 8.20 \end{aligned}$ | $\begin{aligned} & 5.36 \times 10^{-4} \\ & 7.46 \end{aligned}$ | $\begin{aligned} & 7.00 \times 10^{-4} \\ & 8.09 \end{aligned}$ | $\left\{\begin{array}{l} 7.42 \times 10^{-4} \\ 7.88 \end{array}\right.$ | $\begin{aligned} & 7.42 \times 10^{-4} \\ & 7.87 \end{aligned}$ | $\begin{aligned} & 6.95 \times 10^{-4} \\ & 8.04 \end{aligned}$ | $\begin{aligned} & 5.28 \times 10^{-4} \\ & 7.35 \end{aligned}$ | $\begin{aligned} & 4.11 \times 10^{-4} \\ & 8.02 \end{aligned}$ | $\begin{aligned} & 1.35 \times 10^{-4} \\ & 5.06 \end{aligned}$ |
| (D) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.370 \times 10^{-5} \\ & 0.138 \end{aligned}$ | $\begin{aligned} & 14.295 \times 10^{-5} \\ & 2.780 \end{aligned}$ | $\begin{aligned} & 11.857 \times 10^{-5} \\ & 1.650 \end{aligned}$ | $\begin{aligned} & 18.804 \times 10^{-5} \\ & 2.174 \end{aligned}$ | $\begin{aligned} & 18.125 \times 10^{-5} \\ & 1.926 \end{aligned}$ | $\begin{aligned} & 17.954 \times 10^{-5} \\ & 1.908 \end{aligned}$ | $\begin{aligned} & 18.305 \times 10^{-5} \\ & 2.117 \end{aligned}$ | $\left\{\begin{array}{l} 11.070 \times 10^{-5} \\ 1.542 \end{array}\right.$ | $\left\{\begin{array}{l} 13.284 \times 10^{-5} \\ 2.589 \end{array}\right.$ | $\begin{aligned} & 0.784 \times 10^{-5} \\ & 0.294 \end{aligned}$ |
| EXACT SOLUTION |  | 0.002680 | 0.005142 | 0.007187 | 0.008650 | 0.009411 | 0.009409 | 0.008645 | 0.007179 | 0.005132 | 0.002668 |

TABLE (4.11.16)
$r=1.21, \Delta x=1 / 11, \Delta t=0.01, t=0.5$

|  |  | 1/11 | 2/11 | 3/11 | 4/11 | 5/11 | 6/11 | 7/11 | 8/11 | 9/11 | 10/11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $\bigcirc$ | $\bigcirc$ | - | - | - | - | - | - | - |
| (S) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 5.746 \times 10^{-4} \\ & 35.138 \end{aligned}$ | $\begin{aligned} & 1.359 \times 10^{-3} \\ & 43.315 \end{aligned}$ | $\begin{aligned} & 1.803 \times 10^{-3} \\ & 41.116 \end{aligned}$ | $\begin{aligned} & 2.271 \times 10^{-3} \\ & 43.028 \end{aligned}$ | $\begin{aligned} & 2.437 \times 10^{-3} \\ & 42.434 \end{aligned}$ | $\begin{aligned} & 2.436 \times 10^{-3} \\ & 42.423 \end{aligned}$ | $\begin{aligned} & 2.268 \times 10^{-3} \\ & 42.995 \end{aligned}$ | $\begin{aligned} & 1.799 \times 10^{-3} \\ & 41.051 \end{aligned}$ | $\begin{aligned} & 1.353 \times 10^{-3} \\ & 43.204 \end{aligned}$ | $\left\{\begin{array}{l} 5.675 \times 10^{-4} \\ 34.857 \end{array}\right.$ |
| (D) AGE | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 4.732 \times 10^{-4} \\ & 2.894 \end{aligned}$ | $\begin{aligned} & 2.389 \times 10^{-4} \\ & 7.613 \end{aligned}$ | $\begin{aligned} & 1.689 \times 10^{-4} \\ & 3.850 \end{aligned}$ | $\begin{aligned} & 2.000 \times 10^{-4} \\ & 3.790 \end{aligned}$ | $\begin{aligned} & 1.684 \times 10^{-4} \\ & 2.932 \end{aligned}$ | $\begin{aligned} & 1.674 \times 10^{-4} \\ & 2.915 \end{aligned}$ | $\begin{aligned} & 1.970 \times 10^{-4} \\ & 3.734 \end{aligned}$ | $\begin{aligned} & 1.640 \times 10^{-4} \\ & 3.744 \end{aligned}$ | $\begin{aligned} & 2.327 \times 10^{-4} \\ & 7.431 \end{aligned}$ | $\left\{\begin{array}{l} 5.436 \times 10^{-5} \\ 3.339 \end{array}\right.$ |
| EXACT <br> SOLUTION |  | $1.635 \times 10^{-3}$ | $3.138 \times 10^{-3}$ | $4.386 \times 10^{-3}$ | $5.279 \times 10^{-3}$ | $5.743 \times 10^{-3}$ | $5.742 \times 10^{-3}$ | $5.276 \times 10^{-3}$ | $4.381 \times 10^{-3}$ | $3.132 \times 10^{-3}$ | $1.628 \times 10^{-3}$ |

TABLE (4.11.17)

Neumann Problem, Absolute Errors

| Scheme $\quad$ x | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{r}=0.1, \mathrm{t}=0.1$ <br> (D) AGE | 0.0206 | 0.0159 | 0.0123 | 0.0099 | 0.0086 | 0.0085 | 0.0097 | 0.0121 | 0.0159 | 0.0209 | 0.0272 |
| EXACT SOL. | 0.904837 | 0.990650 | 1.066564 | 1.131822 | 1.185771 | 1.227872 | 1.257704 | 1.274970 | 1.279497 | 1.271239 | 1.250280 |
| $r=0.5, t=0.5$ <br> (D) AGE | 0.0568 | 0.0536 | 0.0512 | 0.0500 | 0.0493 | 0.0495 | 0.0505 | 0.0524 | 0.0551 | 0.0585 | 0.0627 |
| EXACT SOL. | 0.606531 | 0.664053 | 0.714939 | 0.758683 | 0.794846 | 0.823067 | 0.843064 | 0.854638 | 0.857672 | 0.852137 | 0.838088 |
| $\mathrm{r}=1.0, \mathrm{t}=1.0$ <br> (D) AGE | 0.0866 | 0.0847 | 0.0831 | 0.0824 | 0.0819 | 0.0822 | 0.0828 | 0.0840 | 0.0856 | 0.0876 | 0.0902 |
| EXACT SOL. | 0.367879 | 0.402768 | 0.433633 | 0.460164 | 0.482098 | 0.499215 | 0.511344 | 0.518364 | 0.520205 | 0.516847 | 0.508326 |
| $r=1.5, t=0.75$ <br> (D) AGE | 0.0368 | 0.0344 | 0.0832 | 0.0896 | 0.0889 | 0.0806 | 0.0335 | 0.0350 | 0.0852 | 0.0959 | 0.0992 |
| EXACT SOL. | 0.4724 | 0.5172 | 0.5568 | 0.5909 | 0.6190 | 0.6410 | 0.6566 | 0.6656 | 0.6680 | 0.6636 | 0.6527 |

TABLE (4.11.18)

| r |  | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | Percentage Error |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\begin{gathered} 0.1 \\ \Delta t=0.001 \\ t=0.1 \end{gathered}$ | (D) AGE <br> EXACT SOL. | $\begin{aligned} & 0.1684 \\ & 0.1709 \end{aligned}$ | $\begin{aligned} & 0.1693 \\ & 0.1718 \end{aligned}$ | $\begin{aligned} & 0.1718 \\ & 0.1739 \end{aligned}$ | $\begin{aligned} & 0.1745 \\ & 0.1763 \end{aligned}$ | $\begin{aligned} & 0.1766 \\ & 0.1781 \end{aligned}$ | $\begin{aligned} & 0.1774 \\ & 0.1788 \end{aligned}$ | $\begin{aligned} & 0.1767 \\ & 0.1781 \end{aligned}$ | $\begin{aligned} & 0.1747 \\ & 0.1763 \end{aligned}$ | $\begin{aligned} & 0.1720 \\ & 0.1739 \end{aligned}$ | $\begin{aligned} & 0.1696 \\ & 0.1718 \end{aligned}$ | $\simeq 1 \%$ |
| $\begin{aligned} & 0.5 \\ & \Delta t=0.005 \\ & t=0.5 \end{aligned}$ | (D) AGE <br> EXACT SOL. | $\begin{aligned} & 0.3517 \\ & 0.3618 \end{aligned}$ | $\begin{aligned} & 0.3556 \\ & 0.3650 \end{aligned}$ | $\begin{aligned} & 0.3639 \\ & 0.3720 \end{aligned}$ | $\begin{aligned} & 0.3724 \\ & 0.3793 \end{aligned}$ | $\begin{aligned} & 0.3784 \\ & 0.3846 \end{aligned}$ | $\begin{aligned} & 0.3808 \\ & 0.3865 \end{aligned}$ | $\begin{aligned} & 0.3788 \\ & 0.3845 \end{aligned}$ | $\begin{aligned} & 0.3730 \\ & 0.3793 \end{aligned}$ | $\begin{aligned} & 0.3650 \\ & 0.3720 \end{aligned}$ | $\begin{aligned} & 0.3567 \\ & 0.3650 \end{aligned}$ | $\simeq 2 \%$ |
| $\begin{gathered} 1.0 \\ \Delta t=0.01 \\ t=1.0 \end{gathered}$ | $\begin{gathered} \text { (D) AGE } \\ \text { EXACT SOL. } \end{gathered}$ | $10.9425$ | $\begin{aligned} & 0.9526 \\ & 0.9795 \end{aligned}$ | $\begin{aligned} & 0.9684 \\ & 0.9937 \end{aligned}$ | $\begin{aligned} & 0.9841 \\ & 1.0088 \end{aligned}$ | $\begin{aligned} & 0.9971 \\ & 1.0197 \end{aligned}$ | $\begin{aligned} & 1.0028 \\ & 1.0237 \end{aligned}$ | $\begin{aligned} & 0.9989 \\ & 1.0197 \end{aligned}$ | $\begin{aligned} & 0.9866 \\ & 1.0088 \end{aligned}$ | $\begin{aligned} & 0.9698 \\ & 0.9937 \end{aligned}$ | $\begin{aligned} & 0.9525 \\ & 0.9795 \end{aligned}$ | $\simeq 3 \%$ |
| $\begin{aligned} & 1.5 \\ & \Delta t=0.015 \\ & t=1.5 \end{aligned}$ | (D) AGE EXACT SOL. | 1.9332 2.0007 | 1.9832 2.0107 | 1.9999 2.0322 | 1.9769 2.0549 | 1.9944 2.0715 | 2.0403 2.0775 | 2.0458 2.0715 | $\begin{aligned} & 2.0097 \\ & 2.0549 \end{aligned}$ | $\begin{aligned} & 1.9685 \\ & 2.0322 \end{aligned}$ | $\begin{aligned} & 1.9350 \\ & 2.0107 \end{aligned}$ | $\simeq 3 \%$ |

TABLE (4.11.19)

The following conclusions can be obtained from the tables:-
(i) For the GER and GEL schemes, the errors near the ungrouped points at the boundaries are higher than anywhere else in the range. This is probably due to the truncation error of the equation approximating the solution at the ungrouped point which is larger than the truncation error terms for the group equations.
(ii) For small values of $r$, the GER and GEL schemes are already sufficiently accurate. For bigger $r$ (D)AGE is the more preferable one.
(iii) The (D)AGE, i.e. (S)AGE with the direction changed, is always more preferable than the (S)AGE scheme. Theoretically, this change of direction is difficult to explain as far as any advantages are concerned. Therefore since this change does not add further to the computational cost, then for any case it is better than the (S)AGE scheme.
(iv) For the fractional schemes of (S)AGE and (D)AGE, the accuracy for small $r$ is about the same. However, for $r \geqslant 1$, (D)AGE is much more accurate.
(v) The advantages of the GE schemes over the explicit, Crank-Nicolson and Du-Fort-Frankel methods can be seen from Table (4.11.12). For $r=1.0$, the accuracy of (D)AGE is still better than Crank-Nicolson and the (D)AGE half-step and (D)AGE quarter step are also superior. For $r>1.0$, the (D)AGE half-step is as accurate as $C N$ but the (D)AGE quarter step is still much more accurate than others (Table (4.11.14)).
(vi) The percentage errors of the (D)AGE scheme at a certain point $x$ always lies between the percentage errors of GER and GEL for any time t. For $x<0.5$, the percentage errors of (D)AGE are lower than those of GEL and higher than those of GER and for $x>0.5$ the percentage
errors of (D)AGE are higher than those of GEL and lower than those of GER. Also for $x=0.5$, the percentage errors of the GEL and GER schemes are approximately equal. The graph also shows that the GER, GEL, (D)AGE schemes and the exact solution are quantitatively in agreement with each other.
(vii) For an odd number of intervals, the GEC, (S)AGE and (D)AGE schemes show the same characteristics in the results for GER/GEL, (S)AGE and (D)AGE as the case of an even number of intervals.
(viii) For the derivative boundary condition it was assumed that a similar stability condition as the Dirichlet condition still holds. The results from the numerical experiments show that the errors near the boundary are higher than anywhere else in the range. This is probably due to the first order approximation in $\Delta x$ used for the boundary condition.
(ix) For the periodic boundary condition, the numerical solution shows that they are quantitatively in agreement with the analytical solution.
(x) Even though most of the schemes are unconditionally stable (except those for the splitting type of schemes) the results indicate reasonable errors up to $r=1.5$. The author feels the effect of cancellation of error term $\frac{\Delta t}{\Delta x}$ and $\Delta x \Delta t$ is quite complex and not as straight forward as would otherwise seem. Further work is required to give a more definitive explanation.

### 4.12 THE CORRECTION PROCEDURES FOR UNGROUPED POINT NEAR BOUNDARY

The numerical evidence obtained from the last section has clearly shown that the solution at the ungrouped point near the boundary is always higher than anywhere else in the range. By error accumulation this will certainly affect the accuracy at other points as the time-level increases. It is clear from Table (4.12.1) that for problem (4.11.1) the percentage error at the left ungrouped point from GEL method is very much higher than anywhere else right from the very first time level. $r=0.5, \Delta t=0.005$

| $j \Delta t$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $j=1$ | 2.241 | 0.122 | 0.002 | 0.015 | 0.000 | 0.015 | 0.002 | 0.122 | 0.849 |
| $=2$ | 2.884 | 0.077 | 0.135 | 0.005 | 0.001 | 0.005 | 0.084 | 0.077 | 2.443 |
| $=10$ | 2.941 | 0.416 | 0.914 | 0.337 | 0.324 | 0.337 | 0.179 | 0.416 | 2.064 |
| $=50$ | 4.572 | 2.078 | 2.664 | 2.078 | 2.075 | 2.078 | 1.486 | 2.078 | 0.421 |

TABLE (4.12.1)
For this reason the author felt it necessary to look at some alternative means of improving the solution at this point, in particular. From the sign of the coefficients of the truncation errors, it is quite logical for us to decide on a process of averaging, even though this is not the only possible way. A few of the strategies which are considered are as follows:
(i) At any ungrouped point, the solution is taken as the average of the solutions of equations (4.2.14) and (4.2.15).
(ii) For the left-ungrouped point instead of using equation (4.5.7) we use the first equation of (4.5.5) and for the right-ungrouped point we use the second equation of (4.5.5) instead of (4.5.6).
(iii) For points near the boundary irrespective of whether it is grouped or not, the solution will again be the average of the solutions of equations (4.2.14) and (4.2.15).
(iv) The average is taken at every point, i.e.,

$$
\begin{align*}
u_{i, j} & =\frac{1}{2}\left\{\left(u_{i, j}\right)_{G E L}+\left(u_{i, j}\right)_{G E R}\right\}  \tag{4.12.1}\\
\text { or } \quad u_{i, j} & =\frac{1}{2}\left\{\left(u_{i, j}\right)_{G E U}+\left(u_{i, j}\right)_{G E C}\right\}
\end{align*}
$$

Tables (4.12.2)-(4.12.4) give a sample of the results where some schemes in GE are implemented using the above strategies. The model problem used is similar to (4.11.1). To see the effectiveness of the correction strategy, we have to compare these results with those in Tables (4.11.2), (4.11.4) and (4.11.6) respectively. For GEL, the use of strategy (ii) and (iv) give little improvement to the solution at the left ungrouped point for $r=0.1$, for other values of $r$ the solutions also do not give any significant improvement.

In the case of (D)AGE, strategy (i) gives a significant improvement only for $r=0.1$, while for $r=0.5$, strategy (iii) seems to give slightly better results when compared with (D)AGE in Table (4.11.4). For r=1.0, (D)AGE is still better than the result from both strategies.

From all tables, the conclusion one can gather is that the effectiveness of these correction procedures are only significant for small values of $r$. For larger values of $r$, the magnitude of the truncation errors overide this effectiveness.

| Scheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GEL <br> strategy <br> (ii) | A.E. | $\begin{aligned} & 0.00111 \\ & 0.572 \end{aligned}$ | $\begin{aligned} & 0.00075 \\ & 0.202 \end{aligned}$ | $\begin{aligned} & 0.00094 \\ & 0.184 \end{aligned}$ | $\left\{\begin{array}{l} 0.001181 \\ 0.197 \end{array}\right.$ | $\begin{aligned} & 0.00169 \\ & 0.268 \end{aligned}$ | $\begin{aligned} & 0.00193 \\ & 0.323 \end{aligned}$ | $\begin{aligned} & 0.00135 \\ & 0.264 \end{aligned}$ | $\begin{aligned} & 0.00156 \\ & 0.421 \end{aligned}$ | $\begin{aligned} & 0.00009 \\ & 0.048 \end{aligned}$ |
| (D) AGE <br> strategy <br> (i) | A.E. | $\begin{aligned} & 0.00010 \\ & 0.053 \end{aligned}$ | $\begin{aligned} & 0.00089 \\ & 0.241 \end{aligned}$ | $\begin{aligned} & 0.00147 \\ & 0.289 \end{aligned}$ | $\left\lvert\, \begin{aligned} & 0.00172 \\ & 0.287 \end{aligned}\right.$ | $\begin{aligned} & 0.00181 \\ & 0.288 \end{aligned}$ | $\begin{aligned} & 0.00174 \\ & 0.291 \end{aligned}$ | $\begin{aligned} & 0.00143 \\ & 0.281 \end{aligned}$ | $\begin{aligned} & 0.00094 \\ & 0.255 \end{aligned}$ | $\begin{aligned} & 0.00007 \\ & 0.038 \end{aligned}$ |
| (D) AGE strategy (iii) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.00086 \\ & 0.438 \end{aligned}$ | $\begin{aligned} & 0.00149 \\ & 0.401 \end{aligned}$ | $\begin{aligned} & 0.00192 \\ & 0.377 \end{aligned}$ | $\begin{aligned} & 0.00207 \\ & 0.346 \end{aligned}$ | $\begin{aligned} & 0.00213 \\ & 0.338 \end{aligned}$ | $\begin{aligned} & 0.00209 \\ & 0.349 \end{aligned}$ | $\begin{aligned} & 0.00188 \\ & 0.369 \end{aligned}$ | $\begin{aligned} & 0.00153 \\ & 0.411 \end{aligned}$ | $\begin{aligned} & 0.00084 \\ & 0.432 \end{aligned}$ |
| GER+GEL average strategy (iv) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.00090 \\ & 0.461 \end{aligned}$ | $\begin{aligned} & 0.00154 \\ & 0.416 \end{aligned}$ | $\begin{aligned} & 0.00194 \\ & 0.380 \end{aligned}$ | $\begin{aligned} & 0.00212 \\ & 0.353 \end{aligned}$ | $\begin{aligned} & 0.00216 \\ & 0.343 \end{aligned}$ | $\begin{aligned} & 0.00212 \\ & 0.353 \end{aligned}$ | $\begin{aligned} & 0.00194 \\ & 0.380 \end{aligned}$ | $\begin{aligned} & 0.00154 \\ & 0.416 \end{aligned}$ | $\begin{aligned} & 0.00090 \\ & 0.461 \end{aligned}$ |
| EXACT SOLUTION |  | 0.195065 | 0.370771 | 0.509872 | 0.598962 | 0.629614 | 0.598962 | 0.509872 | 0.370771 | 0.195065 |

First line is the absolute error (A.E.) and second line is percentage error (P.E.).

TABLE (4.12.2)
$r=0.5, t=0.25$ (50th step)

| Scheme |  |  | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GEL <br> strategy <br> (ii) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 11.289 \times 10^{-4} \\ & 4.174 \end{aligned}$ | $\begin{aligned} & 16.703 \times 10^{-4} \\ & 3.247 \end{aligned}$ | $\begin{aligned} & 10.218 \times 10^{-4} \\ & 1.443 \end{aligned}$ | $\begin{aligned} & 11.585 \times 10^{-4} \\ & 1.392 \end{aligned}$ | $\begin{aligned} & 8.636 \times 10^{-4} \\ & 0.987 \end{aligned}$ | $\begin{aligned} & 6.033 \times 10^{-4} \\ & 0.725 \end{aligned}$ | $\begin{aligned} & 7.967 \times 10^{-4} \\ & 1.125 \end{aligned}$ | $\begin{aligned} & 2.191 \times 10^{-4} \\ & 0.426 \end{aligned}$ | $\begin{aligned} & 7.656 \times 10^{-4} \\ & 2.831 \end{aligned}$ |
| (D) AGE <br> strategy <br> (i) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 8.616 \times 10^{-4} \\ & 3.186 \end{aligned}$ | $\begin{aligned} & 16.605 \times 10^{-4} \\ & 3.228 \end{aligned}$ | $\begin{aligned} & 15.800 \times 10^{-4} \\ & 2.231 \end{aligned}$ | $\begin{aligned} & 21.330 \times 10^{-4} \\ & 2.562 \end{aligned}$ | $\begin{aligned} & 21.025 \times 10^{-4} \\ & 2.402 \end{aligned}$ | $\left\{\begin{array}{l} 19.291 \times 10^{-4} \\ 2.318 \end{array}\right.$ | $\begin{aligned} & 20.322 \times 10^{-4} \\ & 2.870 \end{aligned}$ | $\begin{aligned} & 11.241 \times 10^{-4} \\ & 2.185 \end{aligned}$ | $\begin{aligned} & 11.002 \times 10^{-4} \\ & 4.068 \end{aligned}$ |
| (D) AGE <br> strategy <br> (iii) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 1.700 \times 10^{-4} \\ & 0.629 \end{aligned}$ | $\begin{aligned} & 5.960 \times 10^{-4} \\ & 1.159 \end{aligned}$ | $\begin{aligned} & 4.013 \times 10^{-4} \\ & 0.567 \end{aligned}$ | $\begin{aligned} & 8.742 \times 10^{-4} \\ & 1.050 \end{aligned}$ | $\begin{aligned} & 7.976 \times 10^{-4} \\ & 0.911 \end{aligned}$ | $\begin{aligned} & 6.540 \times 10^{-4} \\ & 0.786 \end{aligned}$ | $\begin{aligned} & 8.626 \times 10^{-4} \\ & 1.218 \end{aligned}$ | $\begin{aligned} & 2.103 \times 10^{-4} \\ & 0.409 \end{aligned}$ | $\begin{aligned} & 2.536 \times 10^{-4} \\ & 0.938 \end{aligned}$ |
| ```GER+GEL average strategy(iv)``` | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 0.516 \times 10^{-4} \\ & 0.191 \end{aligned}$ | $\begin{aligned} & 2.566 \times 10^{-4} \\ & 0.499 \end{aligned}$ | $\begin{aligned} & 3.993 \times 10^{-4} \\ & 0.564 \end{aligned}$ | $\begin{aligned} & 4.959 \times 10^{-4} \\ & 0.596 \end{aligned}$ | $\begin{array}{\|l\|} 5.290 \times 10^{-4} \\ 0.604 \end{array}$ | $\begin{aligned} & 4.959 \times 10^{-4} \\ & 0.596 \end{aligned}$ | $\begin{aligned} & 3.993 \times 10^{-4} \\ & 0.564 \end{aligned}$ | $\begin{array}{\|l\|} 2.566 \times 10^{-4} \\ 0.499 \end{array}$ | $\begin{aligned} & 0.516 \times 10^{-4} \\ & 0.191 \end{aligned}$ |
| EXACT SOLUTION |  | 0.0270461 | 0.0514447 | 0.0708075 | 0.0832392 | 0.0875229 | 0.0832392 | 0.0708075 | 0.0514447 | 0.0270461 |

First line is the absolute error (A.E.) and second line is percentage error (P.E.).
$\underline{r=1.0, t=0.50(50 t h ~ s t e p)}$

| Scheme | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GEL <br> strategy <br> (ii) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 2.042 \times 10^{-4} \\ & 8.904 \end{aligned}$ | $\begin{aligned} & 3.917 \times 10^{-4} \\ & 8.977 \end{aligned}$ | $\begin{aligned} & 3.598 \times 10^{-4} \\ & 5.991 \end{aligned}$ | $\begin{aligned} & 4.019 \times 10^{-4} \\ & 5.694 \end{aligned}$ | $\begin{aligned} & 3.763 \times 10^{-4} \\ & 5.069 \end{aligned}$ | $\begin{aligned} & 3.633 \times 10^{-4} \\ & 5.146 \end{aligned}$ | $\begin{aligned} & 3.598 \times 10^{-4} \\ & 5.991 \end{aligned}$ | $\begin{aligned} & 1.839 \times 10^{-4} \\ & 4.216 \end{aligned}$ | $\begin{aligned} & 2.042 \times 10^{-4} \\ & 8.904 \end{aligned}$ |
| (D) AGE <br> strategy <br> (i) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\left\{\begin{array}{l} 2.873 \times 10^{-4} \\ 12.528 \end{array}\right.$ | $\begin{aligned} & 6.062 \times 10^{-4} \\ & 13.895 \end{aligned}$ | $\begin{aligned} & 6.260 \times 10^{-4} \\ & 10.425 \end{aligned}$ | $\begin{aligned} & 7.974 \times 10^{-4} \\ & 11.297 \end{aligned}$ | $\begin{aligned} & 8.1 .79 \times 10^{-4} \\ & 11.019 \end{aligned}$ | $\begin{aligned} & 7.876 \times 10^{-4} \\ & 11.156 \end{aligned}$ | $\begin{aligned} & 7.733 \times 10^{-4} \\ & 12.877 \end{aligned}$ | $\begin{aligned} & 4.110 \times 10^{-4} \\ & 9.421 \end{aligned}$ | $\begin{aligned} & 3.178 \times 10^{-4} \\ & 13.854 \end{aligned}$ |
| (D) AGE <br> strategy <br> (iii) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 1.580 \times 10^{-4} \\ & 6.888 \end{aligned}$ | $\begin{aligned} & 3.126 \times 10^{-4} \\ & 7.165 \end{aligned}$ | $\begin{aligned} & 2.973 \times 10^{-4} \\ & 4.951 \end{aligned}$ | $\begin{aligned} & 4.650 \times 10^{-4} \\ & 6.586 \end{aligned}$ | $\begin{aligned} & 4.560 \times 10^{-4} \\ & 6.143 \end{aligned}$ | $\begin{aligned} & 4.466 \times 10^{-4} \\ & 6.327 \end{aligned}$ | $\begin{aligned} & 4.870 \times 10^{-4} \\ & 8.110 \end{aligned}$ | $\begin{aligned} & 2.040 \times 10^{-4} \\ & 4.676 \end{aligned}$ | $\begin{aligned} & 1.482 \times 10^{-4} \\ & 6.459 \end{aligned}$ |
| $\begin{aligned} & \text { GER }+ \text { GEL } \\ & \text { average } \\ & \text { strategy (iv) } \end{aligned}$ | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 2.135 \times 10^{-4} \\ & 9.307 \end{aligned}$ | $\begin{aligned} & 4.377 \times 10^{-4} \\ & 10.033 \end{aligned}$ | $\begin{aligned} & 6.087 \times 10^{-4} \\ & 10.137 \end{aligned}$ | $\begin{aligned} & 7.205 \times 10^{-4} \\ & 10.207 \end{aligned}$ | $\begin{aligned} & 7.687 \times 10^{-4} \\ & 10.222 \end{aligned}$ | $\begin{aligned} & 7.205 \times 10^{-4} \\ & 10.207 \end{aligned}$ | $\begin{aligned} & 6.087 \times 10^{-4} \\ & 10.137 \end{aligned}$ | $\begin{aligned} & 4.377 \times 10^{-4} \\ & 10.033 \end{aligned}$ | $\begin{aligned} & 2.135 \times 10^{-4} \\ & 9.307 \end{aligned}$ |
| EXACT <br> SOLUTION |  | $2.2936 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $7.4224 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $2.2936 \times 10^{-3}$ |

First line is the absolute error (A.E.) and the second line is the percentage error (P.E.)

### 4.13 THE WEIGHTED GROUP EXPLICIT (WGE) METHOD

In the general approximation (4.2.1), if we put:

1) $\theta_{2}=\alpha, \theta_{1}=0, \theta_{2}^{\prime}=1-\alpha$ and $\theta_{1}^{\prime}=1,0 \leqslant \alpha \leqslant 1$ we get

$$
\begin{equation*}
-r \alpha u_{i-1, j+1}+(1+\alpha r) u_{i, j+1}=r(1-\alpha) u_{i-1, j}-[2 r-(1+\alpha r)] u_{i, j}+r u_{i+1, j} \tag{4.13.1}
\end{equation*}
$$

and
2) $\theta_{2}=0, \theta_{1}=\alpha, \theta_{1}^{\prime}=1-\alpha, \theta_{2}^{\prime}=1$ we get,

$$
\begin{equation*}
-r \alpha u_{i+1, j+1}+(1+\alpha r) u_{i, j+1}=r(1-\alpha) u_{i+1, j}-[2 r-(1+\alpha r)] u_{i, j}+r u_{i-1, j} \tag{4.13.2}
\end{equation*}
$$

From equation (4.3.3), the truncation errors for these two equations are given by

$$
\begin{align*}
T_{4.13 .1}= & \alpha\left(\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right) \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \\
& +\left.\frac{(1-\alpha)}{2} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{1} \Delta t^{\alpha}\right) \tag{4.13.3}
\end{align*}
$$

$$
\begin{align*}
T_{4.13 .2}= & -\alpha\left[\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right]_{i, j+\frac{1}{2}} \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right) \quad+\frac{(1-\alpha)}{2} \Delta t-\left.\frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}} \\
& +\frac{\Delta t^{2}}{24}{\left.\frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} O\left(\Delta x^{\alpha} \Delta t^{\alpha}\right)} \tag{4.13.4}
\end{align*}
$$

where $\alpha_{1}+\alpha_{2}=5,0, \leqslant \alpha_{1}, \alpha_{2} \leqslant 5$.
The stability condition for equation (4.13.1) and (4.13.2) can easily be derived from (4.4.3) to give the condition

$$
\begin{equation*}
r \leqslant \frac{1}{2(1-\alpha)} \tag{4.13.5}
\end{equation*}
$$


$(j+1)$ th level
(a)

(b)

FIGURE (4.13.1)

(a)

(b)

FIGURE (4.13.2)

These two schemes were first proposed by Saul'yev (1964) and diagrammatically are given in Fig. (4.13.1). For $\alpha=1$, these equations reduce to the equations ( 4.2 .15 ) and (4.2.14) respectively, i.e. the equations which later formed the Group Explicit method. In addition, for $\alpha=0$, they give the explicit equation (4.2.9). It is worth noting that the addition of these two equations will result in the Saul'yev's weighted six point equation (4.2.10). It was from the equations (4.13.1) and (4.13.2) that the author was able to derive the mathematical basis and formulation of the method that will be called the Weighted Group Explicit (WGR) method. Similar to Section (4.5), we consider any group of two points i.e. (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ and $\left((i+1) \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$, in which we use equation (4.13.1) and (4.13.2) respectively to calculate the values $u$ as the approximate solution to equation (4.1.1) at those points. Therefore, at (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ the solution is given by

$$
\begin{equation*}
-r \alpha u_{i+1, j+1}+(1+r \alpha) u_{i, j+1}=r(1-\alpha) u_{i+1, j}-[2 r-(1+\alpha r)] u_{i, j}+r u_{i-1, j}, \tag{4.13.6}
\end{equation*}
$$

whilst at point $\left((i+1) \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ the solution is given by

$$
\begin{equation*}
-r \alpha u_{i, j+1}+(1+r \alpha) u_{i+1, j+1}=r(1-\alpha) u_{i, j}-[2 r-(1+\alpha r)] u_{i+1, j}+r u_{i+2, j} \tag{4.13.7}
\end{equation*}
$$

These two equations are then written simultaneously as

$$
\left[\begin{array}{cc}
1+\alpha r & -r \alpha \\
-r \alpha & 1+\alpha r
\end{array}\right]\left[\begin{array}{ll}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{ll}
(1+\alpha r)-2 r & r(1-\alpha) \\
r(1-\alpha) & (1+\alpha r)-2 r
\end{array}\right]\left[\begin{array}{l}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{l}
r u_{i-1, j} \\
r u_{i+2, j}
\end{array}\right]
$$

or

$$
\begin{gather*}
{\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{|A|}\left[\begin{array}{cc}
1+\alpha r & \alpha r \\
\alpha r & 1+\alpha r
\end{array}\right]\left[\begin{array}{cc}
(1+\alpha r)-2 r & r(1-\alpha) \\
r(1-\alpha) & (1+\alpha r)-2 r
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]}  \tag{4.13.8}\\
\left.+\left[\begin{array}{c}
r u_{i-1, j} \\
r u_{i+2, j}
\end{array}\right]\right\} \tag{4.13.9}
\end{gather*}
$$

where $|A|=1+2 \alpha r$, as an explicit system. This is simplified to

$$
\left[\begin{array}{l}
u_{i, j+1}  \tag{4.13.10}\\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
f_{1} & f_{2} \\
f_{2} & f_{1}
\end{array}\right]\left[\begin{array}{l}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\frac{1}{1+2 \alpha r}\left[\begin{array}{l}
(1+\alpha r) r u_{i-1, j}+\alpha r^{2} u_{i+2, j} \\
\alpha r^{2} u_{i-1, j}+(1+\alpha r) r u_{i+2, j}
\end{array}\right]
$$

where $f_{1}=\left(1+2 \alpha r-2 r-\alpha r^{2}\right) /(1+2 \alpha r)$
and $\quad f_{2}=r(1-\alpha r) /(1+2 \alpha r)$ which is given diagrammatically by Fig. (4.13.2). For the ungrouped points near the right and left boundaries we can use equation (4.13.2) and (4.13.1) respectively, i.e. for the right boundary,

$$
\begin{equation*}
u_{m-1, j+1}=\frac{1}{1+\alpha r}\left\{r \alpha u_{m, j+1}+r(1-\alpha) u_{m, j}-[2 r-(1+\alpha r)] u_{m-1, j}+r u_{m-2, j}\right\} \tag{4.13.11}
\end{equation*}
$$

and for the left boundary

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{1+\alpha r}\left\{r \alpha u_{0, j+1}+r(1-\alpha) u_{0, j}-[2 r-(1+\alpha r)] u_{1, j}+r u_{2, j}\right\} \tag{4.13.12}
\end{equation*}
$$

Before we develop different varieties of the WGE scheme we assume that the boundary conditions are of Dirichlet type and the number of subintervals are even. However, similar procedures are also applicable to periodic conditions, and/or odd number of intervals.
(i) WGER scheme: The scheme denoted Weighted Group Explicit with Right ungrouped point is similar to the GER scheme (4.5.8) as far as the procedure is concerned. The scheme uses equation (4.13.10) for $\frac{1}{2}(\mathrm{~m}-2)$ times for the first (m-2) points and equation (4.13.11) for the last ( $m-1$ ) th point. In matrix form it is given by

$$
\begin{equation*}
\left(I+r G_{3}\right) \underline{u}_{j+1}=\left(I-r G_{4}\right) \underline{u}_{j}+\underline{b}_{5}, \tag{4.13.13}
\end{equation*}
$$



and

$$
b_{5}^{T}=\left[r u_{0, j}, 0, \ldots, r(1-\alpha) u_{m, j}+r \alpha u_{m, j+1}\right]
$$

(ii) WGEL Scheme: This scheme, Weighted Group Explicit with Left ungrouped point corresponds to the GEL scheme (4.5.12). The scheme is written by

$$
\begin{equation*}
\left(I+r G_{5}\right) \underline{u}_{j+1}=\left(I-r G_{6}\right) \underline{u}_{j}+\underline{b}_{6}, \tag{4.13.16}
\end{equation*}
$$

where

and

$$
\underline{b}_{6}^{T}=\left[r \alpha u_{0, j+1}+r(1-\alpha) u_{0, j}, 0, \ldots, 0, r u_{m, j}\right] .
$$

In the case when $\alpha=1, G_{3}=G_{6}$ and $G_{4}=G_{5}$, the equations (4.14.13) and (4.13.16) give the GER and GEL schemes respectively. If $\alpha=0, G_{3}=G_{5}=0$ and $G_{4}=G_{6}$ and the equations (4.13.13) and (4.13.16) reduce to a classical explicit system (2.9.13).
(iii) The (S)WAGE scheme: This scheme called the (Single) Weighted Alternating Group Explicit corresponds to the (S)AGE (4.5.13). It is given by

$$
\left.\begin{array}{l}
\left(I+r G_{3}\right) \underline{u}_{j+1}=\left(I-r G_{4}\right) \underline{u}_{j}+\underline{b}_{-5}  \tag{4.13.19}\\
\left(I+r G_{5}\right) \underline{u}_{j+2}=\left(I-r G_{6}\right) \underline{u}_{j+1}+\underline{b}_{6} .
\end{array}\right\}
$$

The corresponding half-step splitting (S)WAGE is given by

$$
\left.\begin{array}{l}
\left(I+\frac{r_{2}^{2}}{} G_{3} \underline{u}_{j+\frac{1}{2}}=\left(I \frac{r_{2}}{G_{4}}\right) \underline{u}_{j}+\underline{b}_{5}\right.  \tag{4.13.20}\\
\left(I+\frac{r_{2}}{2} G_{5} \underline{u}_{j+1}=\left(I \frac{r_{2}}{2} G_{6}\right) \underline{u}_{j+\frac{1}{2}}+\underline{b}_{6} .\right.
\end{array}\right\}
$$

(iv) The (D)WAGE scheme: This scheme corresponds to (D)AGE scheme (4.5.14) and is termed as the (Double) Weighted Alternating Group Explicit.

It can be written as

$$
\left.\begin{array}{l}
\left(I+r \theta G_{3}\right) \underline{u}_{j+\theta}=\left(I-r \theta G_{4}\right) \underline{u}_{j}+\underline{b}_{5}  \tag{4.13.21}\\
\left(I+r \theta G_{5}\right) \underline{u}_{j+2 \theta}=\left(I-r \theta G_{6}\right) \underline{u}_{j+\theta}+\underline{b}_{6} \\
\left(I+r \theta G_{5}\right) \underline{u}_{j+3 \theta}=\left(I-r \theta G_{6}\right) \underline{u}_{j+2 \theta}+\underline{b}_{6} \\
\left(I+r \theta G_{3}\right) \underline{u}_{j+4 \theta}=\left(I-r \theta G_{4}\right) \underline{u}_{j+3 \theta}+\underline{b}_{5}
\end{array}\right\}
$$

where

$$
\theta= \begin{cases}1, & \text { for ordinary time-level }  \tag{4.13.22}\\ \frac{1}{2}, & \text { for half-step splitting } \\ \frac{1}{4}, & \text { for quarter-step splitting }\end{cases}
$$

## Truncation Errors

If we expand each term of both equations in (4.3.10) as a Taylor series expansion about $\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ and $\left((i+1) \Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ respectively, the following is obtained:

For the first equation of (4.13.10) it gives,

$$
\begin{align*}
\left(\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial x^{2}}\right)_{i, j+\frac{1}{2}}-\left.\frac{\Delta t}{\Delta x} \frac{\alpha}{(1+2 \alpha r)} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}} & +\left.\frac{\Delta x \Delta t}{2} \frac{\alpha r}{(1+2 \alpha r)} \frac{\partial^{4} u}{\partial x^{3} \partial t}\right|_{i, j+\frac{1}{2}} \\
+\frac{\Delta x^{2}}{2}\left[r \frac{\partial^{3} u}{\partial x^{2} \partial t}-\frac{(1+8 \alpha r)}{6(1+2 \alpha r)} \frac{\partial^{4} u}{\partial x^{4}}\right]_{i, j+\frac{1}{2}} & +\frac{\Delta t^{2}}{8}\left[\frac{1}{3} \frac{\partial^{3} u}{\partial t^{3}}-\frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right]_{i, j+\frac{1}{2}} \\
& +\frac{1}{\Delta t} 0\left(\Delta x_{1}^{\alpha_{1}} t^{\alpha}\right)=0 \tag{4.13.23}
\end{align*}
$$

where $0 \leqslant \alpha_{1}, \alpha_{2} \leqslant 5$ and $\alpha_{1}+\alpha_{2}=5$.
For the second equation of (4.13.10) it gives,

$$
\begin{gather*}
\left(\frac{\partial u}{\partial t}-\frac{\partial^{2} u}{\partial t^{2}}\right)_{i+1, j+\frac{1}{2}}+\left.\frac{\Delta t}{\Delta x} \frac{\alpha}{(1+2 \alpha r)} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i+1, j+\frac{1}{2}}-\left.\frac{\Delta x \Delta t}{2} \frac{\alpha r}{(1+2 \alpha r)} \frac{\partial^{4} u}{\partial x^{3} \partial t}\right|_{i+1, j+\frac{1}{2}} \\
+\frac{\Delta x^{2}}{2}\left[r \frac{\partial^{3} u}{\partial x^{2} \partial t}-\frac{(1+8 \alpha r)}{6(1+2 \alpha r)} \frac{\partial^{4} u}{\partial x^{4}}\right]_{i+1, j+\frac{1}{2}}+\frac{\Delta t^{2}}{8}\left[\frac{1}{3} \frac{\partial^{3} u}{\partial t^{3}}-\frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right]_{i+1, j+\frac{1}{2}} \\
+\frac{1}{\Delta t} 0\left(\Delta x_{1} \Delta t^{\alpha}\right)=0 . \tag{4.13.24}
\end{gather*}
$$

Meanwhile, the truncation errors for the ungrouped points are given by

$$
\begin{align*}
T_{1, j+\frac{1}{2}}= & \alpha\left[\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t \frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right]_{1, j+\frac{1}{2}} \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right){ }_{1, j+\frac{1}{2}}+\left.\frac{(1-\alpha)}{2} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{1, j+\frac{1}{2}} \\
& +\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{1, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x_{1}^{\alpha} \Delta t^{2}\right) \tag{4.13.25}
\end{align*}
$$

for the left point and by

$$
\begin{align*}
& \qquad \begin{aligned}
T_{m-1, j+\frac{1}{2}}= & -\alpha\left[\left(\frac{\Delta t}{\Delta x} \frac{\partial^{2} u}{\partial x \partial t}+\frac{1}{6} \Delta x \Delta t-\frac{\partial^{4} u}{\partial x^{3} \partial t}+\frac{1}{24} \frac{\Delta t^{3}}{\Delta x} \frac{\partial^{4} u}{\partial x \partial t^{3}}\right]_{m-1, j+\frac{1}{2}}\right. \\
& -\left(\frac{(\Delta x)^{2}}{12} \frac{\partial^{4} u}{\partial x^{4}}+\frac{(\Delta t)^{2}}{8} \frac{\partial^{4} u}{\partial x^{2} \partial t^{2}}\right)_{m-1, j+\frac{1}{2}}+\frac{(1-\alpha)}{2} \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t} \\
& +\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{m-1, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{1} \Delta t^{\alpha}\right)
\end{aligned} \\
& \text { for the right point. }
\end{align*}
$$

WGER Scheme: The truncation errors for this scheme at any time level are given by the error terms of (4.13.23) and (4.13.24) respectively for $\mathrm{i}=1,2, \ldots, \mathrm{~m}-2$ and for the ( $\mathrm{m}-1$ ) th point is given by (4.13.26). (Fig.4.13.3a)).

WGEL Scheme: The truncation errors for this scheme at any time level are given by equation (4.13.25) for the point closest to the left boundary. For the remaining points, the truncation errors are given by the error terms of (4.13.23) and (4.13.24) respectively, (Fig. 4.13.3b).


Diagram showing the distribution of the truncation errors FIGURE (4.13.3)
1 - Equation 4.13.23
2 - Equation 4.13.24
3 - Equation 4.13 .26
4 - Equation 4.13.25
(S)WAGE Scheme: In general, the distribution of the truncation errors for this scheme are given by the truncation errors of WGER and WGEL schemes respectively. Fig.(4.13.3c) will clearly describe the distribution of the errors.
(D)WAGE Scheme: The error distributions can be more easily described by the diagram given in Fig.(4.13.3d).

From Figs. (4.13.3c) and (4.13.3d), we can see that at all the inner points, the errors (1) and (2) are in sequence as the time level moves forward, and since each of the terms $\left(\frac{\Delta t}{\Delta x}\right)$ and $(\Delta x \Delta t)$ for both equations are opposite in sign, for small enough values of $\Delta t$, some of these error terms tend to cancel themselves. Therefore the only significant error terms which will generate are only of $0\left((\Delta x)^{2}+\Delta t\right)$. For the points near both boundaries, the significant accumulated errors are still of $O\left(\Delta t+(\Delta x)^{2}+\frac{\Delta t}{\Delta x}\right)$. This is one of the reasons why the numerical solution near the boundary is proved not to be as accurate as the solutions far from the boundary.

The Stability
The scheme WGER of (4.13.13) and WGEL of (4.13.16) can be written explicitly, as,

$$
\begin{equation*}
\underline{u}_{j+1}=T \underline{u}_{j}+\underline{b}^{\prime} \tag{4.13.29}
\end{equation*}
$$

where
with

$$
\begin{align*}
& f_{1}=\left(1+2 \alpha r-2 r-\alpha r^{2}\right) /(1+2 \alpha r), \\
& f_{2}=r(1-\alpha r) /(1+2 \alpha r), \\
& f_{3}=r(1+\alpha r) /(1+2 \alpha r), \\
& f_{4}=\alpha r^{2} /(1+2 \alpha r),  \tag{4.13.31}\\
& f_{1}^{\prime}=((1+\alpha r)-2 r) /(1+\alpha r), \\
& f_{4}^{\prime}=r /(1+\alpha r), \\
& b^{\prime}=\left[f_{3} u_{0, j}, f_{4}^{\left.u_{0, j}, 0, \ldots, 0, \frac{r \alpha u_{m, j+1}+r(1-\alpha) u_{m, j}}{1+\alpha r}\right] .}\right.
\end{align*}
$$

and

and

$$
\underline{b}^{T}=\left[\frac{r \alpha u_{0, j+1}+r(1-\alpha) u_{0, j}}{1+\alpha r}, 0, \ldots, 0, f_{4} u_{m, j}, f_{3} u_{m, j}\right]
$$

Now the system (4.13.29) will be stable if we can show that

$$
\begin{equation*}
\|T\|_{\infty}=\max _{1 \leqslant i \leqslant m-1}\left\{\sum_{j=1}^{m-1}\left|t_{i, j}\right|\right\} \leqslant 1 \tag{4.13.33}
\end{equation*}
$$

Hence the following theorem has to be established.

## Theorem 4.6

Let $\alpha r \leqslant 1$ for $0<\alpha \leqslant 1$ and $r>0$, therefore,
(a) $f_{1}$ are non-negative for $r \leqslant 1$ and $a \neq 0$ (for $r \leqslant \frac{1}{2}$ and $\alpha=0$ ),
(b) $\left|f_{1}\right|+\left|f_{2}\right|+\left|f_{3}\right|+\left|f_{4}\right|=1$,
(c) $\left|f_{1}\right|+\left|f_{2}\right|+\left|f_{3}\right|$ and $\left|f_{1}\right|+\left|f_{2}\right|+\left|f_{4}\right|$ are both less (or equal) to unity, and
(d) $\left|f_{1}^{\prime}\right|+\left|f_{4}^{\prime}\right| \leqslant 1$ if $r \leqslant \frac{1}{(2-\alpha)}$.

Thus, from (4.13.29), the WGER scheme and the WGEL schemes are conditionally stable for $r \leqslant \frac{1}{(2-\alpha)}$.

## Proof

(a) Here we need to show that $\left(1+2 \alpha r-2 r-\alpha r^{2}\right) \geqslant 0$. For $\alpha=0$, it is clear that $(1-2 r) \geqslant 0$ for $r \leqslant \frac{1}{2}$. For $\alpha \neq 0, f_{1} \geqslant 0$ if $r$ lies in the range

$$
\begin{aligned}
r & =1-\frac{1}{\alpha}-\frac{1}{\alpha} \sqrt{\left(\alpha^{2}-\alpha+1\right)} \\
& <0 \\
r & =1-\frac{1}{\alpha}+\frac{1}{\alpha} \sqrt{\left(\alpha^{2}-\alpha+1\right)} \leqslant 1 .
\end{aligned}
$$

Therefore values of $f_{1}$ are non-negative for $r \leqslant 1$ and $\alpha \neq 0$ (for $r \leqslant \frac{1}{2}$ and $\alpha=0$ ).
(b) Since $f_{1}, f_{2}, f_{3}$ and $f_{4}$ are non-negative, then

$$
\begin{aligned}
\left|f_{1}\right|+\left|f_{2}\right|+\left|f_{3}\right|+\left|f_{4}\right| & =\frac{\left(1+2 \alpha r-2 r-\alpha r^{2}\right)+\left(r-\alpha r^{2}\right)+\left(r+\alpha r^{2}\right)+\alpha r^{2}}{(1+2 \alpha r)} \\
& =1
\end{aligned}
$$

(c) $\left|f_{1}\right|+\left|f_{2}\right|+\left|f_{3}\right|=\frac{\left(1+2 \alpha r-\alpha r^{2}\right)}{(1+2 \alpha r)}$
$\leqslant 1$ for all r>0.

$$
\left|f_{1}\right|+\left|f_{2}\right|+\left|f_{4}\right|=\frac{\left(1+2 \alpha r-r-\alpha r^{2}\right)}{(1+2 \alpha r)}
$$

$\leqslant 1$ for all r>0.
(d) $f_{l}^{\mathrm{t}}=\left\{\begin{array}{l}\geqslant 0, \text { if } r \leqslant \frac{1}{(2-\alpha)} \\ <0, \text { if } r>\frac{1}{(2-\alpha)} \text {. }\end{array}\right.$

Assuming $f_{1}^{\prime} \geqslant 0$, then

$$
\left|f_{1}^{\prime}\right|+\left|f_{4}^{\prime}\right|=\frac{(1+\alpha r-2 r)+r}{(1+\alpha r)} \leqslant 1 \text { for all } r>0 .
$$

Assuming $\mathrm{f}_{1}<0$, then

$$
\left|f_{1}^{\prime}\right|+\left|f_{4}^{\prime}\right|=\frac{(2 r-1-\alpha r+r)}{(1+\alpha r)} \leqslant 1 \text { for all } r \leqslant \frac{2}{(3-2 \alpha)}
$$

which cannot hold at the same time with $r>\frac{1}{(2-\alpha)}$ for all $0 \leqslant \alpha \leqslant 1$. Therefore, the assumption that $f_{i}^{\prime} \geqslant 0$ is taken into account which then will result in

$$
\left|f_{1}^{\prime}\right|+\left|f_{4}^{\prime}\right| \leqslant 1 \text { if } r \leqslant \frac{1}{(2-\alpha)}
$$

Hence,

$$
\begin{aligned}
\left|\left|\mathrm{T}_{\text {WGER }}\right| l_{\infty}=\right. & \left|\left|\mathrm{T}_{\text {WGEL }}\right|\right|_{\infty} \\
= & \max \left\{\left(\left|\mathrm{f}_{1}\right|+\left|\mathrm{f}_{2}\right|+\left|\mathrm{f}_{3}\right|+\left|\mathrm{f}_{4}\right|\right),\left(\left|\mathrm{f}_{1}\right|+\left|\mathrm{f}_{2}\right|+\left|\mathrm{f}_{3}\right|\right),\right. \\
& \left.\left(\left|\mathrm{f}_{1}\right|+\left|\mathrm{f}_{2}\right|+\left|\mathrm{f}_{4}\right|\right),\left(\left|\mathrm{f}_{1}^{\prime}\right|+\left|\mathrm{f}_{4}^{\prime}\right|\right)\right\} \\
\leqslant & 1 \text { if } r \leqslant \frac{1}{2-\alpha}
\end{aligned}
$$

This completes the proof. For the value $\alpha=1$ this theorem reduces to Theorem 4.3.

To analyse the stability of the two-step (S)WAGE process of equations (4.13.19) and (4.13.20), we write them as

$$
\begin{equation*}
u_{j+2 \theta}=\frac{T u_{j}}{}+\underline{b}^{\prime} \tag{4.13.34}
\end{equation*}
$$

where

$$
\begin{align*}
\theta & =\left\{\begin{array}{l}
1, \text { for equation }(4.13 .19), \\
\frac{1}{2}, \text { for equation }(4.13 .20),
\end{array}\right. \\
T_{(S) W A G E} & =\left(I+r \theta G_{5}\right)^{-1}\left(I-r \theta G_{6}\right)\left(I+r \theta G_{3}\right)^{-1}\left(I-r \theta G_{4}\right) \tag{5}
\end{align*}
$$

and $\underline{b}^{\prime}$ is associated with the appropriate boundary condition.

It can be clearly seen that,

$$
\begin{equation*}
T_{(S) W A G E}=T_{\text {WGEL }} T_{\text {WGER }}, \tag{4,13.36}
\end{equation*}
$$

therefore, we can state that $T_{(S) W A G E} \leqslant 1$ if $r \leqslant 1 /(2-\alpha)$. Evidence gained from the experiments from (S)AGE (if $\alpha=1$ ) shows that (S) WAGE has better stability characteristics. This can be shown using (4.13.30), (4.13.32) and (4.13.35) but for the sake of brevity this will not be proved here.

## Numerical Example

In this example the (S)WAGE half-step splitting scheme has been implemented for the problem (4.11.1) for various values of $\alpha$. The purpose of this study is to see the variation in the numerical solution given by using different $\alpha$. Tables (4.13.1)-(4.13.4) give the effect on the solutions caused by various $\alpha$, for $r=0.1,0.5,1.0,1.5$, in terms of percentage errors.

From the tables we can see that for small values of $r$, the nearly explicit type scheme (i.e. $\alpha \approx 0$ ) give a better solution than other values of $\alpha$. For $r=0.5$, the ranges $0.4 \leqslant \alpha \leqslant 0.6$ give better solutions and for $r \geqslant 1$, $\alpha=1$ is the best value to choose.

This experiment therefore justifies that the explicit method is only best used for very small $r$. For $r \approx \frac{1}{2}$, the method with $\alpha \approx \frac{1}{2}$ is recommended.
$r=0^{\circ} .1, \Delta t=0.001, t=0.1$ Percentage errors

| $\alpha \sim$ | $x$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha=0.1$ | 0.594 | 0.591 | 0.591 | 0.590 | 0.589 | 0.590 | 0.590 | 0.592 | 0.591 |
| 0.2 | 0.619 | 0.614 | 0.615 | 0.613 | 0.613 | 0.614 | 0.614 | 0.617 | 0.614 |
| 0.25 | 0.631 | 0.626 | 0.627 | 0.625 | 0.625 | 0.625 | 0.625 | 0.629 | 0.625 |
| 0.5 | 0.693 | 0.682 | 0.684 | 0.681 | 0.681 | 0.682 | 0.681 | 0.687 | 0.681 |
| 0.75 | 0.754 | 0.736 | 0.739 | 0.735 | 0.735 | 0.737 | 0.736 | 0.744 | 0.736 |
| 1.0 | 0.813 | 0.788 | 0.793 | 0.787 | 0.786 | 0.790 | 0.788 | 0.799 | 0.790 |
| EXACT SOLUTION | 0.118869 | 0.226098 | 0.311192 | 0.365823 | 0.384647 | 0.365823 | 0.311192 | 0.226098 | 0.118869 |

TABLE (4.13.1)
$r=0.5, \Delta t=0.005, t=0.5 \quad$ Percentage Errors

| $\underbrace{\text { a }}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.20 | 0.866 | 1.001 | 0.940 | 0.975 | 0.965 | 0.955 | 0.986 | 0.917 | 1.032 |
| 0.25 | 0.612 | 0.779 | 0.707 | 0.749 | 0.737 | 0.724 | 0.761 | 0.677 | 0.815 |
| 0.4 | 0.084 | 0.175 | 0.075 | 0.138 | 0.122 | 0.103 | 0.154 | 0.027 | 0.218 |
| 0.5 | 0.502 | 0.183 | 0.297 | 0.222 | 0.240 | 0.263 | 0.204 | 0.357 | 0.140 |
| 0.6 | 0.888 | 0.512 | 0.637 | 0.551 | 0.571 | 0.598 | 0.533 | 0.709 | 0.472 |
| 0.75 | 1.419 | 0.959 | 1.098 | 0.996 | 1.019 | 1.050 | 0.978 | 1.186 | 0.928 |
| EXACT SOLUTION | $2.2936 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $7.4224 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $2.2936 \times 10^{-3}$ |

TABLE (4.13.2)
$r=1.0, \Delta t=0.01, t=1.0$ Percentage Errors

| - X | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.25 | 12.384 | 13.559 | 12.991 | 13.359 | 13.262 | 13.146 | 13.452 | 12.762 | 13.708 |
| 0.5 | 10.983 | 12.569 | 11.980 | 12.394 | 12.297 | 12.170 | 12.478 | 11.655 | 12.709 |
| 0.6 | 10.526 | 12.240 | 11.656 | 12.079 | 11.983 | 11.854 | 12.158 | 11.305 | 12.363 |
| 0.8 | 9.710 | 11.648 | 11.081 | 11.518 | 11.424 | 11.291 | 11.589 | 10.691 | 11.727 |
| 0.9 | 9.338 | 11.378 | 10.820 | 11.262 | 11.170 | 11.035 | 11.328 | 10.416 | 11.430 |
| 1.0 | 8.985 | 11.121 | 10.572 | 11.019 | 10.929 | 10.792 | 11.081 | 10.156 | 11.144 |
| EXACT SOLUTION | $1.492 \times 10^{-5}$ | $2.985 \times 10^{-5}$ | $3.981 \times 10^{-5}$ | $4.737 \times 10^{-5}$ | $4.951 \times 10^{-5}$ | $4.696 \times 10^{-5}$ | $4.033 \times 10^{-5}$ | $2.826 \times 10^{-5}$ | $1.619 \times 10^{-5}$ |

TABLE (4.13.3)

## $r=1.5, \Delta t=0.015, t=0.75$ Percentage Errors

| $\underline{x}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.5 | 26.320 | 31.585 | 29.511 | 31.165 | 30.804 | 30.283 | 31.416 | 28.311 | 31.690 |
| 0.6 | 24.301 | 28.999 | 27.351 | 28.651 | 28.374 | 27.971 | 28.850 | 26.304 | 29.127 |
| 0.75 | 22.682 | 27.063 | 25.725 | 26.784 | 26.563 | 26.239 | 26.943 | 24.792 | 27.177 |
| 0.8 | 22.300 | 26.634 | 25.361 | 26.374 | 26.163 | 25.854 | 26.522 | 24.454 | 26.737 |
| 0.9 | 21.658 | 25.944 | 24.772 | 25.715 | 25.521 | 25.233 | 25.849 | 23.807 | 26.021 |
| 1.0 | 21.126 | 25.401 | 24.304 | 25.198 | 25.016 | 24.744 | 25.321 | 23.472 | 25.446 |
| EXACT SOLUTION | $1.945 \times 10^{-4}$ | $3.700 \times 10^{-4}$ | $5.092 \times 10^{-4}$ | $5.987 \times 10^{-4}$ | $6.295 \times 10^{-4}$ | $5.987 \times 10^{-4}$ | $5.092 \times 10^{-4}$ | $3.700 \times 10^{-4}$ | $1.945 \times 10^{-4}$ |

TABLE (4.13.4)

### 4.14 COMPUTATIONAL COMPLEXITY OF THE GROUP EXPLICIT (GE) METHODS

Earlier in the chapter, the author stressed that the Group Explicit method will be able to preserve the simplicity of the explicit method. To show this we have to compare the computational complexity of the schemes with the classical explicit and CN methods.

In the following table we give the number of arithmetic operations involved to evaluate the solution at a point for all schemes in this class of method in comparison with the explicit and CN methods.

| Method Operation | Addition | Multiplication | Division |
| :--- | :---: | :---: | :---: |
| GE (ordinary points) | 3 | 4 | 1 |
| GE (ungrouped points) | 2 | 2 | 2 |
| CN (average per point) | $\frac{5 n-1}{n} \sim 5$ | $\frac{5 n-1}{n} \sim 5$ | $\frac{2 n-1}{n} \sim 2$ |
| Explicit | 2 | 2 | 0 |

TABLE (4.14.1)
For the $C N$ method the figure given is the average from the number for solving the implicit system.

The table shows that very much better stability characteristics are achieved over the explicit method at the cost of some additional computational expense, i.e. approximately double but this is still $30 \%$ less than the CN method. As far as the storage requirement is concerned, all the methods given in Table 4.14 .1 , are similar. For the fractional splitting type of methods, the number of operations are double or quadruple depending on whether it is half-step or quarter-step process respectively.

To evaluate the relationship between the Group Explicit (GE) and Crank-Nicolson (CN) methods, let us consider the matrix forms of each



The scheme (4.15.3) is obtained when the red-circled elements on the lefthand side of (4.15.2) are brought to the right-hand side and given the initial guesses for the solutions. Meanwhile the scheme (4.15.5) is obtained when the blue-circled elements on the left-hand side of (4.15.2) are brought to the right-hand side. Both schemes have now on the lefthand side the block-diagonal coefficient matrix which naturally results in the system similar to the Group Explicit method. Hence we shall refer to this scheme as the Group Iterative Scheme for the $C N$ method. For the convergence of this iteration, from Theorem (3.3), we need the eigenvalues of

$$
\left(I+r \hat{G}_{i}\right)^{-1} r S_{i}<1, \quad i=1,2
$$

It can be easily shown that the eigenvalues are always zero. Therefore the iterative method is convergence for all $\mathrm{r}>0$.

Similarly, equivalent to the (D)AGE scheme of the Group Explicit class, we can also form the scheme as follows:

$$
\left.\begin{array}{l}
\left(I+r \hat{G}_{1}\right) \underline{u}_{j+1}^{(k+1)}=r S_{1} \underline{u}_{j}^{(k)}+\left(I-r \hat{G}_{1}\right) \underline{u}_{j}+r S_{1} \underline{u}_{j}+\underline{b}_{2}  \tag{4.15.5}\\
\left(I+r \hat{G}_{2}\right) \underline{u}_{j}^{(k+1)}=r S_{2} \underline{u}_{j+1}^{(k+1)}+\left(I-r \hat{G}_{2}\right) \underline{u}_{j}+r S_{2} \underline{u}_{j}+\underline{b}_{2} \\
\left(I+r \hat{G}_{2}\right) \underline{u}_{j+1}^{(k+3)}=r S_{2} \underline{u}_{j+1}^{(k+2)}+\left(I-r \hat{G}_{2}\right) \underline{u}_{j}+r S_{2} \underline{u}_{j}+\underline{b}_{2} \\
\left(I+r \hat{G}_{1}\right) \underline{u}_{j+1}^{(k+4)}=r S_{1} \underline{u}_{j+1}^{(k+1)}+\left(I-r \hat{G}_{1}\right) \underline{u}_{j}+r S_{1} \underline{u}_{j}+\underline{b}_{2}
\end{array}\right\}
$$

The clear advantage of these methods is that they are explicit methods of iteration. The evidence of the numerical experiments suggests that the method has no great advantage compared to the (D) AGE type of method already proposed as the number of iterations is quite high, i.e. greater than four iterations (the (D)AGE splitting type of quarter step is considered as equivalent to four iterations as far as the operational work is concerned). However the accuracy of the method is very reasonable, i.e. equivalent to CN itself.

Tables (4.15.1) and (4.15.2) give the comparison of the absolute error and percentage error of this method (equation (4.15.5)) with the (D) AGE method of splitting with quarter step given by equation (4.6.3) and the ordinary $C N$ method. Each table gives the values of the errors when $\Delta t=0.005, r=0.5$ and $t=0.5$ (after 100 steps) and $\Delta t=0.02, r=2.0$ and $t=1.0$ (after 50 steps) for values of $\Delta x=0.1$ respectively. The convergence criteria for the iterative method is defined as follows: If

$$
\varepsilon=\sum_{i=1}^{m-1}\left|u_{i, j+1}^{(k+1)}-u_{i, j+1}^{(k)}\right|
$$

and we define the quantity $\varepsilon_{\max }=1 \times 10^{-6}$ then the method is said to converge if $\varepsilon \leqslant \varepsilon_{\max }$. From the tables, it is clear that the (D)AGE type of splitting with quarter step is always more accurate than the $C N$ and CN Group Explicit iterative methods.
$r=0.5, t=0.5, \Delta t=0.005$

| thod | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CN Group iterative (6 iteration) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 9.24 \times 10^{-5} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 1.76 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 2.42 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 2.84 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 2.99 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 2.84 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 2.42 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 1.76 \times 10^{-4} \\ & 4.028 \end{aligned}$ | $\begin{aligned} & 9.24 \times 10^{-4} \\ & 4.028 \end{aligned}$ |
| (D) AGE quarter step | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 8.12 \times 10^{-5} \\ & 3.566 \end{aligned}$ | $\begin{aligned} & 1.56 \times 10^{-4} \\ & 3.568 \end{aligned}$ | $\begin{aligned} & 2.12 \times 10^{-4} \\ & 3.532 \end{aligned}$ | $\begin{aligned} & 2.50 \times 10^{-4} \\ & 3.539 \end{aligned}$ | $\begin{aligned} & 2.62 \times 10^{-4} \\ & 3.534 \end{aligned}$ | $\begin{aligned} & 2.49 \times 10^{-4} \\ & 3.531 \end{aligned}$ | $\begin{aligned} & 2.13 \times 10^{-4} \\ & 3.548 \end{aligned}$ | $\begin{aligned} & 1.54 \times 10^{-4} \\ & 3.537 \end{aligned}$ | $\begin{aligned} & 8.29 \times 10^{-5} \\ & 3.616 \end{aligned}$ |
| CN | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 9.23 \times 10^{-5} \\ & 4.022 \end{aligned}$ | $\begin{aligned} & 1.76 \times 10^{-4} \\ & 4.022 \end{aligned}$ | $\begin{aligned} & 2.42 \times 10^{-4} \\ & 4.022 \end{aligned}$ | $\begin{aligned} & 2.84 \times 10^{-4} \\ & 4.022 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.98 \times 10^{-4} \\ & 4.022 \end{aligned}$ | $\begin{aligned} & 2.84 \times 10^{-4} \\ & 4.022 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.42 \times 10^{-4} \\ & 4.022 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.76 \times 10^{-4} \\ & 4.022 \\ & \hline \end{aligned}$ | $\begin{aligned} & 9.23 \times 10^{-5} \\ & 4.022 \\ & \hline \end{aligned}$ |
| EXACT SOLUTION |  | $2.2936 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $7.4224 \times 10^{-3}$ | $7.0591 \times 10^{-3}$ | $6.0048 \times 10^{-3}$ | $4.3628 \times 10^{-3}$ | $2.2936 \times 10^{-3}$ |

TABLE (4.15.1)
$r=2.0, t=1.0, \Delta t=0.02$

| Method | x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CN Group iterative (6 iteration) | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 1.02 \times 10^{-6} \\ & 6.205 \end{aligned}$ | $\begin{aligned} & 1.92 \times 10^{-6} \\ & 6.129 \end{aligned}$ | $\begin{aligned} & 2.66 \times 10^{-6} \\ & 6.152 \end{aligned}$ | $\begin{aligned} & 3.11 \times 10^{-6} \\ & 6.132 \end{aligned}$ | $\begin{aligned} & 3.28 \times 10^{-6} \\ & 6.136 \end{aligned}$ | $\begin{aligned} & 3.12 \times 10^{-6} \\ & 6.144 \end{aligned}$ | $\begin{aligned} & 2.65 \times 10^{-6} \\ & 6.127 \end{aligned}$ | $\begin{aligned} & 1.93 \times 10^{-6} \\ & 6.146 \end{aligned}$ | $\begin{aligned} & 1.00 \times 10^{-6} \\ & 6.052 \end{aligned}$ |
| (D) AGE <br> quater step | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 5.4 \times 10^{-7} \\ & 3.297 \end{aligned}$ | $\begin{aligned} & 6.7 \times 10^{-7} \\ & 2.141 \end{aligned}$ | $\begin{aligned} & 1.33 \times 10^{-6} \\ & 3.086 \end{aligned}$ | $\begin{aligned} & 1.37 \times 10^{-6} \\ & 2.700 \end{aligned}$ | $\begin{aligned} & 1.51 \times 10^{-6} \\ & 2.827 \end{aligned}$ | $\begin{aligned} & 1.49 \times 10^{-6} \\ & 2.939 \end{aligned}$ | $\begin{aligned} & 1.09 \times 10^{-6} \\ & 2.535 \end{aligned}$ | $\begin{aligned} & 1.06 \times 10^{-6} \\ & 3.377 \end{aligned}$ | $\begin{aligned} & 2.10 \times 10^{-7} \\ & 1.268 \end{aligned}$ |
| CN | $\begin{aligned} & \text { A.E. } \\ & \text { P.E. } \end{aligned}$ | $\begin{aligned} & 8.4 \times 10^{-7} \\ & 5.066 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.59 \times 10^{-6} \\ & 5.066 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.19 \times 10^{-6} \\ & 5.066 \end{aligned}$ | $\begin{aligned} & 2.57 \times 10^{-6} \\ & 5.066 \end{aligned}$ | $\begin{aligned} & 2.70 \times 10^{-6} \\ & 5.066 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.57 \times 10^{-6} \\ & 5.066 \end{aligned}$ | $\begin{aligned} & 2.19 \times 10^{-6} \\ & 5.066 \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.59 \times 10^{-6} \\ & 5.066 \\ & \hline \end{aligned}$ | $\begin{aligned} & 8.4 \times 10^{-7} \\ & 5.066 \\ & \hline \end{aligned}$ |
| EXACT SOLUTION |  | $0.1650 \times 10^{-4}$ | $0.3138 \times 10^{-4}$ | $0.4319 \times 10^{-4}$ | $0.5077 \times 10^{-4}$ | $0.5338 \times 10^{-4}$ | $0.5077 \times 10^{-4}$ | $0.4319 \times 10^{-4}$ | $0.3138 \times 10^{-4}$ | $0.1650 \times 10^{-4}$ |

### 4.16 THE GROUP EXPLICIT METHOD FOR THE TWO-SPACE DIMENSIONAL PROBLEM

In this section, the concept of the Group Explicit method is to be extended to the case of a two-space dimensional problem of the form,

$$
\begin{equation*}
\frac{\partial U}{\partial t}=\frac{\partial^{2} U}{\partial x^{2}}+\frac{\partial^{2} U}{\partial y^{2}}+g(x, y, t) \tag{4.16.1}
\end{equation*}
$$

The approximate solution $u$ will be required in the cylinder $R \times[0 \leqslant t \leqslant T]$ where $R=\{(x, y) \mid 0 \leqslant x, y \leqslant 1\}$. Appropriate initial condition and boundary data are given on $t=0$ and $B \times[0 \leqslant t \leqslant T], B=\{(x, y) \mid x=0$ or $1, y=0$ or 1$\}$ respectively. For example, in this case we assume,

$$
\begin{equation*}
U(x, y, 0)=e(x, y), \quad x, y \in R \tag{4.16.2}
\end{equation*}
$$

and

$$
\begin{equation*}
U(x, y, t)=f(x, y, t), \quad x, y \in B, \quad 0 \leqslant t \leqslant T, \tag{4.16.3}
\end{equation*}
$$

where $e(x, y)$ and $f(x, y, t)$ are given known values for the prescribed values of $x, y, t$. We let $u_{i, j, k}$ denote the approximate solution of (4.16.1) at the point $(i, j, k) \equiv(i \Delta x, j \Delta y, k \Delta t), i, j=0,1,2, \ldots, m, m \Delta x=m \Delta y=1, k=0,1, \ldots$. For simplicity, we assume that $\Delta x=\Delta y=\Delta s$ and hence $r=\Delta t /(\Delta s)^{2}$.

Consider at any time-level, the group of four points ( $i, j, k),(i+1, j, k)$, $(i, j+1, k)$ and $(i+1, j+1, k)$. Then at each of these points and at $t=\left(k+\frac{1}{2}\right) \Delta t$, we approximate the equation (4.16.1) by

$$
\begin{align*}
\frac{u_{i, j, k+1}^{-u_{i, j, k}}}{\Delta t} & =\frac{u_{i+1, j, k+1}{ }_{i, j, k+1} u_{i, j, k}^{+u_{i-1, j, k}}}{(\Delta x)^{2}} \\
& +\frac{u_{i, j+1, k+1}{ }^{-u} i, j, k+1^{-u_{i}} i, j, k^{+u_{i, j-1, k}}}{(\Delta y)^{2}}+g\left(i, j, k+\frac{1}{2}\right) \tag{4.16.4}
\end{align*}
$$

$$
\begin{align*}
& \frac{u_{i+1, j, k+1^{-u}}{ }_{i+1, j, k}}{\Delta t}=\frac{u_{i, j, k+1} u_{i+1, j, k+1}^{-u_{i+1, j,}}{ }^{+u_{i+2, j, k}}}{(\Delta x)^{2}} \\
& +\frac{u_{i+1, j+1, k+1}{ }^{-u}{ }_{i+1, j, k+1}{ }^{-u}{ }_{i+1, j, k^{+u_{i+1, j-1, k}}}^{(\Delta y)^{2}}}{(\Delta)} \\
& +g\left(i+1, j, k+\frac{1}{2}\right), \tag{4.16.5}
\end{align*}
$$

$$
\begin{align*}
& \frac{u_{i, j+1, k+1}^{-u_{i, j+1, k}}}{\Delta t}=\frac{u_{i+1, j+1, k+1} u_{i, j+1, k+1}^{-u_{i, j+1, k}^{+u}} i-1, j+1, k}{} \\
&(\Delta x)^{2} \frac{u_{i, j, k+1}^{-u} i, j+1, k+1^{-u} i, j+1, k}{}+u_{i, j+2, k}+ \\
&(\Delta y)^{2}+g\left(i, j+1, k+\frac{1}{2}\right) \tag{4.16.6}
\end{align*}
$$

and

$$
\begin{gather*}
\frac{u_{i+1, j+1, k+1} \mathbf{u}_{i+1, j+1, k}}{\Delta t}=\frac{u_{i, j+1, k+1}^{-u} i_{i+1, j+1, k+1}^{-u_{i+1, j+1, k}+u_{i+2, j+1, k}}}{(\Delta x)^{2}} \\
+\frac{u_{i+1, j, k+1} u_{i+1, j+1, k+1}^{-u_{i+1, j+1, k}+u_{i+1, j+2, k}}}{(\Delta y)^{2}}+
\end{gather*}
$$

respectively, (Fig. 4.16.1). Upon simplification the following equations can be obtained,

$$
\begin{align*}
&-r u_{i+1, j, k+1}+(1+2 r) u_{i, j, k+1}-r u_{i, j+1, k+1}= r u_{i-1, j, k}+(1-2 r) u_{i, j, k}+r u_{i, j-1, k} \\
&+\Delta t g\left(i, j, k+\frac{1}{2}\right), \\
&-r u_{i, j, k+1}+(1+2 r) u_{i+1, j, k+1}-r u_{i+1, j+1, k+1}= r u_{i+2, j, k}+(1-2 r) u_{i+1, j, k} \\
&+r u_{i+1, j-1, k}+t \Delta g\left(i+1, j, k+\frac{1}{2}\right),
\end{aligned} \quad \begin{aligned}
(4.16 .9) \tag{4.16.9}
\end{align*}
$$

and

$$
\begin{array}{r}
-r u_{i, j+1, k+1}+(1+2 r) u_{i+1, j+1, k+1}-r u_{i+1, j, k+1}-r u_{i+2, j+1, k}+(1-2 r) u_{i+1, j+1, k} \\
+r u_{i+1, j+2, k}+\Delta \operatorname{tg}\left(i+1, j+1, k+\frac{1}{2}\right), \tag{4.16.11}
\end{array}
$$

Therefore the equations (4.16.8)-(4.16.11) when grouped give the implicit system whose matrix form is given by

$$
\left[\begin{array}{cccc}
(1+2 r) & -r & 0 & -r  \tag{4.16.12}\\
-r & (1+2 r) & -r & 0 \\
0 & -r & (1+2 r) & -r \\
-r & 0 & -r & (1+2 r)
\end{array}\right]\left[\begin{array}{l}
u_{i, j, k+1} \\
u_{i+1, j, k+1} \\
u_{i+1, j+1, k+1} \\
u_{i, j+1, k+1}
\end{array}\right]=\left[\begin{array}{l}
f_{i, j, k} \\
f_{i+1, j, k} \\
f_{i+1, j+1, k} \\
f_{i, j+1, k}
\end{array}\right]
$$



(a)


(c)


The position of the points near the boundary for any time-level $k$

where the values $f$ are defined by the right-hand sides of (4.16.8)(4.16.11) and since $\Delta t$ is normally small enough $g$ can be taken as

$$
g_{i, j, k+\frac{1}{2}}=\left\{\begin{array}{l}
g_{i, j, k} \text { or }  \tag{4.16.13}\\
g_{i, j, k+1} \text { or } \\
\frac{1}{2}\left(g_{i, j, k}+g_{i, j, k+1}\right)
\end{array}\right.
$$

The system (4.16.12) can be written in explicit form as

$$
\begin{equation*}
{\underset{\mathrm{u}}{\mathrm{k}+1}}^{\mathrm{u}^{-1}} \mathrm{f}_{\mathrm{k}} \tag{4.16.14}
\end{equation*}
$$

where $A^{-1}$ is the inverse of the coefficient matrix in (4.16.13) and when evaluated explicitly is given by

$$
A^{-1}=\frac{1}{(1+2 r)(1+4 r)}\left[\begin{array}{cccc}
1+4 r+2 r^{2} & r(1+2 r) & 2 r^{2} & r(1+2 r)  \tag{4.16.15}\\
r(1+2 r) & 1+4 r+2 r^{2} & r(1+2 r) & 2 r^{2} \\
2 r^{2} & r(1+2 r) & 1+4 r+2 r^{2} & r(1+2 r) \\
r(1+2 r) & 2 r^{2} & r(1+2 r) & 1+4 r+2 r^{2}
\end{array}\right]
$$

Special treatment will have to be made for points near the boundary (see Fig. 4.16.2), depending on the position. For the position 1 , the solutions at any two points are approximated by (Fig. 4.16.3a).

$$
\left[\begin{array}{cc}
1+2 r & -r \\
-r & 1+2 r
\end{array}\right]\left[\begin{array}{c}
u_{i, 1, k+1}  \tag{4.16.16a}\\
u_{i+1,1, k+1}
\end{array}\right]=\left[\begin{array}{c}
r_{i, 0, k+1} u_{i-1,1, k}+(1-2 r) u_{i, 1, k}+r u_{i, 2, k} \\
+\Delta t g_{i, 1, k+\frac{1}{2}} \\
r u_{i+1,0, k+1}+r u_{i+2,1, k}+(1-2 r) u_{i+1,1, k}
\end{array}\right]
$$

which can be reduced to

$$
\left[\begin{array}{l}
u_{i, 1, k+1}  \tag{4.16.16b}\\
u_{i+1,1, k+1}
\end{array}\right]=\frac{1}{\left(1+4 r+3 r^{2}\right)}\left[\begin{array}{cc}
1+2 r & r \\
r & 1+2 r
\end{array}\right]\left[\begin{array}{l}
f_{i, 1, k} \\
f_{i+1,1, k}
\end{array}\right]
$$

with $f$ corresponding to the right-hand sides of (4.16.16a).
For position 2 , the system reduces to,
with f corresponding

$$
\left[\begin{array}{l}
u_{m-1, j, k+1}  \tag{4.16.17}\\
u_{m-1, j+1, k+1}
\end{array}\right]=\frac{1}{\left(1+4 r+3 r^{2}\right)}\left[\begin{array}{cc}
1+2 r & r \\
r & 1+2 r
\end{array}\right]\left[\begin{array}{l}
f_{m-1, j, k} \\
f_{m-1, j+1, k}
\end{array}\right]
$$

where
$f_{m-1, j, k}=r u_{m-2, j, k}+(1-2 r) u_{m-1, j, k}+r u_{m-1, j-1, k}+r u_{m, j, k+1}+\Delta t g_{m-1, j, k+\frac{1}{2}}$
and

$$
\begin{array}{r}
f_{m-1, j+1, k}=r u_{m-2, j+1, k}+(1-2 r) u_{m-1, j+1, k}+r u_{m-1, j+2, k}+r u_{m, j+1, k+1}+ \\
\Delta t g_{m-1, j+1, k+\frac{1}{2}},
\end{array}
$$

(see Fig. 4.16.3b).
For position 3, the system reduces to

$$
\left[\begin{array}{c}
u_{i, m-1, k+1}  \tag{4.16.18}\\
u_{i+1, m-1, k+1}
\end{array}\right]=\frac{1}{\left(1+4 r+3 r^{2}\right)}\left[\begin{array}{cc}
(1+2 r) & r \\
r & (1+2 r)
\end{array}\right]\left[\begin{array}{c}
f_{i, m-1, k} \\
f_{i+1, m-1, k}
\end{array}\right]
$$

where

$$
f_{i, m-1, k}=r u_{i, m-2, k}+(1-2 r) u_{i, m-1, k}+r u_{i-1, m-1, k}+r u_{i, m, k+1}+\Delta t g_{i, m-1, k+\frac{1}{2}}
$$

and

$$
\begin{aligned}
& \mathrm{f}_{\mathrm{i}+1, \mathrm{~m}-1, \mathrm{k}}=\mathrm{ru} \mathrm{i}_{\mathrm{i}+1, \mathrm{~m}-2, \mathrm{k}}+(1-2 \mathrm{r}) \mathrm{u}_{\mathrm{i}+1, \mathrm{~m}-1, \mathrm{k}}+\mathrm{ru} \mathrm{i}_{\mathrm{i}+2, \mathrm{~m}-1, k}+\mathrm{ru} \\
& \mathrm{i}+1, \mathrm{~m}, \mathrm{k}+1
\end{aligned}+
$$

(Fig. 4.16.3c).
Finally, for position 4 the system results in

$$
\left[\begin{array}{l}
u_{1, j, k+1}  \tag{4.16.19}\\
u_{1, j+1, k+1}
\end{array}\right]=\frac{1}{\left(1+4 r+3 r^{2}\right)}\left[\begin{array}{cc}
(1+2 r) & r \\
r & (1+2 r)
\end{array}\right]\left[\begin{array}{l}
f_{1, j, k} \\
f_{1, j+1, k}
\end{array}\right]
$$

where,

$$
f_{1, j, k}=r u_{0, j, k+1}+r u_{1, j-1, k}+(1-2 r) u_{1, j, k}+r u_{2, j, k}+\Delta t g_{1, j, k+\frac{1}{2}}
$$

and

$$
f_{1, j+1, k}=r u_{0, j+1, k+1}+r u_{1, j+2, k}+(1-2 r) u_{1, j+1, k}+r u_{2, j+1, k}+\Delta t g_{1, j+1, k+\frac{1}{2}}
$$

(Fig. 4.16.3d).


Position 8


Position 5


Position 7


Position 6

$x-y$ plane at any time-level
FIGURE (4.16.5)

orm(4.16.14)
$x$ or (4.16.17/19)

- or (4.16.16/18)
-(4.16.20/22)


The development of the SAGE method in 2 dimensions

|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 |  | 2 |  | 2 |  | 2 |  | 2 |  |
|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |
| 2 |  | 2 |  | 2 |  | 2 |  | 2 |  |
|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |
| 2 |  | 2 |  | 2 |  | 2 |  | 2 |  |
|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |
| 2 |  | 2 |  | 2 |  | 2 |  | 2 |  |
|  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |
| 2 |  | 2 |  | 2 |  | 2 |  | 2 |  |

1 - GER
2 - GEL

FIGURE (4.16.7)


The development of the DAGE method in 2 dimensions

The solutions at the corner points of the $x-y$ plane, i.e. positions 5,6,7 and 8 are given by

$$
\begin{gather*}
u_{1,1, k+1}=\frac{1}{(1+2 r)}\left\{r u_{0,1, k+1}+r u_{1,0, k+1}+r u_{1,2, k}+(1-2 r) u_{1,1, k}+r u_{2,1, k}+\right. \\
 \tag{4.16.20}\\
\left.+\Delta t g_{1,1, k+\frac{1}{2}}\right\},
\end{gather*}
$$

$$
\begin{align*}
u_{m-1,1, k+1}=\frac{1}{(1+2 r)} & \left\{r u_{m-1,0, k+1}-r u_{m, 1, k+1}+r u_{m-1,2, k}+(1-2 r) u_{m-1,1, k}+\right. \\
& \left.+r u_{m-2,1, k}+\Delta t g_{m-1,1, k+\frac{1}{2}}\right\} \tag{4.16.21}
\end{align*}
$$

$u_{m-1, m-1, k+1}=\frac{1}{(1+2 r)}\left\{r u_{m-2, m-1, k}+(1-2 r) u_{m-1, m-1, k}+r_{m-1, m-2, k}+\right.$ $\left.+r u_{m, m-1, k+1}+r u_{m-1, m, k+1}+\Delta t g_{m-1, m-1, k+\frac{1}{2}}\right\}$,
and

$$
\begin{gather*}
u_{1, \mathrm{~m}-1, \mathrm{k}}=\frac{1}{(1+2 \mathrm{r})}\left\{\mathrm{ru}_{2, \mathrm{~m}-1, \mathrm{k}}+(1-2 \mathrm{r}) \mathrm{u}_{1, \mathrm{~m}-1, \mathrm{k}^{+r u_{1, \mathrm{~m}-2, k}+\mathrm{r} u_{0, \mathrm{~m}-1, k+1}}+}^{+r u_{1, \mathrm{~m}, \mathrm{k}+1}+\Delta \mathrm{tg}}{ }_{1, \mathrm{~m}-1, \mathrm{k}+\frac{1}{2}}\right\}
\end{gather*}
$$

respectively. These are shown in Fig. 4.16.4.

Group Explicit Method: Similar to the one-dimensional problem, now we can develop various types of GE methods using equations (4.16.14),(4.16.16)(4.16.23). To simplify the discussion, we assume that the $x-y$ square is divided into an even number of squares in both directions, hence the number of unknown points in each direction are odd. Therefore corresponding to the one-dimensional case, the possible types of GE are GER, GEL, (S)AGE, (D)AGE and their respective fractional splitting versions.

GER Scheme (x-direction): This scheme at any time-level is obtained by using equation (4.16.14) for the first $\left(\frac{m-2}{2}\right)^{2}$ group of four points, starting from ( $1,1, k$ ), equation (4.16.17) at points ( $m-1, j, k$ ), $j=1,2,3, \ldots, m-2$, equation (4.16.22) at point ( $\mathrm{m}-1, \mathrm{~m}-1, \mathrm{k}$ ) and equation (4.16.18) at points ( $\mathrm{i}, \mathrm{m}-1, \mathrm{k}$ ) , $\mathrm{i}=1,2,3, \ldots, \mathrm{~m}-2$ (see Fig. 4.16.5) .

GEL Scheme ( $x$-direction): This scheme at any time-level is obtained by using equation (4.16.20) at point ( $1,1, k$ ), equation (4.16.16) at points $(i, 1, k), i=2,3, \ldots, m-1$, equation (4.16.19) at points ( $1, j, k$ ), $j=2,3, \ldots, m-1$ and equation (4.16.14) for the remaining $\left(\frac{m-2}{2}\right)^{2}$ group of four points starting from $(2,2, k)$. The ordering for this method is described in Fig. (4.16.6).

The concept of the (S)AGE (Fig. (4.16.7) and (D)AGE (Fig. (4.16.8)) schemes for the two-dimensional problem are similar to the one-dimensional case. For the (S)AGE scheme it is the alternate use of the GER and GEL schemes at alternate time levels. Meanwhile the (D)AGE scheme is the alternate use of the GER and GEL schemes with the direction of the alternation changed at every third-level.

Another alternative AGE scheme is the alternation of four different GE schemes, i.e. GER (x-direction), GEL (x-direction), GER (y-direction) and GEL (y-direction). This scheme is called 4 (D)AGE and 4 (S)AGE, depending on whether the alternating direction is reversed or not, (Fig. 4.16.9).

$\mathrm{y} \uparrow |$| 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 |
| 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 |
| 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 |
| 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 |
| 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 |
| 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 |
| 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 2 |
| 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 |
| 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 |

$$
\begin{aligned}
& 1- \text { GER } \\
& \text { (x-direction) } \\
& 2- \text { GEL } \\
&(x-d i r e c t i o n) ~ \\
& 3- \text { GEL } \\
& \text { (y-direction) } \\
& 4- \text { GER } \\
& \text { (y-direction) }
\end{aligned}
$$

Diagram showing how the alternating scheme takes $\xrightarrow[\text { place }]{x}$
FIGURE (4.16.9)

Matrix Notation: In matrix, form the GER method can be written as

$$
\begin{equation*}
\left(I+r G_{1}\right) u_{k+1}=\left(I-r G_{2}\right) u_{k}+b_{1}+\Delta t g_{k+\frac{1}{2}}, \tag{4.16.24}
\end{equation*}
$$

where $G_{1}$ and $G_{2}$ are matrices of order $(m-1) \times(m-1)$ and $b_{1}$ is the column matrix associated with the boundary conditions. The matrix $G_{1}$ is given by

where each $A_{i}, i=1,2, \ldots, \frac{m-2}{2}$ is a square matrix of order $2(m-1)$ and $\frac{A_{m}-2}{2}+1$ is a matrix of order $(m-1)$ and they are defined by

$$
A_{i}=\left[\begin{array}{lllll}
G & & & &  \tag{4.16.26}\\
& G & & & \\
& \ddots & & 0 & \\
& & \ddots & & \\
& \cap & & \ddots & \\
& & & & G^{\prime}
\end{array}\right] \quad, i=1,2, \ldots, \frac{m-2}{2},
$$

and

The vector $u_{k+1}$ is defined by

$$
\begin{aligned}
& u_{k+1}=\left[\begin{array}{c}
\underline{u}_{1, k+1} \\
\underline{u}_{3, k+1} \\
\vdots \\
\vdots \\
\vdots \\
u_{\mathfrak{u}-3, k+1} \\
u_{m-1, k+1}
\end{array}\right], \underline{u}_{j, k+1}=\left[\begin{array}{c}
\underline{u}_{1, j, k+1} \\
\underline{u}_{3, j, k+1} \\
\vdots \\
\vdots \\
u_{m} \\
u_{m}-3, j, k+1 \\
u_{m-1, j, k+1}
\end{array}\right] j=1,3,5, \ldots, m-3 \\
& \left.\begin{array}{rl}
\underline{u}_{i, j, k+1} & =\left[\begin{array}{l}
u_{i, j, k+1} \\
u_{i+1, j, k+1} \\
u_{i+1, j+1, k+1} \\
u_{i, j+1, k+1}
\end{array}\right] \quad i=1,3, \ldots, m-3 \\
u_{m-1, j, k+1} & =\left[\begin{array}{l}
u_{m-1, j, k+1} \\
u_{m-1, j+1, k+1}
\end{array}\right]
\end{array}\right\} \begin{array}{l}
\text { (4=1,3,5,16.30)}
\end{array} \\
& \text { and } \\
& u_{m-1, k+1}=\left[\begin{array}{c}
u_{1, m-1, k+1} \\
\underline{u}_{3, m-1, k+1} \\
\vdots \\
\vdots \\
u_{-m-3, m-1, k+1} \\
\underline{u}_{m-1, m-1, k+1}
\end{array}\right], \quad u_{i, m-1, k+1}=\left[\begin{array}{l}
u_{i, m-1, k+1} \\
\\
u_{i+1, m-1, k+1}
\end{array}\right], \quad \text { (4.16.31) }
\end{aligned}
$$

$k=0,1,2, \ldots$. The matrix $G_{2}$ is given by
with each $D_{i}, B_{i}, C_{i}, i=1,2, \ldots,\left(\frac{m-2}{2}\right)$ a square matrix of order $2(m-1)$ and defined by

$$
\begin{align*}
& D_{1, i}=\left[\begin{array}{llll}
2 & & & \\
& 2 & & \\
& & 2 & \\
& & & 2
\end{array}\right] \quad, \quad D_{2, i}=\left[\begin{array}{lllr}
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{array}\right] \\
& \mathrm{D}_{3, \mathrm{i}}=\left[\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right] \quad, \quad \mathrm{D}_{4, \mathrm{i}}=\left[\begin{array}{cc}
0 & 0 \\
-1 & 0 \\
0 & -1 \\
0 & 0
\end{array}\right]  \tag{4.16.34}\\
& B_{i}=\left[\begin{array}{lllll}
{ }^{B_{1, i}} & & & & \\
& B_{1, i} & & & \\
& & \ddots & & \\
& 0 & & & \\
& & & B_{1, i} & \\
& & & & B_{2, i}
\end{array}\right],_{B_{1, i}=}=\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right]  \tag{4.16.35}\\
& \text { and } \\
& D_{\left(\frac{m-2}{2}\right)+1}=\left[\begin{array}{rrrrr}
2 & & & & \\
& 2 & -1 & & \\
& -1 & 2 & & 0 \\
\\
& & \ddots & & \\
0 & & \ddots_{2} & -1 \\
& & & & -1
\end{array}\right] \quad \text { of order }(m-1) \text {, } \tag{4.16.36}
\end{align*}
$$


of order $2(\mathrm{~m}-1) \times(\mathrm{m}-1)$ and

$$
C_{\left(\frac{m-2}{2}\right)+1}=B_{\left(\frac{\mathrm{m}-2}{2}\right)+1}^{\mathrm{T}}
$$

The the GEL method, the matrix form is given by

$$
\begin{equation*}
\left(I+r G_{2}\right) u_{k+1}=\left(I-r G_{1}\right) u_{k}+\underline{b}_{2}+\Delta t g_{k+\frac{1}{2}} \tag{4.16.38}
\end{equation*}
$$

with $G_{1}, G_{2}$ defined as before and $\underline{b}_{2}$ is the column matrix associated with the boundary conditions. Similar to the one-dimensional case, the
(S)AGE and (D)AGE schemes are therefore written as,

$$
\begin{aligned}
\left(I+r \theta G_{1}\right) \underline{a}_{k+\theta} & =\left(I-r \theta G_{2}\right) u_{k}+\underline{b}_{1}+\theta \Delta t g_{k+\theta / 2} \\
\left(I+r \theta G_{2}\right) \underline{u}_{k+2 \theta} & =\left(I-r \theta G_{1}\right) u_{k \vdash \theta}+\underline{b}_{2}+\theta \Delta t g_{k+3 \theta / 2} \\
\theta & =\left\{\begin{array}{l}
1, \text { half-step } \\
\frac{1}{2}, \text { half-step splitting }
\end{array}\right.
\end{aligned}
$$

and

$$
\left.\begin{array}{rl}
\left(I+r \theta G_{1}\right) u_{k+\theta} & =\left(I-r \theta G_{2}\right) \underline{u}_{k}+\underline{b}_{1}+\theta \Delta t g_{k}+\theta / 2 \\
\left(I+r \theta G_{2}\right) \underline{u}_{k+2 \theta} & =\left(I-r \theta G_{1}\right) u_{k+\theta}+\underline{b}_{2}+\theta \Delta t g_{k+3 \theta / 2}  \tag{4.16.40}\\
\left(I+r \theta G_{2}\right) u_{k+3 \theta}=\left(I-r \theta G_{1}\right) \underline{u}_{k+2}+\underline{b}_{2}+\theta \Delta t g_{k}+5 \theta / 2
\end{array}\right\} \begin{aligned}
& \left(I+r \theta G_{1}\right) u_{k+4 \theta}=\left(I-r \theta G_{2}\right) \underline{u}_{k+3}+\underline{b}_{1}+\theta \Delta t g_{k+7 / \theta 2} \\
& \theta=\left\{\begin{array}{l}
I, \text { full-step } \\
\frac{1}{2}, \text { half-step splitting } \\
\frac{1}{4}, \text { quarter-step splitting }
\end{array}\right.
\end{aligned}
$$

Matrix $G_{1}$ can easily be shown to have eigenvalues of 2 and 1 each of multiplicity $\frac{m(m-2)}{2}+1$ and $\frac{m(m-2)}{2}$ respectively and matrix $G_{2}$ has eigenvalues of 2 and 1 each of multiplicity $\frac{(m-1)(2 m-1)}{2}+1$ and $\frac{m-2}{2} . \quad$ As $G_{1}$ and $G_{2}$ can be shown to satisfy Lemma 4.1, then using a similar approach to the onedimensional case the schemes GER and GEL can be shown to be conditionally stable for $r \leqslant 1$. Meanwhile, the schemes (S)AGE and (D)AGE can be shown to be unconditionally stable using Lemma 4.2.

In addition, for the truncation errors the orders can be shown to be similar to the one-dimensional case, i.e. $O\left(\Delta t+\frac{\Delta t}{\Delta s}+\Delta t \Delta s+(\Delta s)^{2}\right)$ for the GER and GEL schemes, for (S)AGE and (D)AGE schemes the orders are approximately $O\left(\Delta t+(\Delta s)^{2}\right)$ as both $\Delta t / \Delta s$ and $\Delta t \Delta s$ terms have opposite sign in GER and GEL. The truncation errors also show that the schemes will be consistent and converge to the exact solutions if $\Delta t \rightarrow 0$ faster than $\Delta s \rightarrow 0$.

## Numerical Example

To provide some indication of the accuracy of the Group Explicit method, we consider the equation (4.16.1) with

$$
g(x, y, t)=\sin x \sin y e^{-t}-4, \quad 0 \leqslant x, y \leqslant 1, t \geqslant 0
$$

where the theoretical solution is given by,

$$
u(x, y, t)=\sin x \sin y e^{-t}+x^{2}+y^{2}, 0 \leqslant x, y \leqslant 1, t \geq 0
$$

(Gourlay and MacGuire, 1971). The initial and boundary conditions are defined so as to agree with the exact solution.

| $\underline{r}=0.1, \Delta x=\Delta y=0.1, \Delta t=0.001, t=0.1$ |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x=0.5$ |  |  |  |  |  |  |  |  |  |
| Method ${ }^{\text {y }}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| GER | $\begin{aligned} & 0.303289 \\ & 19 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.376189 \\ & 6 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.468184 \\ & 13 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.578939 \\ & 8 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.707965 \\ & 11 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.854951 \\ & 8 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.019451 \\ & 12 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.201192 \\ & 1 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.399792 \\ & 17 \times 10^{-6} \end{aligned}$ |
| (S) AGE | $\begin{aligned} & 0.303292 \\ & 16 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.376195 \\ & 12 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.468193 \\ & 4 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.578950 \\ & 19 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.707979 \\ & 3 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.854966 \\ & 23 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.019469 \\ & 6 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.201209 \\ & 18 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.399808 \\ & 1 \times 10^{-6} \end{aligned}$ |
| (D) AGE | $\begin{aligned} & 0.303292 \\ & 16 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.376194 \\ & 11 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.468191 \\ & 6 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.578948 \\ & 17 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.707976 \\ & 0.4 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.854963 \\ & 20 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.019465 \\ & 2 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 1.201206 \\ & 15 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.399804 \\ & 5 \times 10^{-6} \end{aligned}$ |
| EXACT SOLUTION | 0.303308 | 0.376183 | 0.468197 | 0.578931 | 0.707976 | 0.854943 | 1.019463 | 1.201191 | 1.399809 |

TABLE (4.16.1)
$r=0.1, \quad \Delta x=\Delta y=0.1, \Delta t=0.001, t=0.1$

|  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GER | $\begin{aligned} & 0.029011 \\ & 7 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.067941 \\ & 5 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.126682 \\ & 13 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.205166 \\ & 11 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.303289 \\ & 19 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.420988 \\ & 18 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.558171 \\ & 23 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.714779 \\ & 22 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.890738 \\ & 22 \times 10^{-6} \end{aligned}$ |
| (S) AGE | $\begin{aligned} & 0.029016 \\ & 2 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.067950 \\ & 4 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.126697 \\ & 2 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.205184 \\ & 7 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.303312 \\ & 4 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.421015 \\ & 9 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.558200 \\ & 6 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.714810 \\ & 9 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.890765 \\ & 5 \times 10^{-6} \end{aligned}$ |
| (D) AGE | $\begin{aligned} & 0.029015 \\ & 3 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.067949 \\ & 3 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.126695 \\ & 0.5 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.205182 \\ & 5 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.303309 \\ & 1 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.421012 \\ & 6 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.558196 \\ & 2 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.714805 \\ & 5 \times 10^{-6} \\ & \hline \end{aligned}$ | $\begin{array}{r} 0.890760 \\ 0.4 \times 10^{-6} \\ \hline \end{array}$ |
| $\begin{gathered} \text { EXACT } \\ \text { SOLUTION } \end{gathered}$ | 0.029018 | 0.067946 | 0.126695 | 0.205177 | 0.303308 | 0.421006 | 0.558194 | 0.714801 | 0.890760 |



| $\mathrm{x}=0.5$ |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\text { Method } y$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| (S) AGE | $\begin{aligned} & 0.288881 \\ & 149 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.347705 \\ & 65 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.425915 \\ & 18 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.523335 \\ & 97 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.639241 \\ & 169 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.774093 \\ & 97 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.927461 \\ & 131 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.098854 \\ & 257 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.287670 \\ & 111 \times 10^{-6} \end{aligned}$ |
| (D) AGE | $\begin{aligned} & 0.288914 \\ & 116 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.347814 \\ & 44 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.425860 \\ & 73 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.523314 \\ & 76 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.639384 \\ & 26 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.774306 \\ & 116 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.927341 \\ & 11 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.098725 \\ & 128 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.287813 \\ & 32 \times 10^{-6} \end{aligned}$ |
| 4(D)AGE | $\begin{aligned} & 0.288739 \\ & 291 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.347444 \\ & 326 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.425340 \\ & 593 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.522662 \\ & 575 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.638648 \\ & 762 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.773576 \\ & 614 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 0.926658 \\ & 672 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.098242 \\ & 355 \times 10^{-6} \end{aligned}$ | $\begin{aligned} & 1.287477 \\ & 304 \times 10^{-6} \end{aligned}$ |
| $\begin{array}{r} \text { EXACT } \\ \text { SOLU'TION } \end{array}$ | 0.289030 | 0.347770 | 0.425933 | 0.523238 | 0.639410 | 0.774190 | 0.927330 | 1.098597 | 1.287781 |

TABLE (4.16.5)
$r=1.0, \Delta x=\Delta y=0.1, \Delta t=0.01, t=1.2$
$\mathrm{x}=0.5$

|  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| (D) AGE | 0.27432 | 0.318665 | 0.382559 | 0.466177 | 0.569098 | 0.691457 | 0.832872 | 0.993487 | 1.172993 |
|  | $87 \times 10^{-6}$ | $23 \times 10^{-6}$ | $114 \times 10^{-6}$ | $55 \times 10^{-6}$ | $131 \times 10^{-6}$ | $77 \times 10^{-6}$ | $153 \times 10^{-6}$ | $99 \times 10^{-6}$ | $120 \times 10^{-6}$ |
| 4 (D) AGE | 0.274208 | 0.318424 | 0.382227 | 0.465742 | 0.569594 | 0. 690946 | 0.832375 | 0.993133 | 1.172729 |
|  | $208 \times 10^{-6}$ | $263 \times 10^{-6}$ | $447 \times 10^{-6}$ | $490 \times 10^{-6}$ | $635 \times 10^{-6}$ | $588 \times 10^{-6}$ | $650 \times 10^{-6}$ | $453 \times 10^{-6}$ | $383 \times 10^{-6}$ |
| EXACT <br> SOLUTION | 0.274416 | 0.318688 | 0.382673 | 0.466232 | 0.569229 | 0.691534 | 0.833025 | 0.993586 | 1.173113 |

The accuracy of the different Group Explicit schemes are compared in Tables (4.16.1)-(4.16.6) for various values of $r$. The tables give the absolute errors along $x=0.5,0.1$ and 0.9 as this is the middle and the end parts of the domain. From the tables, the GER, (S)AGE and (D)AGE scheme are almost similar in accuracy. For the GER schemes, it can be seen that the error near the boundary grows faster than that for the (S)AGE and (D)AGE schemes.

For $r \geqslant 1$, the method can also be implemented using the fractional splitting type of (S)AGE or (D)AGE schemes in order to reduce the effect of the truncation error accumulation.

### 4.17 THE EXTENSION TO MULTI-SPACE DIMENSIONAL PROBLEMS

In this section we will extend the GE formulation to the three space dimensional problem for the heat equation of the form,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}+\frac{\partial^{2} u}{\partial z^{2}}+g(x, y, z, t) \tag{4.17.1}
\end{equation*}
$$

where the appropriate initial and boundary conditions are given. For this case the GE grid points consist of 8 points taken to form a cube instead of the 4 points in a plane as for the two-dimensional problem (Fig. 4.17.1).


FIGURE (4.17.1): A cube which forms a group at any time-level $k$

At each of points $a, b, c, \ldots, g, h$ and $t=k+\frac{\Delta t}{2}$, we approximate equation (4.17.1) by the following equations,

$$
\begin{align*}
& \frac{u_{i, j,}^{k+1}-u_{i, j, \ell}^{k}}{\Delta t}=\frac{u_{i+1, j, \ell^{k+1}}^{u_{i, j, \ell^{-u}}^{k+1}}{ }_{i, j, \ell^{+u_{i-1, j}^{k}}}^{(\Delta x)^{2}}+}{} \\
& +\frac{u_{i, j+1, \ell^{-u_{i, j}^{k+1}} \ell^{-u_{i, j, \ell^{k}}+u_{i, j-1, \ell}^{k}}}^{(\Delta y)^{2}}}{\left(\frac{1}{k}\right.} \\
& +\frac{u_{i, j, \ell+1}^{k+1} u_{i, j, \ell^{-u}}^{k+1, j, \ell^{+}}{ }_{i, j, \ell-1}^{k}}{(\Delta z)^{2}}+g_{i, j, \ell}^{k+\frac{1}{2}},  \tag{4.17.2}\\
& \frac{u_{i+1, j, \ell^{-u_{i+1, j}^{k}}}^{k+1}}{\Delta t}=\frac{u_{i+2, j, \ell^{k}}^{u_{i+1, j}^{k}} \ell^{-u_{i+1, j}^{k+1}} \ell^{-u_{i, j}^{k+1}}, \ell}{(\Delta x)^{2}}+ \\
& +\frac{u_{i+1, j+1, \ell^{-u_{i+1}^{k+1}, j, \ell^{-u_{i+1}^{k}}, j, \ell^{+u_{i+1, j-1, \ell}^{k}}}}^{(\Delta y)^{2}}+}{}
\end{align*}
$$

$$
\begin{align*}
& \text { (4.17.3) } \\
& \frac{u_{i+1, j+1, \ell^{k+1}}^{-u_{i+1, j+1, \ell}^{k}}}{\Delta t}=\frac{u_{i, j+1, \ell^{k+1}}^{-u_{i+1, j+1}^{k+1} \ell^{-u_{i+1, j+1, \ell^{k}}^{k}} i+2, j+1, \ell}}{(\Delta x)^{2}}+ \\
& +\frac{u_{i+1, j+2, \ell^{-u}}^{k} i+1, j+1, \ell^{-u^{k+1}} i+1, j+1, \ell^{-u^{k+1}} i+1, j+1, \ell^{-u^{k+1}} i+1, j, \ell}{(\Delta y)^{2}} \\
& +\frac{u_{i+1, j+1, \ell+1}^{k+1}-u_{i+1, j+1, \ell^{-u_{i+1, j+1, \ell^{k}} \mathrm{u}_{i+1, j+1, \ell-1}^{k}}}^{(\Delta z)^{2}}+}{} \\
& g_{i+1, j+1, \ell}^{k+\frac{1}{2}} \tag{4.17.4}
\end{align*}
$$

$$
\begin{aligned}
& +\frac{u_{i, j+2, \ell^{-u_{i}^{k}}, j+1, \ell^{-u_{i, j+1, \ell^{k+1}} u_{i, j, \ell}^{k+1}}}^{(\Delta y)^{2}}+}{} \\
& +\frac{\stackrel{u}{i, j+1, \ell+1}_{u_{i}+u_{i}^{k+1}}^{i+1, \ell^{-u_{i}^{k}}, j+1, \ell^{+u_{i}^{k}}, j+1, \ell-1}}{(\Delta z)^{2}}+g_{i, j+1, \ell}^{k+\frac{1}{2}}, \\
& \text { (4.17.5) }
\end{aligned}
$$

$$
\begin{align*}
& \frac{{\underset{u}{k+1}}_{k+1, \ell+1}^{-u_{i, j, \ell+1}^{k}}}{\Delta t}=\frac{u_{i+1, j, \ell+1}^{k+1}-u_{i, j, \ell+1}^{k+1}-u_{i, j, \ell+1}^{k} u_{i-1, j, \ell+1}^{k}}{(\Delta x)^{2}}+ \\
& +\frac{u_{i, j+1, \ell+1}^{k+1}-u_{i, j, \ell+1}^{k+1}-u_{i, j, \ell+1}^{k}+u_{i, j-1, \ell+1}^{k}}{(\Delta y)^{2}} \\
& +\frac{u_{i, j, \ell^{k+1}}^{u_{i, j, \ell+1}^{k+1}} u_{i, j, \ell+1}^{k}+u_{i, j, \ell+2}^{k}}{(\Delta z)^{2}}+g_{i, j, \ell+1}^{k+\frac{1}{2}}, \\
& \frac{u_{i+1, j, \ell+1}^{k+1}-u_{i+1, j, \ell+1}^{k}}{\Delta t}=\frac{u_{i, j, \ell+1}^{k+1}-u_{i+1, j, \ell+1}^{k+1} u_{i+1, j, \ell+1}^{k} u_{i+2, j, \ell+1}^{k}}{(\Delta x)^{2}}+ \\
& +\frac{u_{i+1, j+1, \ell+1}^{k+1} u_{i+1, j, \ell+1}^{k+u_{i+1, j, \ell+1}^{k}}{ }^{\text {u }}{ }_{i+1, j-1, \ell+1}}{(\Delta y)^{2}} \\
& +\frac{{ }_{\mathrm{u}+1, j, \ell^{\mathrm{k}+1} \mathrm{u}_{i+1, j, \ell+1}^{\mathrm{k}+1} \mathrm{u}_{i+1, j, \ell+1}^{\mathrm{k}} \mathrm{u}_{i+1, j, \ell+2}^{k}}^{(\Delta z)^{2}}}{k+\frac{1}{2}} \\
& +g_{i+1}^{k+\frac{1}{2}}, j, \ell+1, \\
& \text { (4.17.7) } \\
& \frac{u_{i+1, j+1, \ell+1}^{k+1}-u_{i+1, j+1, \ell+1}^{k}}{\Delta t}=\frac{u_{i, j+1, \ell+1}^{k+1}-u_{i+1, j+1, \ell+1}^{k+1} u_{i+1, j+1, \ell+1}^{k}+u_{i+2 j+1, \ell+1}^{k}}{(\Delta x)^{2}} \\
& +\frac{u_{i+1, j, \ell+1}^{k+1} u_{i+1, j+1, \ell+1}^{k+1} u_{i+1, j+1, \ell+1}^{k}+u_{i+1, j+2, \ell+1}^{k}}{(\Delta y)^{2}} \\
& +\frac{u_{i+1, j+1, e^{k+1}}^{-u_{i+1}^{k+1}, j+1, \ell+1}{ }^{-u_{i+1, j+1, \ell+1}^{k}+u_{i+1, j+1, \ell+2}^{k}}}{(\Delta z)^{2}} \\
& +g_{i+1, j+1, \ell+1}^{k+\frac{1}{2}}, \tag{4.17.8}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{u_{i, j+1, \ell+1}^{k+1}-u_{i, j+1, \ell+1}^{k}}{\Delta t}=\frac{u_{i+1, j+1, \ell+1}^{k+1}-u_{i, j+1, \ell+1}^{k+1}-u_{i, j+1, \ell+1}^{k}+u_{i-1, j+1, \ell+1}^{k}}{(\Delta x)^{2}} \\
& \\
& +\frac{u_{i, j, \ell+1}^{k+1} u_{i, j+1, \ell+1}^{k+u_{i, j+1, \ell+1}^{k}+u_{i, j+2, \ell+1}^{k}}}{(\Delta y)^{2}}  \tag{4.17.9}\\
& \\
& +
\end{align*}
$$

respectively.

Assuming $\Delta x=\Delta y=\Delta z=\Delta s$ and $r=\Delta t /(\Delta s)^{2}$, equations (4.17.2)-(4.17.9) can be reduced to the forms,

$$
-r\left[u_{i+1, j, \ell+1}^{k+1}+u_{i, j+1, \ell+1}^{k+1}+u_{i, j, \ell}^{k+1}\right]+(1+3 r) u_{i, j, \ell+1}^{k+1}=r\left(u_{i-1, j, \ell+1}^{k}+u_{i, j-1, \ell+1}^{k}+\right.
$$

$$
\begin{equation*}
\left.+u_{i, j, \ell+2}^{k}\right)+(1-3 r) u_{i, j, \ell+1}^{k}+g_{i, j, \ell+1}^{k+\frac{1}{2}} \tag{4.17.14}
\end{equation*}
$$

$$
-r\left[u_{i, j, \ell+1}^{k+1}+u_{i+1, j+1, \ell+1}^{k+1}+u_{i+1, j, \ell}^{k+1}\right]+(1+3 r) u_{i+1, j, \ell+1}^{k+1}=r\left(u_{i+2, j, \ell+1}^{k}+\right.
$$

$$
\left.u_{i+1, j-1, \ell+1}^{k}+u_{i+1, j, \ell+2}^{k}\right)+(1-3 r) u_{i+1, j, \ell+1}^{k}+g_{i+1, j, \ell+1}^{k+\frac{1}{2}}
$$

$$
\begin{equation*}
-r\left[u_{i, j+1, \ell+1}^{k+1}+u_{i+1, j, \ell+1}^{k+1}+u_{i+1, j+1, \ell}^{k+1}\right]+(1+3 r) u_{i+1, j+1, \ell+1}^{k+1}=r\left(u_{i+2, j+1, \ell+1}^{k}\right. \tag{4.17.15}
\end{equation*}
$$

$$
\begin{equation*}
\left.+u_{i+1, j+2, \ell+1}^{k}+u_{i+1, j+1, \ell+2}^{k}\right)+(1-3 r) u_{i+1, j+1, \ell+1}^{k}+g_{i+1, j+1, \ell+1}^{k+\frac{1}{2}} \tag{4.17.16}
\end{equation*}
$$

and

$$
\begin{array}{r}
-r\left[u_{i+1, j+1, \ell+1}^{k+1}+u_{i, j, \ell+1}^{k+1}+u_{i, j+1, \ell}^{k+1}\right]+(1+3 r) u_{i, j+1, \ell+1}^{k+1}=r\left(u_{i-1, j+1, \ell+1}^{k}+\right. \\
 \tag{4.17.17}\\
\left.\quad u_{i, j+2, \ell+1}^{k}+u_{i, j+1, \ell+2}^{k}\right)+(1-3 r) u_{i, j+1, \ell+1}^{k}+g_{i, j+1, \ell+1}^{k+\frac{1}{2}}
\end{array}
$$

$$
\begin{align*}
& -r\left[u_{i+1, j, \ell}^{k+1}+u_{i, j+1, \ell}^{k+1}+u_{i, j, \ell+1}^{k+1}\right]+(1+3 r) u_{i, j, \ell}^{k+1}=r\left(u_{i-1, j, \ell}^{k}+u_{i, j-1, \ell}^{k}+u_{i, j, \ell-1}^{k}\right) \\
& +(1-3 r) u_{i, j, \ell}^{k}+g_{i, j, \ell}^{k+\frac{1}{2}},  \tag{4.17.10}\\
& -r\left[u_{i, j, \ell}^{k+1}+u_{\left.i+1, j+1, \ell^{k+1}+u_{i+1, j, \ell+1}^{k+1}\right]+(1+3 r)} u_{i+1, j, \ell}^{k+1}=r\left(u_{i+2, j, \ell}^{k}+u_{i+1, j-1, \ell}^{k}\right.\right. \\
& \left.+u_{i+1, j, \ell-1}^{k}\right)+(1-3 r) u_{i+1, j, \ell^{k}}^{k_{i+1, j}^{k+\frac{1}{2}}, \ell},  \tag{4.17.11}\\
& -r\left[u_{i, j+1, \ell}^{k+1}+u_{i+1, j, \ell}^{k+1}+u_{i+1, j+1, \ell+1}^{k+1}\right]+(1+3 r) u_{i+1, j+1, \ell}^{k+1}=r\left(u_{i+2, j+1, \ell}^{k}{ }^{+}\right. \\
& \left.+u_{i+1, j+2, \ell}^{k}+u_{i+1, j+1, \ell-1}^{k}\right)+(1-3 r) u_{i+1, j+1, \ell}^{k}+g_{i+1, j+1, \ell}^{k+\frac{1}{2}}  \tag{4.17.12}\\
& -r\left[u_{i+1, j+1, \ell}^{k+1}+u_{i, j, \ell}^{k+1}+u_{i, j+1, \ell+1}^{k+1}\right]+(1+3 r) u_{i, j+1, \ell}^{k+1}=r\left(u_{i-1, j+1, \ell}^{k}+u_{i, j+2, \ell}^{k}\right. \\
& \left.+u_{i, j+1, \ell-1}^{k}\right)+(1-3 r) u_{i, j+1, \ell^{k}}^{+g_{i, j+1, \ell}^{k+\frac{1}{2}},} \tag{4.17.13}
\end{align*}
$$

The implicit system formed by equations (4.17.10)-(4.17.17) can then be written in matrix form as

$$
\begin{equation*}
\mathrm{Au}^{\mathrm{k}+1}=\underline{\underline{k}}^{\mathrm{k}} \text {, } \tag{4.17.18}
\end{equation*}
$$

where $\underline{f}^{k}$ is a vector whose elements are defined by the right-hand side of equations (4.17.10)-(4.17.17), A is given by

$$
A=\left[\begin{array}{cccc:cccc}
1+3 r & -r & 0 & -r & -r & 0 & 0 & 0  \tag{4.17.19}\\
-r & 1+3 r & -r & 0 & 0 & -r & 0 & 0 \\
0 & -r & 1+3 r & -r & 0 & 0 & -r & 0 \\
-r & 0 & -r & 1+3 r & 0 & 0 & 0 & -r \\
\hdashline-r & 0 & 0 & 0 & 1+3 r & -r & 0 & -r \\
0 & -r & 0 & 0 & -r & 1+3 r & -r & 0 \\
0 & 0 & -r & 0 & 0 & -r & 1+3 r & -r \\
0 & 0 & 0 & -r & -r & 0 & -r & 1+3 r
\end{array}\right]
$$

and $\underline{u}^{k+1}=\left[u_{i, j, \ell}, u_{i+1, j, \ell},{ }_{i+1, j+1, \ell}, u_{i, j+1, \ell,} u_{i, j, \ell+1}, u_{i+1, j, \ell+1}\right.$,

$$
\left.u_{i+1, j+1, \ell+1}, u_{i, j+1, \ell+1}\right]^{T}
$$

Similar to one and two-dimensional cases to form the GE method we have to invert the matrix A. Due to its special form, the inversion of this matrix is less difficult and using a derivation by W.S. Yousif (1982) the procedure is as follows:

## Inversion of A

Let

$$
A^{\prime}=\left[\begin{array}{ll}
A_{1} & A_{2} \\
A_{2} & A_{1}
\end{array}\right] \quad, A=(1+3 r) A^{\prime}
$$

with
and 1 et

$$
\left(A^{*}\right)^{-1}=\left[\begin{array}{ll}
D & B \\
B & D
\end{array}\right]
$$

From the identity $A^{\prime}\left(A^{\prime}\right)^{-1}=I$ we can obtain the following relationships,

$$
D=-B A_{1} A_{2}^{-1} \quad \text { and } \quad B=\left[A_{2}-A_{1} A_{2}^{-1} A_{1}\right]^{-1}
$$

We evaluate $B$ first from,

$$
A_{1} A_{2}^{-1} A_{1}=\left[\begin{array}{cccc}
\frac{1+2 \alpha^{2}}{\alpha} & 2 & 2 \alpha & 2 \\
2 & \frac{1+2 \alpha^{2}}{\alpha} & 2 & 2 \alpha \\
2 \alpha & 2 & \frac{1+2 \alpha^{2}}{2} & 2 \\
2 & 2 \alpha & 2 & \frac{1+2 \alpha^{2}}{2}
\end{array}\right]
$$

with $\alpha=-r /(1+3 r)$. Therefore,
$A_{2}-A_{1} A_{2}^{-1} A_{1}=\left[\begin{array}{cccr}-\frac{1+\alpha^{2}}{\alpha} & -2 & -2 \alpha & -2 \\ -2 & \frac{1+\alpha^{2}}{\alpha} & -2 & -2 \alpha \\ -2 \alpha & -2 & \frac{1+\alpha^{2}}{\alpha} & -2 \\ -2 & -2 \alpha & -2 & -\frac{1+\alpha^{2}}{\alpha}\end{array}\right]$
and after some algebraic manipulation we obtain the result,

$$
B=\alpha / k \quad\left[\begin{array}{cccc}
3 \alpha^{2}-1 & 2 \alpha & -6 \alpha^{2} & 2 \alpha \\
2 \alpha & 3 \alpha^{2}-1 & 2 \alpha & -6 \alpha^{2} \\
-6 \alpha^{2} & 2 \alpha & \alpha^{2}-1 & 2 \alpha \\
2 \alpha & -6 \alpha^{2} & 2 \alpha & 3 \alpha^{2}-1
\end{array}\right]
$$

and from which $D$ can easily be obtained in the form,

$$
\left[\begin{array}{cccc}
7 \alpha^{2}-1 & \alpha\left(1-3 \alpha^{2}\right) & \frac{-2 \alpha^{2}\left(1+3 \alpha^{2}\right)}{1-\alpha^{2}} & \alpha\left(1-3 \alpha^{2}\right) \\
\alpha\left(1-3 \alpha^{2}\right) & 7 \alpha^{2}-1 & \alpha\left(1-3 \alpha^{2}\right) & \frac{-2 \alpha^{2}\left(1+3 \alpha^{2}\right)}{2} \\
\frac{-2 \alpha^{2}\left(1+3 \alpha^{2}\right)}{1-\alpha^{2}} & \alpha\left(1-3 \alpha^{2}\right) & 7 \alpha^{2}-1 & \alpha\left(1-3 \alpha^{2}\right) \\
\alpha\left(1-3 \alpha^{2}\right) & \frac{-2 \alpha^{2}\left(1+3 \alpha^{2}\right)}{1-\alpha^{2}} & \alpha\left(1-3 \alpha^{2}\right) & 7 \alpha^{2}-1
\end{array}\right]
$$

where $k=\left(1-\alpha^{2}\right)\left(1-9 \alpha^{2}\right)$. Hence the matrix $A^{-1}$ can now be assembled in the form:

Near the boundary, the points are treated either as a group of 4 points or a group of 2 points or an ungrouped point depending on the location of the points in the space, (Fig. (4.17.2). These special cases can be treated in a similar manner as the ordinary points of the two-space dimensional case or the ordinary point of the one-space dimension case. To avoid mathematical repetition, these will not be given here.


How the ungrouped surface is treated


Diagram showing the ungrouped surface

### 4.18 CONCLUSIVE REMARKS

From the experiences gained in carrying out the work in this chapter we can express the following conclusions:-

1. The generalised finite difference approximation (4.2.1) is an equation from which it is possible to derive explicit, implicit, symmetrical and asymmetrical types of approximation.
2. Any existing numerical schemes of order up to $\left((\Delta x)^{2}+(\Delta t)^{2}\right)$ has to fulfill the compulsory conditions (4.2.2)-(4.2.4) and the fulfilment of optional conditions (4.2.5)-(4.2.6) varies according to the chosen scheme.
3. The GE methods are a class of methods which are derived from the combination of two implicit low-order asymmetrical formulae, i.e. Figs. (4.2.6) and (4.2.7) and results in two explicit formulae of low-order accuracy i.e. Fig.(4.5.1). These two formulae are implemented in such a way that sometimes their truncation error signs will tend to cancel each other, therefore upgrading the order of accuracy.
4. In general all the GE schemes have better stability conditions as compared to the classical explicit scheme.
5. The GER, GEL, GEC and GEU schemes are best used for values of $\mathbf{r} \sim 0.5$.
6. For $0.5 \leqslant r \leqslant 1.0$, the alternating formulae are recommended, i.e. (S)AGE and (D)AGE for use.
7. For $1.0 \leqslant r \leqslant 2.0$, the splitting type of half-step is recommended i.e. (S)AGE or (D)AGE half-splitting.
8. For $r \geqslant 2.0$, the fractional splitting type of quarter step (D)AGE is recommended.
9. For $\mathrm{r} \leqslant 0.1$, the classical explicit scheme gives the better accuracy.
10. Most of the numerical results using the recommended schemes are as accurate as the results obtained from the CN method.
11. One important fact from this class of method is its avoidance of solving tridiagonal systems of equations.
12. The extension of the method to two- and higher-space dimensions is fairly straight forward and the unconditional stability of the explicit scheme is obviously a great advantage.
13. To some extent we are able to indicate the best weighting factor to choose for certain values of $r$ in the Weighted Group Explicit method.
14. Although only examples in rectangular space-domains were considered the schemes are usable for non-rectangular domains.
15. Since the schemes are totally explicit, it is also suitable for use on the parallel processing systems of the future.

## Chapter Five

## EXPLICIT METHODS FOR THE SOLUTION

OF DIFFUSION-CONVECTION EQUATIONS

### 5.1 INTRODUCTORY REMARKS

As mentioned earlier in Chapter 3, the diffusion-convection equation needs to be treated separately from the ordinary diffusion equation because of the presence of spatial derivatives of first order. One way of doing that is approximating the terms $\frac{\partial^{2} u}{\partial x^{2}}$ and $\frac{\partial u}{\partial x}$ in the diffusion-convection equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\varepsilon \frac{\partial^{2} u}{\partial x^{2}}-k \frac{\partial u}{\partial x} \tag{5.1.1}
\end{equation*}
$$

by using different types of approximation, i.e. central differences for $\frac{\partial^{2} u}{\partial x^{2}}$ and forward/backward differences for $\frac{\partial u}{\partial x}$ as in the generalized form (5.2.1).

Some stable explicit methods which are actually of similar form to those of Larkin (1964) and Clark and Barakhat (1966) are obtainable from the generalised form (5.2.1). These explicit schemes can then be used to develop the GE scheme for the diffusion-convection equation (5.1.1) and the scheme can also be shown to be stable too.

The upwinding type of approximation for (5.1.1) can also be developed and though it is stable scheme it can be seen that this low-order approximation can really affect the accuracy of the solution.

### 5.2 THE GENERALIZED FINITE-DIFFERENCE APPROXIMATION FOR THE DIFFUSION-

 CONVECTION EQUATIONThe diffusion-convection equation (5.1.1) can generally be approximated at the point $\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t \equiv\left(i, j+\frac{1}{2}\right)\right.$ by the two time-level finite difference representation,

$$
\begin{align*}
\frac{u_{i, j+1}-u_{i, j}}{\Delta t}= & \frac{\varepsilon \delta}{(\Delta x)^{2}}\left[\left(\theta_{1} \delta_{x}^{u}{ }_{i+\frac{1}{2}, j+1}^{\left.-\theta_{2} \delta_{x} u_{i-\frac{1}{2}, j+1}\right)}\right.\right. \\
& \left.+\left(\theta_{1}^{\prime} \delta_{x}^{u}{ }_{i+\frac{1}{2}, j}-\theta_{2}^{\prime} \delta_{x}^{u}{ }_{i-\frac{1}{2}, j}\right)\right]-\frac{k}{2 \Delta x}\left[\alpha_{1} \Delta_{x} u_{i, j+1}+\alpha_{2} \nabla_{x}^{u}{ }_{i, j}\right] \\
& -\frac{k}{2 \Delta x}\left[\alpha_{1}^{\prime} \nabla_{x}^{u}{ }_{i, j+1}+\alpha_{2}^{\prime} \Delta_{x} u_{i, j}\right] \tag{5.2.1}
\end{align*}
$$

where $\delta_{x}, \Delta_{x}$ and $\nabla_{x}$ are the central-difference, forward-difference and backward-difference operators with respect to the $x$ variable respectively. The included parameters $\theta_{1}, \theta_{2} \theta_{1}^{\prime}, \theta_{2}^{\prime}, \alpha_{1}, \alpha_{2}, \alpha_{1}^{\prime}$ and $\alpha_{2}^{\prime}$ are related to each other by the following compulsomy conditions,

$$
\begin{align*}
& \theta_{1}+\theta_{2}+\theta_{1}^{\prime}+\theta_{2}^{\prime}=2  \tag{array}\\
& \theta_{2}-\theta_{1}+\theta_{2}^{\prime}-\theta_{1}^{\prime}=0  \tag{5.2.2}\\
& \alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}+\alpha_{2}^{\prime}=2
\end{align*}
$$

and also by the optional conditions,

$$
\begin{align*}
& \theta_{2}-\theta_{1}-\theta_{2}^{\prime}+\theta_{1}^{\prime}=0 \\
& -\theta_{2}-\theta_{1}+\theta_{2}^{\prime}+\theta_{1}^{\prime}=0 \\
& \alpha_{1}^{\prime}+\alpha_{1}-\alpha_{2}-\alpha_{2}^{\prime}=0  \tag{5.2.3}\\
& -\alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}-\alpha_{2}^{\prime}=0 \\
& -\alpha_{1}^{\prime}+\alpha_{1}-\alpha_{2}+\alpha_{2}^{\prime}=0
\end{align*}
$$

$\left.\begin{array}{l}\text { (a) } \\ \text { (b) } \\ \text { (c) } \\ \text { (d) } \\ \text { (e) }\end{array}\right\}$
The importance of the conditions (5.2.2) and (5.2.3) will be seen to be evident later in the next section.

After the insertion of the values for the difference operators,
equation (5.2.1) will lead to

$$
\begin{align*}
& -\left(E \theta_{2}+K \alpha_{1}^{\prime}\right) u_{i-1, j+1}+\left[1+E\left(\theta_{1}+\theta_{2}\right)-K\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\right] u_{i, j+1}-\left(E \theta_{1}-K \alpha_{1}\right) u_{i+1, j+1}= \\
& \quad\left(E \theta_{2}^{\prime}+K \alpha_{2}\right) u_{i-1, j}+\left[1-E\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)+K\left(\alpha_{2}^{\prime}-\alpha_{2}\right)\right] u_{i, j}+\left(E \theta_{1}^{\prime}-K \alpha_{2}^{\prime}\right) u_{i+1, j} \tag{5.2.4}
\end{align*}
$$

where $E=\varepsilon r, K=k \Delta t / 2 \Delta x=k r \Delta x / 2$ and as usualy $r=\Delta t /(\Delta x)^{2}$.
The resulting approximation (5.2.4) can be regarded as a generalized two-level finite difference representation of (5.1.1) for the several reasons which are outlined as follows:

1. The implicit and explicit type of formulae can be derived from the generalised approximation (5.2.4).
2. A variety of formulae with varying degrees of accuracy up to $O\left\{(\Delta x)^{2}+(\Delta t)^{2}\right\}$ can be derived from this approximation.
3. Most existing formulae for (5.1.1) are derivable from this approximation.
4. A class of asymetric formulae (Saul'yev, 1964, p. 91) can also be obtained from this formulae.

The following choice of parameters are a few examples which will verify the above arguments:-

1. $\theta_{1}=\theta_{2}=0, \alpha_{1}=\alpha_{1}^{\prime}=0, \theta_{1}^{\prime}=\theta_{2}^{\prime}=1, \alpha_{2}=\alpha=1$ gives

$$
\begin{equation*}
u_{i, j+1}=(E+K) u_{i-1, j}+(1-2 E) u_{i, j}+(E-K) u_{i+1, j}, \tag{5.2.5}
\end{equation*}
$$

which is the classical explicit scheme.
2. $\quad \theta_{1}=\theta_{2}=1, \alpha_{1}=\alpha_{1}^{\prime}=1, \theta_{1}^{\prime}=\theta_{2}^{\prime}=0, \alpha_{2}=\alpha_{2}^{\prime}=0$ gives

$$
\begin{equation*}
-(E+K) u_{i-1, j+1}+(1+2 E) u_{i, j+1}-(E-K) u_{i+1, j+1}=u_{i, j}, \tag{5.2.6}
\end{equation*}
$$

which is the classical (fully) implicit scheme.
3. $\theta_{1}=\theta_{2}=\theta_{1}^{\prime}=\theta_{2}^{\prime}=\frac{1}{2}, \alpha_{1}=\alpha_{2}=\alpha_{1}^{\prime}=\alpha_{2}^{\prime}=\frac{1}{2}$ gives

$$
\begin{align*}
- & \frac{1}{2}(E+K) u_{i-1, j+1}+(1+E) u_{i, j+1}-\frac{1}{2}(E-K) u_{i+1, j+1} \\
& =\frac{1}{2}(E+K) u_{i-1, j}+(1-E) u_{i, j}+\frac{1}{2}(E-K) u_{i+1, j} \tag{5.2.7}
\end{align*}
$$

which is the Crank-Nicolson scheme.
4. $\quad \theta_{1}=\theta_{2}=\theta_{1}^{\prime}=\theta_{2}^{\prime}=\frac{1}{2}, \alpha_{2}=\alpha_{1}^{\prime}=1, \alpha_{1}=\alpha_{2}^{\prime}=0$ gives

$$
\begin{align*}
& -\frac{1}{2}\left(E+K^{\prime}\right) u_{i-1, j+1}+(1+E+K) u_{i, j+1}-\frac{E}{2} u_{i+1, j+1}=\frac{1}{2}\left(E+K^{\prime}\right) u_{i-1, j} \\
& \quad+(1-E-K) u_{i, j}+\frac{E}{2} u_{i+1, j} \tag{5.2.8}
\end{align*}
$$

with $K^{\prime}=2 K$, which is the Crank-Nicolson scheme with upwinding. An example for the fourth reason will be discussed later in greater detail as this class of method is the subject of this chapter.

### 5.3 THE TRUNCATION ERROR FOR THE GENERALIZED FINITE-DIFFERENCE APPROXIMATION

To obtain the truncation error, we expand each value of the u's in equation (5.2.4) in terms of a Taylor's series expansion about (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$. That is,

$$
\begin{aligned}
& u_{i,\left(j+\frac{1}{2}\right) \pm \frac{1}{2}}=u_{i, j+\frac{1}{2}}^{ \pm} \pm\left.\frac{\Delta t}{2} \frac{\partial u}{\partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta t^{2}}{8} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j+\frac{1}{2}} \pm\left.\frac{\Delta t^{3}}{48} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+0(\Delta t)^{4}, \\
& u_{i+1,\left(j+\frac{1}{2}\right) \pm \frac{1}{2}}=u_{i, j+\frac{1}{2}}+\left.\Delta x \frac{\partial u}{\partial x}\right|_{i, j+\frac{1}{2}} \pm\left.\frac{\Delta t}{2} \frac{\partial u}{\partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta x^{2}}{2} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j+\frac{1}{2}}+ \\
& \left.+\left.\frac{\Delta t^{2}}{8} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j+\frac{1}{2}} \pm\left.\frac{\Delta x \Delta t}{2} \frac{\partial^{2} u}{\partial x \partial t}\right|_{i, j+\frac{1}{2}}+\frac{(\Delta x)^{3}}{6} \frac{\partial^{3} u}{\partial x^{3}} \right\rvert\, \\
& \pm\left.\frac{(\Delta x)^{2} \Delta t}{12} \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta x(\Delta t)^{2}}{24} \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{i, j+\frac{1}{2}} \pm\left.\frac{\Delta t^{3}}{48} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}} \\
& +O\left(\Delta x^{\gamma} 1_{+\Delta t}{ }^{\gamma}\right) \text {, }
\end{aligned}
$$

with $\gamma_{1}+\gamma_{2}=4, \quad 0 \leqslant \gamma_{1}, \gamma_{2} \leqslant 4$,

$$
\begin{aligned}
& u_{i-1,\left(j+\frac{1}{2}\right) \pm \frac{1}{2}}=u_{i, j+\frac{1}{2}}-\left.\Delta x \frac{\partial u}{\partial x}\right|_{i, j+\frac{1}{2}} \pm\left.\frac{\Delta t}{2} \frac{\partial u}{\partial t}\right|_{i, j+\frac{1}{2}}+\left.\frac{\Delta x^{2}}{2} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j+\frac{1}{2}} \\
& +\left.\left.\frac{\Delta t^{2}}{8} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j+\frac{1}{2}} \mp \frac{\Delta x \Delta t}{2} \frac{\partial^{2} u}{\partial x \partial t}\right|_{i, j+\frac{1}{2}}-\left.\frac{(\Delta x)^{3}}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}} \\
& \pm\left.\frac{(\Delta x)^{2} \Delta t}{12} \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}-\left.\frac{\Delta x(\Delta t)^{2}}{24} \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{i, j+\frac{1}{2}} \pm\left.\frac{\Delta t^{3}}{48} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}} \\
& +0\left(\Delta x^{\gamma} 1 \Delta t^{\gamma}\right) .
\end{aligned}
$$

On substitution in equation (5.2.4) we obtain with further simplification the equation,

$$
\begin{aligned}
\frac{\partial u}{\partial t} & +\left[\frac{\varepsilon}{\Delta x}\left(\theta_{2}-\theta_{1}+\theta_{2}^{\prime}-\theta_{1}^{\prime}\right)+\frac{k}{2}\left(\alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}+\alpha_{2}^{\prime}\right)\right] \frac{\partial u}{\partial x} \\
& +\left[\frac{\varepsilon}{2}\left(-\theta_{2}-\theta_{1}-\theta_{2}^{\prime}-\theta_{1}^{\prime}\right)+\frac{k \Delta x}{4}\left(-\alpha_{1}^{\prime}+\alpha_{1}-\alpha_{2}+\alpha_{2}^{\prime}\right)\right] \frac{\partial^{2} u}{\partial x^{2}} \\
& +\left[\frac{\varepsilon}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(\theta_{2}-\theta_{1}-\theta_{2}^{\prime}+\theta_{1}^{\prime}\right)+\frac{k \Delta t}{4}\left(\alpha_{1}^{\prime}+\alpha_{1}-\alpha_{2}-\alpha_{2}^{\prime}\right)\right] \frac{\partial^{2} u}{\partial x \partial t}
\end{aligned}
$$

$$
\begin{gather*}
+\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}+\left[\frac{\varepsilon \Delta x}{6}\left(\theta_{2}-\theta_{1}+\theta_{2}^{\prime}-\theta_{1}^{\prime}\right)+\frac{k}{12}(\Delta x)^{2}\left(\alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}+\alpha_{2}^{\prime}\right)\right] \\
\frac{\partial^{3} u}{\partial x^{3}}+\left[\frac{\varepsilon}{24} \frac{\Delta t^{2}}{24}\left(\theta_{2}-\theta_{1}+\theta_{2}^{\prime}-\theta_{1}^{\prime}\right)+\frac{k}{48}\left(\Delta t^{2}\right)\left(\alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}+\alpha_{2}^{\prime}\right)\right] \frac{\partial^{3} u}{\partial x \partial t^{2}} \\
+\left[\frac{\varepsilon}{12} \Delta t\left(-\theta_{2}-\theta_{1}+\theta_{2}^{\prime}+\theta_{1}^{\prime}\right)+\frac{k \Delta x \Delta t}{24}\left(-\alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}^{\left.\left.-\alpha_{2}^{\prime}\right)\right]} \frac{\partial^{3} u}{\partial x^{2} \partial t}\right.\right. \\
+\frac{1}{\Delta t} 0\left(\Delta x_{1}^{\gamma_{1}-2} \Delta t^{\gamma_{2}+1}\right)+\frac{1}{\Delta t} 0\left(\Delta t^{\left.\gamma_{2}^{+1} \Delta \gamma_{1}^{-1}\right)}\right. \tag{5.3.1}
\end{gather*}
$$

Thus, it is clearly evident from equation (5.3.1) why the conditions (5.2.2) are compuslory and the conditions (5.2.3) are optional. When the conditions (5.2.2) apply the equation (5.3.1) will reduce to

$$
\begin{equation*}
\frac{\partial u}{\partial t}+k \frac{\partial u}{\partial x}-\varepsilon \frac{\partial^{2} u}{\partial x^{2}}+T_{i, j+\frac{1}{2}}=0 \tag{5.3.2}
\end{equation*}
$$

where

$$
\begin{align*}
T_{i, j+\frac{1}{2}}= & {\left[\frac{\varepsilon}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(\theta_{2}-\theta_{1}-\theta_{2}^{\prime}+\theta_{1}^{\prime}\right)+\frac{k \Delta t}{4}\left(\alpha_{1}^{\prime}+\alpha_{1}-\alpha_{2}-\alpha_{2}^{\prime}\right)\right] \frac{\partial^{2} u}{\partial x \partial t} } \\
& +\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}+\frac{k}{6}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x^{3}}+\frac{k}{24}\left(\Delta t^{2}\right) \frac{\partial^{3} u}{\partial x \partial t^{2}}-\frac{k \Delta x}{4}\left(-\alpha_{1}^{\prime}+\alpha_{1}-\alpha_{2}+\alpha_{2}^{\prime}\right) \frac{\partial^{2} u}{\partial x^{2}} \\
& {\left[\frac{\varepsilon}{12} \Delta t\left(-\theta_{2}-\theta_{1}+\theta_{2}^{\prime}+\theta_{1}^{\prime}\right)+\frac{k \Delta x \Delta t}{24}\left(-\alpha_{1}^{\prime}+\alpha_{1}+\alpha_{2}-\alpha_{2}^{\prime}\right)\right] \frac{\partial^{3} u}{\partial x^{2} \partial t} } \\
& +\frac{1}{\Delta t} O\left(\Delta x_{1}^{\left.\gamma_{1}-2 \gamma_{2}^{+1}\right)+\frac{1}{\Delta t} O\left(\Delta t^{\gamma_{2}+1} \gamma_{1}^{-1}\right)}\right. \tag{5.3.3}
\end{align*}
$$

is the local truncation error, with derivates evaluated at (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$. Using this formulae, we can easily show that the truncation error of examples (5.2.5)-(5.2.8) are respectively given by

$$
\begin{align*}
& T_{5.2 .5}=-\frac{k \Delta t}{2} \frac{\partial^{2} u}{\partial x \partial t}+\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}+\frac{k}{6}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x^{3}}+\frac{k}{24}(\Delta t)^{2} \frac{\partial^{3} u}{\partial x \partial t^{2}} \\
& +\frac{\varepsilon \Delta t}{6} \frac{\partial^{3} u}{\partial x^{2} \partial t^{2}}+\frac{1}{\Delta t} O\left(\Delta x^{\gamma_{1}-2} \Delta t^{\gamma_{2}+1}\right)+\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}-1} \Delta t^{\gamma_{2}^{+1}}\right), \\
& T_{5.2 .6}=\frac{k \Delta t}{2} \frac{\partial^{2} u}{\partial x \partial t}+\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}+\frac{k}{6}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x^{3}}+\frac{k}{24}(\Delta t)^{2} \frac{\partial^{3} u}{\partial x \partial t^{2}} \\
& -\frac{\varepsilon \Delta t}{6} \frac{\partial^{3} u}{\partial x^{2} \partial t}+\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}-2} \Delta t^{\gamma_{2}^{+1}}\right)+\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}-1} \Delta t^{\gamma_{2}^{+1}}\right), \tag{5.3.5}
\end{align*}
$$

$$
\begin{aligned}
& T_{5.2 .7}=\frac{\Delta t^{2}}{24}\left[\frac{\partial^{3} u}{\partial t^{3}}+\right. \frac{k}{\partial \partial^{3} u} \\
& \partial x \partial t^{2}
\end{aligned}+\frac{k}{6}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x} .
$$

and

$$
\begin{gather*}
T_{5.2 .8}=\frac{\Delta t^{2}}{24}\left[\frac{\partial^{3} u}{\partial t^{3}}+k-\frac{\partial^{3} u}{\partial x \partial t^{2}}\right]+\frac{k}{6}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x^{3}}+\frac{k \Delta x}{2}-\frac{x^{2}}{\partial x^{2}} \\
+\frac{1}{\Delta t} 0\left(\Delta x_{1}^{-2} \Delta t^{\gamma+1}\right) \tag{5.3.7}
\end{gather*}
$$

Therefore, equation (5.3.3) gives the generalised formulae for the principal part of the local truncation error to the generalised approximation (5.2.4). This analysis also shows that the compulsory condition (5.2.2) guarantees the consistency of the scheme for the given problem.

### 5.4 STABILITY ANALYSIS OF THE GENERALISED FINITE DIFFERENCE APPROXIMATION

 The stability analysis of equation (5.2.4) will be examined by using the Fourier series method as given in Chapter 2. Assuming the error term $E_{i, j}$ at the point ( $i, j$ ) can be expressed as $e^{i \sqrt{-1} B \Delta x_{\xi} j}$, the error term for the equation (5.2.4) satisfies$$
\begin{aligned}
& -\left(E \theta_{2}+K \alpha_{1}^{\prime}\right) e^{\sqrt{-1}(i-1) \beta \Delta x_{\xi} j+1}+\left[1+E\left(\theta_{1}+\theta_{2}\right)-K\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\right] e^{\sqrt{-1}} i \beta \Delta x_{\xi} j+1 \\
& -\left(E \theta_{1}-K \alpha_{1}\right) e^{\sqrt{-1}(i+1) \beta \Delta x_{\xi} j+1}=\left(E \theta_{2}^{\prime}+K \alpha_{2}\right) e^{\sqrt{-1}(i-1) \beta \Delta x_{k} j}+ \\
& \quad\left[1-E\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)+K\left(\alpha_{2}^{\prime}-\alpha_{2}\right)\right] e^{\sqrt{-1} i \beta \Delta x_{\xi} j}+\left(E \theta_{1}^{\prime}-K \alpha_{2}^{\prime}\right) e^{\sqrt{-1}(i+1) \beta \Delta x_{\xi} j},(5.4 .1)
\end{aligned}
$$

which after division by $e^{\sqrt{-1} i \beta \Delta x} \xi^{j}$ is

$$
\begin{aligned}
& \left.\xi\left[-\left(E \theta_{2}+K \alpha_{1}^{\prime}\right) e^{-\sqrt{-1} \beta \Delta x^{\prime}}\right]+\left\{1+E\left(\theta_{1}+\theta_{2}\right)-K\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\right\}-\left(E \theta_{1}-K \alpha_{1}\right) e^{\sqrt{-1} \beta \Delta x}\right] \\
& \quad=E\left(\theta_{2}^{\prime}+K \alpha_{2}\right) e^{-\sqrt{-1} \beta \Delta x}+\left[1-E\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)+K\left(\alpha_{2}^{\prime}-\alpha_{2}\right)\right]+\left(E \theta_{1}^{\prime}-K \alpha_{2}^{\prime}\right) e^{\sqrt{-1} \beta \Delta x}
\end{aligned}
$$

i.e.,

$$
\xi=\frac{1-2\left[\left(E \theta_{2}^{\prime}+K \alpha_{2}\right)+\left(E \theta_{1}^{\prime}-K \alpha_{2}^{\prime}\right)\right] \sin ^{2} \frac{\beta \Delta x}{2}+\sqrt{-1}\left[\left(E \theta_{1}^{\prime}-K \alpha_{2}^{\prime}\right)-\left(E \theta_{2}^{\prime}+K \alpha_{2}\right)\right] \sin \beta \Delta x}{1+2\left[\left(E \theta_{2}+K \alpha_{1}^{\prime}\right)+\left(E \theta_{1}-K \alpha_{1}\right)\right] \sin ^{2} \frac{\beta \Delta x}{2}+\sqrt{-1}\left[\left(E \theta_{2}+K \alpha_{1}^{\prime}\right)-\left(E \theta_{1}-K \alpha_{1}\right)\right] \sin \beta \Delta x}
$$

Clearing up, we have finally,

$$
\begin{equation*}
\therefore \quad \xi=\frac{1-2\left(A_{1}+A_{2}\right) \sin ^{2} \frac{B \Delta x}{2}+\sqrt{-1}\left(A_{2}-A_{1}\right) \sin \beta \Delta x}{1+2\left(B_{1}+B_{2}\right) \sin ^{2} \frac{\beta \Delta x}{2}+\sqrt{-1}\left(B_{1}-B_{2}\right) \sin \beta \Delta x} \tag{5.4.2}
\end{equation*}
$$

where

$$
\begin{array}{ll}
A_{1}=E \theta_{2}^{\prime}+\mathrm{K} \alpha_{2}, & A_{2}=E \theta_{1}^{\prime}-K \alpha_{2}^{\prime}, \\
B_{1}=E \theta_{2}+K \alpha_{1}^{\prime}, & B_{2}=E \theta_{1}-K \alpha_{1} .
\end{array}
$$

For the stability of the scheme it is necessary that

$$
\begin{equation*}
|\xi|=\frac{\mid \text { numerator } \mid}{\mid \text { denominator } \mid} \leqslant 1 \tag{5.4.3}
\end{equation*}
$$

It can be shown, by further manipulation that,

$$
\begin{equation*}
\mid \text { numerator } \left\lvert\,=\left\{1+4 \sin ^{2} \frac{\beta \Delta x}{2}\left[A_{1}^{2}+A_{2}^{2}+4 A_{1} A_{2} \sin ^{2} \frac{\beta \Delta x}{2}-A_{1}-A_{2}-2 A_{1} A_{2}\right]\right\}^{\frac{1}{2}}\right. \tag{5.4.4}
\end{equation*}
$$

and

$$
\mid \text { denominator } \left\lvert\,=\left\{1+4 \sin ^{2} \frac{\beta \Delta x}{2}\left[B_{1}^{2}+B_{2}^{2}+4 B_{1} B_{2} \sin ^{2} \frac{\beta \Delta x}{2}+B_{1}+B_{2}-2 B_{1} B_{2}\right]\right\}_{(5.4 .5)}^{\frac{1}{2}}\right.
$$

Therefore to fulfill (5.4.3) we need the following relationship to hold,

$$
\begin{align*}
& A_{1}^{2}+A_{2}^{2}+4 A_{1} A_{2} \sin ^{2} \frac{B \Delta x}{2}-A_{1} A_{2}-2 A_{1} A_{2} \\
& \quad \leq B_{1}^{2}+B_{2}^{2}+4 B_{1} B_{2} \sin ^{2} \frac{B \Delta x}{2}+B_{1}+B_{2}-2 B_{1} B_{2} \tag{5.4.6}
\end{align*}
$$

Using the condition derived in (5.4.6), we can show that the stability condition for equations (5.2.5)-(5.2.8) are as follows:
(1) For equation (5.2.5)

$$
A_{1}=E+K, \quad A_{2}=E-K, \quad B_{1}=B_{2}=0,
$$

therefore

$$
4\left(E^{2}-K^{2}\right) \sin ^{2} \frac{\beta \Delta x}{2}-2 E+4 K^{2} \leqslant 0,
$$

i.e.,

$$
\begin{align*}
r & \leqslant \frac{2 \varepsilon}{\left(4 \varepsilon^{2}-k^{2} \Delta x^{2}\right) \sin ^{2} \frac{\beta \Delta x}{2}+k^{2} \Delta x^{2}}  \tag{5.4.7}\\
& =\frac{1}{2 \varepsilon \sin ^{2} \frac{\beta \Delta x}{2}+\left(\frac{k \Delta x}{\sqrt{2 \varepsilon}}\right)^{2} \cos ^{2} \frac{\beta \Delta x}{2}}
\end{align*}
$$

which will reduce to $r \leqslant \frac{1}{2}$ for the case of the pure conduction problem and will be severely restricted for the convection dominated ( $k \gg \varepsilon$ ) problem.
(2) For equation (5.2.6)

$$
A_{1}=0, \quad A_{2}=0, \quad B_{1}=E+K, \quad B_{2}=E-K,
$$

therefore, we need

$$
\mathrm{B}_{1}^{2}+\mathrm{B}_{2}^{2}+2 \mathrm{~B}_{1} \mathrm{~B}_{2}\left(2 \sin ^{2} \frac{\mathrm{~B} \mathrm{\Delta x}}{2}-1\right)+\mathrm{B}_{1}+\mathrm{B}_{2} \geqslant 0
$$

i.e.,

$$
\begin{equation*}
r^{2} k^{2} \Delta x^{2}+2 \varepsilon r \geq 0 \tag{5.4.8}
\end{equation*}
$$

which is unconditionally satisfied for all $r>0$.
(3) For equation (5.2.7)

$$
A_{1}=\frac{1}{2}(E+K)=B_{1}, \quad A_{2}=\frac{1}{2}(E-K)=B_{2}
$$

therefore we need

$$
\begin{equation*}
\varepsilon \mathbf{r} \geqslant 0 \tag{5.4.8}
\end{equation*}
$$

which can be fulfilled by all r>0.
(4) For equation (5.2.8)

$$
A_{1}=\left(\frac{1}{2} E+K\right)=B_{1}, \quad A_{2}=\left(\frac{1}{2} E\right)=B_{2}
$$

This will require,

$$
\begin{equation*}
2(\varepsilon r+k r \Delta x / 2) \geqslant 0 \tag{5.4.9}
\end{equation*}
$$

i.e. $\quad r(2 \varepsilon+k \Delta x) \geqslant 0$,
which can be fulfilled by all r>0 too.
Thus, it is always the case that the implicit type of formula will result in unconditional stability and the explicit type will result in that of restricted stability. It will be seen later that for some explicit algorithms it is possible to reduce the stability restriction without incurring a considerable loss of accuracy.

### 5.5 SOME STABLE EXPLICIT SCHEMES

From the generalised finite difference approximation (5.2.4) some interesting explicit schemes can be obtained, with the truncation error and stability conditions derived from the equations (5.3.3) and (5.4.6) respectively. They are:
(1) $\theta_{1}=\theta_{2}^{\prime}=1, \quad \theta_{2}=\theta_{1}^{\prime}=0, \quad \alpha_{1}=\alpha_{2}=1$ and $\alpha_{1}^{\prime}=\alpha_{2}^{\prime}=0$ to give (Fig. 5.5.1),

$$
\begin{align*}
{[1+(\varepsilon r-} & \left.\left.\frac{k r \Delta x}{2}\right)\right] u_{i, j+1}-\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{i+1, j+1} \\
& =\left(\varepsilon r+\frac{k r \Delta x}{2}\right) u_{i-1, j}+\left[1-\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right] u_{i, j} \tag{5.5.1}
\end{align*}
$$

with L.T.E. given by

$$
\begin{align*}
T_{5.5 .1}=-\varepsilon\left(\frac{\Delta t}{\Delta x}\right) \frac{\partial^{2} u}{\partial x^{2} t}+\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}+\frac{k}{24}(\Delta t)^{2} & \frac{\partial^{3} u}{\partial x \partial t^{2}} \\
& +\frac{k}{6}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x^{3}}+\frac{k \Delta x \Delta t}{12} \frac{\partial^{3} u}{\partial x^{2} \partial t}
\end{aligned} \begin{aligned}
& \frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}-1} \Delta^{\gamma_{2}+1}\right)+ \\
& +\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}^{-2}} \Delta^{\gamma} \gamma^{+1}\right) \tag{5.5.2}
\end{align*}
$$

$$
\begin{gather*}
\gamma_{1}+\gamma_{2}=4, \quad 0 \leqslant \gamma_{1}, \gamma_{2} \leqslant 4, \text { and requires for stability, } \\
0<r \leqslant \frac{1}{k \Delta x} \tag{5.5.3}
\end{gather*}
$$

This condition for stability is always rather favourable since with $\mathrm{k}=1.0, \Delta \mathrm{x}=0.1, \mathrm{k} \Delta \mathrm{x}=1 / 10$ and this is always less restrictive than the condition for the classical explicit formula as in (5.4.7).
(2) $\quad \theta_{2}=\theta_{1}^{\prime}=1, \theta_{1}=\theta_{2}^{\prime}=0, \alpha_{1}^{\prime}=\alpha_{2}^{\prime}=1$ and $\alpha_{1}=\alpha_{2}=0$ will result in (Fig.5.5.2),

$$
\begin{align*}
-(\varepsilon r+ & \left.\frac{k r \Delta x}{2}\right) u_{i-1, j+1}+\left[1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right] u_{i, j+1} \\
& =\left[1-\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\right] u_{i, j}+\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{i+1, j} \tag{5.5.4}
\end{align*}
$$

with the L.T.E. given by


FIGURE (5.5.1)


FIGURE (5.5.2)

For stability it requires

$$
\begin{equation*}
2 \varepsilon r(k r \Delta x+1) \geqslant 0 \tag{5.5.7}
\end{equation*}
$$

which is fulfilled by all $r>0$.
It is worth stressing the opposite signs of the truncation errors in (5.5.2) and (5.5.6) as they will play an important role in the algorithms that will be discussed later in this section. The strategy used in these
algorithms is similar to that suggested by Larkin (1963) for the heat conduction equation. The possible algorithms which can be derived are:

1. By use of equation (5.5.1) from a right-to-left direction (UNE)
2. By use of equation (5.5.4) from a left-to-right direction (UPOS)
3. By use of equation (5.5.1) at the $j^{\text {th }}$ time level from a right-toleft direction and alternatively using equation (5.5.4) at the $(j+1)$ th time level from the left-to-right direction (ALDC).
4. By use of equation (5.5.1) as in (1) and equation (5.5.4) as in (2) at each time-level and then average the results (UAV).

In the case of algorithms (3) and (4) the present stability condition will still hold.

If $u$ and its derivatives are assumed to be smooth such that if ( $x_{A}, t_{A}$ ) and ( $x_{B}, t_{B}$ ) are two neighbouring points, sufficiently close enough in the open-rectangle $\{(x, t): 0 \leqslant x \leqslant 1 \quad t \geqslant 0\}$ then
and

$$
\left.\frac{\partial^{2} u}{\partial x \partial t}\right|_{\left(x_{A}, t_{A}\right)}-\left.\frac{\partial^{2} u}{\partial x \partial t}\right|_{\left(x_{B}, t_{B}\right)} \rightarrow 0
$$

$$
\left.\frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{\left(x_{A}, t_{A}\right)}-\left.\frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{\left(x_{B}, x_{t}\right)} \rightarrow 0
$$

then it can be easily shown that the estimate for the accumulation of the truncation error at a point (i,j+1) for algorithm (3) within two timelevels is

$$
\begin{align*}
& \frac{\Delta t^{2}}{24}\left[\left.\frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\left.\frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{3}{2}}+\left.k \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{i, j+\frac{1}{2}}+\left.k \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{i, j+\frac{3}{2}}\right] \\
& +\frac{k}{6}(\Delta x)^{2}\left[\left.\frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}}+\left.\frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{3}{2}}\right]+\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}^{-1}} \Delta t^{\gamma_{2}^{+1}}\right) \\
& +\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}-1} \Delta t{ }^{\gamma_{2}+1}\right) \\
& \simeq O\left\{(\Delta t)^{2}+(\Delta x)^{2}\right\} . \tag{5.5.8}
\end{align*}
$$

Similarly the estimate for the truncation error at a point for algorithm (4) is given by

$$
\begin{aligned}
& \frac{\Delta t^{2}}{12}\left[\left.\frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\left.k \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{i, j+\frac{1}{2}}\right]+\left.\frac{k}{3}(\Delta x)^{2} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}}+ \\
& +\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}-1} \Delta t^{\gamma_{2}+1}+\Delta x^{\gamma_{1}-2} \Delta t^{\gamma_{2}^{+1}}\right) . \\
& \simeq O\left\{(\Delta t)^{2}+(\Delta x)^{2}\right\} \text {. }
\end{aligned}
$$

From (5.5.2), (5.5.6), (5.5.8) and (5.5.9), it is quite clear that the algorithms (3) and (4) seem to yield more favourable truncation errors than either algorithm (1) or (2). Since computational cost of algorithm (4) is twice that of algorithm (3), then it is clear that algorithm (3) is the most efficient amongst the four. Some results from the numerical experiments to verify this can be found in Section 5.9.

From the fact that these algorithms can be implemented explicitly with a much improved stability condition which is practically unconditionally stable than the classical explicit formulae and with no appreciable loss of accuracy, then it is felt that these algorithms are worthy of recommendation.

### 5.6 GE FORMULATION AND ALGORITHMS

Consider now any two points ( $\mathrm{i}, \mathrm{j}+1$ ) and ( $\mathrm{i}+1, \mathrm{j}+1$ ), and use equation (5.5.1) at point ( $\mathrm{i}, \mathrm{j}+1$ ) and use equation (5.5.4) at point ( $\mathrm{i}+1, \mathrm{j}+1$ ) to give

$$
\begin{align*}
& {\left[1+\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\right] u_{i, j+1}-\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{i+1, j+1}} \\
& \quad=\left(\varepsilon r+\frac{k r \Delta x}{2}\right) u_{i-1, j}+\left[1-\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right]_{i, j} \tag{5.6.1}
\end{align*}
$$

and

$$
\begin{align*}
& -\left(\varepsilon r+\frac{k r \Delta x}{2}\right) u_{i, j+1}+\left[1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right] u_{i+1, j+1} \\
& \quad=\left[1-\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\right] u_{i+1, j}+\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{i+2, j} \tag{5.6.2}
\end{align*}
$$

respectively. Equations (5.6.1) and (5.6.2) will then form a small system of $2 \times 2$ linear equations, i.e.,

$$
\begin{align*}
{\left[\begin{array}{cc}
1+\left(\varepsilon r-\frac{k r \Delta x}{2}\right) & -\left(\varepsilon r-\frac{k r \Delta x}{2}\right) \\
-\left(\varepsilon r+\frac{k r \Delta x}{2}\right) & 1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)
\end{array}\right]\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
1-\left(\varepsilon r+\frac{k r \Delta x}{2}\right) & 0 \\
0 & 1-\left(\varepsilon r-\frac{k r \Delta x}{2}\right)
\end{array}\right.} \\
{\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{c}
\left(\varepsilon r+\frac{k r \Delta x}{2}\right) u_{i-1, j} \\
\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{i+2, j}
\end{array}\right] } \tag{5.6.3}
\end{align*}
$$

Since $\left[\begin{array}{ll}1+\left(\varepsilon r-\frac{k r \Delta x}{2}\right) & -\left(\varepsilon r-\frac{k r \Delta x}{2}\right) \\ -\left(\varepsilon r+\frac{k r \Delta x}{2}\right) & 1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\end{array}\right]^{-1}=\frac{1}{1+2 \varepsilon r}\left[\begin{array}{cc}1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right) & 1 \varepsilon-\frac{k r \Delta x}{2} \\ \varepsilon r+\frac{k r \Delta x}{2} & 1+\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\end{array}\right]$
then (5.6.3) can be explicitly represented by

$$
\left[\begin{array}{l}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{(1+2 \varepsilon r)}\left\{\begin{array}{lc}
1-\left(\varepsilon r+\frac{k r \Delta x}{2}\right)^{2} & \left(\varepsilon r-\frac{k r \Delta x}{2}\right)\left[1-\left(\varepsilon r-\frac{k r \Delta x}{2}\right]\right. \\
\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\left[1-\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right] & 1-\left(\varepsilon r-\frac{k r \Delta x}{2}\right)^{2}
\end{array}\right]
$$

$$
\left.\left.\left[\begin{array}{l}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{l}
\left.\left(\varepsilon r+\frac{k r \Delta x}{2}\right) 1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right] u_{i-1, j}+\left(\varepsilon r-\frac{k r \Delta x}{2}\right)^{2} u_{i+2, j} \\
\left(\varepsilon r+\frac{k r \Delta x}{2}\right)^{2} u_{i-1, j}+\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\left[1+\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\right] u_{i+2, j}
\end{array}\right]\right\} 5.6 .4\right)
$$

This scheme can be diagrammatically shown by Figure (5.6.1).
In the case where there is any ungrouped point near either boundary, we use equation (5.6.2), i.e.,

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{\left[1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right]}\left\{\left(\varepsilon r+\frac{k r \Delta x}{2}\right) u_{0, j+1}+\left[1-\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\right] u_{1, j}+\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{2, j}\right\} \tag{5.6.5}
\end{equation*}
$$

for the left ungrouped point and equation (5.6.1), i.e.,

$$
\begin{gather*}
u_{m-1, j+1}=\frac{1}{\left[1+\left(\varepsilon r-\frac{k r \Delta x}{2}\right)\right]}\left[\left(\varepsilon r-\frac{k r \Delta x}{2}\right) u_{m, j+1}+\left[1-\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right] u_{m-1, j}+\right. \\
\left.\left(\varepsilon r+\frac{k r \Delta x}{2}\right) u_{m-2, j}\right\}, \tag{5.6.6}
\end{gather*}
$$

for the right ungrouped point.
To derive the algorithms which form the class of Group Explicit methods, we use the implicit form (5.6.3). Also we assume that the space interval x is divided into an even number of sub-intervals which implies that the ( $\mathrm{m}-\mathrm{l}$ ) points are odd. With the notations,
and

$$
\begin{align*}
& \alpha_{1}=\varepsilon-\frac{k \Delta x}{2}, \\
& \alpha_{2}=\varepsilon+\frac{k \Delta x}{2} . \tag{5.6.7}
\end{align*}
$$

the following algorithms can be established:-

1. Group Explicit with Right-Ungrouped Point (GER): Use equation (5.6.3) for the first (m-2) points and equation (5.6.6) for the last unknown point. This will give the system,

$$
\begin{equation*}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1, j} \text {, } \tag{5.6.8}
\end{equation*}
$$

where
$\mathrm{i}-1, \mathrm{j}$
$\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\left[1+\left(\varepsilon r+\frac{k r \Delta x}{2}\right)\right]$

$$
\begin{aligned}
& G_{2}=\left[\begin{array}{c:c:c:c}
\alpha_{2} & & & \\
\hdashline & \alpha_{1} & -\alpha_{1} & \\
\hdashline & -\alpha_{2} & \alpha_{2} & 0 \\
\hdashline & 0 & 1 & \\
\hdashline & & & \alpha_{1} \\
\hdashline & & & \alpha_{1} \\
\hdashline & & \alpha_{2}
\end{array}\right] \\
& \underline{b}_{1, j}^{T}=\left[r \alpha_{2} u_{0, j}, 0, \ldots, 0, r \alpha_{1} u_{m, j+1}\right] \text { and } \underline{u}_{j}^{T}=\left[u_{1, j}, u_{2, j}, \ldots, u_{m-1, j}\right]
\end{aligned}
$$

2. Group Explicit with Left-Ungrouped Point (GEL): Use equation (5.6.5) for the first unknown point from the left of the boundary and equation (5.6.3) for the remaining ( $\frac{\mathrm{m}-2}{2}$ ) pairs of points. This will result in the system,

$$
\begin{equation*}
\left(\mathrm{I}+\mathrm{rG} \mathrm{~g}_{2}\right) \underline{\mathrm{u}}_{\mathrm{j}+\mathrm{l}}=\left(\mathrm{I}-r \mathrm{G}_{1}\right) \underline{\mathrm{u}}_{\mathrm{j}}+\underline{b}_{2, \mathrm{j}}, \tag{5.6.11}
\end{equation*}
$$

with $\quad \underline{b}_{2, j}^{T}=\left[r \alpha_{2}{ }_{0, j+1}, 0, \ldots, 0, r \alpha_{1}{ }_{m, j}\right]$.
3. Alternating Group Explicit (S)AGE: Use equation (5.6.8) at the ( $j+1$ ) th time level and equation (5.6.11) at the ( $j+2$ ) th time level, i.e.,

$$
\left.\begin{array}{l}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r\left(G_{2}\right) \underline{u}_{j}+\underline{b}_{1, j}\right.  \tag{5.6.12}\\
\left(I+r G_{2}\right) \underline{u}_{j+2}=\left(I-r G_{1}\right) \underline{u}_{j+1}+b_{2, j+1}
\end{array}\right\}
$$

4. Alternating Group Explicit (D)AGE: In this algorithm, the group explicit formulae are incorporated alternately within the four time-
levels with the direction reversed at the third time level. This will give

$$
\left.\begin{array}{l}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1, j}  \tag{5.6.13}\\
\left(I+r G_{2}\right) \underline{u}_{j+2}=\left(I-r G_{1}\right) \underline{u}_{j+1}+b_{2, j+1} \\
\left(I+r G_{2}\right) \underline{u}_{j+3}=\left(I-r G_{1}\right) \underline{u}_{j+2}+b_{2, j+2} \\
\left(I+r G_{1}\right) \underline{u}_{j+4}=\left(I-r G_{2}\right) \underline{u}_{j+3}+b_{1, j+3}
\end{array}\right\}
$$

5. Two-Level (S)AGE: In this algorithm, the $j$ th and ( $j+1$ ) th level of time are 'artificially' divided into 2 to form the ( $j+\frac{1}{2}$ ) th level. Therefore, the ordinary level is made up into two artificial levels and the (S)AGE formulas (5.6.12) are implemented at each 'leve1' to give

$$
\left.\begin{array}{l}
\left(I+\frac{r_{2}}{G_{1}}\right) \underline{u}_{j+\frac{1}{2}}=\left(I-\frac{r_{2}}{G_{2}}\right) \underline{u}_{j}+\underline{b}_{1, j}  \tag{5.6.14}\\
\left(I+\frac{r_{2}}{} G_{2}\right) \underline{u}_{j+1}=\left(I-\frac{r}{2} G_{1}\right) \underline{u}_{j+\frac{1}{2}}+\underline{b}_{2, j+\frac{1}{2}}
\end{array}\right\}
$$

6. Four-Level (D)AGE: This is similar to the previous one but the ordinary level is made up into four artificial levels and the formulas (5.6.13) are implemented at each 'level' to yield
7. Average of GER and GEL: If $v$ and $w$ are assumed to be the solutions of the GER and GEL algorithms respectively, i.e.,

$$
\left(I+r G_{1}\right) \underline{v}_{j+1}=\left(I-r G_{2}\right) \underline{v}_{j}+\underline{b}_{1, j},
$$

and

$$
\left(I+r G_{2}\right){\underset{-}{w}}_{j+1}=\left(I-r G_{1}\right) \underline{w}_{j}+\underline{b}_{2, j}
$$

therefore $u$ is taken as

$$
\begin{equation*}
\underline{u}_{j+1}=\frac{1}{2}\{{\underset{v}{j+1}}+{\underset{-}{w}+1}\} \tag{5.6.16}
\end{equation*}
$$

These are only a few of the examples of algorithms which can be established from the original formulae (5.6.3), (5.6.5) and (5.6.6). There are a few more algorithms which the author has left unmentioned as they are quite similar to the ones mentioned above. However, from the few algorithms that are mentioned the author has chosen only some of the algorithms for the numerical experiments, the results of which are included in Section (5.9). This is due to the fact that apart from $\alpha_{1}$ and $\alpha_{2}$ which will result in the unsymmetric system (5.6.8), (5.6.11) to (5.6.16), in principle these algorithms are similar to the class of Group Explicit methods which were discussed fully in Chapter 4.

In the use of the number of unknown points being even, i.e. (m-1) is even, the concept of Group Explicit can be implemented if:
(i) the points are grouped into $\left(\frac{m-1}{2}\right)$ groups of two points, or
(ii) the points are grouped into $\left(\frac{m-3}{2}\right)$ groups of two points with the points at the far-left and far-right left ungrouped.

Then the following algorithms are possible:
(1) The Group Explicit Complete (GEC): Use equation (5.6.3) for the $\left(\frac{m-1}{2}\right)$ groups of points in (i).
(2) The Group Explicit with Ungrouped-ends (GEU): Use equation (5.6.5) for the left-ungrouped point, equations (5.6.3) for the $\left(\frac{m-3}{2}\right)$ groups of points and equation (5.6.6) for the far rightungrouped point.

Similarly, from GEC and GEU, the AGE concept can be developed to give the equations corresponding to (5.6.12)-(5.6.16).

### 5.7 ESTIMATE OF THE TRUNCATION ERROR OF THE GE CLASS OF METHODS

To estimate the truncation errors of various algorithms in this class, we expand as a Taylor's series each equation in (5.6.4) about ( $i, j$ ) and ( $i+1, j$ ) respectively, (5.6.5) about (1, j ) and (5.6.6) about ( $\mathrm{m}-1, \mathrm{j}$ ). For equation (6.5.4), the first will result in

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{i, j}=\left.\varepsilon \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}-\left.k \frac{\partial u}{\partial x}\right|_{i, j}+T_{i, j} \tag{5.7.1}
\end{equation*}
$$

where

$$
\begin{align*}
T_{i, j}= & -\left.\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j}+\left.\frac{r k \Delta x}{2} \frac{(k \Delta x-2 \varepsilon)}{(1+2 \varepsilon r)} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j}+ \\
& +\left.\frac{\left[6 r\left(\varepsilon^{2}+\left(\frac{k \Delta x}{2}\right)^{2}\right)-k \Delta x(1-8 r \varepsilon)\right]}{(1+2 \varepsilon r)} \frac{\Delta x}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j}+\frac{1}{\Delta t} 0\left(\Delta x^{4}\right) \\
\approx & -\left.\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i, j}+\left.\frac{k \Delta t}{2 \Delta x}(k \Delta x-2 \varepsilon)\left(1-2 \varepsilon r+\frac{(2 \varepsilon r)^{2}}{2} \ldots\right) \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j} \\
& +\left.\frac{\Delta t}{\Delta x}\left(\varepsilon^{2}+\left(\frac{k \Delta x}{2}\right)^{2}\right)\left(1-2 \varepsilon r+\frac{(2 \varepsilon r)^{2}}{2!}-\ldots\right) \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j} \\
& -\left.\frac{k \Delta x^{2}}{6}(1-8 r \varepsilon)\left(1-2 \varepsilon r+\frac{(2 \varepsilon r)^{2}}{2!}-\ldots\right) \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j}+\frac{1}{\Delta t} 0\left(\Delta x^{4}\right) \tag{5.7.2}
\end{align*}
$$

and the second will result in

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{i+1, j}=\left.\varepsilon \frac{\partial^{2} u}{\partial x^{2}}\right|_{i+1, j}-\left.k \frac{\partial u}{\partial x}\right|_{i+1, j}+T_{i+1, j} \tag{5.7.3}
\end{equation*}
$$

where

$$
\begin{aligned}
T_{i+1, j}= & -\left.\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{i+1, j}+\left.\frac{r k \Delta x}{2} \frac{(k \Delta x+2 \varepsilon)}{(1+2 \varepsilon r)} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i+1, j} \\
& \left.\left.+\frac{\left[-6 r\left(\varepsilon^{2}+\left(\frac{k \Delta x}{2}\right)\right.\right.}{}{ }^{2}\right)-k \Delta x(1+8 \varepsilon \varepsilon)\right] \\
\simeq & -\left.\frac{\Delta x}{6} \frac{\partial^{3} \frac{\partial^{3}}{3}}{\partial x^{3}}\right|_{i+1, j}+\left.\frac{\partial^{2} u}{\partial t^{2}}\right|_{i+1, j} 0\left(\Delta x^{4}\right) \\
& -\frac{\Delta t}{\Delta x}\left(\varepsilon^{2}+\left(\frac{k \Delta x}{2}\right)^{2}\right)\left(1-2 \varepsilon r+\frac{(2 \varepsilon r)^{2}}{2!}-\left.\ldots \frac{\Delta t}{\Delta x}(l \Delta x+2 \varepsilon)\left(1-2 \varepsilon r+\frac{(2 \varepsilon r)^{2}}{2!}-\ldots\right) \frac{\partial^{2} u}{\partial x^{2}}\right|_{i+1, j}\right. \\
& -\left.\left.\frac{k \Delta x^{2}}{6}(1+8 r \varepsilon)\left(1-2 \varepsilon r+\frac{(2 \varepsilon r)^{2}}{2!}-\ldots\right) \frac{\partial^{3} u}{\partial x^{3}}\right|_{i+1, j}\right|_{i+1, j}+\frac{1}{\Delta t} 0\left(\Delta x^{4}\right)(5.7 .4)
\end{aligned}
$$

For equation (5.6.5), the expansion gives

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{I, j}=\left.\varepsilon \frac{\partial^{2} u}{\partial x^{2}}\right|_{1, j}-\left.k \frac{\partial u}{\partial x}\right|_{1, j}+T_{1, j} \tag{5.7.5}
\end{equation*}
$$

where

$$
\begin{align*}
& T_{1, j} \approx-\left.\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{1, j}-\left.\frac{\Delta t}{\Delta x}\left(\varepsilon+\frac{k \Delta x}{2}\right) \frac{\partial^{2} u}{\partial x \partial t}\right|_{1, j}+\left.\frac{\Delta t}{2}\left(\varepsilon+\frac{k \Delta x}{2}\right) \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{1, j} \\
&\left.k \frac{\Delta x^{2}}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{1, j}-\left.r \Delta t \Delta x\left(\varepsilon+\frac{k \Delta x}{2}\right) \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{1, j} \tag{5,7.6}
\end{align*}
$$

and for equation (5.6.6) the expansion results in

$$
\begin{equation*}
\left.\frac{\partial u}{\partial t}\right|_{m-1, j}=\left.\varepsilon \frac{\partial^{2} u}{\partial x^{2}}\right|_{m-1, j}-\left.k \frac{\partial u}{\partial x}\right|_{m-1, j}+T_{m-1, j} \tag{5.7.7}
\end{equation*}
$$

where

$$
\begin{align*}
T_{m-1, j}=- & -\left.\frac{\Delta t}{2} \frac{\partial^{2} u}{\partial t^{2}}\right|_{m-1, j}+\left.\frac{\Delta t}{\Delta x}\left(\varepsilon-\frac{k \Delta x}{2}\right) \frac{\partial^{2} u}{\partial x \partial t}\right|_{m-1, j}+\left.\frac{\Delta t}{2}\left(\varepsilon-\frac{k \Delta x}{2}\right) \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{m-1, j} \\
& -\left.\frac{k \Delta x^{2}}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{m-1, j}+\left.r \Delta t \Delta x\left(\varepsilon-\frac{k \Delta x}{2}\right) \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{m-1, j} \tag{5.7.8}
\end{align*}
$$

Now the GER method of (5.6.8) is estimated with the truncation errors (5.7.2) and (5.7.4) for $i=1,3, \ldots, m-3$ and with the truncation error (5.7.8) for $i=m-1$. For the GEL method (5.6.11), the truncation errors are represented by (5.7.6) for $i=1$ and (5.7.2) and (5.7.4) for $i=2,4, \ldots, m-2$. Meanwhile for the AGE method (5.6.12), the truncation errors are represented alternately at every time-level by either the one for the GER or the one for the GEL method (Fig. 5.7.1).
(5.7.2) (5.7.4) (5.7.2) (5.7.4) (5.7.2) (5.7.4) (5.7.8)

GER jth

GEL ( $\mathrm{j}+1$ ) th

$$
\begin{equation*}
(5.7 .6) \tag{5.7.4}
\end{equation*}
$$

(5.7.2)
(5.7.4)
(5.7.2)
(5.7.4) (5.7.2)

Since the local truncation errors at every point on the line are such that the ith vertical line will normally have alternate errors of (5.7.2) or (5.7.4) and as the coefficients of the ( $\left(\frac{t}{\Delta x}\right)$ term for these errors have opposite signs, therefore we expect the errors to accumulate more slowly than in the GER or GEL methods. Due to this fact the author feels strongly that the AGE type of method is more favourable than the formulae of non-AGE type.

For equations (5.6.13)-(5.6.16), then since they are also of an AGE type, the estimate to the truncation errors can be obtained in a similar manner as the AGE method (5.6.12) although they will be much more complicated to derive. Finally, we can easily deduce that all the methods are approximately of order $\left\{\Delta t+(\Delta x)^{2}\right\}$ with the consistency condition $\left(\frac{\Delta t}{\Delta x}\right) \rightarrow 0$.

### 5.8 STABILITY ANALYSIS OF THE GE CLASS OF METHOD

The stability analysis of the GE class of method for the diffusionconvection equation can be obtained by the matrix method. To do this, equation (5.6.8) is written in the explicit form,

$$
\begin{equation*}
\underline{u}_{j+1}={\underset{A u}{j}}+\hat{\underline{b}}_{1, j} \tag{5.8.1}
\end{equation*}
$$

where

$\widehat{\underline{b}}_{1, j}$ is associated with the boundary conditions, $\alpha_{1}$ and $\alpha_{2}$ are defined by (5.6.7) and $\Delta=1+2 \varepsilon r$.

The stability of the system (5.8.1) can be guaranteed if $\|A\| \leqslant 1$. From the matrix (5.8.2) it can be easily shown that $\left.\|A\|\right|_{\infty} \leqslant 1$ if and only if

$$
r \leqslant \frac{1}{\max \left|\alpha_{i}\right|}=\frac{1}{\max \left\{\left|\varepsilon+\frac{k \Delta x}{2},\left|\varepsilon-\frac{k \Delta x}{2}\right|\right\}\right.}
$$

The GEL scheme can also be written in the form (5.8.1) with,
and a similar stability condition can be shown to be valid. Hence the following theorem can be established.

## Theorem 5.1

The GER and GEL schemes of (5.6.8) and (5.6.11) for the diffusionconvection equation (5.1.1) are stable provided

$$
r \leqslant \frac{1}{\max \left\{\left|\varepsilon-\frac{k \Delta x}{2},\left|\varepsilon+\frac{k \Delta x}{2}\right|\right\}\right.}
$$

(This theorem will reduce to Theorem 4.3 if $\varepsilon=1$ and $k=0$ ).
For the two-step (S)AGE processes given by equations (5.6.12) and (5.6.14), written as

$$
\left.\begin{array}{l}
\left(I+r \theta G_{1}\right) \underline{u}_{j+\theta}=\left(I-r \theta G_{2}\right) \underline{u}_{j}+\underline{b}_{1, j}  \tag{5.8.4}\\
\left(I+r \theta G_{2}\right) \underline{u}_{j+2 \theta}=\left(I-r \theta G_{1}\right) \underline{u}_{j}+\theta+\underline{b}_{2, j+2 \theta}
\end{array}\right\}
$$

where

$$
\theta= \begin{cases}1 & \text { for }(5.6 .12) \\ \frac{1}{2} & \text { for }(5.6 .14)\end{cases}
$$

By eliminating $\underline{u}_{j+\theta}$ from (5.8.4) we obtain

$$
\underline{u}_{j+2 \theta}=\underline{T u}_{j}+\underline{b}^{\prime}
$$

where $\underline{b}^{\prime}$ is independent of the $\underline{u}^{\prime} s$ and

$$
\begin{equation*}
T=\left(I+r \theta G_{2}\right)^{-1}\left(I-r \theta G_{1}\right)\left(I+r \theta G_{1}\right)^{-1}\left(I-r \theta G_{2}\right) \tag{5.8.5}
\end{equation*}
$$

Now we define the matrix

$$
\begin{equation*}
\widetilde{T}=\left(I+r \theta G_{2}\right) T\left(I+r \theta G_{2}\right)^{-1} \tag{5.8.6}
\end{equation*}
$$

which is similar to $T$ and thus has the same eigenvalues as $T$. Therefore, we can express $\widetilde{T}$ as

$$
\begin{equation*}
\widetilde{T}=\left(I-r \theta G_{1}\right)\left(I+r \theta G_{1}\right)^{-1}\left(I-r \theta G_{2}\right)\left(I+r \theta G_{2}\right)^{-1} . \tag{5,8.7}
\end{equation*}
$$

From the definitions of $G_{1}$ and $G_{2}$ in (5.6.9) and (5.6.10) we can determine that their eigenvalues are given by $0,2 \varepsilon((n-1) / 2$ multiplicity of groups) and $\left(\varepsilon-\frac{k \Delta x}{2}\right)$ for $G_{1}$ and $0,2 \varepsilon((n-1) / 2$ multiplicity of groups) and $\left(\varepsilon+\frac{k \Delta x}{2}\right)$ for $G_{2}$. Therefore, we can establish the result,

$$
\begin{align*}
\|\tilde{\mathrm{T}}\|_{2} & \leqslant \|\left(I-r \theta G_{1}\right)\left(I+r \theta G_{1}\right)^{-1}| |_{2}| |\left(I-r \theta G_{2}\right)\left(I+r \theta G_{2} \mid \|_{2}\right. \\
& =\left\{\max \left|\frac{1-r \theta \rho_{1, s}}{1+r \theta \rho_{1, s}}\right|\right\} \max \left\{\left|\frac{1-r \theta \rho_{2, s}}{1+r \theta \rho_{2, s}}\right|\right\}, s=1,2, \ldots, m-1 \tag{5.8.8}
\end{align*}
$$

where $\rho_{1, s}$ and $\rho_{2, s}$ are the eigenvalues of $G_{1}$ and $G_{2}$ respectively.
The inequality (5.8.8) can be shown to be less than or equal to unity for all r>0 provided

$$
\begin{equation*}
\Delta x \leqslant \frac{2 \varepsilon}{k} \tag{5.8.9}
\end{equation*}
$$

For the four-step processes of (D)AGE (5.6.13) and (5.6.15), a similar stability condition will follow. Therefore, the following theorem can be established.

Theorem 5.2
The (S)AGE and (D)AGE schemes of (5.6.12)-(5.6.15) are unconditionally stable for all r>0 provided $\Delta x \leq 2 \varepsilon / k$.

### 5.9 NUMERICAL EXAMPLE

In this example, the equation (5.1.1) together with the initial condition

$$
\begin{equation*}
u(x, 0)=0, \quad 0<x<1, \tag{5.9.1}
\end{equation*}
$$

and the non-homogeneous Dirichlet conditions

$$
\left.\begin{array}{l}
u(0, t)=0,  \tag{5.9.2}\\
u(1, t)=1,
\end{array}\right\} t \geqslant 0
$$

is used as a model problem. This problem can be shown by the method of separation of variables to have the exact solution

$$
\begin{array}{r}
u(x, t)=\frac{e^{k x / \varepsilon}-1}{e^{k / \varepsilon}-1}+\sum_{n=1}^{\infty} \frac{(-1)^{n} n \pi}{(n \pi)^{2}+\left(\frac{k}{2 \varepsilon}\right)^{2}} \tag{5.9.3}
\end{array} e^{\frac{k}{2 \varepsilon}(x-1)} \operatorname{sinn\pi x}
$$

In calculating the exact solution the convergence of the summation term is assumed satisfied if

$$
\operatorname{abs}\left\{\text { summation }_{k+1}-\text { summation] }_{k}\right\} \leqslant 10^{-8} .
$$

The solution of some of the numerical schemes presented earlier are then compared with this exact solution in terms of their absolute errors. A comparison is also made with the Crank-Nicolson upwinding scheme. These results are given in Tables (5.9.1)-(5.9.4) and Figures (5.9.1)-(5.9.3) for cases where a graphical representation is more appropriate.

From the tables and graphs, in general, it can be seen that the schemes in this class of method are more accurate than the CN method. For $r=0.5$ the (D)AGE scheme appears to be better than any other scheme.

From these results we can establish that since the method is explicit and highly stable, it can be recommended as an alternative competitive method for solving the diffusion-convection equation.
$k=1.0, \varepsilon=1.0, \Delta t=0.001, \Delta x=0.1, r=0.1, \Delta_{t}=0.1$

| Method ${ }_{\text {- }}$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CNU | $2.4 \times 10^{-3}$ | $4.9 \times 10^{-3}$ | $7.3 \times 10^{-3}$ | $9.4 \times 10^{-3}$ | $11.0 \times 10^{-3}$ | $12.0 \times 10^{-3}$ | $11.5 \times 10^{-3}$ | $9.4 \times 10^{-3}$ | $5.6 \times 10^{-3}$ |
| 5.5.1 (UNE) | $1.8 \times 10^{-3}$ | $3.2 \times 10^{-3}$ | $4.2 \times 10^{-3}$ | $4.5 \times 10^{-3}$ | $4.2 \times 10^{-3}$ | $3.3 \times 10^{-3}$ | $2.0 \times 10^{-3}$ | $7.3 \times 10^{-4}$ | $0.4 \times 10^{-4}$ |
| 5.5.4 (UPOS) | $1.4 \times 10^{-3}$ | $2.7 \times 10^{-3}$ | $3.7 \times 10^{-3}$ | $4.4 \times 10^{-3}$ | $4.6 \times 10^{-3}$ | $4.3 \times 10^{-3}$ | $3.4 \times 10^{-3}$ | $2.1 \times 10^{-3}$ | $8.5 \times 10^{-4}$ |
| $\begin{aligned} & \text { ALDC } \\ & 5.5 .1 \& 5.5 .4 \end{aligned}$ | $2.0 \times 10^{-4}$ | $3.0 \times 10^{-4}$ | $2.6 \times 10^{-4}$ | $0.4 \times 10^{-4}$ | $2.8 \times 10^{-4}$ | $6.6 \times 10^{-4}$ | $9.0 \times 10^{-4}$ | $8.9 \times 10^{-4}$ | $5.7 \times 10^{-4}$ |
| AVERAGE <br> 5.5.1 \& 5.5.4 | $1.7 \times 10^{-4}$ | $2.7 \times 10^{-4}$ | $2.4 \times 10^{-4}$ | $0.7 \times 10^{-4}$ | $1.9 \times 10^{-4}$ | $5.1 \times 10^{-4}$ | $7.1 \times 10^{-4}$ | $7.0 \times 10^{-4}$ | $4.5 \times 10^{-4}$ |
| (D) AGE | $1.6 \times 10^{-4}$ | $2.7 \times 10^{-4}$ | $2.8 \times 10^{-4}$ | $1.3 \times 10^{-4}$ | $0.9 \times 10^{-4}$ | $4.0 \times 10^{-4}$ | $6.1 \times 10^{-4}$ | $6.2 \times 10^{-4}$ | $4.3 \times 10^{-4}$ |
| (D) AGE <br> TWO-LEVEL | $0.2 \times 10^{-4}$ | $0.4 \times 10^{-4}$ | $0.7 \times 10^{-4}$ | $1.0 \times 10^{-4}$ | $1.4 \times 10^{-4}$ | $1.8 \times 10^{-4.4}$ | $2.0 \times 10^{-4}$ | $2.0 \times 10^{-.4}$ | $1.7 \times 10^{-4}$ |
| EXACT SOLUTION | 0.01895 | 0.04370 | 0.07892 | 0.12982 | 0.20177 | 0.29986 | 0.42794 | 0.58805 | 0.77976 |

TABLE (5.9.1)
$k=1.0, \quad \varepsilon=1.0, \Delta t=0.005, \Delta x=0.1, r=0.5, t=0.5$

| Method | 0,1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CNU | $1.7 \times 10^{-3}$ | $3.2 \times 10^{-3}$ | $4.5 \times 10^{-3}$ | $5.4 \times 10^{-3}$ | $6.1 \times 10^{-3}$ | $6.2 \times 10^{-3}$ | $5.8 \times 10^{-3}$ | $4.7 \times 10^{-3}$ | $2.8 \times 10^{-3}$ |
| 5.5.1 (UNE) | $1.9 \times 10^{-4}$ | $3.5 \times 10^{-4}$ | $4.7 \times 10^{-4}$ | $5.2 \times 10^{-4}$ | $5.1 \times 10^{-4}$ | $4.4 \times 10^{-4}$ | $3.3 \times 10^{-4}$ | $2.0 \times 10^{-4}$ | $0.8 \times 10^{-4}$ |
| 5.5.4 (UPOS) | $3.3 \times 10^{-4}$ | $6.1 \times 10^{-4}$ | $8.1 \times 10^{-4}$ | $9.1 \times 10^{-4}$ | $9.1 \times 10^{-4}$ | $8.2 \times 10^{-4}$ | $6.5 \times 10^{-4}$ | $4.4 \times 10^{-4}$ | $2.2 \times 10^{-4}$ |
| $\begin{aligned} & \text { ALDC } \\ & 5.5 .1 \& 5.5 .4 \end{aligned}$ | $1.1 \times 10^{-4}$ | $2.4 \times 10^{-4}$ | $3.6 \times 10^{-4}$ | $4.6 \times 10^{-4}$ | $5.2 \times 10^{-4}$ | $5.3 \times 10^{-4}$ | $4.8 \times 10^{-4}$ | $3.8 \times 10^{-4}$ | $2.1 \times 10^{-4}$ |
| AVERAGE $5.5 .1 \& 5.5 .4$ | $0.7 \times 10^{-4}$ | $1.3 \times 10^{-4}$ | $1.7 \times 10^{-4}$ | $2.0 \times 10^{-4}$ | $2.0 \times 10^{-4}$ | $1.9 \times 10^{-4}$ | $1.6 \times 10^{-4}$ | $1.2 \times 10^{-4}$ | $0.7 \times 10^{-4}$ |
| (D) AGE | $0.2 \times 10^{-4}$ | $0.6 \times 10^{-4}$ | $0.5 \times 10^{-4}$ | $0.8 \times 10^{-4}$ | $0.8 \times 10^{-4}$ | $0.8 \times 10^{-4}$ | $0.9 \times 10^{-4}$ | $0.6 \times 10^{-4}$ | $0.6 \times 10^{-4}$ |
| (D)AGE <br> TWO-LEVEL | $1.4 \times 10^{-4}$ | $2.9 \times 10^{-4}$ | $4.1 \times 10^{-4}$ | $5.2 \times 10^{-4}$ | $5.9 \times 10^{-4}$ | $6.1 \times 10^{-4}$ | $5.8 \times 10^{-4}$ | $4.7 \times 10^{-4}$ | $4.0 \times 10^{-4}$ |
| EXACT SOLUTION | 0.06043 | 0.12730 | 0.20136 | 0.28345 | 0.37447 | 0.47539 | 0.58724 | 0.71114 | 0.84830 |

TABLE (5.9.2)

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| CNU | $1.5 \times 10^{-3}$ | $2.9 \times 10^{-3}$ | $4.1 \times 10^{-3}$ | $4.9 \times 10^{-3}$ | $5.5 \times 10^{-3}$ | $5.7 \times 10^{-3}$ | $5.3 \times 10^{-3}$ | $4.3 \times 10^{-5}$ | $2.6 \times 10^{-5}$ |
| 5.5 .1 (UNE) | $2.4 \times 10^{-5}$ | $4.5 \times 10^{-5}$ | $6.3 \times 10^{-5}$ | $7.7 \times 10^{-5}$ | $8.6 \times 10^{-5}$ | $8.9 \times 10^{-5}$ | $8.4 \times 10^{-5}$ | $7.0 \times 10^{-5}$ | $4.3 \times 10^{-5}$ |
| 5.5 .4 (UPOS) | $3.3 \times 10^{-5}$ | $6.2 \times 10^{-5}$ | $8.5 \times 10^{-5}$ | $10.3 \times 10^{-5}$ | $11.3 \times 10^{-5}$ | $11.4 \times 10^{-5}$ | $10.4 \times 10^{-5}$ | $8.4 \times 10^{-5}$ | $5.0 \times 10^{-5}$ |
| ALDC <br> S.5.1 \& 5.5.4 | $3.2 \times 10^{-5}$ | $6.2 \times 10^{-5}$ | $8.8 \times 10^{-5}$ | $10.8 \times 10^{-5}$ | $12.1 \times 10^{-5}$ | $12.3 \times 10^{-5}$ | $11.5 \times 10^{-5}$ | $9.3 \times 10^{-5}$ | $5.5 \times 10^{-5}$ |
| AVERAGE <br> $5.5 .1 \& 5.5 .4$ | $2.8 \times 10^{-5}$ | $5.3 \times 10^{-5}$ | $7.4 \times 10^{-5}$ | $9.0 \times 10^{-5}$ | $9.9 \times 10^{-5}$ | $10.1 \times 10^{-5}$ | $9.4 \times 10^{-5}$ | $7.7 \times 10^{-5}$ | $4.6 \times 10^{-5}$ |
| (D)AGE | $2.7 \times 10^{-5}$ | $5.2 \times 10^{-5}$ | $7.2 \times 10^{-5}$ | $8.8 \times 10^{-5}$ | $9.8 \times 10^{-5}$ | $10.0 \times 10^{-5}$ | $9.4 \times 10^{-5}$ | $7.6 \times 10^{-5}$ | $4.6 \times 10^{-5}$ |
| (D)AGE <br> 2-LEVEL | $3.0 \times 10^{-5}$ | $5.0 \times 10^{-5}$ | $8.0 \times 10^{-5}$ | $9.0 \times 10^{-5}$ | $10.0 \times 10^{-5}$ | $11.0 \times 10^{-5}$ | $10.0 \times 10^{-5}$ | $8.0 \times 10^{-5}$ | $5.0 \times 10^{-5}$ |
| EXACT SOLUTION | 0.06120 | 0.12884 | 0.20360 | 0.28621 | 0.37752 | 0.47843 | 0.58996 | 0.71322 | 0.84945 |


| Method $\quad$ x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CNU | $1.5 \times 10^{-3}$ | $2.9 \times 10^{-3}$ | $4.1 \times 10^{-3}$ | $4.9 \times 10^{-3}$ | $5.5 \times 10^{-3}$ | $5.7 \times 10^{-3}$ | $5.3 \times 10^{-3}$ | $4.3 \times 10^{-3}$ | $2.6 \times 10^{-3}$ |
| 5.5.1 (UNE) | $2.7 \times 10^{-5}$ | $5.1 \times 10^{-5}$ | $7.2 \times 10^{-5}$ | $8.8 \times 10^{-5}$ | $9.8 \times 10^{-5}$ | $10.1 \times 10^{-5}$ | $9.4 \times 10^{-5}$ | $7.7 \times 10^{-5}$ | $4.6 \times 10^{-5}$ |
| 5.5.4 (UPOS) | $2.7 \times 10^{-5}$ | $5.1 \times 10^{-5}$ | $7.2 \times 10^{-5}$ | $8.8 \times 10^{-5}$ | $9.8 \times 10^{-5}$ | $10.1 \times 10^{-5}$ | $9.4 \times 10^{-5}$ | $7.7 \times 10^{-5}$ | $4.6 \times 10^{-5}$ |
| $\begin{aligned} & \text { ALDC } \\ & 5.5 .1 \& 5.5 .4 \end{aligned}$ | $2.7 \times 10^{-5}$ | $5.1 \times 10^{-5}$ | $7.2 \times 10^{-5}$ | $8.8 \times 10^{-5}$ | $9.8 \times 10^{-5}$ | $10.1 \times 10^{-5}$ | $9.4 \times 10^{-5}$ | $7.7 \times 10^{-5}$ | $4.6 \times 10^{-5}$ |
| AVERAGE <br> $5.5 .1 \& 5.5 .4$ | $2.7 \times 10^{-5}$ | $5.1 \times 10^{-5}$ | $7.2 \times 10^{-5}$ | $8.8 \times 10^{-5}$ | $9.8 \times 10^{-5}$ | $10.1 \times 10^{-5}$ | $9.4 \times 10^{-5}$ | $7.7 \times 10^{-7}$ | $4.6 \times 10^{-5}$ |
| $\begin{aligned} & \text { (D) AGE } \\ & \text { TWO-LEVEL } \end{aligned}$ | $3.0 \times 10^{-5}$ | $5.0 \times 10^{-5}$ | $7.0 \times 10^{-5}$ | $9.0 \times 10^{-5}$ | $10.0 \times 10^{-5}$ | $10.0 \times 10^{-5}$ | $10.0 \times 10^{-5}$ | $8.0 \times 10^{-5}$ | $5.0 \times 10^{-5}$ |
| EXACT SOLUTION | 0.061207 | 0.128851 | 0.203610 | 0.28631 | 0.377541 | 0.478454 | 0.589980 | 0.713236 | 0.849455 |

TABLE (5.9.4)

SOLUTION U


## SOLUTION U



FIGURE (5.9.2)


$$
\varepsilon=0.1, k=1.0, r=2.0, t=2.0, \Delta t=0.02, \Delta x=0.1
$$

FIGURE (5.9.3)

### 5.10 FORMULATION OF THE GE SCHEMES WITH UPWINDING

In this section, the equation (5.1.1) is approximated at the point (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ by the equation

$$
\begin{align*}
& \frac{u_{i, j+1} \mathrm{u}_{i, j}}{\Delta t}=\frac{\varepsilon}{(\Delta x)^{2}}\left[\left(\theta_{1} \delta_{x} u_{i+\frac{1}{2}, j+1} \theta_{2} \delta_{x} u_{i-\frac{1}{2}, j+1}\right)\right. \\
& \left.+\left(\theta_{1}^{\prime} \delta_{x}^{u}{ }_{i+\frac{1}{2}, j}{ }^{-\theta}{ }_{2}^{\prime} \delta_{x} u_{i-\frac{1}{2}, j}\right)\right]-\frac{k}{2 \Delta x}\left[\alpha_{1} \nabla_{x} u_{i, j}+\alpha_{2} \nabla_{x} u_{i, j+1}\right] \tag{5.10.1}
\end{align*}
$$

where $\delta_{x}$ and $\nabla_{x}$ are the central and backward difference operators with respect to the $x$ variable respectively. Parameters $\theta_{1}, \theta_{2}, q$ and $\theta_{2}^{\prime}, \alpha_{1}$ and $\alpha_{2}$ are related to each other by the following compulsory conditions

$$
\left.\begin{array}{l}
\theta_{1}+\theta_{2}+\theta_{1}^{\prime}+\theta_{2}^{\prime}=2  \tag{5.10.2}\\
\theta_{2}^{-\theta_{1}+\theta_{2}^{\prime}-\theta_{1}^{\prime}=0} \\
\alpha_{1}+\alpha_{2}=2
\end{array}\right\}
$$

and also by the optional conditions

$$
\left.\begin{array}{l}
\theta_{2}-\theta_{1}-\theta_{2}^{\prime}+\theta_{1}^{\prime}=0  \tag{5.10.3}\\
-\theta_{2}-\theta_{1}+\theta_{2}^{\prime}+\theta_{1}^{\prime}=0 \\
\alpha_{2}-\alpha_{1}=0
\end{array}\right\}
$$

The importance of the conditions (5.10.2) and (5.10.3) are similar to the previous conditions given in Section 5.2. After substitution of the difference operators equation (5.10.1) will become,

$$
\begin{align*}
& -\left(E \theta_{2}+k \alpha_{2}\right) u_{i-1, j+1}+\left[1+E\left(\theta_{1}+\theta_{2}\right)+K \alpha_{2}\right] u_{i, j+1}-E \theta_{1} u_{i+1, j+1} \\
& \quad=E\left(\theta_{2}^{\prime}+K \alpha_{1}\right) u_{i-1, j}+\left[1-E\left(\theta_{1}^{\prime}+\theta_{2}^{\prime}\right)-K \alpha_{1}\right] u_{i, j}+E \theta_{1}^{\prime} u_{i+1, j} \tag{5.10.4}
\end{align*}
$$

with $E$ and $K$ as previously defined.
From Section (5.3) it can be easily deduced that approximating (5.10.4)
will give the truncation error term $T_{i, j+\frac{1}{2}}$ as

$$
\begin{align*}
T_{i, j+\frac{1}{2}}= & -\left.\frac{k \Delta x}{2} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j+\frac{1}{2}}+\left[\frac{\varepsilon}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(\theta_{2}-\theta_{1}-\theta_{2}^{\prime}+\theta_{1}^{\prime}\right)+\left.\frac{k \Delta t}{4}\left(\alpha_{2}-\alpha_{1}\right) \frac{\partial^{2} u}{\partial x \partial t}\right|_{i, j+\frac{1}{2}}\right. \\
& +\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\left.\frac{k \Delta x^{2}}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}}+\left.\frac{k}{24} \Delta t^{2} \frac{\partial^{3} u}{\partial x \partial t^{3}}\right|_{i, j+\frac{1}{2}} \\
& \left.+\frac{\varepsilon}{12} \Delta t\left(-\theta_{2}-\theta_{1}+\theta_{2}^{\prime}+\theta_{1}^{\prime}\right)+\frac{k \Delta x \Delta t}{24}\left(\alpha_{1}-\alpha_{2}\right)\right]\left.\frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}} \\
& +\frac{1}{\Delta t} O\left(\Delta x^{\gamma_{1}-2} \Delta t^{\gamma_{2}+1}\right)+\frac{1}{\Delta t} O\left(\Delta t^{\gamma_{2}+1} \gamma_{1}^{\gamma_{1}-1}\right) \tag{5.10.5}
\end{align*}
$$

with $\gamma_{1}+\gamma_{2}=4,0 \leqslant \gamma_{1}, \gamma_{2} \leqslant 4$.
With the ordinary upwinding schemes such as the CN upwinding scheme equation (3.1.8-10) the choice of ( $\theta_{i}=\theta_{i}^{\prime}=\frac{1}{2}, i=1,2 ; \alpha_{1}=\alpha_{2}=1$ ) will result in a T.E. term of

$$
\begin{align*}
& \left(T_{i, j+\frac{1}{2}}\right) C_{C N U}=\left[-\frac{k \Delta x}{2} \frac{\partial^{2} u}{\partial x^{2}}+\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}+\frac{k \Delta x^{2}}{6} \frac{\partial^{3} u}{\partial x^{3}}+\frac{k}{24} \Delta t^{2} \frac{\partial^{3} u}{\partial x \partial t^{2}}\right] \\
& +\frac{1}{\Delta t} O\left(\Delta x^{\gamma_{1}^{-2}} \Delta t^{\gamma_{2}^{+1}}\right)+\frac{1}{\Delta t} O\left(\Delta t^{\gamma_{2}^{+1}} \Delta x^{\gamma_{1}-1}\right) \\
& =0\left(\Delta x+\Delta t^{2}\right) \text {. } \tag{5.10.6}
\end{align*}
$$

However, with the GE schemes, since they are formed from the combination of two 'ladder-step' type equations of different directions', i.e. ( $\qquad$ and $L$ ), we are forced to choose schemes such that $\alpha_{1}-\alpha_{2} \neq 0$ and hence the accuracy is decreased further. Anyway, to complete the investigation, this work was carried out.

We now choose two asymmetric formulae of different directions of the form,
(i) $\theta_{1}=\theta_{2}^{\prime}=1, \quad \theta_{2}=\theta_{1}^{\prime}=0, \quad \alpha_{1}=2, \alpha_{2}=0$ to give

$$
(1+E) u_{i, j+1}-E u_{i+1, j+1}=(E+K) u_{i-1, j}+(1-E-2 K) u_{i, j},(5.10 .7)
$$

with the truncation error,

$$
T_{i, j+\frac{1}{2}}=-\left.\frac{k \Delta x}{2} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j+\frac{1}{2}}-\left[\left.\left[^{\varepsilon}\left(\frac{\Delta t}{\Delta x}\right)+\frac{k \Delta t}{2}\right] \frac{\partial^{2} u}{\partial x \partial t}\right|_{i, j+\frac{1}{2}}\right.
$$

$$
\begin{aligned}
& +\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\frac{k \Delta x^{2}}{6} \frac{\partial^{3} u}{\partial x^{3}}+\left.\frac{k}{12} \Delta t^{2} \frac{\partial^{3} u}{\partial x^{\partial} t^{2}}\right|_{i, j+\frac{1}{2}} \\
& +\left.\frac{k \Delta x \Delta t}{24} \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{1}^{-2}} \Delta t^{\gamma_{2}+1}\right)+\frac{1}{\Delta t} 0\left(\Delta x^{\gamma_{2}+1} \Delta t^{\gamma_{1}-1}\right)
\end{aligned}
$$

and
(ii) $\theta_{2}=\theta_{1}^{\prime}=1, \alpha_{1}=\alpha_{2}^{\prime}=0, \alpha_{1}=0, \alpha_{2}=2$ to give

$$
\begin{equation*}
-(E+2 K) u_{i-1, j+1}+[1+E+2 K] u_{i, j+1}=(1-E) u_{i, j}+E u_{i+1, j} \tag{5.10.9}
\end{equation*}
$$

with the truncation error,

$$
\begin{align*}
T_{i, j+\frac{1}{2}}= & -\left.\frac{k \Delta x}{2} \frac{\partial^{2} u}{\partial x^{2}}\right|_{i, j+\frac{1}{2}}+\left.\left[\varepsilon\left(\frac{\Delta t}{\Delta x}\right)+\frac{k \Delta t}{2}\right] \frac{\partial^{2} u}{\partial x^{\partial} t}\right|_{i, j+\frac{1}{2}} \\
& \left.\left.\frac{\Delta t^{2}}{24} \frac{\partial^{3} u}{\partial t^{3}}\right|_{i, j+\frac{1}{2}}+\left.\frac{k \Delta x^{2}}{6} \frac{\partial^{3} u}{\partial x^{3}}\right|_{i, j+\frac{1}{2}}+\left.\left.\frac{k}{24} \Delta t^{2} \frac{\partial^{3} u}{\partial x \partial t^{2}}\right|_{i, j_{\gamma}+\frac{1}{2}+1}{ }^{\gamma_{2}+\frac{k \Delta x \Delta t}{12}} \frac{\partial^{3} u}{\partial x^{2} \partial t}\right|_{i, j+\frac{1}{2}}+\frac{1}{\Delta t} 0 \Delta x_{1}^{\gamma_{1}-2} \Delta t t^{\gamma_{2}+1}\right)+\frac{1}{\Delta t} 0\left(\Delta x_{2}\right)
\end{align*}
$$

For stability, the equation (5.4.6) i.e.,

$$
\begin{align*}
A_{1}^{2}+ & A_{2}^{2}+4 A_{1} A_{2} \sin ^{2} \frac{B \Delta x}{2}-A_{1}-A_{2}-2 A_{1} A_{2} \\
& \leqslant B_{1}^{2}+B_{2}^{2}+4 B_{1} B_{2} \sin ^{2} \frac{B \Delta x}{2}+B_{1}+B_{2}-2 B_{1} B_{2} \tag{5.10.11}
\end{align*}
$$

still holds but with the quantities $A_{1}, A_{2}, B_{1}, B_{2}$ defined now as

$$
\begin{array}{ll}
A_{1}=E \theta_{2}^{\prime}+K \alpha_{1}, & A_{2}=E \theta_{1}^{\prime}, \\
B_{1}=E \theta_{2}+K \alpha_{2}, & B_{2}=E \theta_{1} .
\end{array}
$$

In the case of scheme (5.10.7), $A_{1}=E+2 K, B_{2}=E, B_{1}=A_{2}=0$ and will require

$$
(k r \Delta x-1)\left(2 \varepsilon r+\frac{k r \Delta x}{2}\right) \leq 0
$$

i.e.

$$
\begin{equation*}
0<r \leqslant \frac{1}{k \Delta x} \tag{5.10.12}
\end{equation*}
$$

for the stability condition to be fulfilled. Meanwhile for scheme (5.9.9) $A_{1}=B_{2}=0, A_{2}=E, B_{1}=E+2 K$ and requires

$$
\begin{equation*}
2(k r \Delta x+1)\left(\varepsilon r+\frac{k r \Delta x}{2}\right) \geqslant 0 . \tag{5.10.3}
\end{equation*}
$$

This requirement is easily seen to be unconditionally satisfied for all $r>0$.

Since the schemes (5.10.7) and (5.10.9) are two explicit schemes of low order accuracy, with their truncation errors possessing terms $\left(\frac{\Delta t}{\Delta x}\right), \Delta t$ and $\Delta x \Delta t$ of opposite signs then similar to the algorithms of Section (5.5) it is possible to improve the truncation errors such that these terms cancel each other out and to leave terms of orders of approximately $\left(\Delta t+(\Delta x)^{2}\right)$. The algorithms are:-
(i) To use schemes (5.10.7) and (5.10.9) at the $j$ th and ( $j+1$ ) th time-level respectively (ALDC)
(ii) To use both schemes (5.10.7) and (5.10.9) at every time level and average the results to obtain the solution (UAV).

Alternative to these algorithms is the development of the GE method from equations (5.10.7) and (5.10.9) as follows:

## Formulation

At any two points $(i, j+1)$ and $(i+1, j+1)$, we now use equations (5.10.7) and (5.10.9) to give,

$$
\begin{equation*}
(1+E) u_{i, j+1}-E u_{i+1, j+1}=(E+2 K) u_{i-1, j}+(1-E-2 K) u_{i, j}, \tag{5.10.14}
\end{equation*}
$$

and

$$
\begin{equation*}
-(E+2 K) u_{i, j}+[1+E+2 K] u_{i+1, j+1}=(1-E) u_{i+1, j}+E u_{i+2, j}, \tag{5.10.15}
\end{equation*}
$$

respectively. Equations (5.10.14) and (5.10.15) will form the small (2×2) implicit system

$$
\begin{align*}
{\left[\begin{array}{cc}
(1+E) & -E \\
-(E+2 K) & (1+E+2 K)
\end{array}\right] } & {\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
(1-E-2 K) & 0 \\
0 & (1-E)
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right] } \\
& +\left[\begin{array}{l}
(E+2 K) u_{i-1, j} \\
E u_{i+2, j}
\end{array}\right] \tag{5.10.16}
\end{align*}
$$

Since

$$
\left[\begin{array}{cc}
(1+E) & -E  \tag{5.10.17}\\
-(E+2 K) & (1+E+2 K)
\end{array}\right]=\frac{1}{(1+2 E+2 K)}\left[\begin{array}{cc}
(1+E+2 K) & E \\
(E+2 K) & 1+E
\end{array}\right]
$$

therefore (5.9.16) can be explicitly represented by the equation

$$
\begin{align*}
& {\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=} \frac{1}{(1+2 E+2 K)}\left\{\left[\begin{array}{ll}
\left(1-(E+2 K)^{2}\right) & E(1-E) \\
(E+2 K)(1-(E+2 K) & \left(1-E^{2}\right)
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]\right. \\
&\left.+\left[\begin{array}{l}
(1+E+2 K)(E+2 K) u_{i-1, j}+E^{2} u_{i+2, j} \\
(E+2 K)^{2} u_{i-1, j}+E(1+E) u_{i+2, j}
\end{array}\right]\right\} . \tag{5.10.18}
\end{align*}
$$

For the left ungrouped point we use the equation

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{(1+E+2 K)}\left\{(E+2 K) u_{0, j+1}+(1-E) u_{1, j}+E u_{2, j}\right\} \tag{5.10.19}
\end{equation*}
$$

and for the right ungrouped point we use

$$
\begin{equation*}
u_{m-1, j+1}=\frac{1}{(1+E)}\left\{E u_{m, j+1}+(E+2 K) u_{m-2, j}+(1-E-2 K) u_{m-1, j}\right\} \tag{5.10.20}
\end{equation*}
$$

Now to form the GE algorithms, we assume that the space interval x is divided into an even number of subintervals, which implies that the unknown (m-1) points are odd in number, and the algorithms are as follows:

1. GER with upwinding: This upwinding algorithm corresponds to the algorithm (5.6.8), i.e.,

$$
\begin{equation*}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1, j}, \tag{5.10.21}
\end{equation*}
$$

where


$$
\begin{aligned}
& \underline{b}_{1, j}^{T}=\left[r \beta u_{0, j}, 0, \ldots, 0, \varepsilon r u_{m, j+1}\right], \underline{u}_{j}^{T}=\left[u_{i, j}, u_{2, j}, \ldots, u_{m-1, j}\right] \\
& \text { and } \quad \beta=\left(\varepsilon+\frac{k \Delta x}{2}\right) \text {. }
\end{aligned}
$$

2. GEL with upwinding: This algorithm corresponds to the algorithm (5.6.11) and given by

$$
\begin{equation*}
\left(\mathrm{I}+\mathrm{rG} \mathrm{~g}_{2}\right) \underline{\mathrm{u}}_{\mathrm{j}+1}=\left(\mathrm{I}-\mathrm{rG} \mathrm{~g}_{1}\right) \underline{\mathrm{u}}_{\mathrm{j}}+\underline{b}_{2, \mathrm{j}}, \tag{5.10.24}
\end{equation*}
$$

with $\quad \underline{b}_{2, j}^{T}=\left[r \beta u_{0, j+1}, 0,0, \ldots, 0, \varepsilon r u_{m, j}\right], G_{1}$ and $G_{2}$ are as defined above.
3. Alternating Group Explicit with upwinding: For the (S) AGE type we use equation (5.10.21) at the ( $\mathrm{j}+1$ ) th time-level and equation (5.10.24) at the ( $\mathrm{j}+2$ ) th time-level. While for the (D)AGE type we then continue for $(j+3)$ th and $(j+4)$ th time-levels with the alternating direction reversed. Similar concepts can be used for the splitting type of schemes with 2 and 4 level splitting.

The (S)AGE scheme with upwinding is now written as

$$
\left.\begin{array}{l}
\left(I+r \theta G_{1}\right) \underline{u}_{j+\theta}=\left(I-r \theta G_{2}\right) \underline{u}_{j}+\underline{b}_{1}, j  \tag{5.10.25}\\
\left(I+r \theta G_{2}\right) \underline{u}_{j+2 \theta}=\left(I-r \theta G_{1}\right) \underline{u}_{j+\theta}+\underline{b}_{2, j+\theta}
\end{array}\right\}
$$

with

$$
\theta= \begin{cases}1, & \text { for ordinary time-level } \\ \frac{1}{2}, & \text { for } 2-1 e v e l ~ s p l i t t i n g ~\end{cases}
$$

The (D)AGE acheme with upwinding is written as
with

$$
\left.\begin{array}{l}
\left(I+r \theta G_{1}\right) \underline{u}_{j+\theta}=\left(I-r \theta G_{2}\right) \underline{u}_{j}+\underline{b}_{1, j}  \tag{5.10.26}\\
\left(I+r \theta G_{2}\right) \underline{u}_{j+2 \theta}=\left(I-r \theta G_{1}\right) \underline{u}_{j+\theta}+\underline{b}_{2, j+\theta} \\
\left(I+r \theta G_{2}\right) \underline{u}_{j+3 \theta}=\left(I-r \theta G_{1}\right) \underline{u}_{j+2 \theta}+\underline{b}_{2}, j+2 \theta \\
\left(I+r \theta G_{1}\right) \underline{u}_{j+4 \theta}=\left(I-r \theta G_{2}\right) \underline{u}_{j+3 \theta}+\underline{b}_{1, j+3}
\end{array}\right\}
$$

4. Average of GER upwinding and GEL upwinding: If $v$ and $w$ are assumed to be the solution of (5.10.21) and (5.10.24) respectively, then $u$ is taken as

$$
\begin{equation*}
\underline{u}_{j+1}=\frac{1}{2}\left\{\underline{v}_{j+1}+\underline{w}_{j+1}\right\} \tag{5.10.27}
\end{equation*}
$$

For the truncation error analysis we can proceed in a similar fashion to section (5.7). But for the sake of brevity it is omitted. However, it can be proved that the order of approximation is $O\left(\Delta t+\Delta x+\frac{\Delta t}{\Delta x}+\Delta x \Delta t\right)$ for the GET and GEL schemes with upwinding. The signs of the coefficients of each of the $\Delta x, \frac{\Delta t}{\Delta x}$ and $\Delta x \Delta t$ terms for both schemes are opposite and due to this the order of approximation for (S)AGE and (D)AGE schemes of any type are approximately $O\left(\Delta t+(\Delta x)^{2}\right)$.

Stability: For an analysis of the stability of scheme GER (5.10.21) is written in the explicit form,

$$
\begin{equation*}
\underline{u}_{j+1}={\underset{A u}{j}}+\hat{\widehat{b}}_{1, j}, \tag{5.10.28}
\end{equation*}
$$

where

with $\quad \Delta=(1+2 E+2 K)$ and $\hat{b}_{1, j}$ is associated with the boundary conditions. For $E \leqslant E+2 K \leqslant 1$, i.e. $r \leqslant \frac{1}{\varepsilon+k \Delta x} \leqslant \frac{1}{\varepsilon}$

$$
\|A\|_{\infty}=\max \left\{\frac{1+E-(E+2 K)^{2}}{1+2 E+2 K}, \frac{1+2(E+K)-(E+2 K)^{2}}{1+2 E+2 K}, 1, \frac{1}{1+E}\right\} \leqslant 1
$$

Therefore, the GER scheme (5.10.21) is stable for $r \leqslant \frac{1}{\varepsilon+k \Delta x} \leqslant \frac{1}{\varepsilon}$. A similar result can be shown for GEL scheme (5.10.24). Hence the following theorem can be established.

Theorem 5.3
The GER and GEL schemes with upwinding given by equations (5.10.21), and (5.10.24) respectively are stable provided,

$$
r \leqslant \frac{1}{\varepsilon+k \Delta x} \leqslant \frac{1}{\varepsilon}
$$

For the (S)AGE and (D)AGE schemes (5.10.25) and (5.10.26) respectively, the stability condition can be shown to be fulfilled for all r>0 if,

$$
\begin{equation*}
\max _{s}\left\{\left|\frac{1-r \rho_{1, s}}{1+r \rho_{1, s}}\right|\right\} \quad \max \left\{\left|\frac{1-r \rho_{2}, s}{1+r \rho_{2, s}}\right|\right\}, s=1,2, \ldots, m-1 \tag{5.10.31}
\end{equation*}
$$

where $\rho_{1, s}$ and $\rho_{2, s}$ are the sth eigenvalues of $G_{1}$ and $G_{2}$ respectively, is less than or equal to unity.

The eigenvalues of $G_{1}$ are given by $\varepsilon, 0$ and $(2 \varepsilon+k \Delta x)$ of multiplicity $\left(\frac{m-2}{2}\right)$ while eigenvalues of $G_{2}$ are given by $(\varepsilon+k \Delta x), 0$ and $(2 \varepsilon+k \Delta x)$ of multiplicity $\left(\frac{m-2}{2}\right)$. Since all the eigenvalues are non-negative, therefore the expression (5.10.31) will be less than (or equal) to unity for all $r>0$. Hence, the (S)AGE and (D)AGE methods are unconditionally stable for all r>0. Thus,

## Theorem 5.4

The (S)AGE scheme of (5.10.25) and the (D)AGE scheme of (5.10.26) are both unconditionally stable for all $r>0$.

## Numerical Experiments

The problem (5.1.1) with conditions similar to (5.9.1)-(5.9.2) is now considered using some of the schemes discussed in this section. The results are also compared with the CN upwinding scheme. These are given in Tables (5.10.1)-(5.10.8).

The results indicate that the upwinding scheme is inferior compared to the GE scheme without upwinding. This is certainly due to the low order terms in the truncation errors. However it is still within reasonable accuracy in comparison with the CN with upwinding. By virtue of its explicitness therefore $G E$ with upwinding is still worthy of recommendation.
$\mathrm{k}=1.0, \mathrm{e}=1.0, \Delta \mathrm{t}=0.001, \mathrm{r}=0.1, \Delta \mathrm{x}=0.1, \mathrm{t}=0.1$

| Method | 0.1 | 0.2 | $0.2)^{3}$ | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| UNEG UPW | 0.02140 | 0.04864 | 0.08627 | 0.13932 | 0.21289 | 0.31167 | 0.43923 | 0.59730 | 0.78527 |
| UPOS UPW | - | - | - | - | - | - | - | - | - |
| UAV UPW - | - | - | - | - | - | - | - | - | - |
| (D) AGE UPW | 0.02136 | 0.04861 | 0.08632 | 0.13943 | 0.21310 | 0.31196 | 0.43954 | 0.59759 | 0.78541 |
| (D) AGE | 0.01911 | 0.04397 | 0.07920 | 0.12995 | 0.20168 | 0.29946 | 0.42733 | 0.58743 | 0.77933 |
| CNU | 0.02138 | 0.04863 | 0.08630 | 0.13940 | 0.21302 | 0.31185 | 0.43944 | 0.59749 | 0.78539 |
| EXACT SOLUTION | 0.01985 | 0.04370 | 0.07892 | 0.12982 | 0.20177 | 0.29986 | 0.42794 | 0.58805 | 0.77976 |

$\mathrm{k}=1.0, \varepsilon=1.0, \Delta t=0.005, \mathrm{r}=0.5, \Delta x=0.1, \mathrm{t}=0.5$ TABLE (5.10.1)

| Meth | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| UNEG UPW | 0.06204 | 0.13035 | 0.20562 | 0.28865 | 0.38024 | 0.48129 | 0.59275 | 0.71562 | 0.85099 |
| UPOS UPW | - | - | - | - | - | - | - | - | - |
| UAV UPW | - | - | - | - | - | - | - | - | - |
| (D) AGE UPW | 0.06213 | 0.13052 | 0.20591 | 0.28899 | 0.28064 | 0.48170 | 0.59311 | 0.71591 | 0.85113 |
| (D) AGE | 0.06041 | 0.12724 | 0.20131 | 0.28337 | 0.27439 | 0.47530 | 0.58715 | 0.71108 | 0.84824 |
| CNU | 0.06210 | 0.13047 | 0.20582 | 0.28890 | 0.38052 | 0.48158 | 0.59301 | 0.71583 | 0.85110 |
| EXACT SOLUTION | 0.06043 | 0.12730 | 0.20136 | 0.28345 | 0.37447 | 0.47539 | 0.58724 | 0.71114 | 0.84830 |

$k=1.0, \varepsilon=1.0, \Delta t=0.01, r=1.0, \Delta x=0.1, t=1.0$

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| UNEG UPW | - | - | - | - | - | - | - | - | - |
| UPOS UPW | 0.06274 | 0.13175 | 0.20767 | 0.29117 | 0.28204 | 0.48409 | 0.59525 | 0.71753 | 0.85204 |
| UAV UPW | - | - | - | - | - | - | - | - | - |
| (D)AGE UPW | 0.06118 | 0.12879 | 0.20352 | 0.28613 | 0.37742 | 0.47833 | 0.58987 | 0.71315 | 0.84940 |
| (D) AGE | 0.06274 | 0.13176 | 0.20768 | 0.29119 | 0.38305 | 0.48411 | 0.59527 | 0.71754 | 0.85204 |
| CNU | 0.06274 | 0.13176 | 0.20768 | 0.29119 | 0.38305 | 0.48411 | 0.59526 | 0.71754 | 0.85204 |
| EXACT SOLUTION | 0.06120 | 0.12884 | 0.20360 | 0.28621 | 0.37752 | 0.47843 | 0.58996 | 0.71322 | 0.84945 |

$k=1.0, t=1.0, \Delta t=0.015, r=1.5, \Delta x=0.1, t=1.5 \quad$ TABLE (5.10.3)

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| UNEG UPW | - | - | - | - | - | - | - | - | - |
| UPOS UPW | 0.06275 | 0.13176 | 0.20769 | 0.29120 | 0.38307 | 0.48412 | 0.59528 | 0.71755 | 0.85205 |
| UAV UPW | - | - | - | - | - | - | - | - | - |
| (D)AGE UPW | 0.04435 | 0.10488 | 0.20433 | 0.29573 | 0.41934 | 0.51617 | 0.57096 | 0.69673 | 0.84412 |
| (D) AGE | 0.06100 | 0.12853 | 0.20349 | 0.28617 | 0.37781 | 0.47868 | 0.58964 | 0.71295 | 0.84932 |
| CNU | 0.06275 | 0.13177 | 0.20769 | 0.29120 | 0.28307 | 0.48412 | 0.59528 | 0.71755 | 0.85205 |
| EXACT SOLUTION | 0.06121 | 0.12885 | 0.20361 | 0.28623 | 0.37754 | 0.47845 | 0.58998 | 0.71324 | 0.84945 |

$\mathrm{k}=1.0, \varepsilon=0.1, \Delta t=0.001, \quad r=0.1, \Delta x=0.1, t=0.1$

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left.\begin{array}{l} \text { UNEG } \\ \text { UPOS } \\ \text { UAV } \end{array}\right\} \text { UPW }$ | $0.00000$ | $0.00000$ | $0.00002$ | $0.00014$ | $0.00093$ | $0.00539$ | $0.02653$ | $0.10892$ | $0.36620$ |
| (D) AGE UPW | 0.00000 | $0.00000$ | 0.00002 | $0.00014$ | $0.00093$ | $0.00538$ | $0.02652$ | $0.10893$ | $0.36 .622$ |
| (D) AGE | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00006 | 0.00061 | 0.00554 | 0.04067 | 0.23229 |
| CNU | 0.00000 | 0.00000 | 0.00002 | 0.00014 | 0.00093 | 0.00538 | 0.02652 | 0.10892 | 0.36621 |
| EXACT SOLUTION | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00003 | 0.00051 | 0.00625 | 0.04939 | 0.26259 |

TABLE (5.10.5)

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left.\begin{array}{l} \text { UNEG } \\ \text { UPOS } \\ \text { UAV } \end{array}\right\} \text { UPW }$ | $0.00033$ | $0.00112$ | $0.00313$ | $0.00807$ | $0.01978$ | $0.04643$ | $0.10478$ | $0.22842$ | $0.48354$ |
| (D) AGE <br> UPW | $0.00033$ | $0.00112$ | $0.00312$ | $0.00806$ | $0.01978$ | $0.04643$ | $0.10481$ | $0.22846$ | $0.48358$ |
| (D) AGE | 0.00001 | 0.00003 | 0.00016 | 0.00064 | 0.00250 | 0.00910 | 0.03137 | 0.10297 | 0.32527 |
| CNU | 0.00033 | 0.00011 | 0.00312 | 0.00805 | 0.01976 | 0.04641 | 0.10477 | 0.22842 | 0.48355 |
| EXACT SOLUTION | 0.00002 | 0.00008 | 0.00031 | 0.00118 | 0.00415 | 0.01364 | 0.04237 | 0.12557 | 0.35890 |


| Method $\quad$ x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $0.00081$ | $0.00248$ | $0.00596$ | $0.01318$ | $0.02808$ | $0.05856$ | $0.12047$ | $0.24546$ | $0.49666$ |
| (D) AGE <br> UPW | $0.00081$ | $0.00248$ | $0.000596$ | $0.01318$ | $0.02808$ | $0.05857$ | $0.12049$ | $0.24549$ | $0.49668$ |
| (D) AGE | 0.00003 | 0.00011 | 0.00037 | 0.00121 | 0.00382 | 0.01184 | 0.03626 | 0.11010 | 0.33239 |
| CNU | 0.00081 | 0.00248 | 0.00595 | 0.01317 | 0.02806 | 0.05854 | 0.12045 | 0.24545 | 0.49666 |
| EXACT SOLUTION | 0.00006 | 0.00024 | 0.00074 | 0.00218 | 0.00624 | 0.01754 | 0.04868 | 0.13399 | 0.36671 |

$k=1.0, \quad \varepsilon=0.1, \Delta t=0.02, r=2.0, \Delta x=0.1, t=2.0$
TABLE (5.10.7)

| Method | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left.\begin{array}{l} \text { UNEG } \\ \text { UPOS } \\ \text { UAV } \end{array}\right\} \text { UPW }$ | $0.00097$ | $0.00291$ | $0.00680$ | $0.01460$ | $0.03021$ | $0.06145$ | $0.12399$ | $0.24910$ | $0.49939$ |
| (D) AGE <br> UPW | $0.00097$ | $0.00291$ | $0.00680$ | $0.01460$ | $0.03021$ | $0.06146$ | $0.12399$ | $0.24911$ | $0.49939$ |
| (D) AGE | 0.00003 | 0.00013 | 0.00044 | 0.00135 | 0.00409 | 0.01232 | 0.03700 | 0.11107 | 0.33330 |
| CNU | 0.00097 | 0.00291 | 0.00680 | 0.01459 | 0.03030 | 0.06144 | 0.12398 | 0.24910 | 0.49939 |
| EXACT SOLUTION | 0.00008 | 0.00029 | 0.00086 | 0.00242 | 0.00668 | 0.01825 | 0.04971 | 0.13526 | 0.36782 |

### 5.11 CONCLUSIVE REMARKS

The generalised form of the approximation (5.2.1) to the convectiondiffusion equation is quite useful as it gives many choices of various approximations. The explicit schemes (5.5.1) and (5.5.4) obtained from it are both very easy and economical to implement. As they are 'practically' unconditionally stable and also accurate, therefore they are strongly recommended.

The GE schemes derived are also comparably accurate and strongly stable. For $r \leqslant 2.0$, the GE schemes ((D)AGE in particular) are still to be recommended against the CN upwinding schemes.

One point worth noting here is that the ordinary GE scheme which is made up of formulae where $\partial u / \partial x$ is approximated by both forward and backward differences at different time-levels is always superior than the GE scheme with upwinding where $\partial u / \partial x$ is approximated by backward differences in this case.

The schemes discussed in this chapter can be easily extended and adapted for the higher space dimensions. This work is suggested for further research.

Finally, it is hoped that the concept of GE schemes for solving diffusion-convection equations can be developed further by using other formulae. This research is underway and will be reported later.

## Chapter Six

## SOME EXPLICIT PROCEDURES FOR THE SOLUTION

OF A FOURTH ORDER PARABOLIC P.D.E.

### 6.1 INTRODUCTION

Consider the equation

$$
\begin{equation*}
\frac{\partial^{2} y}{\partial t^{2}}+\frac{\partial^{4} y}{\partial x^{4}}=0, \quad 0 \leqslant x \leqslant 1, t>0 \tag{6.1.1a}
\end{equation*}
$$

subject to the initial conditions,

$$
\begin{gather*}
y(x, 0)=g_{0}(x), \\
\frac{\partial y}{\partial t}(x, 0)=g_{1}(x), \tag{6.1.1b}
\end{gather*}
$$

for $0 \leqslant x \leqslant 1$, and the boundary conditions,

$$
\begin{align*}
& y(0, t)=f_{0}(t) \\
& y(1, t)=f_{1}(t) \\
& \frac{\partial^{2} y}{\partial x^{2}}(0, t)=p_{0}(x), \frac{\partial^{2} y}{\partial x^{2}}(1, t)=p_{1}(t), t \geqslant 0 . \tag{6.1.1c}
\end{align*}
$$

This problem occurs in the study of the transverse vibrations of a uniform flexible beam of unit length hinged at both ends. Here, y represents the transverse displacement of the beam and $x$ and $t$ the distance and time variables, respectively.

Following Richtmyer (1967), we can introduce new variables $\phi$ and $\psi$ such that:

$$
\begin{equation*}
\phi=\frac{\partial y}{\partial t}, \quad \psi=\frac{\partial^{2} y}{\partial x^{2}} \tag{6.1.2}
\end{equation*}
$$

The equation (6.1.la) can now be written as a system of two second order parabolic equations,

$$
\begin{align*}
& \frac{\partial \phi}{\partial t}=-\frac{\partial^{2} \psi}{\partial x^{2}},  \tag{6.1.3a}\\
& \frac{\partial \psi}{\partial t}=\frac{\partial^{2} \phi}{\partial x^{2}} \tag{6.1.3b}
\end{align*}
$$

or as a second-order system,

$$
\begin{equation*}
\frac{\partial w}{\partial t}=A \frac{\partial^{2} w}{\partial x^{2}} \tag{6.1.4}
\end{equation*}
$$

where

$$
A=\left[\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right] \quad \text { and } \quad w=\left[\begin{array}{l}
\phi \\
\psi
\end{array}\right]
$$

The system (6.1.2) has been solved by Evans, D.J. (1965) using the Du-Fort-Franke1 method and by Danaee, A. and Evans, D.J. (1982) using the Hopscotch method.

Here the system (6.1.2) is solved using the asymmetric finitedifference approximation (Saul'yev, 1964) which leads to an explicit method of solution (Fairweather and Gourlay, 1966). The asymmetric approximation is then reformulated as a Group Explicit method. Theoretical investigations into the truncation errors and stability are presented and a comparison of numerical results given.

### 6.2 THE ASYMMETRIC ALGORITHM

Apart from Saul'yev (1964), similar studies on an asymmetric solution procedure can be found in Larkin (1964), Clark and Barakhat (1966) and Evans D.J. (1966). The extension of the asymmetric formulae (left to right direction) to the fourth-order problem was presented by Gourlay and Fairweather (1966).

In this section, the method will be extended to the fourth order equation (6.1.1) by solving the decomposed form of equation (6.1.3) with a single direction and with alternating directions (left-right and rightleft directions).

At any point (i $\Delta x,\left(j+\frac{1}{2}\right) \Delta t$ ) we approximate the equations (6.1.3a) and (6.1.3b) by

$$
\begin{equation*}
\frac{\phi_{i, j+1}^{-\phi_{i, j}}}{\Delta t}=-\frac{1}{(\Delta x)^{2}}\left\{\psi_{i-1, j+1}-\psi_{i, j+1}-\psi_{i, j}+\psi_{i+1, j}\right\}, \tag{6.2.1a}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\psi_{i, j+1}-\psi_{i, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left\{\phi_{i-1, j+1}-\phi_{i, j+1}^{-\phi_{i, j}}{ }^{+\phi_{i+1, j}}\right\}, \tag{6.2.1b}
\end{equation*}
$$

respectively. These equations can further be simplified into the ( $2 \times 2$ ) implicit system of equations

$$
\left[\begin{array}{cc}
1 & -r  \tag{6.2.2}\\
r & 1
\end{array}\right]\left[\begin{array}{c}
\phi_{i, j+1} \\
\phi_{i, j+1}
\end{array}\right]=\left[\begin{array}{cc}
1 & r \\
-r & 1
\end{array}\right]\left[\begin{array}{c}
\phi_{i, j} \\
\psi \\
i, j
\end{array}\right]+\left[\begin{array}{cc}
0 & -r \\
r & 0
\end{array}\right]\left[\begin{array}{l}
\phi_{i+1, j}{ }^{+\phi_{i-1}, j+1} \\
\psi_{i+1, j}{ }^{+} \psi_{i-1, j+1}
\end{array}\right]
$$

where $r=\Delta t /(\Delta x)^{2}$.
The equation is written explicitly by, (Fig. (6.2.1)),

$$
\left[\begin{array}{c}
\phi_{i, j+1}  \tag{6.2.3}\\
\psi_{i, j+1}
\end{array}\right]=\frac{1}{\Delta}\left\{\left[\begin{array}{cc}
1-r^{2} & 2 r \\
-2 r & 1-r^{2}
\end{array}\right]\left[\begin{array}{c}
\phi_{i, j} \\
\psi_{i, j}
\end{array}\right]+\left[\begin{array}{cc}
r^{2} & -r \\
r & r^{2}
\end{array}\right]\left[\begin{array}{c}
\phi_{i+1, j}+\phi_{i-1, j+1} \\
\psi_{i+1, j}+\psi_{i-1, j+1}
\end{array}\right]\right\}
$$

where $\Delta=1+r^{2}$, if the calculation is assumed to be carried out from left to right (Fairweather and Gourlay, 1966).

Alternatively, at any point (i $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right)$ the equations (6.1.3a) and (6.1.3b) can also be approximated by,

$$
\begin{equation*}
\frac{\phi_{i, j+1}-\phi_{i, j}}{\Delta t}=-\frac{1}{(\Delta x)^{2}}\left\{\psi_{i+1, j+1}-\psi_{i, j+1}-\psi_{i, j}+\psi_{i-1, j}\right\} \tag{6.2.4a}
\end{equation*}
$$

and
respectively. Similarly, these equations result in a similar explicit system, (Fig. (6.2.2)),

$$
\left[\begin{array}{c}
\phi_{i, j+1}  \tag{6.2.5}\\
\psi_{i, j+1}
\end{array}\right]=\frac{1}{\Delta}\left\{\left[\begin{array}{cc}
1-r^{2} & 2 r \\
-2 r & 1-r^{2}
\end{array}\right]\left[\begin{array}{c}
\phi_{i, j} \\
\psi_{i, j}
\end{array}\right]+\left[\begin{array}{cc}
r^{2} & -r \\
r & r^{2}
\end{array}\right]\left[\begin{array}{c}
\phi_{i-1, j}+\phi_{i+1, j+1} \\
\psi_{i-1, j} \psi_{i+1, j+1}
\end{array}\right]\right\}
$$

if the calculation is assumed to be carried out from right to left.
From the equations (6.2.3) and (6.2.5), the following algorithms are possible:
(i) At every time-level j the equations (6.2.3) or (6.2.5) is used to calculate the approximate solution at the $(j+1)$ th time-level.
(ii) For every two time-levels, the equations (6.2.3) and (6.2.5) are used alternately.
(iii) For the first time-level, the equation (6.2.3) is used first then followed by the equation (6.2.5) for the second time-level. Then, the pattern is repeated in reverse order.
(iv) For every time-level, both equations are used and the average of the solutions is considered as the approximate solution to the original problem.

However, in the numerical experiments given later in the chapter only some of these algorithms are considered.

For the truncation error analysis then by using truncated Taylor


Computational molecule for the scheme (6.2.3)
FIGURE (6.2.1)


Computational molecule for the scheme (6.2.5)
FIGURE (6.2.2)
series it is easy to show that the local truncation error of the schemes (6.2.1a)-(6.2.1b) are given by

$$
\begin{align*}
T_{(6.2 .1 a)}= & -\frac{\Delta t^{2}}{24} \phi_{t t t}-\frac{\Delta x^{2}}{12} \psi_{x x x x}+\frac{\Delta t}{\Delta x} \psi_{x t}+\frac{\Delta x \Delta t}{6} \psi_{x x x t}- \\
& \frac{\Delta t^{2}}{16} \psi_{x x t t}+\Delta t^{2} \frac{\Delta t}{\Delta x} \psi_{x t t t}-\Delta t^{2}\left(\frac{\Delta t}{\Delta x}\right)^{2} \psi_{t t t t}  \tag{6.2.6a}\\
= & 0\left(\Delta x^{2}+\Delta t^{2}+\frac{\Delta t}{\Delta x}\right)
\end{align*}
$$

and

$$
\begin{align*}
T_{(6.2 .1 b)}= & -\frac{\Delta t^{2}}{24} \phi_{t t t}+\frac{\Delta x^{2}}{12} \phi_{x x x x}-\frac{\Delta x \Delta t}{6} \phi_{x x x t}-\frac{\Delta t}{\Delta x} \phi_{x t}+\frac{\Delta t^{2}}{16} \phi_{x x t t}- \\
& \Delta t^{2} \cdot \frac{\Delta t}{\Delta x} \phi_{x t t t^{2}}+\Delta t^{2}\left(\frac{\Delta t}{\Delta x}\right)^{2} \phi_{t t t t}  \tag{6.2.6b}\\
= & 0\left(\Delta x^{2}+\Delta t^{2}+\frac{\Delta t}{\Delta x}\right)
\end{align*}
$$

respectively.
Similarly, the truncation errors for the schemes (6.2.4a)-(6.2.4b) are given by,

$$
\begin{align*}
T_{(6.2 .4 a)}= & -\frac{\Delta t^{2}}{24} \phi_{t t t}+\frac{\Delta x^{2}}{12} \psi_{x x x x}-\frac{\Delta t}{\Delta x} \psi_{x t}-\frac{\Delta x \Delta t}{6} \psi_{x x x t} \\
& +\frac{\Delta t^{2}}{16} \psi_{x x t t}+\Delta t^{2} \frac{\Delta t}{\Delta x} \psi_{x t t t}+\Delta t^{2}\left(\frac{\Delta t}{\Delta x}\right) \psi_{t t t t}  \tag{6.2.7a}\\
= & 0\left(\Delta x^{2}+\Delta t^{2}+\frac{\Delta t}{\Delta x}\right)
\end{align*}
$$

and

$$
\begin{align*}
T_{(6.2 .4 b)}= & -\frac{\Delta t^{2}}{24} \psi_{t t t}-\frac{\Delta x^{2}}{12} \phi_{x x x x}+\frac{\Delta x \Delta t}{6} \phi_{x x x t}+\frac{\Delta t}{\Delta x} \phi_{x t} \\
& \frac{\Delta t^{2}}{16} \phi_{x x t t}+\Delta t^{2} \frac{\Delta t}{\Delta x} \phi_{x t t t}-\Delta t^{2}\left(\frac{\Delta t}{\Delta x}\right)^{2} \phi_{t t t t}  \tag{6.2.7b}\\
= & 0\left(\Delta x^{2}+\Delta t^{2}+\frac{\Delta t}{\Delta x}\right)
\end{align*}
$$

respectively.

By observing the signs of the terms in (6.2.6a) and (6.2.6b) in comparison with the signs of the terms in ( 6.2 .7 a ) and ( 6.2 .7 b ) respectively, it seems that algorithms (ii) and (iii) are more accurate than algorithms (i) and (iv).

For the analysis of stability, we consider the implicit form (6.2.2). This system for any time-level $j \Delta_{t}$ is written by

$$
\begin{equation*}
\mathrm{A} \underline{\omega}_{j+1}=\mathrm{B} \underline{\omega}_{j}+\underline{d} \tag{6.2.8}
\end{equation*}
$$

where

$$
\begin{aligned}
& d=\left[\begin{array}{c}
B_{1}{ }_{0} 0, j+1 \\
0 \\
\vdots \\
0 \\
0 \\
B_{1}{ }_{M}{ }_{M, j}
\end{array}\right]_{(M-1) \times 1} \\
& \underline{\omega}_{\mathrm{j}}^{\mathrm{T}}=\left[\underline{\omega}_{1, j}, \underline{\omega}_{2, j}, \cdots, \underline{\omega}_{M-1, j}\right]_{1 \times(M-1)}
\end{aligned}
$$

with

$$
\underline{\omega}_{i, j}=\left[\begin{array}{l}
\phi_{i, j}  \tag{6.2.9}\\
\psi_{i, j}
\end{array}\right] \quad, \quad A_{1}=\left[\begin{array}{rr}
1 & -\bar{r} \\
r & 1
\end{array}\right] \text { and } B_{1}=\left[\begin{array}{rr}
0 & -r \\
r & 0
\end{array}\right]
$$

and $B_{1}^{T}=-B_{1}$.
The matrix $A$ can be shown to have an inverse of the form,

whose eigenvalues are ( $M-1$ ) multiples of the eigenvalues of $A_{1}^{-1}$. Whilst the eigenvalues of $B$ are ( $M-1$ ) multiplies of the eigenvalues of $A_{1}^{T}$. Since the eigenvalues of $A_{1}$ are given by ( $r+i$ ) and ( $r-i$ ), therefore it is clearly evident that the eigenvalues of $A^{-1} B$ are always less than or equal to unity and hence the scheme (6.22) is unconditionally stable.

Similarly, the scheme (6.2.5) is implicitly written as,

$$
\begin{equation*}
\tilde{\mathrm{A}}_{\underline{j}+1}=\tilde{\mathrm{B}}_{\underline{\omega}}+\tilde{\underline{\mathrm{d}}}, \tag{6.2.11}
\end{equation*}
$$

where


This scheme can also be shown to be unconditionally stable too. Since the algorithms mentioned above are derived from a combination of these two schemes then the stability of these algorithms automatically follows.

### 6.3 FORMULATION OF THE GE METHOD

Now we develop the GE method for solving the splitting form of the equations (6.1.3a) and (6.1.3b). To do this, we approximate the equations (6.1.3a) and (6.1.3b) at the point ( $i, j+\frac{1}{2}$ ) by

$$
\begin{equation*}
\frac{\phi_{i, j+1}^{-\phi} i, j}{\Delta t}+\frac{1}{(\Delta x)^{2}}\left\{\psi_{i+1, j+1}-\psi_{i, j+1}-\psi_{i, j}+\psi_{i-1, j}\right\}=0 \tag{6.3.1a}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\psi_{i, j+1}-\psi_{i, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left\{\phi_{i+1, j+1}-\phi_{i, j+1}-\phi_{i, j}+\phi_{i-1, j}\right\} \tag{6.3.1b}
\end{equation*}
$$

respectively, and at the point ( $i+1, j+\frac{1}{2}$ ) by

$$
\begin{equation*}
\frac{\phi_{i+1, j+1}^{-\phi} i+1, j}{\Delta t}+\frac{1}{(\Delta x)^{2}}\left\{\psi_{i, j+1}-\psi_{i+1, j+1}-\psi_{i+1, j}+\psi_{i+2, j}\right\}=0 \tag{6.3.1c}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\psi_{i+1, j+1}-\psi_{i+1, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left\{\phi_{i, j+1}{ }^{-\phi_{i+1, j+1}}{ }_{i+1, j}+\phi_{i+2, j}\right\} \tag{6.3.1d}
\end{equation*}
$$

respectively.
These equations can be written in the matrix form as

$$
\left[\begin{array}{cc:cc}
r & 1 & -r & 0  \tag{6.3.2}\\
-1 & r & 0 & -r \\
\hdashline-r & 0 & r & 1 \\
0 & -r & -1 & r
\end{array}\right]\left[\begin{array}{l}
\phi_{i, j+1} \\
\psi_{i, j+1} \\
\phi_{i+1, j} \\
\psi_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc:cc}
-r & 1 & 0 & 0 \\
-1 & -r & 0 & 0 \\
\hdashline 0 & 0 & -r & 1 \\
0 & 0 & -1 & -r
\end{array}\right]\left[\begin{array}{l}
\phi_{i, j} \\
\psi_{i, j} \\
\phi_{i+1, j} \\
\psi_{i+1, j}
\end{array}\right]+\left[\begin{array}{l}
r \phi_{i-1, j} \\
r \psi_{i-1, j} \\
r \phi_{i+2, j} \\
r \psi_{i+2, j}
\end{array}\right]
$$

The coefficient matrix on the left-hand-side can be easily inverted to yield the form
$\frac{1}{\left(1+4 r^{2}\right)}\left[\begin{array}{lc:cc}r & -\left(1+2 r^{2}\right) & -r & -2 r^{2} \\ 1+2 r^{2} & r & 2 r^{2} & -r \\ \hdashline-r & -2 r^{2} & r & -(1+2 r)^{2} \\ 2 r^{2} & -r & 1+2 r & r\end{array}\right]$
Thus, the equation (6.3.2) can be expressed explicitly by,

$$
\left[\begin{array}{l}
\phi_{i, j+1} \\
\psi_{i, j+1} \\
\phi_{i+1, j+1} \\
\psi_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc:cc}
a_{1} & a_{2} & a_{3} & a_{4} \\
-a_{2} & a_{1} & -a_{4} & a_{3} \\
\hdashline a_{3} & a_{4} & a_{1} & a_{2} \\
-a_{4} & a_{3} & a_{2} & a_{1}
\end{array}\right]\left[\begin{array}{l}
\phi_{i, j} \\
\psi_{i, j} \\
\phi_{i+1, j} \\
\psi_{i+1, j}
\end{array}\right]+\left[\begin{array}{l}
b_{1} \phi_{i-1, j}+b_{2} \psi_{i-1, j}+b_{3} \phi_{i+2, j}+b_{4} \psi_{i+2, j} \\
-b_{2} \phi_{i-1, j}+b_{1} \psi_{i-1, j}-b_{4} \phi_{i+2, j}+b_{3} \psi_{i+2, j} \\
-b_{3} \phi_{i-1, j}{ }^{-b} 4_{i-1, j} \psi_{1}{ }_{1} \phi_{i+2, j}+b_{2} \psi_{i+2, j} \\
b_{i-1, j}{ }^{-b} \psi_{i-1, j}{ }^{-b}{ }_{2} \phi_{i+2, j}+b_{1} \psi_{i+2, j}
\end{array}\right]
$$

where

$$
\begin{array}{ll}
a_{1}=\left(1+r^{2}\right) / \Delta, & a_{2}=2 r\left(1+r^{2}\right) / \Delta,  \tag{6.3.4}\\
a_{3}=3 r^{2} / \Delta, & a_{4}=\left(2 r^{3}-r\right) / \Delta, \\
b_{1}=r^{2} / \Delta, & b_{2}=-r\left(1+2 r^{2}\right) / \Delta, \\
b_{3}=-r^{2} / \Delta, & b_{4}=-2 r^{3} / \Delta,
\end{array}
$$

and $\Delta=1+4 \mathrm{r}^{2}$.
For any ungrouped point on the left-hand side of the region, ie. point ( $\left.\Delta x,\left(j+\frac{1}{2}\right) \Delta t\right), j=0,1, \ldots$, the approximations (6.3.1c) and (6.3.1d) are used. This will result in the matrix equation,

$$
\left[\begin{array}{cc}
r & 1 \\
-1 & r
\end{array}\right]\left[\begin{array}{l}
\phi_{1, j+1} \\
\Psi_{1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
-r & 1 \\
-1 & -r
\end{array}\right]\left[\begin{array}{c}
\phi_{1, j} \\
\Psi_{1, j}
\end{array}\right]+\left[\begin{array}{c}
r \phi_{0, j+1}+r \phi_{2, j} \\
r \psi_{0, j+1}+r \psi_{2, j}
\end{array}\right]
$$

or explicitly, as,

$$
\left[\begin{array}{c}
\phi_{1, j+1}  \tag{6.3.5}\\
\psi_{1, j+1}
\end{array}\right]=\frac{1}{\left(1+r^{2}\right)}\left\{\left[\begin{array}{cc}
1-r^{2} & 2 r \\
-2 r & 1-r^{2}
\end{array}\right]\left[\begin{array}{l}
\phi_{i, j} \\
\psi_{i, j}
\end{array}\right]+\left[\begin{array}{c}
r^{2}\left(\phi_{0, j+1}+\phi_{2, j}\right)-r\left(\psi_{0, j+1}\right. \\
\left.+\psi_{2, j}\right) \\
r\left(\phi_{0, j+1}+\phi_{2, j}\right)+r^{2}\left(\psi_{0, j+1}\right. \\
\left.+\psi_{2, j}\right)
\end{array}\right]\right\}
$$

Similarly, the approximations (6.3.1a) and (6.3.1b) are used for any ungrouped point on the right-hand-side of the region, i.e. point $((M-1) \Delta x$, $\left.\left(j+\frac{1}{2}\right) \Delta t\right)$ and this will result in, $\left[\begin{array}{c}\phi_{M 1, j+1} \\ \psi_{M 1, j+1}\end{array}\right]=\frac{1}{\left(1+r^{2}\right)}\left\{\left[\begin{array}{cc}1-r^{2} & 2 r \\ -2 r & 1-r^{2}\end{array}\right]\left[\begin{array}{c}\phi_{M-1, j} \\ \psi_{M-1, j}\end{array}\right]+\left[\begin{array}{r}r^{2}\left(\phi_{M, j+1} \phi_{M-2, j}\right)-r\left(\psi_{M, j+1}\right. \\ r\left(\phi_{M, j+1} \phi_{M-2, j}\right)+r^{2}\left(\psi_{M, j+1}+\right.\end{array}\right]\right\}$.

For the implementation of the method and without incurring any loss of generality the space interval is assumed to be divided into an even number of intervals, hence the number of unknown internal points is odd. Therefore, at every time-level, there will be one ungrouped point and this is assumed to be alternately placed on the left-hand and right-hand side of the interval.

For the GEL method, the scheme is written as

$$
\begin{align*}
& \tilde{A}_{\underline{j} \omega+1}=\widetilde{B}_{\underline{\omega}}^{j}+\underline{\tilde{d}},  \tag{6.3.7}\\
& \underline{\omega}=\left[\begin{array}{l}
\phi \\
\psi \\
\hline
\end{array}\right] \text {, }
\end{align*}
$$

where the matrices $\tilde{A}, \widetilde{B}$ and $\underline{d}$ are given by,
and

$$
\begin{aligned}
{\underset{\mathrm{d}}{ }}_{\mathrm{T}} & =\left[{\underset{-}{B} \omega, j+1}^{-0, \ldots, 0, B \underset{M j}{ }] \text { with }}\right. \\
A & =\left[\begin{array}{lr}
r & 1 \\
-1 & r
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{ll}
r & 0 \\
0 & r
\end{array}\right]=r I .
\end{aligned}
$$

Similarly the GER method is written by

$$
\begin{equation*}
{\tilde{\mathrm{A}} \underline{\omega}_{j+1}}^{\tilde{\mathrm{B}}_{\underline{\omega}}}+\underline{\tilde{\mathrm{d}}} \tag{6.3.9}
\end{equation*}
$$

where,

and

$$
\underline{\underline{d}}^{T}=\left[{\underset{\sim}{\mathrm{B}}}_{0, j}, 0, \ldots, 0,{\underset{\mathrm{~B}}{\mathrm{M}, j+1}}\right] .
$$

Since $\tilde{A}, \tilde{B}$ (or $\widetilde{\mathbb{A}}, \mathbb{Z}$ ) are two non-commutative block matrices, the analysis of stability is more complicated and left for further work. However, the stability restriction can be verified from the numerical experiments given later.

The truncation errors of this class of method can be shown to have similar order as the asymmetric formulae in the last section, i.e., $0\left(\Delta x^{2}+\Delta t^{2}+\frac{\Delta t}{\Delta x}\right)$.

### 6.4 NUMERICAL EXPERIMENTS

In order to provide a comparison with previous published methods, the problem (6.1.1a) is solved together with the initial conditions,

$$
\begin{align*}
& y(x, 0)=\frac{x}{12}\left(2 x^{2}-x^{3}-1\right),  \tag{6.4.1}\\
& \frac{\partial y}{\partial t}(x, 0)=0, \quad 0 \leq x \leq 1,
\end{align*}
$$

and the boundary conditions:

$$
\begin{align*}
& y(0, t)=y(1, t)=0 \\
& \frac{\partial^{2}}{\partial x^{2}} y(0, t)=\frac{\partial^{2} y}{\partial x^{2}}(1, t)=0, \quad t>0 . \tag{6.4.2}
\end{align*}
$$

The same increment is chosen, i.e. $\Delta x=0.05$ and $\Delta t=0.00125$ (for which $\mathrm{r}=\frac{1}{2}$ ). In addition the problem was tested for $\Delta \mathrm{t}=0.002$, 0.005 (i.e. $\mathrm{r}=0.8,2.0$ ) to demonstrate the stability experimentally and investigate the accuracy of the method.

The theoretical solution of the problem with the given initial and boundary conditions is given by,

$$
\begin{equation*}
y(x, t)=\sum_{s=1}^{\infty} d_{s} \sin (2 s+1) \pi x \cos (2 s+1)^{2} \pi^{2} t, \tag{6.4.3}
\end{equation*}
$$

where

$$
d_{s}=-\frac{8}{(2 s+1)^{5} \pi^{5}}
$$

In Table (6.4.1) the comparisons between the theoretical solution for $y$ given by (6.4.3) and the computed solutions obtained from the schemes given in Sections (6.2)-(6.3) are presented. These are compared with the results from other methods (Danaee and Evans, 1982) for $t=0.02$. Similar results for the bending moment

$$
\psi=\frac{\partial^{2} y}{\partial x^{2}}
$$

are quoted in Table (6.4.2). Tables (6.4.4)-(6.4.5) give the same results for different $r=2.0$ in a larger time domain ( $t=1.0$ ). Table (6.4.3) gives the solution $y$ in comparison with some schemes in Section (6.2)-(6.3) for $\mathrm{t}=0.04$ and $\mathrm{r}=0.8$.
$\Delta x=0.05, \Delta t=0.00125, t=0.02, \quad r=\frac{1}{2}$

| Method $x$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Exact solution <br> of y | -0.00790 | -0.01503 | -0.02069 | -0.02432 | -0.02557 |
| Equation (6.2.3) <br> Algorithm (6.2)(i) | -0.00792 | -0.01506 | -0.02072 | -0.02437 | -0.02532 |
| Algorithm (6.2)(ii) | -0.00790 | -0.01505 | -0.02073 | -0.02439 | -0.02566 |
| GE | -0.00791 | -0.01503 | -0.02075 | -0.02437 | -0.02568 |
| (D)AGE | -0.00791 | -0.01505 | -0.02074 | -0.02440 | -0.02565 |
| Evans Method | -0.00790 | -0.01503 | -0.02071 | -0.02439 | -0.02567 |
| Richtymer | -0.00768 | -0.01468 | -0.02033 | -0.02403 | -0.02532 |
| Hopscotch | -0.00791 | -0.01504 | -0.02071 | -0.02439 | -0.02565 |

TABLE (6.4.1)
$\Delta x=0.05, \Delta t=0.00125, t=0.02, r=\frac{1}{2}$

| Method $x$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Exact Solution <br> of y' | 0.07626 | 0.14770 | 0.20317 | 0.24184 | 0.25570 |
| Equation (6.2.3) <br> Algorithm (6.2)(ii) | 0.07593 | 0.14497 | 0.20142 | 0.23953 | 0.25316 |
| Algorithm (6.2)(iii) | 0.07709 | 0.14741 | 0.20237 | 0.24269 | 0.25642 |
| GE | 0.08081 | 0.13957 | 0.21460 | 0.22978 | 0.26560 |
| (D)AGE | 0.07755 | 0.14968 | 0.20793 | 0.24144 | 0.25183 |
| Evans Method | 0.07576 | 0.14774 | 0.20079 | 0.24227 | 0.25881 |
| Richtmyer | 0.06749 | 0.13657 | 0.19507 | 0.24036 | 0.24887 |
| Hopscotch | 0.07576 | 0.14774 | 0.20079 | 0.24227 | 0.25881 |

TABLE (6.4.2)
$\Delta x=0.05, \quad \Delta t=0.002, t=0.04, r=0.8$

| Method $x$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Exact Solution <br> of $y$ | -0.00746 | -0.01418 | -0.01952 | -0.02295 | -0.02413 |
| Equation (6.2.3) <br> Algorithm (6.2)(i) | -0.00746 | -0.01416 | -0.01949 | -0.02297 | -0.02421 |
| Algorithm (6.2)(iii) | -0.00741 | -0.01412 | -0.01948 | -0.02296 | -0.02417 |
| GE | -0.00984 | -0.00940 | -0.02625 | -0.01393 | -0.03611 |
| (D)AGE | -0.00731 | -0.01407 | -0.01939 | -0.02299 | -0.02436 |

TABLE (6.4.3)
$\Delta x=0.05, \Delta t=0.005, t=1.0, \quad r=2.0$

| Method x | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Exact Solution <br> of y | 0.00729 | 0.01387 | 0.01909 | 0.02244 | 0.02360 |
| Equation(6.2.3) <br> Algorithm(6.2)(i) | 0.00781 | 0.01484 | 0.02051 | 0.02425 | 0.02570 |
| Algorithm (6.2)(iii) | 0.00254 | 0.00432 | 0.00565 | 0.00641 | 0.00660 |
| Evans Method | 0.00701 | 0.01321 | 0.01829 | 0.02153 | 0.02266 |
| Richtmyer | 0.00660 | 0.01257 | 0.01732 | 0.02039 | 0.02147 |
| Hopscotch | 0.00549 | 0.01052 | 0.01451 | 0.01711 | 0.01803 |

TABLE (6.4.4)
$\Delta x=0.05, \Delta t=0.005, t=1.0, r=2.0$

| Method | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Exact Solution <br> of $y^{\prime \prime}$ | -0.06650 | -0.13179 | -0.18619 | -0.22456 | -0.24033 |
| Equation (6.2.3) <br> Algorithm (6.2)(i) | -0.06409 | -0.12766 | -0.18105 | -0.22112 | -0.24781 |
| Algorithm (6.2)(iii) | -0.02911 | -0.5683 | -0.07758 | -0.09006 | -0.08659 |
| Evans Method | -0.06970 | -0.13452 | -0.17639 | -0.21210 | -0.22630 |
| Richtmyer | -0.06377 | -0.12231 | -0.16883 | -0.20158 | -0.21797 |
| Hopscotch | -0.06970 | -0.13452 | -0.17639 | -0.21210 | -0.22630 |

TABLE (6.4.5)

From the above results, for $r=0.5$ the various schemes mentioned in earlier sections are as competitive and accurate as existing methods. For $r=0.8$, the $G E$ scheme starts to portray instability in the solution but the (D)AGE scheme is still stable. However our previous numerical experience shows that the (D) AGE scheme will also be unstable for $r \geqslant 1.0$.

From Tables (6.4.4)-(6.4.5), the unconditional stability of the schemes in Section (6.2) can be clearly seen. For Algorithm (6.2)(iii), the numerical results are in conformity with the remark given by Birtwistle, G.M. (1968), who commented that this type of algorithm is not always better than the basic algorithm i.e. algorithm (6.2)(i).

### 6.5 CONCLUSIVE REMARK

As the numerical experiments have shown the GE class of method are restrictive as far as stability is concerned, therefore the extension of this method to more complex problems in this class is not recommended.

## Chapter 7

EXPLICIT METHODS FOR THE SOLUTION OF

## NON-LINEAR PARABOLIC EQUATIONS

### 7.1 INTRODUCTION

The establishment of new methods of approximation will not be fully justified unless they are implemented on a more realistic situation involving non-linear problems.

In this Chapter, the semi-explicit alternating direction method and the GE method developed earlier will be used in solving a few types of non-linear problems. One problem which we will consider first is the selfadjoint quasi-linear second order equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(K(u) \frac{\partial u}{\partial x}\right) . \tag{7.1.1}
\end{equation*}
$$

Another slightly stronger non-linear problem of the form,

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial t^{2}}=G\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \tag{7.1.2}
\end{equation*}
$$

will be considered also.
An outline investigation into the truncation errors and stability conditions for both problems are also presented.

Finally some brief work on the Korteweg de Vries equation is included.

### 7.2 THE FORMULATION OF GE METHOD FOR SELF-ADJOINT FORM OF QUASI-LINEAR SECOND-ORDER EQUATION

In this section, the formulation of the class of Group Explicit method of the equation (7.1.1) will be discussed. To start with, the equation (7.1.1) is approximated at the point ( $i, j+\frac{1}{2}$ ) by the generalized form,

$$
\begin{align*}
& \frac{u_{i, j+1}{ }^{-u} i_{i, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left[\theta 1^{k}{ }_{i+\frac{1}{2}, j+1} \delta_{x} u_{i+\frac{1}{2}, j+1}{ }^{-\theta} 2^{k} i-\frac{1}{2}, j+1^{\delta} x^{u}{ }_{i-\frac{1}{2}, j+1}+{ }^{\prime}{ }^{k}{ }_{i+\frac{1}{2}, j}\right. \\
& \delta_{x}{ }^{u}{ }_{i+\frac{1}{2}, j^{-\theta}}^{2}{ }_{2}^{k}{ }_{i-\frac{1}{2}, j^{\delta}} x^{u}{ }_{i-\frac{1}{2}, j}{ }^{\}}, \tag{7.2.1}
\end{align*}
$$

where $\theta_{1}, \theta_{2}, \theta_{1}^{\prime}, \theta_{2}^{\prime}$ have to satisfy certain conditions as before and $k_{i+\frac{1}{2}, j+1} \equiv k\left(u_{i+\frac{1}{2}, j+1}\right)$.

Since $k_{i+\frac{1}{2}, j+1}$ does not fall on a grid point, it can be approximated by either

$$
\begin{aligned}
k_{i+\frac{1}{2}, j+1} & \simeq k_{i, j+1} \\
& \simeq k_{i+1, j+1} \\
& \simeq \frac{1}{2}\left(k_{i, j+1}+k_{i+1, j+1}\right)
\end{aligned}
$$

or
or
If the average approximation for $k$ is assumed, then the expansion of (7.2.1) will lead to the non-linear equation,

$$
\begin{align*}
& -\frac{r}{2} \theta_{1}\left(k_{i+1, j+1}+k_{i, j+1}\right) u_{i+1, j+1}+\left[1+\frac{r}{2} \theta_{1}\left(k_{i+1, j+1}^{+k_{i, j+1}}\right)+\frac{r}{2} \theta_{2}\left(k_{i, j+1}\right.\right. \\
& \left.+k_{i-1, j+1}\right) u_{i, j+1}-\frac{r}{2} \theta_{2}\left(k_{i, j+1}^{+k_{i-1, j+1}}\right) u_{i-1, j+1} \\
& =\frac{r}{2} \theta_{1}^{\prime}\left(k_{i+1, j}+k_{i, j}\right) u_{i+1, j}+\left[1-\frac{r}{2} \theta_{1}^{\prime}\left(k_{i+1, j}+k_{i, j}\right)-\frac{r}{2} \theta_{2}^{\prime}\left(k_{i, j}+\right.\right. \\
& \left.\left.k_{i-1, j}\right)\right] u_{i, j}+\frac{r}{2} \theta_{2}^{\prime}\left(k_{i, j}+k_{i-1, j}\right) u_{i-1, j} . \tag{7.2.2}
\end{align*}
$$

With the normal practise of choosing $\theta$, i.e. for example $\theta_{i}=\theta_{i}^{\prime}=\frac{1}{2}$ the case of Crank-Nicolson type formulae it will result in the system of non-linear tridiagonal equations. Of ten this requires a high number of iterations to converge and thus is not considered to be economical. Meanwhile the ordinary explicit type formulae (i.e. $\theta_{i}^{\prime}=1, i=1,2$ ) is normally restricted to small values of $r$ for stability reasons.

To develop the GE method, we choose the following coupled approximations:

## (i) Approximation from negative direction (UNEG)

In this case we choose $\theta_{1}=\theta_{2}^{\prime}=1$ and $\theta_{2}=\theta_{1}^{\prime}=0$ to give the approximation

$$
\begin{gather*}
-\frac{r}{2}\left(k_{i+1, j+1}^{\left.+k_{i, j+1}\right) u_{i+1, j+1}+\left[1+\frac{r}{2}\left(k_{i+1, j+1}+k_{i, j+1}\right)\right] u_{i, j+1}}\right. \\
=\left[1-\frac{r}{2}\left(k_{i, j}+k_{i-1, j}\right)\right] u_{i, j}+\frac{r}{2}\left(k_{i, j}+k_{i-1, j}\right) u_{i-1, j} \tag{7.2.3}
\end{gather*}
$$

This approximation if implemented from right to left along the ( $j+1$ ) th time-level will give the explicit iterative scheme,

$$
\begin{array}{r}
u_{i, j+1}^{(n+1)}=\frac{1}{\{1+} \begin{array}{r}
\left.\frac{r}{2}\left(k_{i+1, j+1}^{(n+1)}+k_{i, j+1}^{(n)}\right)\right\}
\end{array} \frac{r}{2}\left(k_{i+1, j+1}^{(n+1)}+k_{i, j+1}^{(n)}\right) u_{i+1, j+1}^{(n+1)} \\
\left.\quad+\left[1-\frac{r}{2}\left(k_{i, j}+k_{i-1, j}\right)\right] u_{i, j}+\frac{r}{2}\left(k_{i, j}+k_{i-1, j}\right) u_{i-1, j}\right\}, \tag{7.2.4}
\end{array}
$$

where n denotes the iteration number.

## (ii) Approximation from positive direction (UPOS)

For this we choose $\theta_{2}=\theta_{1}^{\prime}=1$ and $\theta_{1}=\theta_{2}^{\prime}=0$ and results in the approximation

$$
\begin{align*}
{[1+} & \left.\frac{r}{2}\left(k_{i, j+1}+k_{i-1, j+1}\right)\right] u_{i, j+1}-\frac{r}{2}\left(k_{i, j+1}+k_{i-1, j+1}\right) u_{i-1, j+1} \\
& =\frac{r}{2}\left(k_{i+1, j}+k_{i, j}\right) u_{i+1, j}+\left[1-\frac{r}{2}\left(k_{i+1, j}+k_{i, j}\right)\right] u_{i, j} \tag{7.2.5}
\end{align*}
$$

This approximation can be implemented explicitly from right to left to give the iterative scheme

$$
\begin{align*}
u_{i, j+1}^{(n+1)}= & \frac{1}{\left[1+\frac{r}{2}\left(k_{i, j+1}^{(n)}+k_{i-1, j+1}^{(n+1)}\right)\right]}\left\{\frac{r}{2}\left(k_{i, j+1}^{(n)}+k_{i-1, j+1}^{(n+1)}\right) u_{i-1, j+1}^{(n+1)}+\right. \\
& \left.\frac{r}{2}\left(k_{i+1, j}+k_{i, j}\right) u_{i+1, j}+\left[1-\frac{r}{2}\left(k_{i+1, j}+k_{i, j}\right)\right] u_{i, j}\right\} . \tag{7.2.6}
\end{align*}
$$

## Formulation

To introduce the GE concept then as before we choose a group of two points ( $\mathrm{i}, \mathrm{j}+1$ ) and ( $\mathrm{i}+1, \mathrm{j}+1$ ) and at each of these points we use equation (7.2.3) and (7.2.5) respectively. This results in the small system,

$$
\left[\begin{array}{cc}
1+\beta_{i, j+1}^{(n)} & -\beta_{i, j+1}^{(n)} \\
-\beta_{i, j+1}^{(n)} & 1+\beta_{i, j+1}^{(n)}
\end{array}\right]\left[\begin{array}{l}
u_{i, j+1}^{(n+1)} \\
u_{i+1, j+1}^{(n+1)}
\end{array}\right]=\left[\begin{array}{l}
\left(1-\beta_{i-1, j}\right) u_{i, j}+\beta_{i-1, j} u_{i-1, j} \\
\left(1-\beta_{i+1, j}\right) u_{i+1, j}+\beta_{i+1, j} u_{i+2, j}
\end{array}\right]
$$

where
and

$$
\begin{align*}
& \beta_{i, j+1}^{(n)}=\frac{r}{2}\left[k_{i+1, j+1}^{(n)}+k_{i, j+1}^{(n)}\right],  \tag{7.2.7}\\
& \beta_{i-1, j}=\frac{r}{2}\left[k_{i, j}+k_{i-1, j}\right]
\end{align*}
$$

$\left.\quad{ }_{i+1, j} \quad{ }^{[ }{ }_{i+1, j}{ }_{i+2, j}\right]$
The inversion of the right-hand side ( $2 \times 2$ ) matrix will then result in the system to have an explicit form, i.e.,

$$
\left[\begin{array}{c}
u_{i, j+1}^{(n+1)} \\
u_{i+1, j+1}^{(n+1)}
\end{array}\right]=\frac{1}{\Delta}\left[\begin{array}{cc}
1+\beta_{i, j+1}^{(n)} & \beta_{i, j+1}^{(n)} \\
\beta_{i, j+1}^{(n)} & 1+\beta_{i, j+1}^{(n)}
\end{array}\right]\left[\begin{array}{l}
\left(1-\beta_{i-1, j}\right) u_{i, j}+\beta_{i-1, j} u_{i-1, j} \\
\left(1-\beta_{i+1, j}\right) u_{i+1, j}+\beta_{i+1, j} \times
\end{array}\right]
$$

where $\Delta=1+2 \beta_{i, j+1}^{(n)}$.
Now the GER scheme is defined as the combination of equations (7.2.6) for $i=1$ and (7.2.8) for $i=2,4,6, \ldots, m-2$ where $m$ is an even number of intervals. Meanwhile the GEL scheme is the combination of equations (7.2.8) for $i=1,3,5, \ldots, m-3$ and (7.2.4) for $i=m-1$. The (S)AGE and (D)AGE are then defined similar to those developed in previous chapters.

For an example of the usage of these types of approximation we consider the equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u^{2}}{\partial x^{2}} \tag{7.2.9}
\end{equation*}
$$

which in self-adjoint form can be written as,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(2 u \frac{\partial u}{\partial x}\right) . \tag{7.2.10}
\end{equation*}
$$

From (7.2.4) and (7.2.6), (7.2.10) is approximated respectively by

$$
\begin{align*}
u_{i, j+1}^{(n+1)}= & \frac{1}{\left\{1+r\left(u_{i+1, j+1}^{(n+1)}+u_{i, j+1}^{(n)}\right)\right\}}\left\{r\left(u_{i+1, j+1}^{(n+1)}+u_{i, j+1}^{(n)}\right) u_{i+1, j+1}^{(n+1)}\right. \\
& \left.+\left[1-r\left(u_{i, j}+u_{i-1, j}\right)\right] u_{i, j}+r\left(u_{i, j}+u_{i-1, j}\right) u_{i-1, j}\right\}, \tag{7.2.11}
\end{align*}
$$

when applied in a negative direction and

$$
\begin{align*}
& u_{i, j+1}^{(n+1)}=\frac{1}{\left[1+r\left(u_{i, j+1}^{(n)}+u_{i-1, j+1}^{(n+1)}\right)\right]}\left\{r\left(u_{i, j+1}^{(n)}+u_{i-1, j+1}^{(n+1)}\right) u_{i-1, j+1}^{(n+1)}+\right. \\
& \left.r\left(u_{i+1, j}+u_{i, j}\right) u_{i+1, j}+\left[1-r\left(u_{i+1, j}+u_{i, j}\right)\right] u_{i, j}\right\}, \tag{7.2.12}
\end{align*}
$$

when applied in a positive direction and are applied in a similar manner to Saul' yev (1964), Larkin (1964), Clark and Barakhat (1966) and Evans (1966). These schemes are generally known as Alternating Direction Explicit (ADE) methods.

For the GER and GEL schemes, these are formed by

$$
\begin{gather*}
{\left[\begin{array}{l}
u_{i, j+1}^{(n+1)} \\
u_{i+1, j+1}^{(n+1)}
\end{array}\right]=} \\
\frac{1}{\Delta}\left[\begin{array}{cc}
1+r\left(u_{i+1, j+1}^{(n)}+u_{i, j+1}^{(n)}\right) & -r\left(u_{i+1, j+1}^{(n)}+u_{i, j+1}^{(n)}\right. \\
-r\left(u_{i+1, j+1}^{(n)}+u_{i, j+1}^{(n)}\right) & 1+r\left(u_{i+1, j+1}^{(n)}+u_{i, j+1}^{(n)}\right.
\end{array}\right] \times  \tag{7.2.13}\\
{\left[\begin{array}{ll}
{\left[1-r\left(u_{i, j}+u_{i-1, j}\right)\right] u_{i, j}+r\left(u_{i, j}+u_{i-1, j}\right) u_{i-1, j}} \\
{\left[1-r\left(u_{i+1, j}+u_{i+2, j}\right)\right] u_{i+1, j}^{+r\left(u_{i+1, j}+u_{i+2, j}\right) u_{i+2, j}}}
\end{array}\right],}
\end{gather*}
$$

where $\Delta=1+2 r\left(u_{i+1, j+1}^{(n)}+u_{i, j+1}^{(n)}\right)$, together with the equation (7.2.11) and (7.2.12) respectively.

In considering the truncation errors of the above formulae, the analysis is quite difficult and complicated. However it is expected that the order of error will be similar to the previously discussed schemes in Chapter 4 and will be of order $0\left(\Delta x^{2}+\Delta t+\frac{\Delta t}{\Delta x}\right)$.

### 7.3 NUMERICAL RESULTS

For confirmation by numerical experiment, the equation (7.2.9) is approximated numerically by some of the schemes mentioned in Section (7.2). The analytical solution is known and given by (Abdullah, A.R., 1976),

$$
\begin{equation*}
(2 u-3)+\ln \left(u-\frac{1}{2}\right)=2(2 t-x) . \tag{7.3.1}
\end{equation*}
$$

The appropriate initial and boundary conditions are given to satisfy above exact solution.

The analytical solution can be solved iteratively using a NewtonRaphson method to give,

$$
u^{(n+1)}=\frac{1}{2}+\left[\frac{3}{2}-\frac{1}{2} \ln \left(u^{(n)}-\frac{1}{2}\right)-x+2 t\right]\left(1-\frac{1}{2 u^{(n)}}\right),
$$

with the initial guesses

$$
\begin{equation*}
u^{(0)}=1+\frac{4-e^{2(x-2 t)}}{4+2 e^{2(x-2 t)}} \tag{7.3.3}
\end{equation*}
$$

The results in this work are compared with some of the earliest methods like Crank-Nicolson and three time-level methods (see Section 3.7). The resulting non-linear system of equations in the $C N$ method is linearised by either the Richtmyer's Linearised Difference Scheme or Newton Linearization (equation (3.8.13)). These comparisons are given in Tables (7.3.1)-(7.3.3) for various values of $r$. The results of the above methods are quoted from Abdullah, A.R. (1976) which need to be consulted for a full comparison.

From the tables, it is clear that the solution of all the methods are as accurate as the analytical solution for $r \leqslant 0.5$. For $r \geqslant 1$, the (D)AGE scheme is no longer stable but the application of the schemes (7.2.11)/ (7.2.12) of alternating direction type is still stable (Table (7.3.4)(7.3.6)) and of reasonable accuracy. This demonstrates one of the advantages of the formulae (7.2.11)/(7.2.12) against GE formulas. Computational experience also shows that equation (7.2.12) of positive
direction always require less number of iterations compared to equation (7.2.11) of negative direction, though the number of both cases are still acceptably small.
$\mathrm{t}=0.05, \Delta \mathrm{x}=0.1, \mathrm{r}=0.05$

| x | Analytical | R.L.D.S./N.L. <br> $(3.18 .13)$ | T.T.L. | $(7.2 .11) /(7.2 .12)$ | (D)AGE |
| :---: | :--- | :---: | :--- | :---: | :--- |
| 0.0 | $1.56739(3)$ | 1.56739 | 1.56739 | 1.56739 | 1.56739 |
| 0.1 | $1.50000(2)$ | 1.50000 | 1.50000 | $1.50000(2)$ | $1.50000(3)$ |
| 0.2 | $1.43409(1)$ | 1.43409 | 1.43409 | $1.43409(2)$ | 1.43409 |
| 0.3 | $1.36977(2)$ | 1.36976 | 1.36976 | $1.36976(2)$ | $1.36976(3)$ |
| 0.4 | $1.30713(3)$ | 1.30713 | 1.30713 | $1.30713(2)$ | 1.30713 |
| 0.5 | $1.24631(3)$ | 1.24630 | 1.24630 | $1.24630(2)$ | $1.24630(3)$ |
| 0.6 | $1.18741(3)$ | 1.18740 | 1.18740 | $1.18741(2)$ | 1.18740 |
| 0.7 | $1.13057(3)$ | 1.13056 | 1.13056 | $1.13056(2)$ | $1.13056(3)$ |
| 0.8 | $1.07591(3)$ | 1.07590 | 1.07590 | $1.07590(2)$ | 1.07590 |
| 0.9 | $1.02356(3)$ | 1.02355 | 1.02355 | $1.02356(2)$ | $1.02356(3)$ |
| 1.0 | $0.97365(4)$ | 0.97365 | 0.97365 | 0.97365 | 0.97365 |

TABLE (7.3.1)
For equation (7.2.11)/(7.2.12), convergence is considered along the whole line and for the (D)AGE scheme convergence is considered for every group of two points. Numbers in brackets are the iteration number required. $t=0.1, \Delta x=0.1, r=0.1$

| x | Analytical | R.L.D.S./N.L. <br> $(3.8 .13)$ | T.T.L. | $(7.2 .11) /$ <br> $(7.2 .12)$ | (D)AGE |
| :---: | :--- | :---: | :--- | :--- | :--- |
| 0.0 | $1.63617(3)$ | 1.63617 | 1.63617 | 1.63617 | 1.63617 |
| 0.1 | $1.56739(2)$ | 1.56739 | 1.56739 | $1.56739(2)$ | $1.56738(3)$ |
| 0.2 | $1.50000(1)$ | 1.50000 | 1.50000 | $1.50000(2)$ | 1.50001 |
| 0.3 | $1.43409(2)$ | 1.43408 | 1.43408 | $1.43409(2)$ | $1.43407(3)$ |
| 0.4 | $1.36977(3)$ | 1.36976 | 1.36976 | $1.36977(2)$ | 1.36976 |
| 0.5 | $1.30713(3)$ | 1.30712 | 1.30712 | $1.30714(2)$ | $1.30710(3)$ |
| 0.6 | $1.24631(3)$ | 1.24630 | 1.24630 | $1.24632(2)$ | 1.24630 |
| 0.7 | $1.18741(3)$ | 1.18740 | 1.18740 | $1.18742(2)$ | $1.18738(3)$ |
| 0.8 | $1.13057(3)$ | 1.13056 | 1.13056 | $1.13058(2)$ | 1.13056 |
| 0.9 | $1.07591(3)$ | 1.07590 | 1.07590 | $1.07592(2)$ | $1.07588(3)$ |
| 1.0 | $1.02356(4)$ | 1.02356 | 1.02356 | 1.02356 | 1.02356 |

$t=0.5, \Delta x=0.1, \quad r=0.5$

| x | Analytical | R.L.D.S./N.L. <br> $(3.8 .13)$ | T.T.L. | $(7.2 .11) /(7.2 .12)$ | (D)AGE |
| :---: | :--- | :---: | :---: | :---: | :--- |
| 0.0 | $2.22685(3)$ | 2.22685 | 2.22685 | 2.22685 | 2.22685 |
| 0.1 | $2.14970(3)$ | 2.14970 | 2.14970 | $2.14953(3)$ | $2.17627(3)$ |
| 0.2 | $2.07339(3)$ | 2.07338 | 2.07338 | $2.07335(3)$ | 2.10847 |
| 0.3 | $1.99795(3)$ | 1.99795 | 1.99795 | $1.99805(3)$ | $2.00417(3)$ |
| 0.4 | $1.92346(3)$ | 1.92346 | 1.92345 | $1.92369(3)$ | 1.92273 |
| 0.5 | $1.84996(3)$ | 1.84996 | 1.84996 | $1.85031(3)$ | $1.81559(3)$ |
| 0.6 | $1.77753(3)$ | 1.77753 | 1.77753 | $1.77798(3)$ | 1.74980 |
| 0.7 | $1.70624(3)$ | 1.70624 | 1.70624 | $1.70677(3)$ | $1.72264(3)$ |
| 0.8 | $1.63617(3)$ | 1.63617 | 1.63617 | $1.63677(3)$ | 1.65024 |
| 0.9 | $1.56739(2)$ | 1.56739 | 1.56739 | $1.56805(3)$ | $1.57221(3)$ |
| 1.0 | $1.50000(1)$ | 1.50000 | 1.50000 | 1.50000 | 1.50000 |

TABLE (7.3.3)
$\mathrm{t}=1.5, \Delta \mathrm{x}=0.1, \mathrm{r}=1.0$

| $\mathbf{x}$ | Analytical | $(7.2 .11) /(7.2 .12)$ |
| :---: | :---: | :---: |
| 0.0 | 3.88964 | 3.88964 |
| 0.1 | 3.80264 | $3.79990(2)$ |
| 0.2 | 3.71594 | $3.71407(2)$ |
| 0.3 | 3.62955 | $3.62849(2)$ |
| 0.4 | 3.54350 | $3.54319(2)$ |
| 0.5 | 3.45778 | $3.45815(2)$ |
| 0.6 | 3.37242 | $3.37341(2)$ |
| 0.7 | 3.28744 | $3.28896(2)$ |
| 0.8 | 3.20285 | $3.20481(2)$ |
| 0.9 | 3.11867 | $3.12098(2)$ |
| 1.0 | 3.03492 | 3.03492 |

TABLE (7.3.4)
$t=2.0, \Delta x=0.1, r=2.0$

| $x$ | Analytical | $(7.2 .11) /(7.2 .12)$ |
| :---: | :---: | :---: |
| 0.0 | 4.7738 | 4.7738 |
| 0.1 | 4.6843 | $4.6750(4)$ |
| 0.2 | 4.5951 | $4.5892(4)$ |
| 0.3 | 4.5061 | $4.5034(4)$ |
| 0.4 | 4.4173 | $4.4177(4)$ |
| 0.5 | 4.3287 | $4.3320(4)$ |
| 0.6 | 4.2404 | $4.2463(4)$ |
| 0.7 | 4.1523 | $4.0751(4)$ |
| 0.8 | 4.0645 | $3.9895(4)$ |
| 0.9 | 3.9769 | 3.8896 |

TABLE (7.3.5)
$t=4.0, \Delta x=0.1, \quad r=4.0$

| $x$ | Analytical | $(7.2 .11) /(7.2 .12)$ |
| :---: | :---: | :---: |
| 0.0 | 8.4626 | 8.4626 |
| 0.1 | 8.3686 | $8.3288(4)$ |
| 0.2 | 8.2746 | $8.2466(4)$ |
| 0.3 | 8.1806 | $8.1641(4)$ |
| 0.4 | 8.0868 | $8.0815(4)$ |
| 0.5 | 7.9930 | $7.9986(4)$ |
| 0.6 | 7.8993 | $7.8155(4)$ |
| 0.7 | 7.8057 | $7.7483(4)$ |
| 0.8 | 7.7121 | $7.6643(4)$ |
| 0.9 | 7.6186 | 7.5252 |
| 1.0 | 7.5252 |  |

TABLE (7.3.6)

### 7.4 THE ANALYSIS OF STABILITY OF THE GE CLASS OF METHODS FOR NON-LINEAR

## EQUATIONS

The equation (7.2.11) in matrix form can be written as

$$
\begin{equation*}
\mathrm{Au}_{j+1}=\mathrm{Bu}_{\mathrm{j}}+\mathrm{b}_{1}, \tag{7.4.1}
\end{equation*}
$$



and

$$
\begin{aligned}
b_{1}^{T} & =\left[\beta_{0, j} u_{0, j}, 0, \ldots, 0, \beta_{m-1, j+1}^{(n)}\right] \\
\underline{u}_{j}^{T} & =\left[u_{1, j}, u_{2, j}, \ldots, u_{m-2, j}, u_{m-1, j}\right]
\end{aligned}
$$

Now the matrix A since in triangular form can be easily inverted to give

where $\alpha_{i}=1+\beta_{i}$ with all $\alpha^{\prime} s$ and $\beta^{\prime}$ s calculated at $(j+1)$ th time-level. The $\beta^{\prime}$ s are defined by (7.2.13).

Therefore the equation (7.2.11) is written explicitly

$$
\begin{equation*}
\underline{u}_{j+1}=A^{-1} \underline{B u}_{j} . \tag{7.4.6}
\end{equation*}
$$

The matrix $A^{-1} B$ is therefore an upper Hessenberg matrix and is given by

with $\gamma_{i}=1-\beta_{i, j}$ and $\beta_{i}$ means $\beta_{i, j+1}$.

In the case of problem (7.2.9) it can be seen after further algebraic manipulation that if $u$ is positive, monotonic increasing with respect to $t$ and monotonic decreasing w.r.t. $x$, the approximation (7.2.11) is unconditionally stable for all $r>0$.

Similarly, we are also able to achieve unconditional stability for the scheme (7.2.12) provided that $u$ is again positive, monotonic increasing w.r.t. $t$ and monotonic decreasing w.r.t. $x$.

The unconditional stability of the schemes (7.2.11) and (7.2.12) was confirmed experimentally in the previous section.

An alternative approximate method for investigating the stability is described as follows:

Let

$$
\begin{aligned}
\beta_{j+1}^{(n)} & =\max _{i}\left\{\beta_{i, j+1}^{(n)}\right\} \\
\hat{\beta}_{j} & =\min _{i}\left\{\beta_{i, j}\right\} \\
\alpha_{j} & =\min _{i}\left\{\beta_{i-1, j}\right\} \\
\hat{\alpha}_{j+1} & =\max _{i}\left\{\beta_{i-1, j+1}\right\}
\end{aligned}
$$

where

$$
\beta_{i}=r\left(u_{i+1}+u_{i}\right), \quad i=1,2, \ldots, m-1
$$

Then the system (7.2.11) can be approximated by the matrix equation,

$$
\begin{equation*}
\left(I+\beta_{j+1}^{(n)} C\right) \underline{u}_{j+1}^{n+1}=\left(I-\alpha_{j} C^{T}\right) \underline{u}_{j}+\underline{b}_{1} \tag{7.4.8}
\end{equation*}
$$

where

For the stability of the scheme, we need the eigenvalues $\lambda$ which are given by

$$
\begin{equation*}
\lambda_{s}^{(n)}=\frac{1-\alpha_{j}}{1+\beta_{j+1}^{(n)}} \quad, \quad s=1,2, \ldots, m-1, \quad \lambda^{(n)} \rightarrow \lambda \tag{7.4.9}
\end{equation*}
$$

are less than or equal to unity.

From the analytical solution (7.3.1), it can be shown that
and

$$
\left.\begin{array}{l}
\frac{\partial u}{\partial t}=2-\frac{1}{u},  \tag{7.4.10}\\
\frac{\partial u}{\partial x}=\frac{1}{2 u}-1
\end{array}\right\}
$$

for $u>\frac{1}{2}$. Therefore $u$ is monotonic increasing with respect to $t$ and monotonic decreasing with respect to $x$. Thus if,

$$
\begin{equation*}
\alpha_{j} \leqslant r\left(u_{i, j}^{+u_{i-1, j}}\right)<r\left(u_{i+1, j+1}^{(n+1)}+u_{i, j+1}^{(n)}\right) \leqslant B_{j+1}^{(n)}, \tag{7.4.11}
\end{equation*}
$$

(7.4.9) will be satisfied for all $r>0$.

On the other hand equation (7.2.12) is approximated by

$$
\begin{equation*}
\left(I+\hat{\alpha}_{j+1}^{(n)} C^{T}\right)_{\underline{u}+1}=\left(I-\hat{\beta}_{j} C\right) \underline{u}_{j}+\underline{b}_{2} \tag{7.4.12}
\end{equation*}
$$

where $\underline{b}_{2}$ are the boundary values. The eigenvalues of the amplification matrix are therefore given by,

$$
\begin{equation*}
\lambda_{s}=\frac{1-\hat{\beta}_{j}}{1+\hat{\alpha}_{j+1}^{(n)}} \tag{7.4.13}
\end{equation*}
$$

and since $\quad \hat{\beta}_{j} \leqslant r\left(u_{i+1, j}^{+u} u_{i, j}\right)<r\left(u_{i, j+1}^{(n)}+u_{i-1, j+1}^{(n)}\right) \leqslant \hat{\alpha}_{j+1}^{(n)}$,
for all $r>0$ and $u>\frac{1}{2}$. Therefore the scheme (7.2.12) is unconditionally stable. Hence we can state:

Hypothesis (7.1):
Given the quasi-linear self-adjont p.d.e.,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k(u) \frac{\partial u}{\partial x}\right), \quad t>0, \quad 0 \leqslant x \leqslant 1, \tag{7.4.15}
\end{equation*}
$$

where $k(u)$ is positive (non-negative) monotonic increasing w.r.t. t and monotonic decreasing w.r.t. $x$, and the two schemes of numerical approximation are given by equation (7.2.4) and (7.2.6). Then the equation (7.2.6) is unconditionally globally stable for all $r>0$ and the equation (7.2.4) is locally stable for all r>0 if

$$
\begin{equation*}
\alpha_{j} \leqslant \frac{r}{2}\left(k_{i, j}+k_{i-1, j}\right)<\frac{r}{2}\left(k_{i+1, j+1}^{(n+1)}+k_{i, j+1}^{(n)}\right) \leqslant \beta_{j+1}^{(n)}, \tag{7.4.16}
\end{equation*}
$$

for given $\mathrm{i}, \mathrm{j}$ and n .

On the other hand, if $k(u)$ is monotonic increasing w.r.t. both $x$ and $t$, then equation (7.2.4) is unconditionally globally stable for all $r>0$ and equation (7.2.6) is locally stable for $r>0$ if

$$
\begin{equation*}
\hat{\beta}_{j} \leqslant \frac{r}{2}\left(k_{i+1, j}+k_{i, j}\right)<\frac{r}{2}\left(k_{i, j+1}^{(n)}+k_{i-1, j+1}^{(n)}\right) \leqslant \alpha_{j+1}^{(n)} . \tag{7.4.17}
\end{equation*}
$$

This hypothesis is very important as it enables us to choose either scheme (7.2.4) or (7.2.6) depending on the value of $k(u)$ in the problem.

In the case of problem (7.2.9), since (7.4.16) and (7.4.17) are fulfilled, unconditional stability is achieved.

Now the GER and GEL schemes can be written as
and

$$
\begin{equation*}
\underline{u}_{j+1}=G_{1} \underline{u}_{j}+b_{1} \tag{7.4.18}
\end{equation*}
$$

$$
\begin{equation*}
\underline{u}_{j+1}=G_{2} \underline{u}_{j}+b_{2} \tag{7.4.19}
\end{equation*}
$$

respectively. The matrices $G_{1}$ and $G_{2}$ are defined by (7.2.4) and the r.h.s. of (7.2.8) by (7.2.6) and the r.h.s. of (7.2.8) respectively.

In both cases, if $r \leqslant \frac{2}{k_{i, j}^{+k_{i+1, j}}}$ (i.e. $1-\beta_{i, j} \geqslant 0$ ) and $r \leqslant \frac{2}{k_{i-1, j}+k_{i, j}}$, (i.e. $1-\beta_{i-1, j} \geqslant 0$ ), it can be shown that the $\left\|G_{1}\right\|_{\infty}$ and $\left\|G_{2}\right\|_{\infty}$ are both less than or equal to unity (in the case of problem (7.2.9) $r \leqslant \frac{1}{2 u} \simeq \frac{1}{4}$ ).

For the (D)AGE scheme, it is expected that a slightly better condition will apply and we found that for problem (7.2.9), (D)AGE is not stable for $r \geqslant 1$.

### 7.5 FORMULATION OF THE GE METHOD FOR A GENERAL TYPE OF NON-LINEAR EQUATION

## OF SECOND ORDER

Consider the general non-linear equation of second order of the form,

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}=G\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \tag{7.5.1}
\end{equation*}
$$

in the region $R=\{(x, t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$.
The implicit finite difference approximation to (7.5.1) will give rise to a system of non-linear algebraic equations which have to be solved at each time step and whose convergence to the solution may be difficult. This can be avoided by choosing to use instead explicit schemes which are known to have inferior stability conditions. Thus, the desire to develop explicit schemes with improved stability restrictions are paramount.

The equation (7.5.1) at point $\left(i, j+\frac{1}{2}\right)$ is approximated by

$$
\begin{align*}
& \frac{1}{(\Delta x)^{2}}\left[\theta_{1} \delta_{x} u_{i+\frac{1}{2}, j+1}{ }^{-\theta} 2^{\delta} x_{i-\frac{1}{2}, j+1}\right]+\frac{1}{(\Delta x)^{2}}\left[\theta_{1}^{\prime} \delta_{x} u_{i+\frac{1}{2}, j^{-\theta}}^{2} \delta^{\prime} x_{i-\frac{1}{2}, j}\right] \\
& =G\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{u_{i, j+1}{ }^{-u} i, j}{2},\right. \\
& \frac{\alpha_{1} \Delta_{x} u_{i, j+1}+\alpha_{2} \nabla_{x} u_{i, j}{ }^{+\alpha_{1}^{\prime}} \nabla_{x} u_{i, j+1}+\alpha_{2}^{\prime} \Delta_{x} u_{i, j}}{2 \Delta x}, \frac{u_{i, j+1} u_{i, j}}{\Delta t}, \tag{7.5.2}
\end{align*}
$$

where $\delta_{x}, \Delta_{x}$ and $\nabla_{x}$ are the central-, forward- and backward-difference operators w.r.t. the x variable respectively.

The parameters $\theta$ and $\alpha$ have to satisfy the relations,

$$
\left.\begin{array}{l}
\sum_{i=1}^{2}\left(\theta_{i}+\theta_{i}^{\prime}\right)=2  \tag{7.5.3}\\
\sum_{i=1}^{2}\left(\alpha_{i}+\alpha_{i}^{\prime}\right)=2 \\
\theta_{2}-\theta_{1}+\theta_{2}^{\prime}-\theta_{1}^{\prime}=0
\end{array}\right\}
$$

In this study the two 'ladder-step' type formulae will be considered again. For this, the following choices for $\theta$ and $\alpha$ are made:
(i) $\theta_{1}=\theta_{2}^{\prime}=1, \theta_{2}=\theta_{1}^{\prime}=0, \alpha_{1}=\alpha_{2}=1, \alpha_{1}^{\prime}=\alpha_{2}^{\prime}=1, \quad \alpha_{1}^{\prime}=\alpha_{2}^{\prime}=0$ to result in

$$
\begin{align*}
& \frac{1}{(\Delta x)^{2}}\left(u_{i+1, j+1}-u_{i, j+1}\right)-\frac{1}{(\Delta x)^{2}}\left(u_{i, j}-u_{i-1, j}\right) \\
& =G\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{u_{i, j+1}+u_{i, j}}{2},\right. \\
& \frac{u_{i+1, j+1}-u_{i, j+1}+u_{i, j}-u_{i-1, j}}{2 \Delta x},  \tag{7.5.4}\\
& \left.\frac{u_{i, j+1}-u_{i, j}}{\Delta t}\right),
\end{align*}
$$

and
(ii) $\theta_{1}=\theta_{2}^{\prime}=0, \theta_{2}=\theta_{1}^{\prime}=1, \alpha_{1}=\alpha_{2}=0, \alpha_{1}^{\prime}=\alpha_{2}^{\prime}=1$ to give

$$
\begin{align*}
& \frac{1}{(\Delta x)^{2}}\left(u_{i+1, j}-u_{i, j}\right)-\frac{1}{(\Delta x)^{2}}\left(u_{i, j+1}^{\left.-u_{i-1, j+1}\right)}\right. \\
& =G\left(i \Delta x,\left(j+\frac{1}{2}\right) \Delta t, \frac{u_{i, j+1}+u_{i, j}}{2},\right. \\
& \frac{u_{i, j+1}-u_{i-1, j+1}+u_{i+1, j}-u_{i, j}}{2(\Delta x)},  \tag{7.5.5}\\
& \\
& \frac{\left.u_{i, j+1}-u_{i, j}\right)}{\Delta t},
\end{align*}
$$

The equation (7.5.4) is an implicit equation in terms of the unknown values $u_{i+1, j+1}$ and $u_{i, j+1}$ whilst the equation (7.5.5) is an implicit equation in terms of $u_{i, j+1}$ and $u_{i-1, j+1}$. Therefore these equations can be solved as a semi-explicit type of scheme if the calculations are carried out from right to left in the case of the equation (7.5.4) and from left to right in the case of the equation (7.5.5). On the other hand, the concept of the Group Explicit method can be developed if the equations (7.5.4) and (7.5.5) are used at every group of two points ( $\mathrm{i}, \mathrm{j}+1$ ) and ( $\mathrm{i}+1, \mathrm{j}+1$ ) respectively, as previously discussed.

For mathematical simplicity, the implementation of these concepts will now be described for the Burger's equation (1949),

$$
\begin{equation*}
\varepsilon \frac{\partial^{2} u}{\partial x^{2}}=\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x} \quad, \quad 0 \leqslant x \leqslant 1, t \geqslant 0, \quad \varepsilon>0, \tag{7.5.6}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
u(x)=f(x), \quad 0<x<1, \tag{7.5.6a}
\end{equation*}
$$

and the boundary conditions

$$
\begin{align*}
& u(0, t)=g_{1}(t), \quad t \geqslant 0 .  \tag{7.5.6b}\\
& u(1, t)=g_{2}(t) .
\end{align*}
$$

Substitution of (7.5.4) and (7.5.5) into (7.5.6) will result in the equations,
and

$$
\begin{align*}
& -r a u_{i+1, j+1}+(1+r a) u_{i, j+1}=r b u_{i-1, j}+(1-r b) u_{i, j}  \tag{7.5.7}\\
& -r b u_{i-1, j+1}+(1+r b) u_{i, j+1}=(1-r a) u_{i, j}+r a u_{i+1, j} \tag{7.5.8}
\end{align*}
$$

respectively, where $a$ and $b$ are given by

$$
\left.\begin{array}{l}
a=\varepsilon-\frac{\Delta x}{4}\left(u_{i, j+1}+u_{i, j}\right),  \tag{7.5.9}\\
b=\varepsilon+\frac{\Delta x}{4}\left(u_{i, j+1}+u_{i, j}\right)
\end{array}\right\}
$$

The equations (7.5.7) and (7.5.8) can therefore be solved iteratively as a semi-explicit or $A D E$ method. For (7.5.7) it is solved from right to left using,
with

$$
\begin{equation*}
u_{i, j+1}^{(n+1)}=\frac{1}{\left(1+r a^{(n)}\right)}\left\{r a^{(n)} u_{i+1, j+1}^{(n+1)}+r b u_{i-1, j}^{(n)}+\left(1-r b^{m}\right) u_{i, j}\right\} \tag{7.5.10}
\end{equation*}
$$

$$
a^{(n)}=\varepsilon-\frac{\Delta x}{4}\left(u_{i, j+1}^{(n)}+u_{i, j}\right)
$$

and for (7.5.8) it is solved from left to right by

$$
\begin{equation*}
\left.u_{i, j+1}^{(n+1)}=\frac{1}{(1+r b}{ }^{(n)}\right) \quad\left\{r b^{(n)} u_{i-1, j+1}^{(n+1)}+(1-r a) u_{i, j}^{(n)}+r a_{i+1, j}^{(n)}\right\} \tag{7.5.11}
\end{equation*}
$$

with $b^{(n)}=\varepsilon+\frac{\Delta x}{4}\left(u_{i, j+1}^{(n)} u_{i, j}\right)$ and $n$ is the iteration number.
For the GE method, we use the equation (7.5.7) at point (i, $j+1$ ) and the equation (7.5.8) at the point ( $i+1, j+1$ ). This will result in a ( $2 \times 2$ ) system of equations,


$$
\begin{array}{ll}
a_{1}^{(n)}=\varepsilon-\frac{\Delta x}{4}\left(u_{i, j+1}^{(n)}+u_{i, j}\right) ; & a_{2}^{(n)}=\varepsilon-\frac{\Delta x}{4}\left(u_{i+1, j+1}^{(n)}+u_{i+1, j}\right) \\
b_{1}^{(n)}=\varepsilon+\frac{\Delta x}{4}\left(u_{i, j+1}^{(n)}+u_{i, j}\right) ; & b_{2}^{(n)}=\varepsilon+\frac{\Delta x}{4}\left(u_{i+1, j+1}^{(n)}+u_{i+1, j}\right) . \tag{7.5.12}
\end{array}
$$

The explicit form of which is given by,

$$
\begin{align*}
& \left.\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{l}
\left.{ }^{\left(1+b_{2}\right.}{ }_{2}^{(n)} r\right) b_{1}^{(n)} r_{u_{i-1, j}}{ }^{+a_{1}^{(n)}} a_{2}^{(n)} r_{r}^{2} u_{i+2, j} \\
b_{1}^{(n)} b_{2}^{(n)} r_{r}^{2} u_{i-1, j}+\left(1+a_{1}^{(n)} r\right) a_{2}^{(n)} r_{i+2, j}
\end{array}\right]\right\} \tag{7.5.13}
\end{align*}
$$

where $\Delta=1+a_{1}^{(n)} r+b_{2}^{(n)} r$. For the left ungrouped point we use the equation (7.5.11) and for the right ungrouped point we use the equation (7.5.12). From the equations (7.5.11)-(7.5.13), schemes such as GER, GEL, GEC, GEU, (S)AGE and (D)AGE can be developed as we have shown previously. However, to avoid repetition these will not be described in detail here.

Similar to the problems previously discussed the above schemes can be shown to have the truncation error of $O\left(\Delta x^{2}+\Delta t+\frac{\Delta t}{\Delta x}\right)$ and thus will be consistent if $\frac{\Delta t}{\Delta x} \rightarrow 0$ when $\Delta t \rightarrow 0$ and $\Delta x+0$.

For the chosen numerical examples, the problem (7.5.6) of Burger's equation with:
(a) the exact solution

$$
\begin{equation*}
\phi(x, t)=\frac{0.1 e^{-A}+0.5 e^{-B}+e^{-C}}{e^{-A}+e^{-B}+e^{-C}}, 0 \leqslant x \leqslant 1, t \geqslant 0, \tag{7.5.14}
\end{equation*}
$$

where $A=\frac{0.05}{\varepsilon}(x-0.5+4.95 t)$

$$
B=\frac{0.25}{\varepsilon}(x-0.5+0.75 t)
$$

$$
c=\frac{0.5}{\varepsilon}(x-0.375)
$$

(N.K. Madsen and R.F. Sincovec, 1976) and
(b) the exact solution

$$
\begin{equation*}
u(x, t)=\frac{2 \pi \sum_{m=1}^{\infty} m A_{m} \sin m \pi x \exp \left(-\varepsilon m^{2} \pi^{2} t\right)}{\frac{1}{\varepsilon}\left\{A_{0}+\sum_{m=1}^{\infty} A_{m} \cos m \pi x \exp \left(-\varepsilon m^{2} \pi^{2} t\right)\right\}} \tag{7.5.15}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{m}=2 \int_{0}^{1} \cos m \pi x \exp \left\{-\frac{1}{2 \varepsilon} \int_{0}^{x} f\left(x^{\prime}\right) d x^{\prime}\right\} d x,(m=1,2,3, \ldots) \\
& A_{0}=\int_{0}^{1} \exp \left\{-\frac{1}{2 \varepsilon} \int_{0}^{x} f\left(x^{\prime}\right) d x^{\prime}\right\} d x
\end{aligned}
$$

(Cole, J.D. 1951), with (i) $f(x)=4 x(1-x)$ and (ii) $f(x)=\sin \pi x$, were considered using one of the above schemes.

These problems were tested for several values of $\varepsilon$ (see Tables (7.5.1)(7.5.3)). For problem (b) no attempt was made to evaluate the exact solution. However the numerical results obtained were compared (Tables (7.5.4)-(7.5.5)) with the results from the existing methods quoted from Caldwell, J. and Smith, P. (1982). Furthermore for problem (a) and (b) (ii), the behaviour of the solutions (Figs. (7.5.1)-(7.5.2)) were qualitatively compared with those in Graney, L. and Richardson A.A. (1981) and Madsen and Sinovec (1976).

For problem (a) a large number of iterations were required to achieve
convergence for small values of $\varepsilon$ such as $\varepsilon=0.003$. However this problem does not occur for $\varepsilon<0.01$.

For problem (b) no difficulty in achieving the convergence criterion arises even for small values of $\varepsilon$. But the inaccuracy in the solution as $\varepsilon$ becomes smaller is the prime problem. From Figure (7.5.2) we can see how a disturbance appears at $x=0.5$ for $\operatorname{small} t$ and moves towards $x=1$ with a steeper front as $t$ increases. After this the disturbance reaches a maximum near $x=1$ for some time $t$ and all values of $u$ tend to decrease in a uniform manner.
$\varepsilon=1.0, \quad r=1.0, \Delta t=0.01, \Delta x=0.1, t=1.0$

| $x$ | EXACT SOLUTION | (D)AGE | $\mid$ ERROR $\mid$ |
| :---: | :--- | :--- | :--- |
| 0.0 | 0.59602645 | 0.59602645 | - |
| 0.1 | 0.58918851 | $0.58918844(2)$ | $7 \times 10^{-8}$ |
| 0.2 | 0.58233803 | 0.58233828 | $25 \times 10^{-8}$ |
| 0.3 | 0.57547863 | $0.57547826(2)$ | $37 \times 10^{-8}$ |
| 0.4 | 0.56861393 | 0.56861422 | $29 \times 10^{-8}$ |
| 0.5 | 0.56174757 | $0.56174692(2)$ | $65 \times 10^{-8}$ |
| 0.6 | 0.55488319 | 0.55488354 | $35 \times 10^{-8}$ |
| 0.7 | 0.54802440 | $0.54802379(2)$ | $61 \times 10^{-8}$ |
| 0.8 | 0.54117482 | 0.54117578 | $96 \times 10^{-8}$ |
| 0.9 | 0.53433803 | $0.53433681(2)$ | $122 \times 10^{-8}$ |
| 1.0 | 0.52751756 | 0.52751756 | - |

TABLE (7.5.1)
The numbers in brackets indicate the number of iterations requires to achieve an accuracy of $10^{-6}$.
$\varepsilon=0.1, r=1.0, \Delta t=0.01, \Delta x=0.1, t=1.0$

| $x$ | EXACT SOLUTION | $(\mathrm{D})$ AGE | $\|l\| l \mid$ |
| :--- | :--- | :--- | :--- |
| 0.0 | 0.949067 | 0.949067 | - |
| 0.1 | 0.932745 | $0.932928(3)$ | $18.3 \times 10^{-5}$ |
| 0.2 | 0.911271 | 0.911597 | $32.6 \times 10^{-5}$ |
| 0.3 | 0.883314 | $0.883703(3)$ | $38.8 \times 10^{-5}$ |
| 0.4 | 0.847514 | 0.847753 | $23.9 \times 10^{-5}$ |
| 0.5 | 0.802758 | $0.802663(3)$ | $9.6 \times 10^{-5}$ |
| 0.6 | 0.748601 | 0.747952 | $64.9 \times 10^{-5}$ |
| 0.7 | 0.685736 | $0.684544(3)$ | $119.2 \times 10^{-5}$ |
| 0.8 | 0.616304 | 0.614798 | $150.6 \times 10^{-5}$ |
| 0.9 | 0.543775 | $0.542556(3)$ | $121.9 \times 10^{-5}$ |
| 1.0 | 0.472298 | 0.472298 | - |

TABLE (7.5.2)
$\varepsilon=0.003, \mathrm{r}=1.0, \Delta \mathrm{x}=0.01, \Delta \mathrm{t}=0.0001, \mathrm{t}=0.1$

| $x$ | EXACT SOLUTION | (D)AGE | $x$ | EXACT SOLUTION | (D)AGE |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0.0 | 1.000000 | 1.000000 | 0.55 | 0.183443 | 0.181832 |
| 0.05 | 1.000000 | 1.000000 | 0.60 | 0.103726 | 0.103999 |
| 0.10 | 1.000000 | 1.000000 | 0.65 | 0.100134 | 0.100148 |
| 0.15 | 1.000000 | 1.000000 | 0.70 | 0.100004 | 0.100005 |
| 0.20 | 0.999985 | 0.999993 | 0.75 | 0.100000 | 0.100000 |
| 0.25 | 0.999037 | 0.999572 | 0.80 | 0.100000 | 0.100000 |
| 0.3 | 0.944636 | 0.952565 | 0.85 | 0.100000 | 0.100000 |
| 0.35 | 0.555361 | 0.557343 | 0.90 | 0.100000 | 0.100000 |
| 0.4 | 0.500894 | 0.501467 | 0.95 | 0.100000 | 0.100000 |
| 0.45 | 0.498093 | 0.498438 | 1.00 | 0.100000 | 0.100000 |
| 0.5 | 0.452319 | 0.454613 |  |  |  |

TABLE (7.5.3)

## SOLUTION U


$\Delta x=0.01, \Delta t=0.0001, \quad \varepsilon=0.003$

## Comparison of results for case $\mathrm{f}(\mathrm{x})=\sin \pi \mathrm{x}$ with $\varepsilon=1.0$

$\Delta t=0.01, \Delta x=0.25$

| $\mathrm{x}=0.25$ | Analytical | Implicit | Explicit | (D) AGE |
| :---: | :---: | :---: | :---: | :---: |
| t |  |  |  |  |
| 0.01 | 0.6290 | 0.6377 | 0.6267 | 0.6259 |
| 0.05 | 0.4131 | 0.4339 | 0.4099 | 0.4168 |
| 0.10 | 0.2536 | 0.2768 | 0.2525 | 0.2639 |
| 0.15 | 0.1566 | 0.1784 | 0.1565 | 0.1681 |
| 0.20 | 0.0964 | NA | 0.0967 | 0.1047 |
| 0.25 | 0.0592 | NA | NA | 0.0651 |
| $\frac{x=0.5}{t}$ | Analytical | Implicit | Explicit | (D) AGE |
| 0.01 | 0.9057 | 0.9141 | 0.9063 | 0.9082 |
| 0.05 | 0.6091 | 0.6380 | 0.6100 | 0.6222 |
| 0.10 | 0.3716 | 0.4075 | 0.3729 | 0.3876 |
| 0.15 | 0.2268 | 0.2604 | 0.2281 | 0.2417 |
| 0.20 | 0.1385 | NA | 0.1395 | 0.1509 |
| 0.25 | 0.0845 | NA | NA | 0.0942 |
| $x=0.75$ | Analytical | Implicit | Explicit | (D) AGE |
| 0.01 | 0.6524 | 0.6556 | 0.6550 | 0.6612 |
| 0.05 | 0.4502 | 0.4702 | 0.4556 | 0.4668 |
| 0.10 | 0.2726 | 0.3007 | 0.2762 | 0.2871 |
| 0.15 | 0.1644 | 0.1904 | 0.1663 | 0.1753 |
| 0.20 | 0.0994 | NA | 0.1006 | 0.1094 |
| 0.25 | 0.0603 | NA | NA | 0.0684 |

TABLE (7.5.4)

Comparison of results for case $f(x)=4 x(1-x)$ with $\varepsilon=0.01$
$\Delta t=0.01, \Delta x=0.25$

| $x=0.25$ | Analytical | Implicit | Explicit | (D) AGE |
| :---: | :---: | :---: | :---: | :---: |
| t |  |  |  |  |
| 0.01 | 0.7492 | 0.7346 | 0.7342 | 0.7342 |
| 0.05 | 0.7460 | 0.6766 | 0.6748 | 0.6755 |
| 0.10 | 0.7420 | 0.6122 | 0.6087 | 0.6104 |
| 0.15 | 0.7380 | 0.5562 | 0.5512 | 0.5537 |
| 0.20 | 0.7340 | NA | 0.5016 | 0.5046 |
| 0.25 | 0.7300 | NA | NA | 0.4625 |
| $\mathrm{x}=0.5$ | Analytical | Implicit | Explicit | (D) AGE |
| 0.01 | 0.9992 | 0.9986 | 0.9992 | 0.9989 |
| 0.05 | 0.9960 | 0.9873 | 0.9901 | 0.9887 |
| 0.10 | 0.9920 | 0.9611 | 0.9662 | 0.9636 |
| 0.15 | 0.9880 | 0.9233 | 0.9299 | 0.9263 |
| 0.20 | 0.9840 | NA | 0.8835 | 0.8796 |
| 0.25 | 0.9800 | NA | NA | 0.8256 |
| $x=0.75$ | Analytical | Implicit | Explicit | (D) AGE |
| 0.01 | 0.7492 | 0.7644 | 0.7642 | 0.7645 |
| 0.05 | 0.7460 | 0.8241 | 0.8232 | 0.8240 |
| 0.10 | 0.7420 | 0.9027 | 0.9012 | 0.9020 |
| 0.15 | 0.7380 | 0.9843 | 0.9828 | 0.9832 |
| 0.20 | 0.7340 | NA | 1.0065 | 1.0671 |
| 0.25 | 0.7300 | NA | NA | 1.1513 |

## SOLUTION



FIGURE (7.5.2)

### 7.6 THE STABILITY ANALYSIS FOR THE ADE AND GE SCHEMES IN (7.5)

For the purpose of analysing stability, we now rewrite the equations (7.5.7) and (7.5.8) in matrix form as,

$$
\begin{equation*}
[\operatorname{I}+r(\varepsilon-\alpha) C]_{u_{j}}=\left[\operatorname{I}-r(\varepsilon+\alpha) C^{T}\right]_{\underline{u}_{j}}+\underline{b}_{1}, \tag{7.6.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[I+r(\varepsilon+\alpha) C^{T}\right]_{\underline{u}_{j+1}}=[I-r(\varepsilon-\alpha) C] \underline{u}_{j}+\underline{b}_{2}, \tag{7.6.2}
\end{equation*}
$$

respectively. The matrix $C$ is defined by

and $\alpha=\frac{\Delta x}{4}\left(u_{i, j+1}+u_{i, j}\right)$. Whilst the ADE scheme which uses (7.5.7) and (7.5.8) at alternating time levels is given by,

$$
\left.\begin{array}{l}
{[I+r(\varepsilon-\alpha) C] \underline{u}_{j+1}=\left[I-r(\varepsilon+\alpha) C^{T}\right]_{\underline{u}_{j}}+\underline{b}_{1}}  \tag{7.6.4}\\
{\left[I+r(\varepsilon+\alpha) C^{T}\right]_{\underline{u}+2}=[I-r(\varepsilon-\alpha) C] \underline{u}_{j+1}+\underline{b}_{2}}
\end{array}\right\}
$$

The schemes (7.6.1) and (7.6.2) will be stable if the eigenvalues of their amplification matrices are less than or equal to unity. These are defined by

$$
\begin{equation*}
\lambda_{(7.6 .1), \mathrm{s}}=\frac{1-\mathrm{r}(\varepsilon+\alpha)}{1+\mathrm{r}(\varepsilon-\alpha)} \tag{7.6.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{(7.6 .2), s}=\frac{1-r(\varepsilon-\alpha)}{1+r(\varepsilon+\alpha)} \tag{7.6.6}
\end{equation*}
$$

for $s=1,2, \ldots,(m-1)$.
If $\alpha$ is positive in the solution domain, it can be easily seen that (7.6.5) and (7.6.6) are both equal or less than unity. Thus the schemes (7.5.7)-(7.5.8) are unconditionally stable for all r>0. Similarly, the eigenvalues of the amplification matrix of the scheme (7.6.4) is given by

$$
\begin{equation*}
\lambda_{(7.6 .4), \mathrm{s}}=\frac{1-r(\varepsilon-\alpha)}{1+r(\varepsilon-\alpha)} \frac{1-r(\varepsilon+\alpha)}{1+r(\varepsilon+\alpha)} \tag{7.6.7}
\end{equation*}
$$

for $s=1,2, \ldots, m-1$. Again if $\alpha$ is positive, these eigenvalues are less than or equal to unity for all $r>0$.

The analysis of stability of the GE type of schemes is very complicated for this case. However since these schemes are derived from unconditionally stable semi-explicit formulas, the probability of them being stable are very high. This can be seen from the numerical results in the last section.

The analysis of stability of the approximations (7.5.2), (7.5.4) and (7.5.5) has been left for further research.

### 7.7 THE KORTEWEG DE YRIES EQUATION - AN ITERATIVE EXPLICIT METHOD

The Korteweg-De Vries (KdV) equation is a non-linear wave equation which has been used to model dispersive, non-linear, long wave phenomena (see Jeffrey and Kakutarii, 1970, 1972; Miura, 1968; 1976). The form of the equation is

$$
\begin{equation*}
\frac{\partial u}{\partial t}=u \frac{\partial u}{\partial x}+\delta \frac{\partial^{3} u}{\partial x^{3}} \tag{7.7.1}
\end{equation*}
$$

where $u$ is a function of the independent variables $x$ and $t$. The coefficient $\delta$ is called the 'dispersive parameter' and in this case is taken to be a positive quantity.

Historically, this type of equation was first derived by D.G. Korteweg and G. de Vries (1895). The equation was formulated as a model for the profile of surface waves on shallow water. The form of the Korteweg and de Vries original derivation is given by,

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}-\frac{3}{2} \sqrt{\frac{g}{\ell}} \frac{\partial\left(\frac{1}{2} \eta^{2}+\frac{2}{3} \alpha \eta+\frac{\sigma}{3} \frac{\partial^{2} \eta}{\partial x^{2}}\right)}{\partial x}=0, \tag{7.7.2}
\end{equation*}
$$

where $\ell=$ depth of water measured from the undisturbed surface,
$\eta=$ elevation of the surface at a distance $x$ from the origin of coordinates,
$\sigma=$ function of the physical properties of the water, $g=$ acceleration due to gravity $\alpha=a \operatorname{small}$ arbitrary constant.

In order to obtain the equation (7.7.1) from (7.7.2) we set $\alpha=0$, $\mu=\sigma / 3$ and rescale the time variable from $t$ to $\frac{2 t}{3} \sqrt{\frac{\ell}{g}}$. The detailed historical account about early development of this equation can be found in M. McKinnell (1980).

## The Problem

In this work, the equation (7.7.1) is defined in the region $R=\{(x, t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$ with the initial condition,

$$
\begin{equation*}
u(x, 0)=f(x), \quad 0 \leqslant x \leqslant 1 \tag{7.7.3}
\end{equation*}
$$

where $f$ is a sufficiently smooth function on $[0,1]$ and the boundary conditions are such that,

$$
\begin{equation*}
u(x+1, t)=u(x, t) \tag{7.7.4}
\end{equation*}
$$

The solution of this initial-boundary value problem exists and is unique. (Sjoberg, 1970; see also Lax, 1968).

The range $[0,1]$ is now divided into $M$ subintervals of size $\Delta x$ and we consider intervals in the open $t$ direction of size $\Delta t$. Then, by using the following implicit difference approximation to (7.7.1) of the form,

$$
\begin{equation*}
\left.\left(I-\Delta t \delta D_{+} D_{-} D_{0}\right) u_{i, j+1}=\frac{2 \Delta t}{3}\left(u_{i, j} D_{0} u_{i, j}+D_{0} u_{i, j}^{2}\right)+\delta I+\Delta t \delta D_{+} D_{-} D_{0}\right) u_{i, j-1}, \tag{7.7.5}
\end{equation*}
$$

$i=1,2, \ldots, M$, where $D_{+}, D_{-}$and $D_{0}$ are the forward, backward and centraldifferences operator respectively (Sjoberg, 1969).

The use of the implicit difference scheme for the solution of equation (7.7.1) together with the equations (7.7.3)-(7.7.4) will result in a cyclic quindiagonal system of equations to be solved at each time step for the solution. Even though there are direct algorithms for solving this system such as Evans, D.J. (1980), an explicit scheme is normally more economical.

## The Method

The finite difference formula for solving the equation (7.7.1) at point ( $i, j+\frac{1}{2}$ ) can generally be given by

$$
\begin{gather*}
\frac{1}{\Delta t} \delta_{t} u_{i, j+\frac{1}{2}}=\frac{1}{2 \Delta x}\left(\frac{u_{i, j+1}+u_{i, j}}{2}\right)_{\left[\alpha_{1} \Delta_{x} u_{i, j+1}+\alpha_{2} \nabla_{x} u_{i, j}+\alpha_{1}^{\prime} \nabla_{x} u_{i, j+1}+\alpha_{2}^{\prime} \Delta_{x} u_{i, j}\right]+} \begin{array}{c}
\frac{\delta}{2(\Delta x)^{3}}\left[\theta_{1} \tilde{\delta}_{x}^{2} u_{i+1, j+1}-\theta_{2} \tilde{\delta}_{x}^{2} u_{i-1, j+1}^{+\theta_{1}^{\prime}} \tilde{\delta}_{x}^{2} u_{i+1, j}^{-\theta} \tilde{\delta}_{x}^{2} u_{i-1, j}\right]
\end{array}, l
\end{gather*}
$$

where $\tilde{\delta}, \Delta$ and $\nabla$ are the central-, forward- and backward-difference
operators respectively. The parameters $\theta_{1}, \theta_{2}, \theta_{1}^{\prime}, \theta_{2}^{\prime}, \alpha_{1}, \alpha_{2}, \alpha_{1}^{\prime}$ and $\alpha_{2}^{\prime}$ are constant parameters which are related to consistency relations and the order of the approximation. This approximation results in the system,

$$
\begin{align*}
& \frac{\delta \Delta t}{2(\Delta x)^{3}} \theta_{2} u_{i-2, j+1}+\left\{\frac{\alpha_{1}^{\prime} \Delta t}{4 \Delta x}\left(u_{i, j+1}+u_{i, j}\right)-\frac{\delta \Delta t}{(\Delta x)^{3}}\right\}_{2} u_{i-1, j+1} \\
& +\left\{1+\frac{\left(\alpha_{1}-\alpha_{1}^{\prime}\right)}{4 \Delta x}\left(u_{i, j+1}+u_{i, j}\right) \Delta t-\frac{\delta \Delta t}{2(\Delta x)^{3}}\left(\theta_{1}-\theta_{2}\right)\right\}_{i, j+1}+ \\
& \quad\left\{\frac{-\alpha_{1} \Delta t}{4 \Delta x}\left(u_{i, j+1}+u_{i, j}\right)+\frac{\delta \Delta t}{(\Delta x)^{3}} \ell\right\} u_{i+1, j+1}-\frac{\delta \Delta t}{2(\Delta x)^{3}} \theta_{1} u_{i+2, j+1}= \\
& -\frac{\delta \Delta t}{2(\Delta x)^{3}} \theta_{2}^{\prime} u_{i-2, j}+\left\{-\frac{\alpha_{2}}{4 \Delta x}\left(u_{i, j+1}+u_{i, j}\right) \Delta t+\frac{\delta \Delta t}{(\Delta x)^{3}} \theta_{2}^{\prime}\right\} u_{i-1, j}+ \\
& \left\{1+\frac{\Delta t}{4 \Delta x}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(u_{i, j+1}^{+u_{i, j}}\right)+\frac{\delta \Delta t}{2 \Delta x^{3}}\left(\theta_{1}^{\prime}-\theta_{2}^{\prime}\right)\right\} u_{i, j}+ \\
& \left\{\frac{\alpha_{2}^{\prime} \Delta t}{4 \Delta x}\left(u_{i, j+1}+u_{i, j}\right)-\frac{\theta_{1}^{\prime} \delta \Delta t}{(\Delta x)^{3}}\right\} u_{i+1, j}+\frac{\delta \Delta t}{2(\Delta x)^{3}} \theta_{1}^{\prime} u_{i+2, j} . \tag{7.7.6a}
\end{align*}
$$

The objective of this work is to derive methods of approximation which are semi-explicit and with improved stability conditions. Therefore, the following schemes are expected to serve this purpose:
(i) We set $\theta_{1}=\theta_{2}^{\prime}=1, \theta_{2}=\theta_{1}^{\prime}=0, \alpha_{1}=\alpha_{2}=1$ and $\alpha_{1}^{\prime}=\alpha_{2}^{\prime}=0$ which results in the equation,

$$
\begin{gather*}
\left\{1+p_{1}\left(u_{i, j+1}+u_{i, j}\right)-\frac{\delta p_{2}}{2}\right\} u_{i, j+1}+\left\{-p_{1}\left(u_{i, j+1}+u_{i, j}\right)+\delta p_{2}\right\} u_{i+1, j+1} \\
-\frac{\delta p_{2}}{2} u_{i+2, j+1}=- \\
\frac{\delta p_{2}}{2} u_{i-2, j}+\left\{-p_{1}\left(u_{i, j+1}+u_{i, j}\right)+\delta p_{2}\right\} u_{i-1, j}+  \tag{7.7.7}\\
+\left\{1+p_{1}\left(u_{i, j+1}+u_{i, j}\right)-\frac{\delta p_{2}}{2}\right\} u_{i, j},
\end{gather*}
$$

where $p_{1}=\frac{\Delta t}{4 \Delta x}$ and $p_{2}=\frac{\Delta t}{(\Delta x)^{3}}$.
(ii) Similarly, we can set $\theta_{1}=\theta_{2}^{\prime}=0, \theta_{2}=\theta_{1}^{\prime}=1, \alpha_{1}=\alpha_{2}=0$ and $\alpha_{1}^{\prime}=\alpha_{2}^{\prime}=1$ and obtain

$$
\begin{align*}
& \left\{1-p_{1}\left(u_{i, j+1}+u_{i, j}\right)+\frac{\delta p_{2}}{2}\right\} u_{i, j+1}+\left\{p_{1}\left(u_{i, j+1}+u_{i, j}\right)-\delta p_{2}\right\} u_{i-1, j+1} \\
& +\frac{\delta p_{2}}{2} u_{i-2, j+1}=\left\{1-p_{1}\left(u_{i, j+1}+u_{i, j}\right)+\frac{\delta p_{2}}{2}\right\} u_{i, j}+\left\{p_{1}\left(u_{i, j+1}+u_{i, j}\right)-\right. \\
& \left.\quad \delta p_{2}\right\} u_{i+1, j}+\frac{\delta p_{2}}{2} u_{i+2, j} \tag{7.7.8}
\end{align*}
$$

The equation (7.7.7) can then be written in the matrix form as

$$
\begin{equation*}
K_{1}^{T} \underline{u}_{j+1}=K_{1} \underline{u}_{j} \tag{7.7.9}
\end{equation*}
$$

where

$$
\begin{align*}
& K_{1}=\left[\begin{array}{ccccc}
a_{1}^{1} & & & & a_{3} \\
a_{2}^{1} & a_{1}^{2} & a_{2}^{m-1} \\
a_{3} & a_{2}^{2} & a_{1}^{3} & & a_{3} \\
& & \ldots & a_{3} & a_{2}^{m-3} \\
& & a_{1}^{m-2} & \\
& & & a_{3} & a_{2}^{m-2} \\
& & a_{1}^{m-1}
\end{array}\right],  \tag{7.7.10}\\
& \text { with } \\
& a_{1}=1+p_{1}\left(u_{i, j+1}+u_{i, j}\right)-\frac{\delta p_{2}}{2}, \\
& a_{2}^{i}=-p_{1}\left(u_{i, j+1}+u_{i, j}\right)+\delta p_{2} \text {, } \\
& a_{3}=-\frac{\delta p_{2}}{2}, \\
& \text { and } \quad u_{j+1}^{T}=\left[u_{1, j+1}, u_{2, j+1}, \cdots, u_{M-1, j+1}, u_{M, j+1}\right] \text {. }
\end{align*}
$$

On the other hand equation (7.7.8) can be written as

$$
\begin{equation*}
\mathrm{K}_{2 \underline{\mathrm{u}}}^{\mathrm{j}+1} \mathrm{~T}=\mathrm{K}_{2} \underline{u}_{\mathrm{j}} \tag{7.7.11}
\end{equation*}
$$

where
with $b_{1}^{i}=1-p_{1}\left(u_{i, j+1}+u_{i, j}\right)+\frac{\delta p_{2}}{2}$,

$$
b_{2}^{i}=p_{1}\left(u_{i, j+1}+u_{i, j}\right)-\delta p_{2}
$$

and $\quad b_{3}=\frac{\delta p_{2}}{2}$.
The systems (7.7.9) and (7.7.11) are non-linear and have to be solved
iteratively. A practically explicit method can be obtained if the matrices $\mathrm{K}_{1}^{\mathrm{T}}$ and $\mathrm{K}_{2}^{\mathrm{T}}$ are written respectively as

$$
\begin{equation*}
\mathrm{K}_{1}^{\mathrm{T}}=\mathrm{L}_{1}+\mathrm{U}_{1} \quad \text { and } \quad \mathrm{K}_{2}^{\mathrm{T}}=\mathrm{L}_{2}+\mathrm{U}_{2} \tag{7.7.13}
\end{equation*}
$$

where
and
and the iterative processes are implemented in the usual way. Hence, we obtain the standard iterative processes;

$$
\begin{equation*}
L_{1} \underline{u}_{j+1}^{(n+1)}=-U_{1} \underline{u}_{j+1}^{(n)}+K_{1} \underline{u}_{j} \tag{7.7.14}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{2}{\underset{\mathrm{u}}{\mathrm{j}}+1}_{(\mathrm{n}+1)}=-\mathrm{U}_{2{\underset{\mathrm{u}}{\mathrm{j}}}^{(\mathrm{n})}}+\mathrm{K}_{2} \underline{\mathrm{u}}_{\mathrm{j}} \tag{7.7.15}
\end{equation*}
$$

which are of the Gauss-Seidel type. Finally the SOR method of iteration can be defined by,

$$
\begin{equation*}
\hat{\underline{u}}_{j+1}^{(n+1)}=(1-\omega) \hat{\underline{u}}_{j+1}^{(n)}+\omega \underline{u}_{j+1}^{(n+1)}, \tag{7.7.16}
\end{equation*}
$$

where $\underline{u}_{j+1}^{(n+1)}$ are as obtained from either (7.7.14) or (7.7.15) and $\hat{u}_{j+1}$ is the required solution.

## Consistency

In this part, we will study the consistency of the approximations (7.7.7) and (7.7.8) to the original equation (7.7.1) at the point ( $\mathrm{i}, \mathrm{j}+\frac{1}{2}$ ). As this is only a brief introduction to this topic, the investigation on the generalised form (7.7.6) will not be carried out here and will be left for further work.

We now expand each of the terms in (7.7.7) and (7.7.8) involving $u$ in a Taylor series expansion about ( $i, j+\frac{1}{2}$ ) and assume the terms ( $u_{i, j+1}+u_{i, j}$ ) is approximated as $2 u_{i, j+\frac{1}{2}}+0\left(\Delta t^{2}\right)$. Both equations are found to approximate the original equation at that point with the truncation error of $0\left(\Delta x^{2}+\Delta t^{2}+\frac{\Delta t}{\Delta x}+\Delta x \Delta t\right)$. Therefore, the schemes (7.7.7) and (7.7.8) are consistent if $\Delta t \rightarrow 0$ faster than $\Delta x \rightarrow 0$.

## Stability

For the stability analysis we consider the matrices $K_{1}$ and $K_{2}$. Since these matrices vary with respect to time, then we need only consider the system at any one particular $j$. Since by definition the eigenvalues of $\left(\mathrm{K}_{1}^{\mathrm{T}}\right)^{-1} \mathrm{~K}_{1}$ and $\left(\mathrm{K}_{2}^{\mathrm{T}}\right)^{-1} \mathrm{~K}_{2}$ equal to unity then the system is neutrally stable at that particular $j$. Since this is true successively for $j=1,2, \ldots$, therefore the system is stable as j tends to infinity for all $\mathrm{r}>0$.

Also from the definition of the matrices $K_{1}$ and $K_{2}$ in (7.7.10) and (7.7.12) respectively, the convergence of the iterative systems (7.7.14) and (7.7.15) are obvious.

## Numerical Results

The method was tested for the problem (7.7.1) with the initial condition $\mathrm{f}(\mathrm{x})=\cos (2 \pi \mathrm{x})$ and the periodic boundary condition as defined in (7.7.4). However since the exact solution was not known only a comparison
of the solutions at various times with different values of $r$ (Tables (7.7.1)(7.7.3)) was conducted.

From the results it was found that for small values of $\delta$ and $t>0.4$, a substantially large number of iterations was required for the convergence of the scheme when a reasonable value of $\Delta t$ was used.

This short study has shown that the use of the unsymmetric formulae for this type of problem is feasible. The investigation for the GE class of methods is under way and will enable us to solve the KdV problem with a wider class of boundary conditions, i.e. Dirichlet and Neumann boundary conditions.
$\mathrm{t}=0.2, \delta=0.000484$

| x | $\mathrm{r}=0.4$ <br> $\Delta t=0.001$ <br> $\Delta \mathrm{x}=0.05$ | $\mathrm{r}=0.1$ <br> $\Delta t=0.001$ <br> $\Delta \mathrm{x}=0.1$ | $\mathrm{r}=1.0$ <br> $\Delta \mathrm{t}=0.01$ <br> $\mathrm{x}=0.1$ | $\mathrm{r}=0.5$ <br> $\Delta \mathrm{t}=0.005$ <br> $\Delta \mathrm{x}=0.1$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.0 | 1.088953 | 0.911448 | 0.937031 | 0.922577 |
| 0.1 | 0.302829 | 0.385939 | 0.382416 | 0.384489 |
| 0.2 | 0.052844 | 0.074303 | 0.073965 | 0.074134 |
| 0.3 | -0.275796 | -0.248238 | -0.248775 | -0.248513 |
| 0.4 | -0.580173 | -0.560399 | -0.561353 | -0.560804 |
| 0.5 | -0.838216 | -0.820093 | -0.824968 | -0.822339 |
| 0.6 | -0.981140 | -1.081161 | -1.079715 | -1.080679 |
| 0.7 | -0.735449 | -0.676847 | -0.668568 | -0.672734 |
| 0.8 | 1.151313 | 0.934798 | 0.940470 | 0.937356 |
| 0.9 | 0.746821 | 0.501937 | 0.490732 | 0.496906 |

TABLE (7.7.1)
$t=0.2, \delta=0.0001$

| x | $\mathrm{r}=0.4$ <br> $\Delta \mathrm{t}=0.001$ <br> $\Delta \mathrm{x}=0.05$ | $\mathrm{r}=0.1$ <br> $\Delta \mathrm{t}=0.001$ <br> $\Delta \mathrm{x}=0.1$ | $\mathrm{r}=1.0$ <br> $\Delta \mathrm{t}=0.01$ <br> $\Delta \mathrm{x}=0.1$ | $\mathrm{r}=0.5$ <br> $\Delta \mathrm{t}=0.005$ <br> $\Delta \mathrm{x}=0.1$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.0 | 1.300519 | 0.930576 | 0.960374 | 0.943509 |
| 0.1 | 0.405001 | 0.365017 | 0.359306 | 0.362628 |
| 0.2 | 0.056461 | 0.075840 | 0.076520 | 0.076095 |
| 0.3 | -0.265641 | -0.250932 | -0.251326 | -0.251142 |
| 0.4 | -0.572658 | -0.563516 | -0.564638 | -0.564002 |
| 0.5 | -0.837582 | -0.832208 | -0.837697 | -0.834775 |
| 0.6 | -0.994970 | -1.116117 | -1.113290 | -1.114876 |
| 0.7 | -0.695014 | -0.585910 | -0.578308 | -0.582210 |
| 0.8 | 0.959302 | 0.907028 | 0.911208 | 0.908853 |
| 0.9 | 0.535578 | 0.457211 | 0.445135 | 0.451788 |

TABLE (7.7.2)
$t=0.3, \delta=0.001089$

| $\mathbf{x}$ | $\mathrm{r}=0.1$ <br> $\Delta t=0.001$ <br> $\Delta x=0.1$ | $\mathrm{r}=1.0$ <br> $\Delta t=0.01$ <br> $\Delta x=0.1$ | $\mathrm{r}=0.5$ <br> $\Delta t=0.005$ <br> $\Delta \mathrm{x}=0.1$ |
| :---: | :---: | :---: | :---: |
| 0.0 | 0.775966 | 0.797780 | 0.785533 |
| 0.1 | 0.317351 | 0.316643 | 0.317101 |
| 0.2 | 0.028081 | 0.025965 | 0.027148 |
| 0.3 | -0.220497 | -0.221433 | -0.220923 |
| 0.4 | -0.472996 | -0.473605 | -0.473449 |
| 0.5 | -0.726348 | -0.725103 | -0.725006 |
| 0.6 | -0.719630 | -0.737190 | -0.729384 |
| 0.7 | -1.973267 | -1.951119 | -1.956457 |
| 0.8 | 1.862570 | 1.908264 | 1.882759 |
| 0.9 | 0.462465 | 0.450942 | 0.457431 |

TABLE (7.7.3)

### 7.8 CONCLUSIVE REMARK

In the solution to non-linear problems our work has revealed that there are still many unresolved issues concerning the use of the general approximation. In particular, the relation of $\theta^{\prime} s$ and $\alpha^{\prime} s$ with the consistency and stability of the approximation.

The semi-explicit alternating direction formulae of 'ladder-step' form appears to show its superiority in solving the self-adjoint form of the parabolic equation. However, this does not mean that (D)AGE is unacceptable for this class of problems.

For the Burger's equation, the solution obtained from the (D)AGE scheme seems to fit with the exact solution very well. However, at small values of $t$, oscillations in the solution occur and appear to be unavoidable. Fig. (7.5.1) shows this behaviour at $t=0.2$ but as $t$ increases this phenomena slowly decays.

Since the KdV problem was only briefly investigated no definitive results can be established. The fact that the numerical solutions obtained with varying $r$ are in good qualitative agreement is a reassuring sign of consistency, even the values of $\Delta x$ are comparatively large (see Sjoberg, 1969).

In all the cases considered, the stability analysis for most of $G E$ types of schemes were found to be intractable.

Finally, for strongly non-linear parabolic equations the need for an explicit type of approximation is paramount as sometimes the number of iterations required for convergence is prohibitive.

Chapter Eight

## THE FINITE ELEMENT METHOD AND

BOUNDARY VALUE TECHNIQUES FOR THE SOLUTION OF

## THE DIFFUSION-CONVECTION EQUATION

### 8.1 INTRODUCTION

The study of the finite element method for solving parabolic p.d.e.'s has attracted much attention due to its wide ranging and f1exible applications. In this chapter, the work on a finite element method for the solution of the diffusion convection equation,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\varepsilon \frac{\partial^{2} u}{\partial x^{2}}-k \frac{\partial u}{\partial x} \tag{8.1.1}
\end{equation*}
$$

will be presented.
The equation (8.1.1) will be considered in an open-rectangular domain $\{(x, t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$ with the initial condition,

$$
\begin{equation*}
u(x, 0)=f(x), \quad 0 \leqslant x \leqslant 1, \tag{8.1.1a}
\end{equation*}
$$

and boundary conditions,

$$
\begin{align*}
& u(0, t)=g_{1}(t),  \tag{8.1.1b}\\
& u(1, t)=g_{2}(t) . \tag{8.1.1c}
\end{align*}
$$

The formulation of a centred nine-point finite element equation for the diffusion-convection equation is also presented. This equation is then used in a manner similar to Greenspan (1974) whereby regarding the problem as a boundary-value problem leads to a large linear system which is usually solved by an appropriate iterative method.

### 8.2 FINITE ELEMENT FORMULATION

Since the majority of problems in this topic arise from parabolic equations with initial boundary value conditions then the region of interest is usually rectangular. Hence for this formulation, the most suitable finite element discretisation i.e., the rectangular element (Fig. (8.2.1)) is chosen. The open rectangle $R=\{(x, t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$ is now imagined to be divided into small rectangular elements.


FIGURE (8.2.1)
The approximate solution $\phi$ in every element is now assumed to be represented by a linear combination of the solution at every node-point of the element. A global coordinate system ( $x, t$ ) is assumed to be transformed to a local coordinate $\operatorname{system}(\xi, n)$. For every element, these are related by the formulae (Hinton and Owen, 1979, p.245).

$$
\begin{equation*}
\xi=\frac{2(x-\bar{x})}{\Delta x}, \eta=\frac{4(t-\bar{t})}{\Delta t} \tag{8.2.1}
\end{equation*}
$$

with ( $\bar{x}, \bar{t}$ ) corresponding to the origin of the local coordinate system. Therefore, a suitable linear combination is chosen, viz.

$$
\begin{equation*}
\phi \simeq \sum_{i=1}^{4} N_{i} \phi_{i} \tag{8.2.2}
\end{equation*}
$$

where $N_{i}$ is the $i$ th node basis function and the $N_{i}, i=1,2,3,4$ are given by

$$
\begin{array}{ll}
N_{1}=\frac{1}{4}(1+\xi)(1+n), & N_{2}=\frac{1}{4}(1-\xi)(1+n), \\
N_{3}=\frac{1}{4}(1-\xi)(1-n), & N_{4}=\frac{1}{4}(1+\xi)(1-n),
\end{array}
$$

with $\phi_{i}$ the value of $\phi$ at the $i$ th node.

In every element let the residual $R$ be defined by

$$
\begin{equation*}
\mathrm{R}=\mathrm{L}(\phi), \tag{8.2.4}
\end{equation*}
$$

where $L$ is the given operator such that

$$
\begin{equation*}
\mathrm{L}(\mathrm{u}) \equiv 0 . \tag{8.2.5}
\end{equation*}
$$

Now if $\phi$ is an exact approximation to the exact solution of (8.1.1), the residual $R$ would vanish. In the method of weighted residuals, a weighting function $w$ is chosen such that the residual is forced to zero, in an average sense, i.e.,

$$
\begin{equation*}
\iint_{R} L(\phi) w(x, t) d x d t=0 \tag{8.2.6}
\end{equation*}
$$

Therefore integrating (8.2.6) by parts gives,

$$
\begin{align*}
\iint_{R}\left(\frac{\partial \phi}{\partial t}\right. & \left.-\varepsilon \frac{\partial^{2} \phi}{\partial x^{2}}+k \frac{\partial \phi}{\partial x}\right) w(x, t) d x d t \\
& =\int_{R} \int_{R} \frac{\partial \phi}{\partial t} w(x, t) d x d t-\varepsilon \int_{R} \frac{\partial}{\partial x}\left(w \frac{\partial \phi}{\partial x}\right) d x d t \\
& +\varepsilon \int_{R} \int \frac{\partial w}{\partial x} \frac{\partial \phi}{\partial x} d x d t+k \iint \frac{\partial \phi}{\partial x} w d x d t=0 . \tag{8.2.7}
\end{align*}
$$

If the integration is assumed to be in the closed region $R_{c}$, with the boundary $\partial R_{c}$, then by using Green's theorem (K. Rektory's, 1969), we have
i.e.

$$
\begin{aligned}
& \iint_{R_{c}} \frac{\partial \phi}{\partial t} w d x d t-\varepsilon \oint_{\partial R_{c}} w \frac{\partial \phi}{\partial x} d t+\varepsilon \int_{R_{c}} \int_{w} \frac{\partial w}{\partial x} \frac{\partial \phi}{\partial x} d x d t \\
&+k \int_{R} \int_{0} \frac{\partial \phi}{\partial x} w d x d t=0,
\end{aligned}
$$

$$
\begin{equation*}
\int_{R} \int_{c} \frac{\partial \phi}{\partial t} w d x d t+\varepsilon \int_{R_{c}}^{\int_{c}} \frac{\partial w}{\partial x} \frac{\partial \phi}{\partial x} d x d t+k \int_{R_{c}} \int_{c} \frac{\partial \phi}{\partial x} w d x d t=0 . \tag{8.2.8}
\end{equation*}
$$

Since this result is valid for every closed region $R_{c}$, the integration of (8.2.8) can take place in every element. Thus if we let $w \not \mathrm{~N}_{\mathrm{i}}, i=1,2,3,4$, then (8.2.8) can be written as

$$
\begin{equation*}
\int_{-1}^{1} \int_{-1}^{1}\left[\frac{\partial \phi}{\partial t} N_{i}+\varepsilon \frac{\partial N_{i}}{\partial x} \frac{\partial \phi}{\partial x}+k \frac{\partial \phi}{\partial x} N_{i}\right]|J| \mathrm{d} \xi \mathrm{~d} \eta=0, \tag{8.2.9}
\end{equation*}
$$

where

$$
\begin{align*}
& \frac{\partial \phi}{\partial t}=\sum_{i=1}^{4} \phi_{i} \frac{\partial N_{i}}{\partial \eta} \frac{\partial \eta}{\partial t}=\frac{4}{\Delta t} \sum_{i=1}^{4} \phi_{i} \frac{\partial N_{i}}{\partial \eta}  \tag{8.2.10}\\
& \frac{\partial \phi}{\partial x}=\sum_{i=1}^{4} \phi_{i} \frac{\partial N_{i}}{\partial \xi} \frac{\partial \xi}{\partial x}=\frac{2}{\Delta x} \sum_{i=1}^{4} \phi_{i} \frac{\partial N_{i}}{\partial \xi} \tag{8.2.11}
\end{align*}
$$

$$
\left.\begin{array}{ll}
\frac{\partial N_{1}}{\partial \xi}=\frac{1}{4}(1+\eta), & \frac{\partial N_{1}}{\partial \eta}=\frac{1}{4}(1+\xi) \\
\frac{\partial N_{2}}{\partial \xi}=-\frac{1}{4}(1+\eta), & \frac{\partial N_{2}}{\partial \eta}=\frac{1}{4}(1-\xi)  \tag{8.2.13}\\
\frac{\partial N_{3}}{\partial \xi}=-\frac{1}{4}(1-\eta), & \frac{\partial N_{3}}{\partial \eta}=-\frac{1}{4}(1-\xi) \\
\frac{\partial N_{4}}{\partial \xi}=\frac{1}{4}(1-\eta), & \frac{\partial N_{4}}{\partial \eta}=-\frac{1}{4}(1+\xi)
\end{array}\right\}
$$

$$
|J|=\left|\begin{array}{ll}
\frac{\partial x}{\partial \xi} & \frac{\partial t}{\partial \xi}  \tag{8.2.12}\\
\frac{\partial x}{\partial \eta} & \frac{\partial t}{\partial \eta}
\end{array}\right|=\frac{\Delta x \Delta t}{8}
$$

Let us now consider a group of four elements as given in Fig.(8.2.2). In these elements, the integration (8.2.9) are performed such that the $N_{i}$ are chosen to correspond with ith element, $i=1,2,3,4$.


For the 1st element, the integration (8.2.9) leads to

$$
\begin{equation*}
A_{1}^{(1)} \phi_{1}^{(1)}+A_{2}^{(1)} \phi_{2}^{(1)}+A_{3}^{(1)} \phi_{3}^{(1)}+A_{4}^{(1)} \phi_{4}^{(1)}=0 \tag{8.2.14}
\end{equation*}
$$

where,

$$
\left.\begin{array}{l}
A_{1}^{(1)}=\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{12} \\
A_{2}^{(1)}=\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}-\frac{k \Delta t}{12} \\
A_{3}^{(1)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{12} \frac{\Delta t}{\Delta x}-\frac{k \Delta t}{24}  \tag{8.2.15}\\
A_{4}^{(1)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{12} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{24}
\end{array}\right\} \text { for element } 1
$$

Similarly for the second, third and fourth elements, the integration (8.2.9) results in similar finite element equations of the form:

$$
\begin{equation*}
\sum_{i=1}^{4} A_{i}^{(k)} \phi_{i}^{(k)}=0, k=2,3,4 \tag{8.2.16}
\end{equation*}
$$

where

$$
\left.\begin{array}{l}
A_{1}^{(2)}=\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{12}, \\
A_{2}^{(2)}=\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{12}, \\
A_{3}^{(2)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{12} \frac{\Delta t}{\Delta x}-\frac{k \Delta t}{24}, \\
A_{4}^{(2)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{12} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{24},
\end{array}\right\} \text { for element } 2
$$

and finally,

$$
\left.\begin{array}{l}
A_{1}^{(4)}=\frac{\Delta x}{6}+\frac{\varepsilon}{12} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{24},  \tag{8.2.16c}\\
A_{2}^{(4)}=\frac{\Delta x}{12}-\frac{\varepsilon}{12} \frac{\Delta t}{\Delta x}-\frac{k \Delta t}{24}, \\
A_{3}^{(4)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}-\frac{k \Delta t}{12}, \\
A_{4}^{(4)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{12},
\end{array}\right\} \text { for element } 4
$$

The totality of these finite element equations for the four elements is now given in molecular form by Fig. (8.2.3). If these elements are suitably chosen to be within the $j$ th, $\left(j+\frac{1}{2}\right)$ th and $(j+1)$ th time-levels and ( $i-1$ ) th, ith and ( $i+1$ ) th vertical space-lines, we can obtain from (8.2.9) the following relationship which can be regarded as a grouped finite element equation centered at point (i, $j+\frac{1}{2}$ ).


FIGURE (8.2.3): Group Finite Element Equation

$$
\begin{aligned}
& \left(\frac{1}{12}-\frac{r \varepsilon}{12}-\frac{k r \Delta x}{24}\right) \phi_{i-1, j+1}+\left(\frac{1}{3}+\frac{r \varepsilon}{6}\right) \phi_{i, j+1}+\left(\frac{1}{12}-\frac{r \varepsilon}{12}+\frac{k r \Delta x}{24}\right) \phi_{i+1, j+1} \\
+ & \left(-\frac{r \varepsilon}{3}-\frac{k r \Delta x}{6}\right) \phi_{i-1, j+\frac{1}{2}}+\frac{2}{3} r \varepsilon \phi_{i, j+\frac{1}{2}}+\left(-\frac{r \varepsilon}{3}+\frac{k r \Delta x}{6}\right) \phi_{i+1, j+\frac{1}{2}} \\
+ & \left(-\frac{1}{12}-\frac{r \varepsilon}{12}-\frac{k r \Delta x}{24}\right) \phi_{i-1, j}+\left(-\frac{1}{3}+\frac{r \varepsilon}{6}\right) \phi_{i, j}+\left(-\frac{1}{12}-\frac{r \varepsilon}{12}+\frac{k r \Delta x}{24}\right)_{\phi_{i+1, j}}=0,
\end{aligned}
$$

$$
\begin{equation*}
r=\Delta t /(\Delta x)^{2} \tag{8.2.17}
\end{equation*}
$$

The formulae (8.2.17) will be meaningful if the components on the $\left(j+\frac{1}{2}\right)$ th line are weighted with factors $\theta$ and $(1-\theta)$ to $(j+1)$ th and $j$ th time-levels respectively which will then produce the six-point formulae, i.e.,

$$
\begin{align*}
& {\left[\frac{1}{12}-\frac{r \varepsilon}{12}(1+4 \theta)-\frac{k r \Delta x}{24}(1+4 \theta)\right] \phi_{i-1, j+1}+\left[\frac{1}{3}+\frac{r \varepsilon}{6}(1+4 \theta)\right] \phi_{i, j+1}} \\
& \quad+\left[\frac{1}{12}-\frac{r \varepsilon}{12}(1+4 \theta)+\frac{k r \Delta x}{24}(1+4 \theta)\right] \phi_{i+1, j+1}+\left[-\frac{1}{12}-\frac{r \varepsilon}{12}(5-4 \theta)-\frac{k r \Delta x}{24}(5-4 \theta)\right] \phi_{i-1, j} \\
& \quad+\left[-\frac{1}{3}+\frac{r \varepsilon}{6}(5-4 \theta)\right] \phi_{i, j}+\left[-\frac{1}{12}-\frac{r \varepsilon}{12}(5-4 \theta)+\frac{k r \Delta x}{24}(5-4 \theta)\right] \phi_{i+1, j}=0 . \tag{8.2.18}
\end{align*}
$$

Therefore the six-point formulae (8.2.18) which is obtained from the integration (8.2.9) in any four adjacent elements can be regarded as a finite element system for approximating the solution of (8.1.1). This system will lead to a tridiagonal system of equations which is easily solved by any of the standard algorithms for solving banded linear systems.

By a Taylor's series expansion, the formula (8.2.18) can be shown to approximate the equation (8.1.1) at point ( $i, j+\frac{1}{2}$ ) with the error term,

$$
\begin{align*}
T_{(8.2 .18)}=\frac{\mathrm{kr} \mathrm{\Delta x}}{2} & (2 \theta-1) \phi_{x t}+\frac{\Delta t^{2}}{24} \phi_{t t t}+\frac{k \Delta x^{2}}{6} \phi_{x x x}+\frac{\Delta t^{2}}{8} k \phi_{x t t}+ \\
& +\frac{\Delta x^{2}}{6}[1+2 r \varepsilon(1-2 \theta)] \phi_{x x t}+0\left(\Delta x^{\alpha} \Delta t^{\alpha_{2}}\right) \tag{8.2.19}
\end{align*}
$$

$0 \leqslant \alpha_{1}, \alpha_{2} \leqslant 4, \alpha_{1}+\alpha_{2}=4$.
Thus, we have the result,

$$
\begin{equation*}
T_{(8.2 .18)}=O\left(\Delta t+(\Delta x)^{2}\right), \tag{8.2.19a}
\end{equation*}
$$

whilst for $\theta=\frac{1}{2}$ or $r=1$, it can be easily seen that

$$
\begin{equation*}
T_{(8.2 .18)}=O\left((\Delta t)^{2}+(\Delta x)^{2}\right) . \tag{8.2.20}
\end{equation*}
$$

For the pure diffusion equation (i.e. $\varepsilon=1, \mathrm{k}=0$ ), the equation (8.2.18) reduces to

$$
\begin{align*}
& \frac{1}{12}[1-r(1+4 \theta)] \phi_{i-1, j+1}+\frac{1}{6}[2+r(1+4 \theta)] \phi_{i, j+1}+\frac{1}{12}[1-r(1+4 \theta)] \phi_{i+1, j+1} \\
& \quad=\frac{1}{12}[1+r(5-4 \theta)] \phi_{i-1, j}+\frac{1}{6}[2-r(5-4 \theta)] \phi_{i, j}+\frac{1}{12}[1+r(5-4 \theta)] \phi_{i+1, j} \tag{8.2.21}
\end{align*}
$$

with the corresponding truncation error term given by,

$$
\begin{align*}
T_{(8.2 .21)} & =\frac{\Delta t^{2}}{24} \phi_{t t t}+\frac{\Delta x^{2}}{6}[1+2 r(1-2 \theta)] \phi_{x x t}+o\left(\Delta x^{\alpha_{1}} \Delta t^{\alpha_{2}-1}\right) .  \tag{8.2.22}\\
& =0\left(\Delta x^{2}+\Delta t^{2}\right)
\end{align*}
$$

Again, we have that for $r=1, \theta=3 / 4$,

$$
\begin{equation*}
T_{(8.2 .21)}=O\left(\Delta t^{2}+\Delta x^{4}\right) \tag{8.2.23}
\end{equation*}
$$

To analyse the stability of equation (8.2.18) then rewriting in the matrix form,

$$
\begin{align*}
& {\left[\frac{1}{3} \mathrm{I}+\frac{\mathrm{r} \mathrm{\varepsilon}}{12}(1+4 \theta) \mathrm{T}_{\mathrm{M}-1}+\frac{\mathrm{kr} \Delta \mathrm{x}}{24}(1+4 \theta) \mathrm{S}_{\mathrm{M}-1}+\frac{1}{12} \mathrm{~V}_{\mathrm{M}-1}\right]_{-j+1}=} \\
& {\left[\frac{1}{3} \mathrm{I}+\frac{1}{12} \mathrm{~V}_{\mathrm{M}-1}-\frac{\mathrm{r} \varepsilon}{12}(5-4 \theta) \mathrm{T}_{\mathrm{M}-1}-\frac{\mathrm{kr} \Delta \mathrm{x}}{24}(5-4 \theta) \mathrm{S}_{\mathrm{M}-1}\right]_{\Phi_{j}}+\underline{b}_{1}} \tag{8.2.24}
\end{align*}
$$

and the equation (8.2.21) as

$$
\begin{equation*}
\left[\frac{1}{3} I+\frac{r}{12}(1+4 \theta) T_{M-1}+\frac{1}{12} V_{M-1}\right] \phi_{j+1}=\left[\frac{1}{3} I-\frac{r}{12}(5-4 \theta) T_{M-1}+\frac{1}{12} V_{M-1}\right] \phi_{j}+\underline{b}_{2}, \tag{8.2.25}
\end{equation*}
$$

where,
$T_{M-1}=\left[\begin{array}{lllll}2 & -1 & & 0 \\ -1 & 2 & -1 & & \\ & 0 & & -1 & 2 \\ & & & -1 & 2\end{array}\right]$,

$V_{M-1}=\left[\begin{array}{lllll}0 & 1 & & & \\ 1 & 0 & 1, & 0 & \\ & & \ddots & \ddots & \\ & 0 & \ddots & 0 & 1 \\ & & & 1 & 0\end{array}\right]$,
$\Phi_{j}^{T}=\left[\phi_{1, j}, \phi_{2, j}, \ldots, \phi_{M-2, j}, \phi_{M-1, j}\right]$ and $\underline{b}_{1}, \underline{b}_{2}$ are the column vector associated with the boundary values.

Since $T$ and $V$ are commutative, the modulus of the eigenvalues of the matrix of coefficients of the system (8.2.25) is given by,

$$
\begin{equation*}
\left|\frac{\frac{1}{3}-\frac{\mathrm{r}}{12}(5-4 \theta) \lambda_{\mathrm{s}, \mathrm{~T}_{\mathrm{M}-1}}+\frac{1}{12} \lambda_{\mathrm{s}, \mathrm{~V}_{\mathrm{M}-1}}}{\frac{1}{3}+\frac{\mathrm{r}}{12}(1+4 \theta) \lambda_{\mathrm{s}, \mathrm{~T}_{\mathrm{M}-1}}+\frac{1}{12} \lambda_{\mathrm{s}, \mathrm{~V}_{\mathrm{M}-1}}}\right| \quad, \mathrm{s}=1,2, \ldots, \mathrm{M}-1 \tag{8.2.29}
\end{equation*}
$$

where $\lambda_{s, T_{M-1}}$ and $\lambda_{s, V_{M-1}}$ are the sth eigenvalues of the matrices $T_{M-1}$ and $V_{M-1}$ respectively. These eigenvalues are given by

$$
\begin{equation*}
\lambda_{s, T_{M-1}}=2+2 \sin \frac{s \pi}{M}=4-4 \sin ^{2} \frac{s \pi}{2 M}, \tag{8.2.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{s, V_{M-1}}=2 \sin \frac{s \pi}{M}=2-4 \sin ^{2} \frac{s \pi}{2 M} \tag{8.2.31}
\end{equation*}
$$

Thus, from (8.2.29)-(8.2.31) it can be easily shown that the scheme (8.2.25) is unconditionally stable for $\frac{1}{2} \leqslant \theta \leqslant 1$ and conditionally stable if

$$
\begin{equation*}
r \leqslant \frac{1}{4(1-2 \theta)} \quad \text { for } 0 \leqslant \theta \leqslant \frac{1}{2} . \tag{8.2.32}
\end{equation*}
$$

For the scheme (8.2.24) of the diffusion-convection equation, since $S$ is not commutative with $T$ and $V$, a theoretical investigation on the stability condition is rather difficult to accomplish. However, from the stability of the scheme (8.2.25) and the results obtained from the numerical experiments (see Section 8.7) improved stability range for the scheme (8.2.24) is indicated.

### 8.3 THE BOUNDARY VALUE TECHNIQUE

## Introduction:

So far we have been studying the solution of parabolic p.d.e.'s by means of step-by-step procedures as an initial value problem. In the case of an elliptic problem, however, boundary vatue procedures are the usual ones to use. Here, the technique constitutes a system of algebraic equations to be solved in as many unknowns as there are interior node points in the region of interest. Clearly, one disadvantage which arises in this case is the amount of computer storage required. However the recent development of the high-speed computer with a large storage facility has, to a certain extent, overcome this problem.

Recently, attention has been focused on converting a parabolic equation with known boundary conditions to an equivalent elliptic form and to apply boundary value techniques rather than the usual step-by-step procedures which are associated with initial value problems. The work related to this technique can be found in Greenspan, D. (1967) and (1974), Carasso, A. (1968), Carasso, A. and Parter S.V. (1970) and Evans, D.J. (1979). Implementation of the B.V.T. in solving some parabolic problem can be found in Danaee, A. (1980) and Danaee, A. and Evans D.J. (1981).

The aim of this work is to apply this strategy to a finite element formulation for solving the diffusion-convection equation. In the following example we illustrate the technique for a simple finite difference formulation of the diffusion convection equation.

## Procedure:

Consider the parabolic initial-boundary value problem,

$$
\begin{equation*}
\frac{\partial \phi}{\partial t}=\varepsilon \frac{\partial^{2} \phi}{\partial x^{2}}+f\left(x, t, \phi, \frac{\partial \phi}{\partial x}\right), \quad \varepsilon>0 \tag{8.3.1}
\end{equation*}
$$

to be solved over the region $R=\left\{(x, t): 0 \leqslant x \leqslant 1,0 \leqslant t \leqslant T_{\infty}\right\}$, where the solution
$\phi(x, t)$ of (8.3.1) attains a known steady-state value $\tilde{\phi}(x)$ as $t \rightarrow \infty$. The appropriate initial and boundary conditions are given by,
and

$$
\left.\begin{array}{ll}
\phi(x, 0)=f_{0}(x) & , 0 \leqslant x \leqslant 1, t=0  \tag{8.3.2}\\
\phi(0, t)=\phi_{1}(t) & , t>0 \\
\phi(1, t)=\phi_{2}(t), & t>0
\end{array}\right\}
$$

The approach proposed by Greenspan (1967) is as follows.
Assume that the problem (8.3.1) is approximated by a finite-difference formulae in the region $R$. Interpret these formulae as a system of algebraic equations for the approximation to $\phi(x, t)$ at all the interior mesh points of R. We now solve this system, subject to the given initial and boundary conditions and given data $\phi=\widetilde{\phi}(x)$ at $t=T_{\infty}$. If the scheme is consistent with the differential equation, then it is possible that the solution of the finite difference equations is an approximation to $\phi(x, t)$ at the mesh points (Carasso, A. 1968, p.3).

Greenspan chose the well-known unstable Richardson scheme to apply the BVT. For this application the formula does not suffer from the well known instability when used to solve a linear parabolic problem with timeindependent coefficients.

The complete analysis of the technique was given later by Carasso, A. (1968). He discussed the convergence of the technique and evaluated the rate of convergence of the method for linear problems with timeindependent coefficients.

Carasso, A. (1968) also shows the failure of the BVT to solve the parabolic equation,

$$
\frac{\partial \phi}{\partial t}=\frac{\partial^{2} \phi}{\partial x^{2}}+\pi^{2} \phi+\sin n x . \cos t, 0<x<1, t>0
$$

which had been attempted by Greenspan earlier.
Finally, as the technique normally leads to a large linear or nonlinear system of equations, iterative methods of solution to the problem are essential. Various iterative procedures using the BVT can be found in Danaee, A., (1980).

### 8.4 ITERATIVE METHODS OF SOLUTION

Consider the problem specified by (8.3.1)-(8.3.2) and use a five-point finite difference approximation to represent (8.3.1)-(8.3.2) as follows:

$$
\begin{align*}
\phi_{i, j+1}-\phi_{i, j-1}= & 2 r \varepsilon\left(\phi_{i-1, j}-2 \phi_{i, j}+\phi_{i+1, j}\right) \\
& +2 \Delta t . f\left(x, t, \phi_{i, j}, \frac{\phi_{i+1, j}-\phi_{i-1, j}}{2 \Delta x}\right) \tag{8.4.1}
\end{align*}
$$

where $r=\Delta t /(\Delta x)^{2}, i=1,2, \ldots, M-1, j=1,2, \ldots, N-1$.
We now consider the case when $f$ is a linear function, in particular when $\mathrm{f}=-\mathrm{k} \frac{\partial \phi}{\partial \mathrm{x}}, \mathrm{k}>0$ and rewrite (8.4.1) accordingly. Then using a row-wise ordering of the mesh points in the region $R$, the totality of the difference equations (8.4.1) produced in this way yields an ( $\mathrm{N}-1$ ) $\times(\mathrm{N}-1)$ block linear system of the form,

$$
\begin{equation*}
\mathrm{L} \phi=\underline{\mathrm{b}}, \tag{8.4.2}
\end{equation*}
$$

where
with


$$
C_{i}=-I_{(M-1) \times(M-1)}, \quad A_{i}=I_{(M-1) \times(M-1)}, i=1,2, \ldots, N-1
$$

and $\underline{b}$ is obtained by inserting the known boundary values when applied to equation (8.4.1).

Similarly, a column-wise ordering of the mesh points in the region $R$
leads to a $(M-1) \times(M-1)$ block linear system,

$$
\tilde{L}_{\phi}=\tilde{\underline{b}}
$$

where,

$$
\widetilde{L}=\left[\begin{array}{llllll}
\tilde{B}_{1} & \tilde{C}_{1} & & & &  \tag{8.4.6}\\
\tilde{A}_{2} & \tilde{B}_{2} & \tilde{C}_{2} & & \\
& & & & \\
& & & & & \\
& & & \tilde{A}_{M-3} & \widetilde{B}_{M-2} & \widetilde{C}_{M-2} \\
& & & & \tilde{A}_{M-2} & \widetilde{B}_{M-1}
\end{array}\right]_{(M-1) \times(M-1)}=\tilde{A}+\widetilde{B}+\widetilde{C}
$$

with
where $I$ is the unit matrix of order $(N-1) \times(N-1)$.
From the block tridiagonal structure of the matrices $L$ and $\tilde{L}$, we can consider block iterative schemes for the solutions of (8.4.2) and (8.4.5) respectively. Here each block of unknowns consists of all the points $u_{i, j}$ in a row (for (8.4.2)) or a column (for (8.4.5)) of the grid.

Then, the block Jacobi method for (8.4.2) is given by,

$$
\begin{equation*}
B_{\phi}(\mathrm{n}+1)=-(\mathrm{A}+\mathrm{C}) \underline{\phi}^{\mathrm{n}}+\underline{\mathrm{b}}, \tag{8.4.8}
\end{equation*}
$$

and the block successive overrelaxation method by,

$$
(B+\omega A) \underline{\phi}^{(n+1)}=-[\omega C+(\omega-1) B] \underline{\phi}^{(n)}+\omega \underline{b},
$$

or

$$
\begin{equation*}
\Phi^{(n+1)}=-(B+\omega A)^{-1}\{\omega C+(\omega-1) B\} \Phi^{(n)}+\omega(B+\omega A)^{-1} \underline{b}, \tag{8.4.9}
\end{equation*}
$$

where the subscript $n$ denotes the iteration cycle and $\omega$ the block overrelaxation parameter.

For the convergence of the block Jacobi method we require $\rho\left(-B^{-1}(A+C)\right) \leqslant 1$. From (8.4.3) and (8.4.4) the eigenvalues of $\left(-B^{-1}(A+C)\right)$ are given by

$$
\begin{equation*}
\mu_{p, q}=\frac{i \cos \frac{q \pi}{N}}{r\left[-\sqrt{(2 \varepsilon)^{2}-(k \Delta x)^{2}} \cos \frac{p \pi}{M}+2 \varepsilon\right]}, \quad q=1(1) N-1, \tag{8.4.10}
\end{equation*}
$$

Therefore the maximum eigenvalue

$$
\begin{equation*}
\mu_{\max }=\frac{i \cos \frac{\pi}{N}}{r\left[2 \varepsilon-\sqrt{(2 \varepsilon)^{2}-(k \Delta x)^{2}} \cos \pi / M\right]} \tag{8.4.11}
\end{equation*}
$$

Alternatively we have for the system (8.4.5), the block Jacobi simultaneous method is given by

$$
\begin{equation*}
{\widetilde{\mathrm{B}} \phi^{(\mathrm{n}+1)}}^{(\tilde{\mathrm{A}}+\tilde{\mathrm{C}}) \phi+\widetilde{\mathrm{b}},} \tag{8.4.12}
\end{equation*}
$$

and block successive overrelaxation method by,

$$
\begin{equation*}
(\tilde{\mathrm{B}}+\omega \tilde{\mathrm{A}}) \Phi^{(\mathrm{n}+1)}=-[\omega \tilde{\mathrm{C}}+(\omega-1) \tilde{B}] \underline{+}+\omega \tilde{\underline{b}} . \tag{8.4.13}
\end{equation*}
$$

From (8.4.6)-(8.4.7) the eigenvalues of $\left[\tilde{-B}^{-1}(\tilde{A}+\widetilde{C})\right]$ are given by

$$
\begin{equation*}
\mu_{p, q}=\frac{\left[(2 \varepsilon r)^{2}-(k r \Delta x)^{2}\right]^{\frac{1}{2}} \cos \frac{p \pi}{M}}{2 r-i \cos \frac{q \pi}{N}} \tag{8.4.14}
\end{equation*}
$$

The maximum eigenvalue is then given by

$$
\begin{equation*}
\tilde{\mu}_{\max }=\frac{\left[(2 \varepsilon r)^{2}-(k r \Delta x)^{2}\right]^{\frac{1}{2}} \cos \pi / M}{2 r-i \cos \pi / N} \tag{8.4.15}
\end{equation*}
$$

From the S.O.R. theory (Young, 1971) we can find the optimum overrelaxation parameter $\omega_{\text {opt }}$ so that the spectral radius of the block SOR methods, i.e.,

$$
\begin{equation*}
\rho\left[-(B+\omega A)^{-1}\{\omega C+(\omega-D B)\}\right], \tag{8.4.16}
\end{equation*}
$$

(for the equation (8.4.9)) and

$$
\begin{equation*}
\tilde{\rho}\left[-(\tilde{B}+\omega \tilde{A})^{-1}\{\omega \tilde{C}+(\omega-1) \tilde{B}\}\right] . \tag{8.4.17}
\end{equation*}
$$

(for the equation (8.4.13)) are minimised. According to the theorem given by Young (1971), we have,

$$
\begin{equation*}
\omega_{\text {opt }}=\frac{2}{\left\{1+\sqrt{1-\left(\gamma^{2}-\delta^{2}\right)}\right\}} \tag{8.4.18}
\end{equation*}
$$

where $\lambda=\gamma+i \delta$ are the complex eigenvalues of $\left[-B^{-1}(A+C)\right]$ or $\left[-\tilde{B}^{-1}(\widetilde{A}+\widetilde{C})\right]$ contained in an elliptical region with major and minor axes $\gamma$ and $\delta$. Also if (8.4.17a) or (8.4.17b) are denoted by $\rho\left(L_{\omega}\right)$, then

$$
\begin{align*}
\rho\left(L_{\omega_{\text {opt }}}\right) & =\left[\frac{\gamma+\delta}{1+\sqrt{1-\left(\gamma^{2}-\delta^{2}\right)}}\right]^{2} \\
& =\frac{(\gamma+\delta)\left(\omega_{\text {opt }}-1\right)}{(\gamma-\delta)} \tag{8.4.19}
\end{align*}
$$

and the approximate number of iterations $n$ is given by,

$$
\mathrm{n} \approx \frac{\log (\varepsilon p s)}{\log \left[\rho\left(L_{\omega_{\mathrm{opt}}}\right)\right]}
$$

where $\varepsilon p$ is the convergence criterion.
The system (8.4.2) was tested with several diffusion-convection problems. These experiments can be found in Section (8.7) later in the Chapter. The main purpose for these experiments is for comparison with the finite element formulation of the B.V.T. discussed later in this chapter.

### 8.5 A NINE-POINT FINITE ELEMENT FORMULA

The nine-point formula is obtained by using a similar approach to that given in Section (8.2) except that a different size of rectangular element (Fig. 8.5.1) is chosen.


## FIGURE (8.5.1)

When such elements are grouped together as shown in Fig. (8.5.2) then a similar formulation to that in (8.2) leads to a different series of values for $A_{i}^{(j)}, i, j=1,2,3,4(\operatorname{see}(8.2 .14)-(8.2 .16))$ namely,

$$
\left.\begin{array}{l}
A_{1}^{(1)}=\frac{\Delta x}{6}+\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}+\frac{k}{6} \Delta t \\
A_{2}^{(1)}=\frac{\Delta x}{12}-\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}-\frac{k}{6} \Delta t \\
A_{3}^{(1)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}-\frac{k}{12} \Delta t \\
A_{4}^{(1)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k}{12} \Delta t \\
A_{2}^{(2)}=\frac{\Delta x}{6}+\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}-\frac{k}{6} \Delta t  \tag{8.5.1}\\
A_{3}^{(2)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}-\frac{k}{12} \Delta t \\
A_{4}^{(2)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k}{12} \Delta t \\
A_{1}^{(2)}=\frac{\Delta x}{12}-\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}+\frac{k}{6} \Delta t \\
A_{3}^{(3)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}-\frac{k}{6} \Delta t \\
A_{4}^{(3)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}+\frac{k}{6} \Delta t \\
A_{1}^{(3)}=\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k}{12} \Delta t \\
A_{2}^{(3)}=\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}-\frac{k}{12} \Delta t
\end{array}\right\} \quad \text { first element }
$$

$$
\begin{aligned}
& A_{4}^{(4)}=-\frac{\Delta x}{6}+\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}+\frac{k \Delta t}{6} \\
& A_{1}^{(4)}=\frac{\Delta x}{6}+\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}+\frac{k}{12} \Delta t \\
& A_{2}^{(4)}=\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{\Delta x}-\frac{k}{12} \Delta t \\
& A_{3}^{(4)}=-\frac{\Delta x}{12}-\frac{\varepsilon}{3} \frac{\Delta t}{\Delta x}-\frac{k}{6} \Delta t
\end{aligned}
$$

$$
A_{2}^{(4)}=\frac{\Delta x}{12}-\frac{\varepsilon}{6} \frac{\Delta t}{10}-\frac{k}{10} \Delta t \quad\{\quad \text { fourth element }
$$



FIGURE (8.5.2)
Since the time-interval for each element is now taken as $\Delta t$, then when the four elements are used as indicated a formula involving three time-levels (Fig. (8.5.2)) is obtained.

The totality of these four elements leads to a nine-point formula, centred at $(i, j)$ which is

$$
\begin{align*}
& {\left[\frac{1}{12}-\frac{1}{6} \varepsilon r-\frac{k}{12} r \Delta x\right] \phi_{i-1, j+1}+\frac{1}{3}(1+\varepsilon r) \phi_{i, j+1}+\left[\frac{1}{12}-\frac{1}{6^{\varepsilon r}}+\frac{k r \Delta x}{12}\right] \phi_{i+1, j+1}} \\
& \quad+\left[-\frac{2}{3} \varepsilon r-\frac{k r \Delta x}{3}\right] \phi_{i-1, j}+\frac{4}{3} \varepsilon \phi_{i, j}+\left[-\frac{2}{3} \varepsilon r+\frac{k r \Delta x}{3}\right] \phi_{i+1, j} \\
& +\left[-\frac{1}{12}-\frac{1}{6} \varepsilon r-\frac{k}{12} r \Delta x\right] \phi_{i-1, j-1}+\frac{1}{3}(-1+\varepsilon r) \phi_{i, j-1}+\left[-\frac{1}{12}-\frac{1}{6^{\varepsilon} r}+\right. \\
& \left.\frac{k r \Delta x}{12}\right] \phi_{i+1, j-1}=0 \tag{8.5.2}
\end{align*}
$$

Using Taylor's series expansion, the equation (8.5.2) can be shown to approximate (8.1.1) at point $(i, j)$ with the resulting error term,

$$
\begin{equation*}
T_{(8.5 .2)}=\frac{\Delta t^{2}}{6} \phi_{t t t}+\frac{k \Delta x^{2}}{6} \phi_{x x x}+\frac{\Delta t^{2}}{6} k_{\phi_{x t t}}+\frac{\Delta x^{2}}{6} \phi_{x t t}+o\left(\Delta x^{\alpha_{1}}{ }_{\Delta t}^{\alpha_{2}^{-1}}\right) \tag{8.5.3}
\end{equation*}
$$

$0 \leqslant \alpha_{1}, \alpha_{2} \leqslant 4, \alpha_{1}+\alpha_{2}=4$.

This nine-point formulae can be used to solve a parabolic problem using a boundary value technique when considering the solution over the closed region and assuming the existence of the steady-state solution $\phi(x)$ as $t \rightarrow \infty$.

### 8.6 THE BOUNDARY VALUE TECHNIQUE FOR A FINITE ELEMENT NINE-POINT FORMULA

We now consider in detail the B.V.T. formulation of the nine-point formula (8.5.2). Both row-wise and column-wise ordering of the mesh points will be considered.

For the row-wise ordering, a ( $\mathrm{N}-1$ ) $\times(\mathrm{N}-1)$ block linear system similar to (8.4.2)-(8.4.3) is obtained with,

$C_{i}=\left[\begin{array}{llll}\frac{1}{3}(1+\varepsilon r) & \left(\frac{1}{12}-\frac{1}{6} \varepsilon r+\frac{k r \Delta x}{12}\right) \\ \left(\frac{1}{12}-\frac{1}{6} \varepsilon r\right. & \left.-\frac{\mathrm{kr} \mathrm{\Delta x}}{12}\right) & \frac{1}{3}(1+\varepsilon r) & \left(\frac{1}{12}-\frac{1}{6} \varepsilon r+\frac{\mathrm{kr} \mathrm{\Delta x}}{12}\right) \\ & \ddots & 0 \\ 0 & \left(\frac{1}{12}-\frac{1}{6} \varepsilon r-\frac{k r \Delta x}{12}\right) & \frac{1}{3}(1+\varepsilon r) & \left(\frac{1}{12}-\frac{1}{6} \varepsilon r+\frac{k r \Delta x}{12}\right) \\ & & \left(\frac{1}{12}-\frac{1}{6} \varepsilon r-\frac{k r \Delta x}{12}\right) & \frac{1}{3}(1+\varepsilon r)\end{array}\right]$,
and
$A_{i}=\left[\begin{array}{llll}\frac{1}{3}(\varepsilon r-1) & \left(-\frac{1}{12}-\frac{1}{6} \varepsilon r+\frac{k}{12} r \Delta x\right) \\ \left(-\frac{1}{12}-\frac{1}{6} \varepsilon r-\frac{k r \Delta x}{12}\right) & \frac{1}{3}(r \varepsilon-1) & \left(-\frac{1}{12}-\frac{1}{6} \varepsilon r+\frac{k}{12} r \Delta x\right) & 0 \\ 0 & \left(-\frac{1}{12}-\frac{1}{6} \varepsilon r-\frac{k r \Delta x}{12}\right) & \frac{1}{3}(\varepsilon r-1) & \left(-\frac{1}{12}-\frac{1}{6} \varepsilon r+\frac{k}{12} r \Delta x\right) \\ 0 & & \left(-\frac{1}{12}-\frac{1}{6} r-\frac{k r \Delta x}{12}\right) & \frac{1}{3}(\varepsilon r-1)\end{array}\right]$ (8.6.3)

Similarly a column-wise ordering of the mesh points leads to a ( $M-1$ ) $\times(M-1)$ block system of equations as given in (8.4.5)-(8.4.6) with



for $i=1,2, \ldots, M-1$.
For the row-wise ordering the block simultaneous method and block SOR method are given by equations (8.4.8) and (8.4.9), respectively, whilst for the column-wise ordering they are given by equations (8.4.12) and (8.4.13).

The matrices $A_{i}, B_{i}, C_{i}$ and $\widetilde{A}_{i}, \widetilde{B}_{i}, \widetilde{C}_{i}$ in (8.6.1)-(8.6.6) are much more complex and consequently more difficult to handle theoretically. Hence resort to numerical experiments was considered sufficient for our purposes.

### 8.7 NUMERICAL EXPERIMENTS

The following numerical experiments were carried out for computational justification of the theoretical analysis:-

## Experiment 1

The equation (8.1.1) subject to the initial condition,

$$
\begin{equation*}
\phi(x, 0)=0 \tag{8.7.1}
\end{equation*}
$$

and the boundary conditions,

$$
\begin{align*}
& \phi(0, t)=0,  \tag{8.7.2}\\
& \phi(1, t)=1,
\end{align*}
$$

was considered. The various schemes considered in this chapter were tested and the results are given in Tables (8.7.1)-(8.7.3). The solutions from these schemes are compared with the exact solution which is given by

$$
\begin{align*}
\phi(x, t)= & {\left[\frac{e^{k x / \varepsilon}-1.0}{e^{k / \varepsilon}-1.0}+2 \sum_{n=1}^{\infty}(-1)^{n} \frac{n \pi}{(n \pi)^{2}+(k / 2 \varepsilon)^{2}} e^{k(x-1) / 2 \varepsilon}\right.} \\
& \left.\sin (n \pi x) e^{-\left\{n^{2} \pi^{2} \varepsilon^{2}+k^{2} / 4 \varepsilon\right\} t}\right] \tag{8.7.3}
\end{align*}
$$

A comparison is also made with the Crank-Nicolson scheme. An observation on the optimum relaxation parameter $\omega_{\text {opt }}$ for FDS and FEM9 with respect to the ratio $\mathrm{k} / \varepsilon$ is also made (Table (8.7.4)). The convergence criteria is denoted by eps.

In these experiments it can be observed that the results of FEM and FEM9 are in good agreement with the exact solution and are more accurate than those obtained from FD5. It can also be seen that the FEM9 requires a slightly higher iteration number than FD5. The variation of $\omega_{\text {opt }}$ with respect to $\mathrm{k} / \varepsilon$ is noticeable.

## Experiment 2

The equation (8.1.1) subject to the initial condition,

$$
\begin{equation*}
\phi(x, 0)=e^{k x / 2 \varepsilon} x(1-x), 0 \leqslant x \leqslant 1, \tag{8.7.4}
\end{equation*}
$$

and the homogeneous Dirichlet boundary conditions

$$
\begin{equation*}
\phi(0, t)=\phi(1, t)=0, \quad t>0, \tag{8.7.5}
\end{equation*}
$$

was considered. The exact solution to the problem can be shown to be

$$
\begin{equation*}
\phi(x, t)=\sum_{n=1}^{\infty} \frac{4}{(n \pi)^{3}}\left(1-(-1)^{n}\right) e^{k x / 2 \varepsilon} e^{-\left[(n \pi)^{2}+k^{2} / 4 \varepsilon\right] t} \sin (n \pi x) \tag{8.7.6}
\end{equation*}
$$

(Appendix 3).
A comparison of the results obtained from the exact solution with the solutions obtained from the schemes discussed in this chapter are made (Tables (8.7.5)-(8.7.7)). Similar observations on the relation of the ratio $\mathrm{k} / \varepsilon$ with the optimum relaxation parameter are also made (Table (8.7.8)).

In these experiments, the results of FEM and FEM9 are consistent with the exact solution. The solutions of FD5 are however not in close agreement with the exact solution. This is due to the linear system becoming almost singular and hence gives a non-unique numerical solution. As expected due to the homogeneous nature of the problem the solutions are a multiple of an arbitrary constant. For $k=1, \varepsilon=1$, the solutions of FD5 are a multiple of 2 in comparison with the exact solution and for $k=1, \varepsilon=0.5$, they are the multiples of 1.16. Therefore for this particular problem FEM9 possesses clear advantages over FD5.

## Experiment 3

The equation (8.1.1) subject to the initial condition,

$$
\begin{equation*}
\phi(x, 0)=0, \quad 0<x \leqslant 1, \tag{8.7.7}
\end{equation*}
$$

and the mixed boundary conditions,

$$
\begin{align*}
\phi(0, t) & =1, \quad t \geqslant 1  \tag{8.7.8a}\\
\phi_{x}(1, t) & =0, \quad t>0 \tag{8.7.8b}
\end{align*}
$$

was considered. The exact solution to the problem is given by,

$$
\begin{align*}
u(x, t)= & \frac{1}{2}\{\operatorname{erfc}((x-\lambda \varepsilon t) / 2 \sqrt{\varepsilon t})+\exp (\lambda x) \operatorname{erfc}((x+\lambda \varepsilon t) / 2 \sqrt{\varepsilon t}) \\
& +\exp (\lambda)\left\{\left(1+\frac{1}{2} \lambda(2-x+\lambda \varepsilon t)\right) \operatorname{erfc}((2-x+\lambda \varepsilon t) / 2 \sqrt{\varepsilon t})\right. \\
& \left.-\lambda \sqrt{(\varepsilon t / \pi)} \exp \left(-(2-x+\lambda \varepsilon t)^{2} / 4 \varepsilon t\right)\right\}, \tag{8.7.9}
\end{align*}
$$

where $\lambda=k / \varepsilon$ (Siemieniuch J.L, and Gladwell, I., 1976).
The finite element formula (8.2.18) was tested and the results obtained are compared with the exact solution (Tables (8.7.9)-(8.7.11)). For small $\varepsilon$, oscillations in the solution at small time-levels occur. This oscillation slowly decays as the time is increased.
$\mathrm{k}=1.0, \mathrm{r}=1.0, \mathrm{t}=0.5, \Delta \mathrm{x}=0.1$

| $\mathbf{x}$ | EXACT sOL. | CNCD <br> $\Delta \mathrm{t}=0.01$ | FEM <br> $\Delta \mathrm{t}=0.01$ | FD5 <br> $\Delta \mathrm{t}=0.1$ | FEM9 <br> $\Delta \mathrm{t}=0.1$ |
| :---: | :--- | :---: | :---: | :---: | :---: |
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0604 | 0.0604 | 0.0604 | 0.5972 | 0.0604 |
| 0.2 | 0.1273 | 0.1272 | 0.1273 | 0.1259 | 0.1273 |
| 0.3 | 0.2014 | 0.2012 | 0.2014 | 0.1993 | 0.2014 |
| 0.4 | 0.2835 | 0.2833 | 0.2835 | 0.2809 | 0.2835 |
| 0.5 | 0.3745 | 0.3745 | 0.3745 | 0.3717 | 0.3746 |
| 0.6 | 0.4754 | 0.4752 | 0.4754 | 0.4726 | 0.4754 |
| 0.7 | 0.5873 | 0.5870 | 0.5873 | 0.5847 | 0.5872 |
| 0.8 | 0.7111 | 0.7110 | 0.7112 | 0.7092 | 0.7111 |
| 0.9 | 0.8483 | 0.8482 | 0.8483 | 0.8472 | 0.8486 |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

FD5 is the block SOR method with a row-wise ordering of the Richardson's 5-point F.D. formula.

FEM is the block SOR method with a row-wise ordering of the 9-point FEM formula.
$k=1.0, \quad \varepsilon=0.1, \quad t=0.5, \quad \Delta x=0.1$

| $\mathbf{x}$ | EXACT SOL | CNCD <br> $\Delta t=0.01$ | FEM <br> $\Delta t=0.01$ | FD5 <br> $\Delta t=0.1$ | FEM9 <br> $\Delta t=0.1$ |
| :---: | :--- | :---: | :---: | :---: | :---: |
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0299 | 0.0298 | 0.0298 | 0.0291 | 0.0298 |
| 0.2 | 0.0669 | 0.0664 | 0.0667 | 0.0653 | 0.0668 |
| 0.3 | 0.1134 | 0.1126 | 0.1131 | 0.1108 | 0.1132 |
| 0.4 | 0.1719 | 0.1710 | 0.1716 | 0.1685 | 0.1717 |
| 0.5 | 0.2458 | 0.2447 | 0.2454 | 0.2418 | 0.2456 |
| 0.6 | 0.3389 | 0.3376 | 0.3384 | 0.3344 | 0.3386 |
| 0.7 | 0.4554 | 0.4541 | 0.4549 | 0.4511 | 0.4551 |
| 0.8 | 0.6004 | 0.5993 | 0.6000 | 0.5968 | 0.6001 |
| 0.9 | 0.7797 | 0.7790 | 0.7795 | 0.7776 | 0.7797 |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

TABLE (8.7.2)
$k=1.0, \quad \varepsilon=0.5, \quad t=0.5, \quad \Delta x=0.1$

| $\mathbf{x}$ | EXACT SOL. | CNCD <br> $\Delta t=0.01$ | FEM <br> $\Delta t=0.01$ | FD5 <br> $\Delta t=0.1$ | FEM9 <br> $\Delta t=0.1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.3 | 0.0003 | 0.0002 | 0.0001 | 0.0002 | 0.0001 |
| 0.4 | 0.0012 | 0.0006 | 0.0005 | 0.0007 | 0.0005 |
| 0.5 | 0.0042 | 0.0025 | 0.0021 | 0.0025 | 0.0021 |
| 0.6 | 0.0136 | 0.0091 | 0.0085 | 0.0090 | 0.0085 |
| 0.7 | 0.0424 | 0.0314 | 0.0308 | 0.0309 | 0.0308 |
| 0.8 | 0.1256 | 0.1030 | 0.1027 | 0.1020 | 0.1027 |
| 0.9 | 0.3589 | 0.3253 | 0.3255 | 0.3241 | 0.3255 |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

TABLE (8.7.3)
$\mathrm{k}=1.0, \quad \Delta \mathrm{x}=0.1, \quad \Delta \mathrm{t}=0.1$

| ratio <br> k/ $\varepsilon$ | FD5 ( $\varepsilon \mathrm{ps}=10^{-7}, \mathrm{~T}=5.0$ ) |  | FEM9 ( $\varepsilon$ ¢s $\left.=10^{-7}, \mathrm{~T}=5.0\right)$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Iteration Number | $\omega_{\text {opt }}$ | Iteration Number | $\omega_{\text {opt }}$ |
| 1.0 | 20 | 0.8 | 26 | 0.725 |
| 2.0 | 34 | 0.6 | 45 | 0.50 |
| 10.0 | 53 | 0.5 | 69 | 0.36 |

TABLE (8.7.4)
$k=1.0, \quad \varepsilon=1.0, \quad t=0.5, \Delta x=0.1$

| $x$ | EXACT SOL. | CNCD <br> $\Delta t=0.01$ | FEM <br> $\Delta t=0.01$ | FD5 <br> $\Delta t=0.1$ | FEM9 <br> $\Delta t=0.1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.000529 | 0.000555 | 0.000513 | 0.001012 | 0.000497 |
| 0.2 | 0.001059 | 0.001110 | 0.001025 | 0.002024 | 0.001001 |
| 0.3 | 0.001532 | 0.001606 | 0.001483 | 0.002929 | 0.001449 |
| 0.4 | 0.001893 | 0.001984 | 0.001831 | 0.003620 | 0.001790 |
| 0.5 | 0.002092 | 0.002193 | 0.002022 | 0.004002 | 0.001977 |
| 0.6 | 0.002091 | 0.002193 | 0.002020 | 0.004001 | 0.001975 |
| 0.7 | 0.001869 | 0.001961 | 0.001805 | 0.003578 | 0.001765 |
| 0.8 | 0.001427 | 0.001498 | 0.001378 | 0.002733 | 0.001344 |
| 0.9 | 0.000787 | 0.000828 | 0.000761 | 0.001511 | 0.000737 |

TABLE (8.7.5)
$k=1.0, \quad \varepsilon=0.5, \quad t=0.5, \quad \Delta x=0.1$

| $x$ | EXACT SOL. | CNCD <br> $\Delta t=0.01$ | FEM <br> $\Delta t=0.01$ | FD5 | FEM9 <br> $\Delta t=0.1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.005803 | 0.005996 | 0.005801 | 0.006730 | 0.005796 |
| 0.2 | 0.012199 | 0.012609 | 0.012178 | 0.014152 | 0.012171 |
| 0.3 | 0.018555 | 0.019186 | 0.018497 | 0.021533 | 0.018486 |
| 0.4 | 0.024104 | 0.024935 | 0.023996 | 0.027985 | 0.023981 |
| 0.5 | 0.028005 | 0.028985 | 0.027843 | 0.032531 | 0.027825 |
| 0.6 | 0.029429 | 0.030476 | 0.029223 | 0.034204 | 0.029204 |
| 0.7 | 0.027655 | 0.028660 | 0.027432 | 0.032167 | 0.027416 |
| 0.8 | 0.022190 | 0.023021 | 0.021995 | 0.025837 | 0.021982 |
| 0.9 | 0.012865 | 0.013380 | 0.012761 | 0.015017 | 0.012749 |

TABLE (8.7.6)
$\mathrm{k}=1.0, \varepsilon=0.1, \mathrm{t}=0.5, \Delta \mathrm{x}=0.1$

| x | EXACT SOL. | CNCD <br> $\Delta \mathrm{t}=0.01$ | FEM <br> $\Delta \mathrm{t}=0.01$ | FD5 | FEM9 <br> $\Delta \mathrm{t}=0.1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.0230 | 0.0196 | 0.0227 | 0.0205 | 0.0227 |
| 0.2 | 0.0721 | 0.0640 | 0.0720 | 0.0668 | 0.0720 |
| 0.3 | 0.1635 | 0.1506 | 0.1650 | 0.1566 | 0.1650 |
| 0.4 | 0.3165 | 0.3016 | 0.3230 | 0.3127 | 0.3230 |
| 0.5 | 0.5485 | 0.5400 | 0.5657 | 0.5584 | 0.5657 |
| 0.6 | 0.8601 | 0.8453 | 0.8968 | 0.9038 | 0.8968 |
| 0.7 | 1.2067 | 1.2722 | 1.2726 | 1.3125 | 1.2727 |
| 0.8 | 1.4456 | 1.5848 | 1.5438 | 1.6347 | 1.5438 |
| 0.9 | 1.2513 | 1.4342 | 1.3560 | 1.4794 | 1.3560 |

TABLE (8.7.7)
$\mathrm{k}=1.0, \quad \Delta \mathrm{x}=0.1, \Delta \mathrm{t}=0.1$

| ratio <br> $\mathrm{k} / \varepsilon$ | FD5 ( $\left.\varepsilon \mathrm{cps}=10^{-7}, \mathrm{~T}=5.0\right)$ |  | FEM9 ( $\left.\varepsilon \mathrm{ps}=10^{-7}, \mathrm{~T}=20.0\right)$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Iteration <br> Number | $\omega_{\text {opt }}$ | Iteration <br> Number | $\omega_{\text {opt }}$ |
|  | 21 | 0.75 | 53 | 0.70 |
| 2.0 | 35 | 0.60 | 94 | 0.50 |
| 10.0 | 52 | 0.50 | 156 | 0.35 |

$\mathrm{k}=1.0, \varepsilon=1.0, \mathrm{t}=0.5, \Delta \mathrm{x}=0.1$

| $\mathbf{x}$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| EXACT | 1.0000 | 0.9957 | 0.9618 | 0.9276 | 0.8940 | 0.8622 |
| FEM | 1.0000 | 0.9721 | 0.9423 | 0.9116 | 0.8811 | 0.8518 |


| x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| EXACT | 0.8335 | 0.8090 | 0.7901 | 0.7780 | 0.7737 |
| FEM | 0.8251 | 0.8022 | 0.7844 | 0.7728 | 0.7687 |

TABLE (8.7.9)
$\mathrm{k}=1.0, \varepsilon=0.5, \mathrm{t}=0.5, \Delta \mathrm{x}=0.1$

| x | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| EXACT | 1.0000 | 0.9603 | 0.9115 | 0.8574 | 0.7998 | 0.7411 |
| FEM | 1.0000 | 0.9587 | 0.9106 | 0.8572 | 0.8001 | 0.7418 |


| x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| EXACT | 0.6844 | 0.6330 | 0.5909 | 0.5623 | 0.5518 |
| FEM | 0.6853 | 0.6340 | 0.5918 | 0.5629 | 0.5518 |

TABLE (8.7.10)
$\mathrm{k}=1.0, \varepsilon=0.1, \mathrm{t}=0.5, \Delta \mathrm{x}=0.1$

| x | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| EXACT | 1.0000 | 0.9756 | 0.9278 | 0.8511 | 0.7449 | 0.6162 |
| FEM | 1.0000 | 0.9765 | 0.9305 | 0.8569 | 0.7550 | 0.6306 |


| x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| EXACT | 0.4777 | 0.3448 | 0.2313 | 0.1482 | 0.1121 |
| FEM | 0.4952 | 0.3630 | 0.2469 | 0.1568 | 0.1073 |

TABLE (8.7.11)

### 8.8 CONCLUSIONS

From the above numerical experiments and accompanying analysis, it is clear that the six-point FEM formula, as far as stability and accuracy are concerned, are computationally acceptable. The applicability of this method to other problems such as the Burger's equation are considered promising but need further detailed investigations before a final conclusion can be given.

For the B.V.T. using the 9 -point FEM formula, beside being slightly computationally expensive (in terms of the iteration number and storage requirement), its consistency to provide the solution to model problems as opposed to FD5 need to be taken into account in the final analysis.

## Chapter Nine

### 9.1 NUMERICAL SOLUTION OF THE DIFFUSION-CONVECTION EQUATION (BY ARTIFICIAL DIFFUSION)

Consider initially the first-order hyperbolic equation in one space dimension

$$
\begin{equation*}
\frac{\partial u}{\partial t}+v \frac{\partial u}{\partial x}=0,0 \leqslant x \leqslant 1, t \geqslant 0 \tag{9.1.1}
\end{equation*}
$$

where $v>0$ is a constant. One finite-difference method for approximating this problem at the point $\left(i, j+\frac{1}{2}\right)$ is given by (Peaceman, D.W., 1977, p.75)

$$
\frac{u_{i, j+1}-u_{i, j}}{\Delta t}+\frac{v}{\Delta x}\left[\theta\left(u_{i+\frac{1}{2}, j+1}^{-u_{i-\frac{1}{2}, j+1}}\right)+(1-\theta)\left(u_{i+\frac{1}{2}, j}-u_{i-\frac{1}{2}, j}\right)\right]=0
$$

where

$$
\begin{equation*}
u_{i+\frac{1}{2}}=w u_{i}+(1-w) u_{i+1} \tag{9.1.2}
\end{equation*}
$$

and $0 \leqslant \theta, w \leqslant 1$ are weighting parameters.
The substitution of (9.1.3) into (9.1.2) gives the difference approximation for equation (9.1.1) as

$$
\begin{align*}
& -v \frac{\Delta t}{\Delta x}\left[\theta\left\{(1-w) u_{i+1, j+1}+(2 w-1) u_{i, j+1}-w u_{i-1, j+1}\right\}\right] \\
&  \tag{9.1.4}\\
& \quad+(1-\theta)\left\{(1-w) u_{i+1, j}+(2 w-1) u_{i, j}-w u_{i-1, j}\right\}=u_{i, j+1}-u_{i, j}
\end{align*}
$$

The special equation of interest is for the case when $w=\frac{1}{2}$, i.e.,

$$
\begin{align*}
& -\frac{v \Delta t}{2 \Delta x} \theta u_{i-1, j+1}+u_{i, j+1}+\frac{v \Delta t}{2 \Delta x} u_{i+1, j+1}= \\
& \quad \frac{v \Delta t}{2 \Delta x}(1-\theta) u_{i-1, j}+u_{i, j}-\frac{v \Delta t}{2 \Delta x}(1-\theta) u_{i+1, j} \tag{9.1.5}
\end{align*}
$$

An analysis of the stability of (9.1.5) by Fourier series will after some algebraic simplification finally require the condition,

$$
\begin{equation*}
|\gamma|^{2}=\frac{1+\lambda^{2}(1-\theta)^{2} \sin ^{2}(\beta \Delta x)}{1+\lambda^{2} \theta^{2} \sin ^{2}(\beta \Delta x)} \leqslant 1 \tag{9.1.6}
\end{equation*}
$$

where $\lambda=v \Delta t / \Delta x$.
This requirement is satisfied only for all $\theta>\frac{1}{2}$ and not for $\theta<\frac{1}{2}$. For $\theta=\frac{1}{2}, \gamma^{2}=1$. Therefore, equation (9.1.5) is unconditionally stable for all $\lambda$ for $\theta<\frac{1}{2}$ and 'neutrally stable' for $\theta=\frac{1}{2}$.

## Truncation Error

To analyse the truncation error associated with the difference approximation (9.1.5), we expand each of the terms about the point $m \Delta t$ where,

$$
\begin{equation*}
m=(1-\theta) j+\theta(j+1) . \tag{9.1.7}
\end{equation*}
$$

Now from Taylor's series, we have,

$$
\begin{aligned}
& u_{i, j+1}=u_{i, m}+(1-\theta) \Delta t\left(\frac{\partial u}{\partial t}\right)_{i, m}+\frac{1}{2}(1-\theta)^{2} \Delta t^{2}\left(\frac{\partial^{2} u}{\partial t^{2}}\right)_{i, m}+O\left(\Delta t^{3}\right) \\
& u_{i, j}=u_{i, m}-\theta \Delta t\left(\frac{\partial u}{\partial t}\right)_{i, m}+\frac{1}{2} \theta^{2} \Delta t^{2}\left(\frac{\partial^{2} u}{\partial t^{2}}\right)_{i, m}+O\left(\Delta t^{3}\right)
\end{aligned}
$$

so that,

$$
\begin{equation*}
\frac{u_{i, j+1}^{-u} i, j}{\Delta t}=\left(\frac{\partial u}{\partial t}\right)_{i, m}+\left(\frac{1}{2}-\theta\right) \Delta t\left(\frac{\partial^{2} u^{2}}{\partial t^{2}}{ }_{i, m}+o\left(\Delta t^{2}\right)\right. \tag{9.1.8}
\end{equation*}
$$

In addition,

$$
\begin{aligned}
& \left(\frac{\partial u}{\partial x}\right)_{i, j+1}=\left(\frac{\partial u}{\partial x}\right)_{i, \dot{m}}+(1-\theta) \Delta t\left(\frac{\partial^{2} u}{\partial t \partial x}\right)_{i, m}+o\left(\Delta t^{2}\right) \\
& \left(\frac{\partial u}{\partial x}\right)_{i, j}=\left(\frac{\partial u}{\partial x}\right)_{i, m}-\theta \Delta t\left(\frac{\partial^{2} u}{\partial t \partial x}\right)_{i, m}+o\left(\Delta t^{2}\right)
\end{aligned}
$$

so that,

$$
\begin{equation*}
\theta\left(\frac{\partial u}{\partial x}\right)_{i, j+1}+(1-\theta)\left(\frac{\partial u}{\partial x}\right)_{i, j}=\left(\frac{\partial u}{\partial x}\right)_{i, m}+0\left(\Delta t^{2}\right) . \tag{9.1.9}
\end{equation*}
$$

From the definition of local truncation error (see Section 2.6 )

$$
\begin{align*}
& \varepsilon_{L}=-v\left[\theta\left\{\frac{u_{i+1, j+1}^{-u_{i-1, j+1}}}{2 \Delta x}\right\}+(1-\theta)\left\{\frac{\left.\left.u_{i+1, j^{-u_{i-1, j}}}^{2 \Delta x}\right\}\right]}{}\right.\right. \\
&-\frac{u_{i, j+1}^{-u_{i, j}}}{\Delta t}+v\left(\frac{\partial u}{\partial x}\right)_{i, m}+\left(\frac{\partial u}{\partial t}\right)_{i, m}, \tag{9.1.10}
\end{align*}
$$

and as,

$$
\begin{equation*}
\frac{u_{i+1}-u_{i-1}}{2 \Delta x}=\left(\frac{\partial u}{\partial x}\right)_{i}+O\left(\Delta x^{2}\right), \tag{9.1.11}
\end{equation*}
$$

therefore,

$$
\begin{gather*}
\varepsilon_{L}=-v\left[\theta\left(\frac{\partial u}{\partial x}\right)_{i, j+1}+(1-\theta)\left(\frac{\partial u}{\partial x}\right)_{i, j}\right]-\frac{u_{i, j+1}-u_{i, j}}{\Delta t}+ \\
v\left(\frac{\partial u}{\partial x}\right)_{i, m}+\left(\frac{\partial u}{\partial t}\right)_{i, m}+0\left(\Delta x^{2}\right) . \tag{9.1.12}
\end{gather*}
$$

Substitution of equation (9.1.8) and (9.1.9) into (9.1.12) yields,

$$
\begin{align*}
\varepsilon_{L}= & -v\left(\frac{\partial u}{\partial x}\right)_{i, m}-\left(\frac{\partial u}{\partial t}\right)_{i, m}-\left(\frac{1}{2}-\theta\right) \Delta t\left(\frac{\partial^{2} u}{\partial t^{2}}\right)_{i, m}+v\left(\frac{\partial u}{\partial x}\right)_{i, m}+ \\
& \left(\frac{\partial u}{\partial t}\right)_{i, m}+0\left(\Delta t^{2}\right)+O\left(\Delta x^{2}\right) \\
= & \left(\theta-\frac{1}{2}\right) \Delta t\left(\frac{\partial^{2} u}{\partial t^{2}}\right)_{i, m}+O\left(\Delta x^{2}\right)+O\left(\Delta t^{2}\right) \tag{9.1.13}
\end{align*}
$$

Now, differentiation of equation (9.1.1) w.r.t. t gives,

$$
\frac{\partial^{2} u}{\partial t^{2}}=-v \frac{\partial^{2} u}{\partial t \partial x}=-v \frac{\partial^{2} u}{\partial x \partial t}
$$

while differentiation w.r.t. $x$ gives,

$$
\frac{\partial^{2} u}{\partial x \partial t}=-v \frac{\partial^{2} u}{\partial x^{2}}
$$

so that,

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial t^{2}}=v^{2} \frac{\partial^{2} u}{\partial x^{2}} \tag{9.1.14}
\end{equation*}
$$

Then, a substitution into equation (9.1.13) gives the final form for the local truncation error as

$$
\begin{equation*}
\varepsilon_{L}=\operatorname{v\Delta x\lambda }\left(\theta-\frac{1}{2}\right)\left(\frac{\partial^{2} u}{\partial x^{2}}\right)_{i, m}+O\left(\Delta x^{2}\right)+O\left(\Delta t^{2}\right) \tag{9.1.15}
\end{equation*}
$$

Thus, by solving the difference equation (9.1.5) we are, in effect, solving the diffusion-convection problem,

$$
\begin{equation*}
D \frac{\partial^{2} u}{\partial x^{2}}-v \frac{\partial u}{\partial x}=\frac{\partial u}{\partial t} \tag{9.1.16}
\end{equation*}
$$

where

$$
D=v \Delta x \lambda\left(\theta-\frac{1}{2}\right), \quad \lambda=\frac{v \Delta t}{\Delta x}
$$

with the local truncation error of $0\left(\Delta t^{2}+\Delta x^{2}\right)$, rather than the pure convection problem (9.1.1).

Hence, an alternative strategy of solving the diffusion-convection problem

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\varepsilon \frac{\partial^{2} u}{\partial x^{2}}-k \frac{\partial u}{\partial x}, \tag{9.1.16a}
\end{equation*}
$$

is by choosing $\Delta t$ such that,

$$
\begin{gather*}
\frac{v}{k}=\frac{D}{\varepsilon} \\
\text { i.e. } \left.\Delta t=\frac{\varepsilon}{k v\left(\theta-\frac{1}{2}\right)} \quad \text { (for simplicity } v \text { is assumed } 1\right) . \tag{9.1.17}
\end{gather*}
$$

Since the diffusion constant $\varepsilon$ in the difference approximation (9.1.5) is artificially defined from the truncation error, the method for solving
the diffusion-convection equation in this way is called the artificial diffusion method.

From (9.1.17) we can see that if $\varepsilon \ll k$ (for example, $\varepsilon=0.01, k=1.0$, $\theta=1, \Delta t=0.02)$, the time step $\Delta t$ required is reasonably small and acceptable. However, if $\varepsilon$ and $k$ are of a similar order the $\Delta t$ required is no longer reasonable as the terms $O\left(\Delta x^{2}+\Delta t^{2}\right)$ from the truncation error will contribute to error build up, and then the 'physical' diffusion term has to be taken into account. Hence the following statement can be stated:

Given the diffusion-convection problem (9.1.16a) and $\varepsilon \ll k$, then the finite-difference analogue (9.1.5) of the hyperbolic equation (9.1.1) with

$$
\Delta t^{\cdot}=\frac{\varepsilon}{\operatorname{kv}\left(\theta-\frac{1}{2}\right)} \quad, \quad v=1
$$

actually solves the equation (9.1.16a) with a truncation error of $O\left(\Delta x^{2}+\Delta t^{2}\right)$ rather than solving equation (9.1.1) with a truncation error of $O\left(\Delta t+\Delta x^{2}\right)$.

## Numerical Example

Numerical experiments have been carried out to justify the above analysis and the equation (9.1.16a) was solved with the initial condition,

$$
\begin{equation*}
u(x, 0)=0, \quad 0<x<1 \tag{9.1.18}
\end{equation*}
$$

and boundary conditions,

$$
\begin{align*}
u(0, t) & =1, \quad t>0,  \tag{9.1.19a}\\
\frac{\partial u}{\partial x}(1, t) & =0 . \quad t>0 . \tag{9.1.19b}
\end{align*}
$$

The analytical solution is given by

$$
\begin{align*}
u(x, t)= & \left.\frac{1}{2}\{\operatorname{erfc}((x-\lambda \varepsilon t) / 2 \sqrt{\varepsilon t})+\exp (\lambda x) \operatorname{erfc}(x+\lambda \varepsilon t) / 2 \sqrt{\varepsilon t})\right\} \\
& +\exp (\lambda)\left\{\left(1+\frac{1}{2} \lambda(2-x+\lambda \varepsilon t)\right) \operatorname{erfc}((2-x+\lambda \varepsilon t) / 2 \sqrt{\varepsilon t})\right. \\
& \left.-\lambda \sqrt{(\varepsilon t / \pi)} \exp \left(-(2-x+\lambda \varepsilon t)^{2} / 4 \varepsilon t\right)\right\} \tag{9.1.20}
\end{align*}
$$

where $\lambda=k / \varepsilon$,
as given in a previous chapter. Some of the results obtained are shown
in Figs. (9.1.1)-(9.1.4). The results were found to be qualitatively acceptable for this problem.

The method was also tested using the model problem with Dirichlet boundary conditions. However results are not included here as they are unacceptable. Further results are required to explore whether this strategy is applicable for a wider range of problems.

## SOLUTION U



## SOLUTION U



$$
k=1.0, \quad \varepsilon=0.008, \Delta t=0.016, \Delta x=0.1, \quad t=0.32
$$

SOLUTION U


$$
\mathrm{k}=1.0, \varepsilon=0.01, \Delta \mathrm{t}=0.02, \Delta \mathrm{x}=0.1, \mathrm{t}=0.4
$$

SOLUTION U

$\mathrm{k}=1.0, \quad \varepsilon=0.05, \Delta t=0.1, \Delta \mathrm{x}=0.1, \quad \mathrm{t}=0.5$

### 9.2 THE GE METHOD FOR THE HYPERBOLIC EQUATIONS OF SECOND ORDER

Consider the simple hyperbolic problem of the vibrating string,

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial t^{2}}-\frac{\partial^{2} u}{\partial x^{2}}=0 \tag{9.2.1}
\end{equation*}
$$

in the domain $R=\{(x, t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$ satisfying the following initial conditions,

$$
\left.\begin{array}{rl}
u(x, 0) & =f_{1}(x)  \tag{9.2.2}\\
\frac{\partial u}{\partial t}(x, 0) & =f_{2}(x)
\end{array}\right\} \quad, \quad 0 \leq x \leq 1
$$

and boundary conditions,

$$
\left.\begin{array}{l}
u(0, t)=g_{1}(t)  \tag{9.2.3}\\
u(1, t)=g_{2}(t)
\end{array}\right\} \text { for all } t>0
$$

We shall assume that these initial and boundary conditions are given with sufficient smoothness to maintain the order of accuracy of the difference scheme under consideration.

This type of problem can be solved using the classical explicit scheme (Gerald, C.F., 1978, p.435) to result in the formula,

$$
\begin{equation*}
u_{i, j+1}=2\left(1-p^{2}\right) u_{i, j}+p^{2}\left(u_{i-1, j}+u_{i+1, j}\right)-u_{i, j-1}, p=\Delta t / \Delta x \tag{9.2.4}
\end{equation*}
$$

which is stable for $0<p \leqslant 1$ and with truncation error,

$$
\begin{equation*}
T_{i, j}=(\Delta t \Delta x)^{2}\left[\left.\frac{1}{12}\left(p^{2}-1\right) \frac{\partial^{4} u}{\partial x^{4}}\right|_{i, j}+\left.\frac{1}{360}\left(p^{4}-1\right) \frac{\partial^{6} u}{\partial x^{4}}\right|_{i, j}+\ldots\right] \tag{9.2.5}
\end{equation*}
$$

For $p=1$, the truncation error vanishes and so an exact difference representation of (9.2.1) is obtained as,

$$
\begin{equation*}
u_{i, j+1}=u_{i-1, j}+u_{i+1, j}-u_{i, j-1} \tag{9.2.6}
\end{equation*}
$$

Of the unconditionally stable schemes, the implicit approximation,

$$
\begin{equation*}
\frac{1}{(\Delta t)^{2}} \delta_{t}^{2} u_{i, j}=\frac{1}{(\Delta x)^{2}}\left(\frac{1}{4} \delta_{x}^{2} u_{i, j+1}+\frac{1}{2} \delta_{x}^{2} u_{i, j}+\frac{1}{4} \delta_{x}^{2} u_{i, j-1}\right) \tag{9.2.7}
\end{equation*}
$$

is most attractive. Its truncation error is,

$$
\begin{equation*}
(\Delta x)^{2}\left\{-\left.\frac{1}{12}\left(4 p^{2}+1\right) \frac{\partial^{4} u}{\partial x^{4}}\right|_{i, j}-\left.\frac{1}{720}(\Delta x)^{2}\left(13 p^{4}+15 p^{2}+1\right) \frac{\partial^{6} u}{\partial x^{6}}\right|_{i, j}+\ldots\right\} \tag{9.2.8}
\end{equation*}
$$

Other implicit schemes are due to Lees (1962), Fairweather and Mitchell (1965), Samarskii (1964) and D'Yakonov (1963). In all these cases, the solution of a tridiagonal system of equations is required at each time step.

Now, we explore the possible extension of the G.E. methods to other types of problems in p.d.e.'s in order to avoid the solution of implicit systems of equations. Thus the GE method will be applied to this problem.

Consider the points ( $i, j$ ) and ( $i+1, j$ ) and at these points approximate the equation (9.2.1) by

and

$$
\begin{equation*}
\frac{u_{i+1, j+1}^{-2 u_{i+1, j}}{ }^{+u_{i+1, j-1}}}{(\Delta t)^{2}}=\frac{u_{i, j+1}-u_{i+1, j+1}-u_{i+1, j-1}+u_{i+2, j-1}}{(\Delta x)^{2}} \tag{9.2.10}
\end{equation*}
$$

respectively.
The combination of equations (9.2.9) and (9.2.10) leads to the system

$$
\begin{array}{r}
{\left[\begin{array}{cc}
1+p^{2} & -p^{2} \\
-p^{2} & 1+p^{2}
\end{array}\right]\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right]\left[\begin{array}{l}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]-\left[\begin{array}{cc}
1+p^{2} & 0 \\
0 & 1+p^{2}
\end{array}\right]\left[\begin{array}{l}
u_{i, j-1} \\
u_{i+1, j-1}
\end{array}\right]} \\
+\left[\begin{array}{c}
p^{2} u_{i-1, j-1} \\
p^{2} u_{i+2, j-1}
\end{array}\right] \tag{9.2.11}
\end{array}
$$

which can be expressed explicitly by,

$$
\begin{align*}
{\left[\begin{array}{l}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=} & \frac{1}{\Delta}
\end{align*} \begin{cases}\left\{\left[\begin{array}{cc}
2\left(1+p^{2}\right) & 2 p^{2} \\
2 p^{2} & 2\left(1+p^{2}\right)
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]-\left[\begin{array}{cc}
\left(1+p^{2}\right)^{2} & p^{2}\left(1+p^{2}\right) \\
p^{2}\left(1+p^{2}\right) & \left(1+p^{2}\right)^{2}
\end{array}\right]\right. \\
\left.\left[\begin{array}{l}
u_{i, j-1}^{4} \\
u_{i+1, j-1}
\end{array}\right]+\left[\begin{array}{cc}
p^{2}\left(1+p^{2}\right) u_{i-1, j-1}+u_{i+2, j-1} \\
p^{4} u_{i-1, j-1}+p^{2}\left(1+p^{2}\right) u_{i+2, j-1} \tag{9.2.12}
\end{array}\right]\right\}\end{cases}
$$

where $\Delta=1+2 p^{2}$, (Fig. (9.2.1)).


Computational molecule of scheme (9.2.9)
FIGURE (9.2.1)

At every time-level, if we assume there are an odd number of unknown points, we will always have a single ungrouped point, i.e. either $(1, j+1)$ or $(m-1, j+1)$ at each end of the range of integration. As the value of the boundary points are known, these points can be approximated by,

$$
\begin{align*}
& u_{1, j+1}=\frac{1}{\left(1+p^{2}\right)}\left\{p^{2} u_{0, j+1}+2 u_{1, j}-\left(1+p^{2}\right) u_{1, j-1}+p^{2} u_{2, j-1}\right\},  \tag{9.2.13}\\
& \text { or } \\
& u_{m-1, j+1}=\frac{1}{\left(1+p^{2}\right)}\left\{p^{2} u_{m, j+1}+2 u_{m-1, j}+p^{2} u_{m-2, j-1}-\left(1+p^{2}\right) u_{m-1, j-1}\right\} . \tag{9.2.14}
\end{align*}
$$

Now the GE concept is developed from equations (9.2.1D-(9.2.14).
The GEL is the combination of equation (9.2.13) (for ( $1, \mathrm{j}+1$ ) ) and equation (9.2.12) (for $i=2,4, \ldots, m-2$ ). This scheme is expressed compactly by

$$
\begin{equation*}
\left(I+p_{G_{1}}^{2}\right){\underset{\mathrm{u}}{j+1}}=2 \underline{I}_{j}-\left(I+p_{G_{2}}^{2}\right)_{j-1}+\underline{b}_{1}, \tag{9.2.15}
\end{equation*}
$$

where,

with

$$
\begin{aligned}
& \underline{b}_{1}^{T}=\left[p^{2} u_{0, j+1}, 0, \ldots, 0, p^{2} u_{m, j-1}\right] \quad \text { and } \\
& \underline{u}_{j}^{T}=\left[u_{1, j}, u_{2, j}, \ldots, u_{m-2, j}, u_{m-1, j}\right] .
\end{aligned}
$$

Meanwhile the GER scheme is the combination of equation (9.2.12) (for $i=1,3,5, \ldots, m-3$ ) and equation (9.2.14) (for ( $\mathbb{m}-1, j+1$ ). This leads to the system,

$$
\begin{equation*}
\left(I+p^{2} G_{2}\right) \underline{u}_{j+1}=2 \underline{I}_{j}-\left(I+p^{2} G_{1}\right) \underline{u}_{j-1}+\underline{b}_{2}, \tag{9.2.18}
\end{equation*}
$$

where

$$
\underline{b}_{2}^{T}=\left[p^{2} u_{0, j-1}, 0, \ldots, 0, p^{2} u_{m, j+1}\right] .
$$

Finally, the (S)AGE and (D) AGE are then defined by,

$$
\left.\begin{array}{l}
\left(I+p^{2} G_{1}\right) \underline{u}_{j+1}=2 I_{u_{j}}-\left(I+p^{2} G_{2}\right) \underline{u}_{j-1}+b_{1},  \tag{9.2.19}\\
\left(I+p^{2} G_{2}\right) \underline{u}_{j+2}=2 \underline{u}_{j+1}-\left(I+p^{2} G_{1}\right) \underline{u}_{j}+b_{2},
\end{array}\right\}
$$

and

$$
\left.\begin{array}{r}
\left(I+p^{2} G_{1}\right) \underline{u}_{j+1}=2 I_{\underline{u}}-\left(I+p^{2} G_{2}\right) \underline{u}_{j-1}+\underline{b}_{1},  \tag{9.2.20}\\
\left(I+p^{2} G_{2}\right) \underline{u}_{j+2}=2 \underline{I}_{j+1}-\left(I+p^{2} G_{1}\right) \underline{u}_{j}+\underline{b}_{2}, \\
\left(I+p^{2} G_{2}\right) \underline{u}_{j+3}=2 \underline{I}_{j+2}-\left(I+p^{2} G_{1}\right) \underline{u}_{j+1}+\underline{b}_{2}, \\
\left(I+p^{2} G_{1}\right) \underline{u}_{j+4}=2 \underline{I}_{j+3}-\left(I+p^{2} G_{2}\right) \underline{u}_{j+2}+\underline{b}_{1},
\end{array}\right\}
$$

respectively.

## Truncation Error

The estimate of the truncation errors of the above method can be found by expanding the terms in (9.2.12)-(9.2.14) as a Taylor series expansion centred at $(i, j)$ and $(i+1, j),(1, j)$ and ( $m-1, j$ ) respectively.

For the first equation of (9.2.12) the truncation error is given by

$$
\begin{gather*}
T_{9.2 .9(i)}=\left(\frac{-2 p}{1+2 p^{2}}\right) \frac{\partial^{2} u}{\partial x \partial t}+\left(\frac{2 p^{2}}{1+2 p^{2}}\right) \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}+\left(\frac{1}{1+2 p^{2}}\right) \frac{1}{\left(\Delta t^{2}\right)} \\
0\left(\Delta x^{\alpha} \Delta^{\alpha} t^{\alpha}\right), \tag{9.2.21}
\end{gather*}
$$

and for the second equation of (9.2.12) the truncation error is given by

$$
\begin{gather*}
T_{9.2 .9(i i)}=\left(\frac{2 p}{1+2 p^{2}}\right) \frac{\partial^{2} u}{\partial x \partial t}+\left(\frac{2 p^{2}}{1+2 p^{2}}\right) \Delta t \frac{\partial^{3} u}{\partial x^{2} \partial t}+\left(\frac{1}{1+2 p^{2}}\right) \frac{1}{\Delta t^{2}} \\
0\left(\Delta x^{\alpha} \Delta t^{2}\right), \tag{9.2.22}
\end{gather*}
$$

where $\alpha_{1}+\alpha_{2}=4,0 \leq \alpha_{1}, \alpha_{2} \leq 4$.
The expression for the truncation errors above indicate a possible inconsistency of the method in approximating the solution of (9.2.1) if the derivative $\frac{\partial^{2} u}{\partial x \partial t}$ in the solution domain is significant. This is supported by the remark in Saul'yev (1964, p.86) where the asymmetrical type of approximation for the solution of second order hyperbolic equation is stated to be a class of divergence approximations. This conclusion
therefore excludes the possibility of extending the strategy further for hyperbolic equations. Due to this the stability analysis of the method was abandoned.

## Numerical Examples

For computational confirmation the (D)AGE method was tested with the following problems:-
(a) $f_{1}(x)=100 x^{2}, f_{2}(x)=200 x, g_{1}(t)=100 t^{2}$ and $g_{2}(t)=100(1+t)^{2}$. The exact solution can be easily verifed as (Ames, 1977, p.197)

$$
\begin{equation*}
u(x, t)=100(x+t)^{2} \tag{9.2.23}
\end{equation*}
$$

(b) $f_{1}(x)=\sin \pi x, f_{2}(x)=0, g_{1}(t)=g_{2}(t)=0$. The theoretical solution is given by (G.D. Smith, 1978, p.199),

$$
\begin{equation*}
u(x, t)=\sin \pi x \cos \pi t \tag{9.2.24}
\end{equation*}
$$

For both cases, the solution for various values of $p$ are computed and they are displayed in Tables (9.2.1)-(9.2.2). In both cases the method is unstable when $p \geqslant 1$.

Problem (a)
$\Delta x=0.1, \quad t=1.0$

| $x$ | EXACT | $p=0.1$ <br> $\Delta t=0.01$ | $p=0.2$ <br> $\Delta t=0.02$ | $p=0.5$ <br> $\Delta t=0.05$ | $p=1.0$ <br> $\Delta t=0.1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| 0.1 | 121.00 | 120.97 | 120.97 | 120.55 | 121.73 |
| 0.2 | 144.00 | 143.98 | 143.92 | 144.77 | 137.10 |
| 0.3 | 169.00 | 169.01 | 169.04 | 168.39 | 167.97 |
| 0.4 | 196.00 | 196.01 | 195.98 | 196.14 | 187.37 |
| 0.5 | 225.00 | 224.99 | 225.02 | 224.65 | 231.52 |
| 0.6 | 256.00 | 256.01 | 255.98 | 256.75 | 256.64 |
| 0.7 | 289.00 | 289.01 | 289.04 | 288.50 | 291.11 |
| 0.8 | 324.00 | 323.98 | 323.92 | 324.25 | 321.07 |
| 0.9 | 361.00 | 360.97 | 350.97 | 360.40 | 363.64 |
| 1.0 | 400.00 | 400.00 | 400.00 | 400.00 | 400.00 |

$\Delta x=0.1, \quad t=2.0$

| x | EXACT | $\mathrm{p}=0.2$ <br> $\Delta t=0.02$ | $\mathrm{p}=0.5$ <br> $\Delta t=0.05$ | $\mathrm{p}=1.0$ <br> $\Delta \mathrm{t}=0.1$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.0 | 400.00 | 400.00 | 400.00 | 400.00 |
| 0.1 | 441.00 | 441.03 | 441.06 | 472.59 |
| 0.2 | 484.00 | 484.08 | 484.13 | 439.75 |
| 0.3 | 529.00 | 529.05 | 528.72 | 486.15 |
| 0.4 | 576.00 | 575.98 | 576.60 | 588.27 |
| 0.5 | 625.00 | 624.95 | 624.51 | 627.83 |
| 0.6 | 676.00 | 675.98 | 675.95 | 684.51 |
| 0.7 | 729.00 | 729.05 | 728.98 | 729.41 |
| 0.8 | 784.00 | 784.08 | 784.75 | 822.08 |
| 0.9 | 841.00 | 841.03 | 840.58 | 873.84 |
| 1.0 | 900.00 | 900.00 | 900.00 | 900.00 |

TABLE (9.2.1)

## Problem (b)

$\Delta x=0.1, t=1.0$

| $x$ | EXACT | $p=0.1$ <br> $\Delta t=0.01$ | $p=0.2$ <br> $\Delta t=0.02$ | $p=0.5$ <br> $\Delta t=0.05$ | $p=1.0$ <br> $\Delta t=0.1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | -0.30902 | -0.30893 | -0.30891 | -0.31473 | -0.59725 |
| 0.2 | -0.58779 | -0.58761 | -0.58741 | -0.53656 | -0.70066 |
| 0.3 | -0.80902 | -0.80877 | -0.80858 | -0.80630 | -1.48030 |
| 0.4 | -0.95106 | -0.95077 | -0.95044 | -0.94524 | -1.51631 |
| 0.5 | -1.00000 | -0.99970 | -0.99943 | -1.00654 | -1.83855 |

$\Delta x=0.1, t=2.0$

| x | EXACT | $\mathrm{p}=0.2$ <br> $\Delta \mathrm{t}=0.02$ | $\mathrm{p}=0.5$ <br> $\Delta \mathrm{t}=0.05$ |
| :---: | :---: | :---: | :---: |
| 0.1 | 0.30902 | 0.30854 | 0.32242 |
| 0.2 | 0.58779 | 0.58703 | 0.57551 |
| 0.3 | 0.80902 | 0.80779 | 0.80196 |
| 0.4 | 0.95106 | 0.94982 | 0.95034 |
| 0.5 | 1.00000 | 0.99861 | 1.01464 |

TABLE (9.2.2)

### 9.3 A NEW STRATEGY OF SOLVING SECOND ORDER HYPERBOLIC EQUATIONS USING ASSYMMETRIC FORMULAE

In this section we will explore how the assymmetry formulae are able to approximate the solution of the equation (9.2.1) in different ways in order to overcome the difficulties experienced in the previous section.

To do this, we define a new variable $q=q(x, t)$ such that,

$$
\begin{equation*}
q=\frac{\partial u}{\partial t} . \tag{9.3.1}
\end{equation*}
$$

Therefore, the equation (9.2.1) is now written by,

$$
\begin{equation*}
\frac{\partial g}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}} \tag{9.3.2}
\end{equation*}
$$

with the initial conditions,

$$
\begin{align*}
& u(x, 0)=f_{1}(x),  \tag{9.3.3}\\
& q(x, 0)=f_{2}(x),
\end{align*}
$$

and boundary conditions,

$$
\begin{align*}
& q(0, t)=g_{1}^{\prime}(t),  \tag{9.3.4}\\
& q(1, t)=g_{2}^{\prime}(t),
\end{align*}
$$

in addition to the conditions (9.2.3).
The equation (9.3.2) is now in almost a parabolic form and hence the assymmetrical formulae discussed in the previous chapters are now possible.

The Formulation
At the point $\left(i, j+\frac{1}{2}\right)$ the equations (9.3.2) and (9.3.1) are approximated by,

$$
\begin{equation*}
\frac{q_{i, j+1}-q_{i, j}}{\Delta t}=\frac{1}{(\Delta x)^{2}}\left\{u_{i+1, j+1}-u_{i, j+1}-u_{i, j}+u_{i-1, j}\right\}, \tag{9.3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{q_{i, j+1}+q_{i, j}}{2}=\frac{u_{i, j+1}-u_{i, j}}{\Delta t} \tag{9.3.6}
\end{equation*}
$$

respectively. Whilst at point ( $i+1, j+\frac{1}{2}$ ) they are approximated by,

$$
\begin{equation*}
\frac{q_{i+1, j+1}-q}{\Delta t} \frac{1}{\Delta t, j}=\frac{1}{(\Delta x)^{2}}\left\{u_{i, j+1}-u_{i+1, j+1}^{-u_{i+1, j}}+u_{i+2, j}\right\}, \tag{9.3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{q_{i+1, j+1}+q_{i+1, j}}{2}=\frac{u_{i+1, j+1}^{-u_{i+1, j}}}{\Delta t} \tag{9.3.8}
\end{equation*}
$$

respectively.
The equations (9.3.5)-(9.3.6) can be simplified to (Fig. (9.3.1)),

$$
\begin{equation*}
-p^{2} u_{i+1, j+1}+\left(2+p^{2}\right) u_{i, j+1}=\left(2-p^{2}\right) u_{i, j}+p^{2} u_{i-1, j}+2 \Delta t q_{i, j} \tag{9.3.9}
\end{equation*}
$$

and the equations (9.3.7)-(9.3.8) are simplified to,

$$
-p^{2} u_{i, j+1}+\left(2+p^{2}\right) u_{i+1, j+1}=\left(2-p^{2}\right) u_{i+1, j}+p^{2} u_{i+2, j}+2 \Delta t q_{i+1, j}, \text { (9.3.10) }
$$

From these formulations, the following algorithms are possible.

## The Algorithms

1. The equations (9.3.9) and (9.3.10) can be implemented individually as a semi-explicit method. That is,

$$
\begin{equation*}
u_{i, j+1}=\frac{1}{\left(2+p^{2}\right)}\left\{p^{2} u_{i+1, j+1}+\left(2-p^{2}\right) u_{i, j}+p^{2} u_{i-1, j}+2 \Delta t q_{i, j}\right\} \tag{9.3.11}
\end{equation*}
$$

for the negative direction calculation, and

$$
\begin{equation*}
u_{i, j+1}=\frac{1}{\left(2+p^{2}\right)}\left\{p^{2} u_{i-1, j+1}+\left(2-p^{2}\right) u_{i, j}+p^{2} u_{i+1, j}+2 \Delta t q_{i, j}\right\} \tag{9.3.12}
\end{equation*}
$$

for the positive direction calculation. In both cases, $q$ is given by

$$
\begin{equation*}
q_{i, j+1}=\frac{2}{\Delta t}\left(u_{i, j+1}^{-u_{i, j}}\right)-q_{i, j} . \tag{9.3.13}
\end{equation*}
$$

2. The equations (9.3.11) and (9.3.12) are implemented alternately.
3. At every time-level, the average to the solutions of (9.3.12) and (9.3.13) is regarded as the approximate solution to (9.2.1).
4. The equations (9.3.9) and (9.3.10) can be implemented as a Group Explicit method whose implicit system is given by,

$$
\begin{align*}
\left(\begin{array}{cc}
2+p^{2} & -p^{2} \\
-p^{2} & 2+p^{2}
\end{array}\right)\binom{u_{i, j+1}}{u_{i+1, j+1}} & =\left(\begin{array}{cc}
2-p^{2} & 0 \\
0 & 2-p^{2}
\end{array}\right)\binom{u_{i, j}}{u_{i+1, j}}+2 \Delta t\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)\binom{q_{i, j}}{q_{i+1, j}} \\
& +\binom{p^{2} u_{i-1, j}}{p^{2} u_{i+2, j}}, \tag{9.3.14}
\end{align*}
$$



$$
q=\frac{\partial u}{\partial t}
$$



Computational molecule for schemes (9.3.9) and (9.3.10) FIGURE (9.3.1)

and explicitly written as (Fig. (9.3.2)),

$$
\begin{align*}
& \binom{u_{i, j+1}}{u_{i+1, j+1}}=\frac{1}{\Delta}\left[\left(\begin{array}{cc}
4-p^{4} & p^{2}\left(2-p^{2}\right) \\
p^{2}\left(2-p^{2}\right) & 4-p^{4}
\end{array}\right)\binom{u_{i, j}}{u_{i+1, j}}+\right. \\
& \left.\quad+2 \Delta t\left(\begin{array}{cc}
\left(2+p^{2}\right) & p^{2} \\
p^{2} & \left(2+p^{2}\right)
\end{array}\right)\binom{q_{i, j}}{q_{i+1, j}}+\binom{p^{2}\left(2+p^{2}\right) u_{i-1, j}+p^{4} u_{i+2, j}}{p^{4} u_{i-1, j}+p^{2}\left(2+p^{2}\right) u_{i+2, j}}\right] \tag{9.3.15}
\end{align*}
$$

with $\Delta=4+4 p^{2}$, for any group of 2 -points. For the ungrouped point near the boundary, equation (9.3.11) or (9.3.12) is used, depending on the position of the point. From (9.3.15), (9.3.11) and (9.3.12), the GEL, GER, (S)AGE and (D)AGE scheme as previously discussed can also be developed.

For truncation errors, then by using Taylor's expansion it can be shown that the approximations (9.3.5) and (9.3.7) are of $O\left(\frac{\Delta t}{\Delta x}+(\Delta x)^{2}+(\Delta t)^{2}\right)$ with the coefficients to $\left(\frac{\Delta t}{\Delta x}\right)$ being of opposite sign. Therefore, for algorithms (2) and (3), the accuracy is approximately of $O\left((\Delta x)^{2}+(\Delta t)^{2}\right)$.

As this technique was discovered at the end of the research programme and the time for a detailed analysis was limited, therefore the theoretical analysis of stability was not vigorously pursued. However, the following numerical results were considered sufficient to indicate the stability regions.

## The Numerical Experiments

The two model problems considered in the previous section were again used for testing the various schemes in this section. Among the schemes chosen was the schemes, GEL, GER, (S)AGE and algorithm (2). Numerically, for problem (a), we found that algorithm (2) are unconditionally stable and the GEL, GER, (S)AGE schemes are stable for $p=\frac{\Delta t}{\Delta x} \leqslant 1$. Some of the numerical results are given in Tables (9.3.1)-(9.3.4). Furthermore the
(S)AGE and Algorithm (2) schemes yield more accurate results compared to those given by the GEL and GER schemes.

For problem (b), the results of which are given in Tables (9.3.5)(9.3.6), the inaccuracy of GEL and GER schemes are clearly seen as they only give acceptable results for $p=0.1$ and the results show signs of divergence for $p=0.5$. The reason for this divergence is that the contribution of the truncation errors affects the consistency of the solution to the original problem. Hence, the condition for convergence (Lax Theorem) is not fulfilled.

From the above results, we can conclude that the use of assymmetric formulae for the solution of second order hyperbolic equations is possible (as opposed to Saul' yev's remark mentioned earlier) provided the problem is reformulated as a parabolic equation.

## Problem (a)

$\mathrm{x}=0.1, \Delta \mathrm{t}=0.01, \mathrm{r}=1.0, \mathrm{p} \frac{\Delta \mathrm{t}}{\Delta \mathrm{x}}=0.1, \mathrm{t}=1.0$

| x | EXACT | ALGORITHM 2 | GEL | GER | (S)AGE |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 |
| 0.1 | 121.000 | 121.004 | 120.855 | 121.086 | 120.985 |
| 0.2 | 144.000 | 144.010 | 143.856 | 144.052 | 144.105 |
| 0.3 | 169.000 | 169.014 | 168.814 | 169.062 | 168.892 |
| 0.4 | 196.000 | 196.016 | 195.834 | 196.002 | 196.026 |
| 0.5 | 225.000 | 225.017 | 224.803 | 225.029 | 224.992 |
| 0.6 | 256.000 | 256.020 | 255.835 | 256.005 | 256.039 |
| 0.7 | 289.000 | 289.017 | 288.794 | 289.046 | 288.893 |
| 0.8 | 324.000 | 324.010 | 323.840 | 324.041 | 324.098 |
| 0.9 | 361.000 | 361.005 | 360.850 | 361.094 | 360.972 |
| 1.0 | 400.000 | 400.000 | 400.000 | 400.000 | 400.000 |

TABLE (9.3.1)

Problem (a)
$\Delta x=0.1, \Delta t=0.05, r=5.0, p=0.5, t=5.0$

| $\mathbf{x}$ | EXACT | ALGORITHM (2) | GEL | GER | (S)AGE |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 2500 | 2500 | 2500 | 2500 | 2500 |
| 0.1 | 2601 | 2601 | 2600 | 2602 | 2601 |
| 0.2 | 2704 | 2704 | 2704 | 2704 | 2704 |
| 0.3 | 2809 | 2916 | 2916 | 2808 | 2810 |
| 0.4 | 3025 | 2916 | 2916 | 2809 |  |
| 0.5 | 3025 | 3136 | 3024 | 3026 | 2916 |
| 0.6 | 3136 | 3249 | 3136 | 3136 | 3025 |
| 0.7 | 3249 | 3364 | 3248 | 3250 | 3249 |
| 0.8 | 3364 | 3481 | 3364 | 3364 | 3364 |
| 0.9 | 3481 | 3600 | 3480 | 3482 | 3481 |
| 1.0 | 360 |  | 3600 | 3600 | 3600 |

TABLE (9.3.2)

## Problem (a)

$\Delta x=0.1, \Delta t=0.1, r=10.0, p=1.0, \quad t=5.0$

| $\mathbf{x}$ | EXACT | ALGORITHM (2) | GEL | GER | (S)AGE |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 2500 | 2500 | 2500 | 2500 | 2500 |
| 0.1 | 2601 | 2600 | 2608 | 2606 | 2601 |
| 0.2 | 2704 | 2703 | 2698 | 2706 | 2704 |
| 0.3 | 2809 | 2808 | 2797 | 2798 | 2809 |
| 0.4 | 2916 | 3915 | 2925 | 2914 | 2916 |
| 0.5 | 3025 | 3135 | 3027 | 3035 | 3025 |
| 0.6 | 3249 | 3363 | 3126 | 3139 | 3136 |
| 0.7 | 3364 | 3481 | 3369 | 3256 | 3249 |
| 0.8 | 3481 | 3600 | 3488 | 3469 | 3364 |
| 0.9 |  | 3600 | 3600 | 3481 |  |
| 1.0 |  |  | 3600 |  |  |

Problem (a), ALGORITHM (2)
$\Delta x=0.1, t=10.0$

| x | EXACT | $\mathrm{r}=20, \mathrm{p}=2.0$ <br> $\Delta \mathrm{t}=0.2$ | $\mathrm{r}=50.0, \mathrm{p}=5.0$ <br> $\Delta t=0.5$ | $\mathrm{r}=100, \mathrm{p}=10.0$ <br> $\Delta \mathrm{t}=1.0$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.0 | 10000 | 10000 | 10000 | 10000 |
| 0.1 | 10201 | 10196 | 10188 | 10154 |
| 0.2 | 10404 | 10398 | 10384 | 10355 |
| 0.3 | 10609 | 10600 | 10585 | 10560 |
| 0.4 | 10816 | 10804 | 10790 | 10768 |
| 0.5 | 11025 | 11011 | 10999 | 10981 |
| 0.6 | 11236 | 11222 | 11211 | 11197 |
| 0.7 | 11449 | 11438 | 11427 | 11417 |
| 0.8 | 11664 | 11656 | 11647 | 11641 |
| 0.9 | 11881 | 11877 | 11872 | 11869 |
| 1.0 | 12100 | 12100 | 12100 | 12100 |

TABLE (9.3.4)

## Problem (b)

$\Delta x=0.1, \Delta t=0.01, r=1.0, p=0.1, t=1.0$

| x | EXACT | ALGORITHM (2) | GEL | GER | (S)AGE |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | -0.3090 | -0.3091 | -0.3086 | -0.3095 | -0.3091 |
| 0.2 | -0.5878 | -0.5879 | -0.5877 | -0.5881 | -0.5879 |
| 0.3 | -0.8090 | -0.8091 | -0.8093 | -0.8091 | -0.8092 |
| 0.4 | -0.9511 | -0.9511 | -0.9513 | -0.9512 | -0.9512 |
| 0.5 | -1.0000 | -1.0001 | -1.0002 | -1.0003 | -0.0002 |

$\Delta x=0.1, \Delta t=0.05, r=5.0, p=0.5, t=5.0$

| $x$ | EXACT | ALGORITHM (2) | GEL | GER | (S)AGE |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | -0.3090 | -0.3067 | -0.4701 | 0.0355 | -0.3084 |
| 0.2 | -0.5878 | -0.5840 | -0.6741 | -0.5002 | -0.5855 |
| 0.3 | -0.8090 | -0.8043 | -0.8868 | -1.2080 | -0.8062 |
| 0.4 | -0.9511 | -0.9458 | -0.8061 | -1.0940 | -0.9472 |
| 0.5 | -1.0000 | -0.9946 | -0.7060 | -0.7059 | -0.9961 |

## Problem (b)

$\Delta x=0.1, \Delta t=0.1, r=10.0, \quad p=1.0, t=10.0$

| $\mathbf{x}$ | EXACT | ALGORITHM (2) | (S) AGE |
| :---: | :---: | :---: | :---: |
| 0.1 | 0.3090 | 0.2381 | 0.3125 |
| 0.2 | 0.5878 | 0.4580 | 0.5906 |
| 0.3 | 0.8090 | 0.6325 | 0.8069 |
| 0.4 | 0.9511 | 0.7456 | 0.9455 |
| 0.5 | 1.0000 | 0.7853 | 0.9928 |
| 0.6 | 0.9511 | 0.7464 | 0.9452 |
| 0.7 | 0.8090 | 0.6341 | 0.8077 |
| 0.8 | 0.5878 | 0.4638 | 0.5895 |
| 0.9 | 0.3090 | 0.2437 | 0.3122 |

TABLE (9.3.6)

### 9.4 THE GROUP EXPLICIT METHOD INCORPORATING THE CRANK-NICOLSON FORMULAE

The GE method can also be developed using different types of formulae. In this section, the GE method will be developed utilising the CrankNicolson six-point formulae. For the discussion, consider the one-dimensional heat-conduction equation,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}, 0 \leq x \leq 1, t \geqslant 0, \tag{9.4.1}
\end{equation*}
$$

in the specified open-rectangle. Without loss of generality, it is assumed that the space domain is divided into $m$, an even number of sub-intervals and hence the number of unknown points (i.e. m-1) along the time-level is odd if the boundary points are considered known.

$$
\begin{aligned}
& \text { Initially, let us consider the Crank-Nicolson formulae, } \\
& -\frac{r}{2} u_{i-1, j+1}+(1+r) u_{i, j+1}-\frac{r}{2} u_{i+1, j+1}=\frac{r}{2} u_{i-1, j}+(1-r) u_{i, j}+\frac{r}{2} u_{i+1, j},
\end{aligned}
$$

and the assymetric formulae,

$$
\begin{align*}
-r u_{i+1, j+1}+(1+r) u_{i, j+1} & =(1-r) u_{i, j}+r u_{i-1, j},  \tag{9.4.3}\\
\text { and } \quad-r u_{i-1, j+1}+(1+r) u_{i, j+1} & =(1-r) u_{i, j}+r u_{i+1, j},
\end{align*}
$$

$$
\text { Now, at points }(i, j+1) \text { and }(i+1, j+1), i=1,3,5, \ldots, m-3 \text {, the equation }
$$

(9.4.1) is approximated by the CN formulae (9.4.2) and the assymmetric formulae (9.4.4) respectively to form the system,

$$
\left[\begin{array}{cc}
1+r & -\frac{r}{2}  \tag{9.4.5}\\
-r & 1+r
\end{array}\right]\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
1-r & r / 2 \\
0 & 1-r
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{c}
\frac{r}{2} u_{i-1, j+1}+\frac{r}{2} u_{i-1, j} \\
r u_{i+2, j}
\end{array}\right]
$$

and at point ( $m-1, j+1$ ) it is approximated by,

$$
\begin{equation*}
u_{m-1, j+1}=\frac{1}{(1+r)}\left\{\frac{r}{2} u_{m-2, j+1}+\frac{r}{2} u_{m, j+1}+\frac{r}{2} u_{m-2, j}+\frac{r}{2} u_{m, j}+(1-r) u_{m-1, j}\right\} \tag{9.4.6}
\end{equation*}
$$

resulting from the CN formulae.

The equation (9.4.5) can be easily inverted to the explicit system (Fig. (9.4.1))
where $\Delta=1+2 r+\frac{r^{2}}{2}$, i.e.,

$$
\left[\begin{array}{l}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{\Delta}\left[\begin{array}{c}
\left(1-r^{2}\right) u_{i, j}+r u_{i+1, j}+\frac{r}{2}(1+r)\left(u_{i-1, j+1}+u_{i-1, j}\right)+\frac{r^{2}}{2} u_{i+2, j}  \tag{9.4.8}\\
r(1-r) u_{i, j}+\left(1-\frac{r^{2}}{2}\right) u_{i+1, j}+\frac{r^{2}}{2}\left(u_{i-1, j+1}+u_{i-1, j}\right)+r(1+r)
\end{array}\right]
$$

Therefore the system (9.4.8) $(i=1,3, \ldots, m-3)$ and (9.4.6) are used at every time-level to form the Group Explicit method with Left ungrouped point (CEL) as in Chapter 4.

Now if at every time level, the equations (9.4.2) and (9.4.3) are used at points $(i+1, j+1)$ and $(i, j+1), i=m-2, m-4, \ldots, 5,3$ respectively, they will result in the implicit system,

$$
\left[\begin{array}{cc}
1+r & -r  \tag{9.4.9}\\
-\frac{r}{2} & 1+r
\end{array}\right]\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\left[\begin{array}{cc}
1-r & 0 \\
\frac{r}{2} & 1-r
\end{array}\right]\left[\begin{array}{c}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\left[\begin{array}{c}
r u_{i-1, j} \\
\frac{r}{2}\left(u_{i+2, j+1}+u_{i+2, j}\right)
\end{array}\right]
$$

while at point $(1, j+1)$ it is approximated by,

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{(1+r)}\left\{\frac{r}{2}\left(u_{0, j+1}+u_{0, j}\right)+\frac{r}{2}\left(u_{2, j+1}+u_{2, j}\right)+(1-r) u_{1, j}\right\} \tag{9.4.10}
\end{equation*}
$$

from the CN formula.
Similarly, the equation (9.4.9) can be inverted into the explicit system (Fig. (9.4.2)),


Computational molecule for the scheme (9.4.8)
FIGURE (9.4.1)


Computational molecule for the scheme (9.4.12)
FIGURE (9.4.2)

$$
\begin{align*}
& {\left[\begin{array}{c}
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{\Delta}\left[\begin{array}{cc}
1+r & r \\
r / 2 & 1+r
\end{array}\right]\left\{\left[\begin{array}{cc}
1-r & 0 \\
r / 2 & 1-r
\end{array}\right]\left[\begin{array}{l}
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+\right.} \\
& {\left.\left[\begin{array}{c}
r u_{i-1, j} \\
\frac{r}{2}\left(u_{i+2, j+1}+u_{i+2, j}\right.
\end{array}\right]\right\} } \tag{9.4.11}
\end{align*}
$$

ie.,

$$
\left[\begin{array}{c}
u_{i, j+1}  \tag{9.4.12}\\
u_{i+1, j+1}
\end{array}\right]=\frac{1}{\Delta}\left[\begin{array}{c}
\left(1-\frac{r^{2}}{2}\right) u_{i, j}+r(1-r) u_{i, j}+r(1+r) u_{i-1, j}+\frac{r^{2}}{2}\left(u_{i+2, j}+\right. \\
\left.u_{i+2, j+1}\right) \\
r u_{i, j}+\left(1-r^{2}\right) u_{i+1, j}+\frac{r^{2}}{2} u_{i-1, j}+\frac{r}{2}(1+r)\left(u_{i+2, j}+u_{i+2, j+1}\right)
\end{array}\right]
$$

The system (9.4.12) ( $\mathrm{i}=2,4, \ldots, \mathrm{~m}-2$ ) and (9.4.10) when used at every time-
level forms the Group Explicit method with Right ungrouped point (GER).
The GEL method, in implicit form can be written as,

$$
\begin{equation*}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1}, \tag{9.4.13}
\end{equation*}
$$

where,
and

$$
\underline{b}_{1}^{T}=\left[\frac{r}{2}\left(u_{0, j+1}+u_{0, j}\right), 0, \ldots, 0, \frac{r}{2}\left(u_{m, j+1}+u_{m, j}\right)\right],
$$

while the GER method is given by,

$$
\begin{equation*}
\left(I+r G_{2}\right) \underline{u}_{j+1}=\left(I-r G_{1}\right) \underline{u}_{j}+\underline{b}_{1} \tag{9.4.16}
\end{equation*}
$$

Similar to the previous discussion the concept can now be developed to the (Single) Alternating Group Explicit ((S)AGE) and (Double) Alternating Group Explicit Methods, i.e.,
and

$$
\left.\begin{array}{l}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1},  \tag{9,4.17}\\
\left(I+r G_{2}\right) \underline{u}_{j+2}=\left(I-r G_{1}\right) \underline{u}_{j+1}+\underline{b}_{1},
\end{array}\right\}
$$

$$
\left.\begin{array}{l}
\left(I+r G_{1}\right) \underline{u}_{j+1}=\left(I-r G_{2}\right) \underline{u}_{j}+\underline{b}_{1},  \tag{9.4.18}\\
\left(I+r G_{2}\right) \underline{u}_{j+2}=\left(I-r G_{1}\right) \underline{u}_{j+1}+\underline{b}_{1}, \\
\left(I+r G_{2}\right) \underline{u}_{j+3}=\left(I-r G_{1}\right) \underline{u}_{j+2}+\underline{b}_{1}, \\
\left(I+r G_{1}\right) \underline{u}_{j+4}=\left(I-r G_{2}\right) \underline{u}_{j+3}+\underline{b}_{1},
\end{array}\right\}
$$

respectively.
For the stability of $(9.4 .16)$, it is necessary to show that the spectral radius of

$$
\begin{equation*}
\left(I+r G_{2}\right)^{-1}\left(I-r G_{1}\right)<1 \tag{9.4.19}
\end{equation*}
$$

in modulus. Since $G_{2}$ and $G_{1}$ are unsymmetric matrices not sharing a common eigenvector basis, then no further definitive analytical results can be stated. However, the analysis of stability for the schemes (9.4.17) and (9.4.18) is possible but due to the limited time available, this will appear later in separate reports.

The truncation errors of the equations in this class of methods, which is determined by a Taylor's series expansion, can be shown to approximate the equation (9.4.1) with error term $T$ given by,
and

The results of the numerical experiments will be given at the end of the next section.

$$
\begin{align*}
& T_{(9.4 .12 a)} \approx-\frac{(1+r)}{(2+3 r)} \frac{\Delta t}{\Delta x} u_{x t}-\frac{\Delta t^{2}}{24} \mu_{t t t}-\frac{\Delta x^{2}}{2} \frac{r(1+2 r)}{(2+3 r)} u_{x x t}+ \tag{9.4,20}
\end{align*}
$$

$$
\begin{align*}
& 2\left(\frac{\Delta t}{\Delta \mathbf{x}}\right) \frac{1}{(1+2 r)} u_{x x x} \tag{9.4.21}
\end{align*}
$$

### 9.5 SOME EXPLICIT PROCEDURES FOR SOLVING THE DIFFUSION EQUATION

Apart from the above mentioned schemes of the GE class of methods there are a few more explicit schemes which are derivable from (9.4.6), (9.4.8) , (9.4.10) and (9.4.12) respectively. They are as follows:-
(i) For $i=1,2, \ldots, m-2$, the first equation of (9.4.8), i.e.,

$$
\begin{equation*}
u_{i, j+1}=\frac{1}{\Delta}\left[\left(1-r^{2}\right) u_{i, j}+r_{i+1, j}+\frac{r}{2}(1+r)\left(u_{i-1, j+1}+u_{i-1, j}\right)+\frac{r^{2}}{2}{ }_{i+2, j}\right\}, \tag{9.5.1}
\end{equation*}
$$

is used and for $i=m-1$ the equation (9.4.6) i.e.,

$$
\begin{equation*}
u_{m-1, j+1}=\frac{1}{(1+r)}\left\{\frac{r}{2}\left(u_{m-2, j+1}+u_{m, j+1}+u_{m-2, j}+u_{m, j}\right)+(1-r) u_{m-1, j}\right\} \tag{9.5.2}
\end{equation*}
$$

is used.
(ii) For $i=1$, the solution is approximated by the equation (9.5.1) to give,

$$
u_{1, j+1}=\frac{1}{\Delta}\left\{\left(1-r^{2}\right) u_{1, j}+r u_{2, j}+\frac{r}{2}(1+r)\left(u_{0, j+1}+u_{0, j}\right)+\frac{r^{2}}{2} u_{3, j}\right\},(9 \cdot 5.3)
$$

and for $i=2,3, \ldots, m-1$ the second equation of (9.4.8) i.e.,

$$
\begin{align*}
u_{i, j+1}=\frac{1}{\Delta}\left\{r(1-r) u_{i-1, j}+\left(1-\frac{r^{2}}{2}\right) u_{i, j}+\right. & \frac{r^{2}}{2}\left(u_{i-2, j+1}+u_{i-2, j}\right)+ \\
& \left.+r(1+r) u_{i+1, j}\right\} \tag{9.5.4}
\end{align*}
$$

is used.
(iii) For $i=m-1, m-2, \ldots, 3,2$, the second equation of (9.4.12), i.e.,

$$
\begin{equation*}
u_{i, j+1}=\frac{1}{\Delta}\left\{r u_{i-1, j}+\left(1-r^{2}\right) u_{i, j}+\frac{r^{2}}{2} u_{i-2, j}+\frac{r}{2}(1+r)\left(u_{i+1, j}+u_{i+1, j+1}\right)\right\} \tag{9.5.5}
\end{equation*}
$$

is used and for $i=1$ the equation (9.4.10), i.e.,

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{(1+r)}\left\{\frac{r}{2}\left(u_{0, j+1}+u_{0, j}+u_{2, j+1}+u_{2, j}\right)+(1-r) u_{1, j}\right\} \tag{9.5.6}
\end{equation*}
$$

is used.
(iv) For $i=m-1$, the solution is approximated by the equation (9.5.5) to result in,

$$
\begin{gather*}
u_{m-1, j+1}=\frac{1}{\Delta}\left\{r u_{m-2, j}+\left(1-r^{2}\right) u_{m-1, j}+\frac{r^{2}}{2} u_{m-3, j}+\frac{r}{2}(1+r)\right. \\
\left.\left(u_{m, j}+u_{m, j+1}\right)\right\} \tag{9.5.7}
\end{gather*}
$$

and for $i=m-2, m-3, \ldots, 3,2,1$ the first equation of (9.4.12) i.e.,

$$
\begin{gather*}
u_{i, j+1}=\frac{1}{\Delta}\left\{\left(1-\frac{r^{2}}{2}\right) u_{i, j}+r(1-r) u_{i+1, j}+r(1+r) u_{i-1, j}+\right. \\
\left.\frac{r^{2}}{2}\left(u_{i+2, j}+u_{i+2, j+1}\right)\right\}, \tag{9.5.8}
\end{gather*}
$$

is employed.
(v) Another strategy is to use any combination of two of the above four strategies at alternating time levels. This alternating use of two strategies along the line will normally tend to cancel some truncation error terms and will give a better approximation. Examples of possible combinations are between strategies (i) and (iii) or strategies (ii) and (iv). (vi) An average between two schemes is also an alternative strategy which sometimes gives a better approximation. In this case this approach is also possible.

In the numerical experiments, some schemes discussed in this section and the previous section were implemented for the problem (9.4.1) with the initial condition

$$
\begin{equation*}
u(x, 0)=4 x(1-x) \tag{9.5.9a}
\end{equation*}
$$

and the boundary conditions,

$$
\begin{equation*}
u(0, t)=u(1, t)=0 . \tag{9.5.9b}
\end{equation*}
$$

These results are compared with the analytical solution which is given in Section (4.15). Tables (9.5.1)-(9.5.4) show some of the results obtained using various values of r. From these tables, we can see that the schemes in this section and those of the previous section are stable and acceptably accurate. Hence, the extension of these methods to a wide class of problems especially for many space-dimensions is recommended before more definitive conclusions can be given.
$\Delta \mathrm{x}=0.1, \quad \Delta \mathrm{t}=0.001, \quad \mathrm{r}=0.1, \mathrm{t}=0.1$

| x | EXACT | (D)AGE <br> $(9.4 .18)$ | ALGORITHM <br> $(9.5)(\mathrm{ii})$ | ALGORITHM <br> $(9.5)(\mathrm{iv})$ | AVERAGE <br> (ii) \& (iv) | ALGORITHM <br> (v) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.11887 | 0.11983 | 0.12083 | 0.11769 | 0.11926 | 0.11925 |
| 0.2 | 0.22610 | 0.22788 | 0.22984 | 0.22481 | 0.22732 | 0.22736 |
| 0.3 | 0.31119 | 0.31370 | 0.21540 | 0.31070 | 0.31305 | 0.31317 |
| 0.4 | 0.36582 | 0.36875 | 0.36948 | 0.36670 | 0.36809 | 0.36827 |
| 0.5 | 0.38465 | 0.38774 | 0.38706 | 0.38706 | 0.38706 | 0.38726 |
| 0.6 | 0.36582 | 0.36879 | 0.36670 | 0.36949 | 0.36809 | 0.36830 |
| 0.7 | 0.31119 | 0.31370 | 0.31070 | 0.31540 | 0.31305 | 0.31323 |
| 0.8 | 0.22610 | 0.22797 | 0.22481 | 0.22984 | 0.22732 | 0.22745 |
| 0.9 | 0.11887 | 0.11983 | 0.11769 | 0.12083 | 0.11926 | 0.11931 |

TABLE (9.5.1)
$\Delta x=0.1, \quad \Delta t=0.005, \quad r=0.5, t=0.5$

| $\mathbf{x}$ | EXACT | (D)AGE <br> $(9.4 .18)$ | ALGORITHM <br> $(9.5)(\mathrm{ii})$ | ALGORITHM <br> $(9.5)(\mathrm{iv})$ | AVERAGE <br> $(\mathrm{ii}) \&(\mathrm{iv})$ | ALGORITHM <br> (v) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.00229 | 0.00239 | 0.00233 | 0.00206 | 0.00219 | 0.00231 |
| 0.2 | 0.00436 | 0.00451 | 0.00443 | 0.00399 | 0.00421 | 0.00445 |
| 0.3 | 0.00600 | 0.00624 | 0.00599 | 0.00559 | 0.00579 | 0.00616 |
| 0.4 | 0.00706 | 0.00733 | 0.00693 | 0.00669 | 0.00681 | 0.00726 |
| 0.5 | 0.00742 | 0.00772 | 0.00716 | 0.00716 | 0.00716 | 0.00765 |
| 0.6 | 0.00706 | 0.00735 | 0.00669 | 0.00693 | 0.00681 | 0.00728 |
| 0.7 | 0.00600 | 0.00624 | 0.00559 | 0.00599 | 0.00579 | 0.00619 |
| 0.8 | 0.00436 | 0.00456 | 0.00399 | 0.00443 | 0.00421 | 0.00450 |
| 0.9 | 0.00229 | 0.00239 | 0.00206 | 0.00233 | 0.00219 | 0.00234 |

TABLE (9.5.2)
$\Delta x=0.1, \quad \Delta t=0.01, \tau=1.0, t=0.5$

| x | EXACT | (D)AGE <br> (9.4.18) | ALGORITHM <br> (9.5)(ii) | ALGORITHM <br> (9.5) (iv) | AVERAGE <br> (ii) \& (iv) | ALGORITHM <br> $(9.5)(v)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.00229 | 0.00237 | 0.00220 | 0.00175 | 0.00197 | 0.00236 |
| 0.2 | 0.00436 | 0.00440 | 0.00417 | 0.00344 | 0.00380 | 0.00465 |
| 0.3 | 0.00600 | 0.00620 | 0.00555 | 0.00489 | 0.00522 | 0.00652 |
| 0.4 | 0.00706 | 0.00724 | 0.00634 | 0.00594 | 0.00614 | 0.00769 |
| 0.5 | 0.00742 | 0.00766 | 0.00645 | 0.00645 | 0.00645 | 0.00813 |
| 0.6 | 0.00706 | 0.00732 | 0.00594 | 0.00634 | 0.00614 | 0.00775 |
| 0.7 | 0.00600 | 0.00619 | 0.00489 | 0.00555 | 0.00522 | 0.00660 |
| 0.8 | 0.00436 | 0.00460 | 0.00344 | 0.00417 | 0.00380 | 0.00484 |
| 0.9 | 0.00229 | 0.00236 | 0.00175 | 0.00220 | 0.00197 | 0.00249 |

TABLE (9.5.3)
$\Delta x=0.1, \Delta t=0.015, r=1.5, t=0.75$

| x | EXACT | (D)AGE <br> $(9.4 .18)$ | ALGORITHM <br> $(9.5)($ ii) | ALGORITHM <br> (9.5) (iv) | AVERAGE <br> (ii) \& (iv) | ALGORITHM <br> $(9.5)(v)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | 0.00019 | 0.00020 | 0.00015 | 0.00011 | 0.00013 | 0.00024 |
| 0.2 | 0.00037 | 0.00035 | 0.00028 | 0.00021 | 0.00025 | 0.00056 |
| 0.3 | 0.00051 | 0.00051 | 0.00037 | 0.00031 | 0.00034 | 0.00083 |
| 0.4 | 0.00060 | 0.00059 | 0.00042 | 0.00038 | 0.00040 | 0.00079 |
| 0.5 | 0.00063 | 0.00063 | 0.00042 | 0.00042 | 0.00042 | 0.00101 |
| 0.6 | 0.00060 | 0.00060 | 0.00038 | 0.00042 | 0.00040 | 0.00097 |
| 0.7 | 0.00051 | 0.00051 | 0.00031 | 0.00037 | 0.00034 | 0.00068 |
| 0.8 | 0.00037 | 0.00039 | 0.00021 | 0.00028 | 0.00025 | 0.00064 |
| 0.9 | 0.00019 | 0.00019 | 0.00011 | 0.00015 | 0.00013 | 0.00033 |

TABLE (9.5.4)

### 9.6 THE 3-POINT GROUP EXPLICIT METHOD

## Formulation

To date the derivation of the GE methods for the solution of onedimensional diffusion equations was only limited to a group of 2-points. In this section, the method will be extended and developed further by using a group of 3-points.

This 3-point GE method is motivated by higher order formulae, i.e.,

$$
\begin{align*}
& (3 r+2) u_{i}, j+1{ }^{-4 r u_{i-1, j+1}+r u_{i-2, j+1}} \\
& =r u_{i-1, j}+3 r u_{i+1, j}+(2-4 r) u_{i, j} \tag{9.6.1}
\end{align*}
$$

and

$$
\begin{align*}
(3 r+2) u_{i} & , j+1 \\
& =r r_{i+1, j+1}+r u_{i+2, j+1}  \tag{9.6.2}\\
& =r u_{i+1}+3 r u_{i-1, j}+(2-4 r) u_{i, j}
\end{align*}
$$

as suggested by Liu (1969, Fig. (9.6.1).


FIGURE (9.6.1)

Consider now any three points $(i-1, j+1),(i, j+1)$ and $(i+1, j+1)$
along the $(j+1)$ th time-level. At points $(i-1, j+1)$ and $(i+1, j+1)$, we approximate the solution of the diffusion equation by the equation (9.6.2) and (9.6.1) respectively. Whilst at point $(i, j+1)$, the solution is
approximated by the Crank-Nicolson six-points formulae. This approximation will lead to an implicit system of ( $3 \times 3$ ) equations,

$$
\begin{align*}
& {\left[\begin{array}{ccc}
3 r+2 & -4 r & r \\
-r & 2+2 r & -r \\
r & -4 r & 3 r+2
\end{array}\right]\left[\begin{array}{l}
u_{i-1, j+1} \\
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=} {\left[\begin{array}{ccc}
2-4 r & r & 0 \\
r & 2-2 r & r \\
0 & r & 2-4 r
\end{array}\right]\left[\begin{array}{l}
u_{i-1, j} \\
u_{i, j} \\
u_{i+1, j}
\end{array}\right]+} \\
& {\left[\begin{array}{c}
3 r u_{i-2, j} \\
0 \\
3 r u_{i+2, j}
\end{array}\right], } \tag{9.6.3}
\end{align*}
$$

Since the ( $3 \times 3$ ) matrix on the left-hand side can be inverted easily and has the form,

$$
\frac{1}{4(3 r+1)(r+1)}\left[\begin{array}{ccc}
r^{2}+5 r+2 & 4 r(r+1) & r(r-1)  \tag{9.6.4}\\
r(r+1) & 2(2 r+1)(r+1) & r(r+1) \\
r(r-1) & 4 r(r+1) & r^{2}+5 r+2
\end{array}\right] \text {, }
$$

then (9.6.3) can be explicitly represented by (Fig. (9.6.2)),

$$
\begin{align*}
{\left[\begin{array}{l}
u_{i-1, j+1} \\
u_{i, j+1} \\
u_{i+1, j+1}
\end{array}\right]=} & \frac{1}{4(3 r+1)(r+1)}\left\{\left[\begin{array}{lll}
2 r-14 r^{2}+4 & -6 r^{3}+4 r^{2}+10 r & 10 r^{2}-2 r \\
4 r(r+1) & 2(r+1)\left(2+2 r-3 r^{2}\right) & 4 r(r+1) \\
10 r^{2}-2 r & -6 r^{3}+4 r^{2}+10^{r} & 2 r-14 r+4
\end{array}\right]\right. \\
{\left[\begin{array}{l}
u_{i-1, j} \\
u_{i, j} \\
u_{i+1, j}
\end{array}\right] } & \left.+\left[\begin{array}{l}
3 r\left(r^{2}+5 r+2\right) u_{i-2, j}+3 r^{2}(r-1) u_{i+2, j} \\
3 r^{2}(r+1) u_{i-2, j}+3 r^{2}(r+1) u_{i+2, j} \\
3 r^{2}(r-1) u_{i-2, j}+3 r\left(r^{2}+5 r+2\right) u_{i+2, j}
\end{array}\right]\right\} \tag{9.6.5}
\end{align*}
$$

The Method
The equation (9.6.5) forms the basis for the 3-point GE method. For the complete discussion on the 3 -point GE method, a few cases of the number of unknown points along the time-level will be considered. These are as follows:-
(i) The number of unknown points is an exact multiple of 3 ,


Computational molecule for the scheme (9.6.5)
FIGURE (9.6.2)
(ii) The number of unknown points is a multiple of 3 with one ungrouped point,
(iii) The number of unknown points is a multiple of 3 with two ungrouped points.

## Case (i):

In this case, for every group of 3 -points we use equation (9.6.5).

For the whole line of points at any time-level the solution is given by,

$$
\begin{equation*}
\underline{u}_{j+1}=\underline{A u}_{j}+\underline{b}, \tag{9.6.6}
\end{equation*}
$$

where,

$$
\begin{aligned}
& \underline{b}^{T}=\left[\alpha_{8} u_{0, j}, \alpha_{7} u_{0, j}, \alpha_{6}{ }_{0, j}, 0, \ldots, 0, \alpha_{6}{ }_{M, j}, \alpha_{7}{ }^{u_{M, j}, \alpha_{8}}{ }_{u_{M, j}}\right],
\end{aligned}
$$

with

$$
\begin{aligned}
& \Delta=4(3 r+1)(r+1) \\
& \alpha_{1}=2 r-14 r^{2}+4, \quad \alpha_{2}=-6 r^{3}+4 r^{2}+10 r, \alpha_{3}=10 r^{2}-2 r \text {, } \\
& \alpha_{4}=4 r(r+1), \quad \alpha_{5}=2(r+1)\left(2+2 r-3 r^{2}\right), \\
& \alpha_{6}=3 r^{2}(r-1), \quad \alpha_{7}=3 r^{2}(r+1) \quad \text { and } \alpha_{8}=3 r\left(r^{2}+5 r+2\right) .
\end{aligned}
$$

## Case (ii):

In this case the ungrouped point can be assumed either on the leftmost or right-most boundary at any of each or alternating time-levels. For each assumption, the solution at this ungrouped point is approximated by equations (9.6.1),

$$
\begin{equation*}
u_{M-1, j+1}=\frac{1}{(3 r+2)}\left[4 r u_{M-2, j+1}^{\left.-r u_{M-3, j+1}+r u_{M-2, j}+3 r u_{M, j}+(2-4 r) u_{M-1, j}\right]}\right. \tag{9.6.8}
\end{equation*}
$$

for the left-to-right (LR) calculation or by equation (9.6.2),

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{(3 r+2)}\left[4 r u_{2, j+1}-r u_{3, j+1}+r u_{2, j}+3 r u_{0, j}+(2-4 r) u_{1, j}\right] \tag{9.6.9}
\end{equation*}
$$

for the right-to-left (RL) calculation.
Case (iii):
In this case, the ungrouped points are assumed to be each on the left-most and right-most boundary at any time-level j. Initially all solutions at grouped points are calculated and then the solutions at both ungrouped points are calculated using (9.6.9) and (9.6.8).

The detailed analysis of stability and the truncation error were left for later analysis and further work on this topic will be reported 1ater.

Two of the three cases considered above were tested numerically using the problem given in the last section and some of the numerical results are presented in Table (9.6.1)-(9.6.3). From the experiments, it can be seen that the solutions are in good agreement with the analytical solution for $r \leqslant 1$. For $1.0<r<2$ the (D)AGE solutions are in qualitative agreement with the analytical decreasing solution, but for $r \geqslant 2.0$, the solutions are clearly unstable. For algorithm (i), the clear signals of instability immediately occur when $r>1$. Hence, in this case the (D)AGE method of algorithm (ii) is more preferable than algorithm (i).

For further work the (D) AGE scheme is recommended and the analysis on stability and convergence are worthy of further investigation.

Algorithm (i)
$\Delta \mathrm{x}=0.1, \mathrm{t}=0.1$

| $x$ | EXACT | $r=0.1$ | $r=0.5$ | $r=1.0$ |
| :---: | :--- | :--- | :--- | :--- |
| 0.1 | 0.11887 | 0.12065 | 0.11724 | 0.11356 |
| 0.2 | 0.22610 | 0.22669 | 0.22323 | 0.21879 |
| 0.3 | 0.31119 | 0.31033 | 0.30688 | 0.30068 |
| 0.4 | 0.36582 | 0.36570 | 0.36077 | 0.35570 |
| 0.5 | 0.38465 | 0.38469 | 0.37976 | 0.37329 |

$\Delta x=0.1, t=0.5$

| $x$ | EXACT | $r=0.5$ | $r=1.0$ |
| :---: | :--- | :--- | :--- |
| 0.1 | 0.00229 | 0.00214 | 0.00196 |
| 0.2 | 0.00436 | 0.00408 | 0.00375 |
| 0.3 | 0.00600 | 0.00561 | 0.00513 |
| 0.4 | 0.00706 | 0.00660 | 0.00606 |
| 0.5 | 0.00742 | 0.00694 | 0.00639 |

TABLE (9.6.1)

## Algorithm (ii) (D)AGE

$t=0.1$

| x | EXACT | $\Delta \mathrm{x}=0.2$ <br> $\Delta \mathrm{t}=0.005$ <br> $\mathrm{r}=0.125$ | $\Delta \mathrm{x}=0.2$ <br> $\Delta \mathrm{t}=0.01$ <br> $\mathrm{r}=0.25$ |
| :---: | :---: | :---: | :---: |
| 0.2 | 0.22610 | 0.23819 | 0.22751 |
| 0.4 | 0.36582 | 0.37787 | 0.36438 |
| 0.6 | 0.36582 | 0.37769 | 0.36551 |
| 0.8 | 0.22610 | 0.23828 | 0.22731 |

$t=0.4$

| x | EXACT | $\Delta \mathrm{x}=0.2$ <br> $\Delta \mathrm{t}=0.01$ <br> $\mathrm{r}=0.25$ | $\Delta \mathrm{x}=0.2$ <br> $\Delta t=0.04$ <br> $\mathrm{r}=1.0$ |
| :---: | :--- | :--- | :--- |
| 0.2 | 0.01171 | 0.01189 | 0.00251 |
| 0.4 | 0.018940 | 0.01911 | 0.00474 |
| 0.6 | 0.018940 | 0.01906 | 0.00540 |
| 0.8 | 0.01171 | 0.01190 | 0.00318 |

TABLE (9.6.2)

## Algorithm (ii) (D)AGE

$t=0.5, \Delta x=0.0909$

| $x$ | EXACT | $\Delta t=0.005$ <br> $r=0.605$ | $\Delta t=0.01$ <br> $r=1.21$ |
| :---: | :---: | :---: | :---: |
| 0.0909 | 0.001642 | 0.001911 | 0.001180 |
| 0.1818 | 0.003152 | 0.003682 | 0.002266 |
| 0.2727 | 0.004406 | 0.005153 | 0.003376 |
| 0.3636 | 0.005303 | 0.006205 | 0.004847 |
| 0.4545 | 0.005770 | 0.006760 | 0.003914 |
| 0.5455 | 0.005770 | 0.006751 | 0.004324 |
| 0.6364 | 0.005303 | 0.006211 | 0.004762 |
| 0.7273 | 0.004406 | 0.005157 | 0.002994 |
| 0.8182 | 0.003152 | 0.003676 | 0.002397 |
| 0.9091 | 0.001642 | 0.001907 | 0.001263 |

TABLE (9.6.3)

## Summary and Final Remarks

With the increasingly high standard of technology used in society today and the accelerating use of digital computers, the numerical solution of parabolic partial differential equations remains an important subject for research for the future. The discovery of the Group Explicit method is an important landmark in methods of solving partial differential equations as in principle the method is simple, stable, accurate and flexible in the sense that it allows the calculation of more than one point at a time - a new and original idea!

The evidence of the applicability of the method to simple parabolic problems has been discussed and described in detail. The discussion on the applicability of the method to various boundary conditions is also demonstrated successfully.

The method progresses a step further when it was extended successfully to more specific and difficult equations like the diffusion-convection equation, the non-linear self-adjoint equation and the Burger's equation. In most cases, the method performed better than the existing standard methods. However, the theoretical definitive conclusions for some of the particular investigations on specific problems are left unsolved as their theory was too difficult to handle comprehensively during the short period of research.

The implementation of the Group Explicit method to a larger variety of problems is left for future work. Furthermore, with the availability of parallel computers, the transformation of all the algorithms and the schemes of the Group Explicit type to run on these machines remains a necessary and important area of work still to do.

Another area of research remaining is the discovery of new explicit schemes as a by-product of the work developed on Group Explicit Method. Appendix 4 gives one example of a new stable explicit scheme as a byproduct from the 3-point Group Explicit method and there remains many more undiscovered.

Another important task carried out in this work is location of a Group Explicit Method with a higher order of accuracy. It is important to discover whether such a method exists or not.

For the work on the Finite Element method, the author plans to extend the work to higher order element and this will be reported later.

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## APPENDIX 1

## The matrices $r\left(G_{s}+G_{s}^{*}\right)$ and $r\left(\hat{G}_{s}+\hat{G}_{s}^{*}\right)$ are non-negative definite

To prove $r\left(G_{S}+G_{S}^{*}\right)$ and $r\left(\hat{G}_{S}+\hat{G}_{S}^{*}\right) s=1,2$ are non-negative definite matrices (* denotes the transpose matrix), where,

with integer $M$ even.
Proof:
We consider the case when all of the above-mentioned matrices are symmetric. The eigenvalues of these matrices are:
(i) 1 and 0 and 2 with multiplicity $(M-2) / 2$ for matrices $G_{1}$ and $G_{2}$.
(ii) 0 and 2 with multiplicity ( $M-2$ )/2 and 1 with multiplicity 2 for $\operatorname{matrix} \hat{G}_{1}$.
(iii) 0 and 2 with multiplicity $M / 2$ for matrix $\hat{G}_{2}$.

As all eigenvalues of the matrices $G_{s}$ and $\hat{G}_{s}, s=1,2$ and their transpose are non-negative, therefore the eigenvalues of $r\left(G_{S}+G_{S}^{*}\right)$ and $r\left(\hat{G}_{S}+\hat{G}_{S}^{*}\right)$ are also non-negative. Hence, from Theorem (1.1), it is proven that the matrices $r\left(G_{S}+G_{S}^{*}\right)$ and $r\left(\hat{G}_{S}+\hat{G}_{S}^{*}\right)$ are non-negative definite.

## APPENDIX 2

Matrix $r\left(\hat{\hat{G}}_{1}+\hat{\mathbf{G}}_{1}\right)$ is non-negative definite

To prove $r\left(\hat{\hat{G}}_{1}+\hat{\mathrm{G}}_{\mathrm{l}}^{\mathrm{t}}\right)$ is a non-negative definite matrix where,

with integer $M$ even.
Proof:
It can be easily shown that,

$$
\begin{aligned}
& =\left(\lambda^{2}-2 \lambda\right)\left(\lambda^{2}-2 \lambda\right)^{(M-2 / 2)} \\
& =(\lambda-2 \lambda)^{M}
\end{aligned}
$$

where the $\lambda$ are eigenvalues of the matrix $\hat{\hat{G}}_{1}$ when det $\hat{\hat{G}}_{1}=0$. Therefore, the eigenvalues of matrix $\hat{\hat{G}}_{1}$ of order $M$ are 0 and 2 with multiplicity $M / 2$. Since the matrix $\hat{\hat{G}}_{1}$ is a symmetric matrix and its eigenvalues are nonnegative, therefore the matrix $r\left(\hat{\hat{G}}_{1}+\hat{\hat{G}}_{1}^{\star}\right)$ which is also a symmetric matrix having non-negative eigenvalues is clearly a non-negative definite matrix.

## APPENDIX 3

Analytical solution of the Diffusion Convection Equation

To verify that the analytical solution for

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\varepsilon \frac{\partial^{2} u}{\partial x^{2}}-k \frac{\partial u}{\partial x} \tag{A3.1}
\end{equation*}
$$

where $k / \varepsilon \gg 0$ is a constant and the initial condition,

$$
\begin{equation*}
u(x, 0)=f(x) \neq 0,0 \leqslant x \leq 1, \tag{A3.2}
\end{equation*}
$$

and the boundary conditions,

$$
\begin{equation*}
u(0, t)=u(1, t)=0, \quad t \geqslant 0, \tag{A3.3}
\end{equation*}
$$

is given by,

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty} \beta_{n} e^{\left(-n^{2} \pi^{2} \varepsilon-\frac{k^{2}}{4 \varepsilon}\right) t} e^{\frac{k x}{2 \varepsilon}} \sin (n \pi x), \tag{A3.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\beta_{n}=2 \int_{0}^{1} f(s) e^{-\frac{k s}{2 \varepsilon}} \sin (n \pi s) d s \tag{A3.5}
\end{equation*}
$$

In particular,

$$
u(x, t)=\sum_{n=1}^{\infty} \frac{4}{(n \pi)^{3}}\left(1-(-1)^{n}\right) e^{\frac{k x}{2 \varepsilon}} e^{-\left[(n \pi)^{2} \varepsilon+k^{2} / 4 \varepsilon\right] t}
$$

$$
\sin (n \pi x),(A 3.6)
$$

if $f(x)=x(1-x) e^{k x / 2 \varepsilon}$.

## Proof:

Using the separation of variables, we define the solution as,

$$
\begin{equation*}
u(x, t)=X(x) T(t), \tag{A3.7}
\end{equation*}
$$

where $X$ and $T$ are functions of $x$ and $t$ respectively. The substitution of (A3.7) into (A3.1) will lead to two ordinary differential equations, i.e.,

$$
\begin{equation*}
\frac{d T}{d t}-\lambda T=0 \tag{A3.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\varepsilon \frac{d^{2} x}{x^{2}}-k \frac{d X}{d x}-\lambda x=0, \tag{A3.9}
\end{equation*}
$$

The equation (A3.8) and (A3.9) will result in the general solutions,

$$
\begin{equation*}
T(t)=A e^{\lambda t} \tag{A3.10}
\end{equation*}
$$

and $\quad X(x)=\left(B e^{\sqrt{(k / \varepsilon)^{2}+4 \lambda / \varepsilon}} x / 2+C e^{-\sqrt{(k / \varepsilon)^{2}+4 \lambda / 2}} \quad x / 2\right) e^{k x / 2 \varepsilon}$
for arbitrary constant $\mathrm{A}, \mathrm{B}$ and C .

Using definition (A3.7), the boundary condition (A3.3) will be given by,

$$
\begin{equation*}
x(0)=x(1)=0 \tag{A3.12}
\end{equation*}
$$

Thus, from (A3.11), for $X(0)=0$ we need $B+C=0$ and hence,

$$
\begin{equation*}
X(x)=B e^{k x / 2 \varepsilon}\left(e^{\sqrt{(k / \varepsilon)^{2}+4 \lambda / \varepsilon}} x / 2-e^{-\sqrt{(k / \varepsilon)^{2}+4 \lambda / \varepsilon}} x / 2\right) \tag{A3.13}
\end{equation*}
$$

Also for $X(1)=0$,
i.e. $\quad B e^{k / 2 \varepsilon}\left(e^{\sqrt{(k / \varepsilon)^{2}+4 \lambda / \varepsilon}} 1 / 2-e^{-\sqrt{(k / \varepsilon)^{2}+4 \lambda / \varepsilon}} 1 / 2\right)=0$,
and to obtain the non-trivial solution (i.e. $B \neq 0$ ) we need to choose

$$
(k / \varepsilon)^{2}+4 \lambda / \varepsilon<0
$$

Therefore, we have,

$$
\begin{align*}
X(x) & =B e^{k x / 2 \varepsilon}\left(e^{i x / 2 \sqrt{-(k / \varepsilon)^{2}-4 \lambda / \varepsilon}-e^{-i x / 2 \sqrt{\left.-(k / \varepsilon)^{2}-4 \lambda / \varepsilon\right)}}}\right.  \tag{A3.14}\\
& =2 i B e^{k x / 2 \varepsilon} \sin \left(x / 2 \sqrt{\left.-(k / \varepsilon)^{2}-4 \lambda / \varepsilon\right)}\right.
\end{align*}
$$

As $\bar{X}(1)=0$, then,

$$
2 i B e^{k / 2} \sin \left(\frac{1}{2} \sqrt{\left.-(k / \varepsilon)^{2}-4 \lambda / \varepsilon\right)}=0\right.
$$

if and only if

$$
\frac{1}{2} \sqrt{-(k / \varepsilon)^{2}-4 \lambda / \varepsilon}=n \pi, \quad n=1,2, \ldots
$$

i.e.,

$$
\begin{equation*}
-(k / \varepsilon)^{2}-4 \lambda / \varepsilon=4 n^{2} \pi^{2} \tag{A3.15}
\end{equation*}
$$

i.e., $\quad \lambda=-n^{2} \pi^{2} \varepsilon-k^{2} / 4 \varepsilon$.

By using (A3.10), (A3.14) and (A3.15) we have,

$$
\begin{equation*}
u_{n}(x, t)=\beta_{n} e^{\left(-n^{2} \pi^{2} \varepsilon-k^{2} / 4 \varepsilon\right) t} \quad e^{k x / 2 \varepsilon} \sin (n \pi x), n=1,2, \ldots, \tag{A3.16}
\end{equation*}
$$

where $\beta_{n}$ is a constant. As the equation is linear and using the principle of superposition, the formal solution $u(x, t)$ can be represented by,

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty} \beta_{n} e^{-\left(n^{2} \pi^{2} \varepsilon+k^{2} / 4 \varepsilon\right) t} e^{k x / 2 \varepsilon} \sin (n \pi x) \tag{A3.17}
\end{equation*}
$$

To determine the $\beta_{n}$, we use the initial condition (A3.2) and take $\beta_{n}$ as the coefficient of the Fourier sine series, i.e.,

$$
\begin{equation*}
B_{n}=2 \int_{0}^{1} f(s) e^{-k s / 2 \varepsilon} \sin (n \pi s) d s \tag{A3.18}
\end{equation*}
$$

For $f(s)=s(1-s) e^{k s / 2 \varepsilon}$, then,

$$
\begin{aligned}
\beta_{\mathrm{n}} & =2 \int_{0}^{1} \mathrm{~s}(1-\mathrm{s}) \sin (\mathrm{n} \pi \mathrm{~s}) \mathrm{ds} \\
& =2 \int_{0}^{1} \mathrm{~s} \sin (\mathrm{n} \pi \mathrm{~s}) \mathrm{ds}-2 \int_{0}^{1} \mathrm{~s}^{2} \sin (\mathrm{n} \pi \mathrm{~s}) \mathrm{ds} \\
& =\frac{4}{(\mathrm{n} \pi)^{3}}\left(1-(-1)^{n}\right) .
\end{aligned}
$$

Therefore, from (A3.17), we have the analytical solution,

$$
u(x, t)=\sum_{n=1}^{\infty} \frac{4}{(n \pi)^{3}}\left(1-(-1)^{n}\right) e^{k x / 2 \varepsilon} e^{-\left[(n \pi)^{2} \varepsilon+k^{2} / 4 \varepsilon\right] t} \sin (n \pi x)
$$

## APPENDIX 4

The 3 Point Group Explicit Method

Consider the equation from Section (9.6) for calculating the finite difference solution at the mid point of any group of 3 -points, namely,

$$
\begin{gather*}
u_{i, j+1}=\frac{1}{4(3 r+1)}\left\{3 r^{2}\left(u_{i-2, j}+u_{i+2, j}\right)+4 r\left(u_{i+1, j}+u_{i-1, j}\right)+\right. \\
 \tag{A4.1}\\
\left.+2\left(2+2 r-3 r^{2}\right) u_{i, j}\right\},
\end{gather*}
$$

which is given in molecular form by Fig. (A4.1). The equation (A4.1) is an explicit formula using the values at 5 -node points from the previous time-level.


FIGURE (A4.1)
The equation (A4.1) can be shown to approximate the diffusion equation,

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}, \tag{A4.2}
\end{equation*}
$$

with the order of accuracy $O\left(\Delta t+\frac{\Delta t^{3}}{\Delta x^{2}}+\Delta x^{2}\right)$. This type of approximation is clearly consistent with the given problem.

Now for $2+2 r-3 r^{2} \geq 0$, i.e., $0<r \leqslant \frac{1+\sqrt{7}}{3}$, using the matrix norm, it can be easily shown that the eigenvalues of the matrix,
which is the coefficient matrix to the system (A4.1) is less than or equal to unity. Hence the formula (A4.1) is stable for all $r \leqslant \frac{1+\sqrt{7}}{3}$.

As the formula (A4.1) involves 5 -node points along the $j$ th time-level, the following computational procedures are proposed:

Along the time-level $j$, at point ( $i \Delta x, j \Delta t$ ), $i=2,3, \ldots,(m-2)$ where $m$ is the number of intervals, the equation (A4.1) is used. At points ( $\Delta x, j \Delta t$ ) and $((m-1) \Delta x, j \Delta t)$, the fully implicit equation can be used explicitly as,

$$
\begin{equation*}
u_{1, j+1}=\frac{1}{(1+2 r)}\left\{u_{1, j}+r u_{0, j+1}+r u_{2, j+1}\right\} \tag{A4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{m-1, j+1}=\frac{1}{(1+2 r)}\left\{u_{m-1, j}+r u_{m-2, j+1}+r u_{m, j+1}\right\} \tag{A4.5}
\end{equation*}
$$

The numerical experiments for the use of this method will be reported in a later report.

## APPENDIX 5

## Some Selected Computer Programs

MASTER AOEN 1
C
C
C
c $X$ representing the space variatle
C T REPRESENTING THE TIME VARTABLE
C U REPRESENTING THE DEPEndemt variable 'traperature'
C DX AND DT ARE THE SPACE AND Tlife INTERVALS RESPECTIVELY
C TTHE IS THE HAXIMIH TIME COMSIDERED
C $V$ IS TIE EYACT SOLUTION
C ABERR IS THE AESOLUTE ERROR IN THE WMERICAL SOLUTTON
C PERR IS THE PEFCENTAGE ERROR
C U IS TUE MHPER OF SURINTERVAL IN SPACE DOAAIN
C JMAX IS THE RUMDER OF TIME-LEVEL IN TIAE DOMAIN
C
C THIS is the hain progran
C
C
C
DAFASIONING
C
C
PEAL EX, DT, TIME,U(11,201),V(11,201),X(11),T(201),P
RFAL ADERP(11,201), PERR(11,201)
C
C
C
C
5
C
C start idth individual data
C
!DATA $=0$
NDATA =:IDATA +1
IF (IIDATA. (IT.IAXDAT) STOP
C
C READING THE DATAS FOR INDIVIDUAL SET
C
READ (1, 15) DX, DT, TME
15 FOKMAT (3FO.0)
©
C THE SPACE-HODE
C
$M=1.0 / 0 X+1.0$

```
    DO 20 I= 1,14
20
    X(I)=(I-1.0)*DX
C
C THE TIME INTERVAL
C
C
C
C
        INC=1
        DO 45 J=1,JHAX
        IF(J.GE.11) INC=10
        DO 40 I=1,M
        ABERR(I,J)=ABS(U(I,J)-V(I,J))
        IF(ABS(V(I,J)).LE.(10.0**(-10))) GO TO 35
        PERR(I,J)=(ASERR}(I,J)/ABS(V(T,J)))*100.0 
        GO TO 40
        PERR(I,J)=0.0
40 CONTIMUE
45 conTINuE
C
C PRINT THE RESULT
C
50 FORMAT(////15X,'THE SOLUTION OF CONOUCTION EOUATION USTHG AGE ALGO
        1RITHA'//10X,'SPACE-INTERVAL=',F6.4,'TIME-INTERVAL=',F6.4,'11AY-TIME
        2=',F6.4)
            INC=1
            DO 110 J=1,JMAX,INC
            IF(J.GE.11) INC=10
            WRITE(2.55) T(J)
        00 30 I=1,11
        U(I, 1) =SIM(X(I))+\operatorname{Cos}(X(T))
        V(I, T)=STM}(X(I))+\operatorname{cos}(X(I)
        COUTINUE
        CALLING THE AGE ALGORITHM
        CALL AGEN(DX,DT,TIME,R,U,T)
        CALLIMG THE AHALYTICAL SOLUTION
        CALL ANALN(DX,DT,TI:AE,T,X,V)
        COHPARING THE RESULT WITH ANALYTICAL SOLUTION
        WPITE(2,50) DX,DT,TI'IE
            FORMAT(/5X,'THE TIME=',F6.4)
```

```
    WRTTE(2,60)(X(I),I=1,6)
GO TO 10
STOP
END
C
C
C ANALYTICAL SOLUTION FOR THE :IEUMAMA PRORLEA
C
C
C
C
C
REAL PI, DX, DT, T(201),V(11,201), Y(11),THE
IWTEGER M, JIAX
C
C SOLUTION EOR EVERY FOTMT OU THE GRID
C
CONTINUE
CONTINIE
RETURN
END
\(11=1.0 / 0 X+1.0\)
\(\mathrm{J}: \mathrm{AX}=\mathrm{TI} 1 \mathrm{E} / \mathrm{DT}+1.0\)
DO \(120 \mathrm{~J}=2\), JMAX
DO \(115 \mathrm{I}=1\), M
\(V(I, J)=\operatorname{ExP}(-T(J)) *(S I I(X(I))+\operatorname{CoS}(X(I)))\)
```

```
C
C
C
C
C
C
C
C THE frocrai4 FOR CALCulattNG tHE SOLuTION OF the heat conduction
C
C
C
C
C
C
    REAL F(210),G(201),DK,OT,DETA,リ(11,201),P,T(201),TIME,S(11,201)
    IMTEGER 4,:11,H2,:3, J:AAX, J1AX1,J
C THE REGUIRED CONSTANT AND ITEAS
C
    H=1.0/DX+1.0
    J:AX=TIME/DT+1.0
    J\AX1= JHAX-1
    \1=\1-1
    M2=1-2
    M3=1-3
    R=OT/(DX**2)
    DETA=1.n+2.0*R
C
C IMITIALIZE TIE TINE-LEVEL
C
C
C
C
C
125 J=J+1
c
C THE FIRST GROUP
C
    G(J+1)=EXP(-T(J+1))*(COS(1.0)-BIn(1.0))
    F(J+1)=EXP(-T(J+1))
    DO 150 I=2,13,?
    S(I,J+1)=((1.+R)*(R*U(I-1,J)+(1.0-R)*U(I,J))+R*((1.0-R)*!J(I+1,J)+
    1N*リ(I+2,J)))/DETA
    S(I+1,J+1)=(R*(R*U(I-1,J)+(1.-R)*U(T,J))+(1.0+R)*((1.0-R)*U(I+1,J)
    1+R*U(I+2,J)))/DETA
130 CONTTMUE
    S(111,J+1)=R*U(:12,J)+(1, -R)*U(41,J)+R*G(J+1)*DX
C
C
    S(1,J+1)=S(2,J+1)-DX*F(J+1)
    S(M,J+1)=S(M1,J+1)+G(J+1)*DX
```

```
C
C
C
        J(2,J+1)=R*S(3,J+1)+(1,-R)*S(2,J+1)-R*DX*F(J+1)
        D0 135 I = 3.:2,2
        U(I,J+1)=((1.+R)*(R*S(I-1,J+1)+(1.0-R)*S(I,J+1)) +R*((1.O-R)*S(I+1,
    1!+1)+R*S(T+2,J+1)))/DETA
    U(I+1,J+1)=(R* (R*S(I-1,J+1)+(1.-R)*S(I,J+1))+(1.0+R)*((1.C-R)*S(I
    1+1.J+1)+n*3(I+2,U+1)))/DETA
135
C
    U(1,J+1)=U(2,J+1)-DX*F(J+1)
    U(14,J+1)=U(141,J+1)+G(J+1)*DX
C
C TE.STING THE TIME
C
    IF(J.EQ.JIAX1) RETURN
c
C
C
C Calculatilig TiE j+1 time level
C
        G(J+1)=EXP}(-T(J+1))*(\operatorname{Cos}(1.0)-3TN(1.0)
    F(J+1)=FXP(-T(J+1))
C
C SECOND altermate group
C
        S(2,J+1)=R*J(3,J)+(1, -R)*O(2,J)-R*DX*F(J+1)
        DO 140 I=3,:2,2
        S(I,J+1)=((1.+R)*(R*U(I-1,J)+(1.0-R)*IJ(I,J))+R*((1.0-R)*IJ(I+1,J)+
        1R*'J(I+2,J)))/DETA
        S(I+1,J+1)=(Q*(R*IJ (I-1,J)+(1.-R)*U(I,J))+(1.C+R)*((1.0-R)*U(I+1,J)
        1+R*(J(I+2,J)))/DETA
140 COMTTHUE
C
    S(1,J+1)=S(2,J+1)-2X*F(J+1)
    S(:1,J+1)=S(:11,J+1)+0X*G(J+1)
C
c. Ftost altermate gromp
C
    DO 145 I=2,43.2
    U(I,J+1)=((1.+R)*(R*S(I-1,J+1)+(1.0-R)*S(I,J+1))+R*((1.0-R)*S(I+1,
    1J+1)+F*S(I+2,J+1)))/DETA
    I(I+1,J+1)=(R*(R*S(I-1,J+1)+(1. -R)*S(I,J+1))+(1.0+R)*((1.0-P)*S(I
    1+1,j+1)+R*S(I+2,j+1)))/DETA
145 CONTINUE
    U(111,J+1)=R*S(1!2,J+1)+(1.-R)*S(M1,J+1)+R*G(J+1)*DX
C
C
    U(1,J+1)=U(2,J+1)-DX*F(J+1)
    IJ(II,J+1)=U(M1,J+1)+C(J+1)*DX
C
C
```

```
            IF(J.EQ.JHAX1) RETURN
            GO TO 125
C
C
C THIS PROGRAM WAS COHPILE,TESTED AIS RUN ISIUG ICL 1900
C AT THE COHPUTER CENTRE,UHTVERSITY OF TECHHOLOGY,
C LOUGHBOROUOH,LEICESTERSHIRE LE 11 3TU.
C
C
C
    END
    FT:IIS:I
```

MASTER AGE2DH

```
C C THE PROGRAM FOR CALCULATING U USI:GG AGE ALGORITHM
C
C FOR TWO DIMENSION PROBLEM
C
    REAL DX,DY,DT,G(11,11,41),U(11,11,41),UAUA(11,11,41),X(11)
    REAL Y(11),T(41),R,ERR(11,11,41),TBASE
    REAL PER(11,11,41)
C
    INTEGER KMAX,MX,MY,MX1,:YY1,IMC
    IMTEGER NDATA,MAXDAT
    IHTEGER KTRUE,NCIR,NCIRC
C
C
C READING THE DATA
C
C
    READ(1,678) MAXDAT
678 FORMAT(IO)
    NDATA=0
6789 NDATA=NDATA +1
    IF(NDATA.GT.MAXDAT) STOP
    READ(1,10) DT,DX,DY,KTRUE,NCIR
    FOR:1AT(3F0.0,2IO)
    MX=1.0/DX+1.0
    MY=1.C/DY+1.0
    K:1AX=(KTRUE-1)/NCIR+1
    MX1=1X-1
    MY1=MY-1
C
C
C
C
C
    DO 100 I=1,MX
    X(I)=(I-1)*DX
100 continue
C
C TIE Y-MODE
C
    DO 101 J=1,MY
    Y(J)=(J-1)*DY
101 CONTINUE
C
C
    R=DT/(DX**2)
C
C
C WRITE THE HEADING
C
        WRITE(2,200) DX,DT, KIAX,R,KTRUE
200 FORMAT(////15X,'THE SOLUTION OF THE TNO DIMENSIONAL PARABOLIC EQUA
```

1TION'/15X,'USING THE AGE ALGORITHM'//10X,'X-SPACE INTERVAL=',F6.3/ 210X,'m-LEVEL INTERVAL=', F8.5/10X,'MAX-TIME STEP =', I6. $10 \mathrm{X},{ }^{\prime} \mathrm{R}={ }^{\prime}, \mathrm{F} 6.3$ 3/10x,'TRUE TIME STEP=', I6)

C
C
$N C I R C=0$
TYASE=0
320 CONTINUE
C
C
C
TUE TIME IMTERVAL
C
DO $102 K=1$, K1AX
$T(K)=T B A S E+(K-1) * D T$
102 CONTINUE
C
C THE LEFT AND RIGHT EOUNDARY CONDITION
$\stackrel{C}{C}$
DO $103 \mathrm{~K}=1$, K1AX
DO $104 \mathrm{~J}=1, \mathrm{MY}$
$U(1, J, K)=Y(J) * * 2$.
U(14X,J,K)=SIH(X(MX))*SIN(Y(J))*EXP(-?(K))+X(MX)**2+Y(J)**2
104 CONTINJE
103 CONTINUE
C
C THE FUNCTION G
C
DO $13 \mathrm{~K}=1$, K1AX
DO $12 \mathrm{~J}=2, \mathrm{MY} 1$
DO $11 \mathrm{I}=2, \mathrm{HX} 1$
$G(I, J, K)=3 \operatorname{IN}(K(I)) * \operatorname{STN}(Y(J)) * E X P(-T(K))-4.0$
11 Continue
12 CONTINUE
13 Continue
C THE INITIAL CONDITION AND BGTTOM AHD TOP BOUNDARY CONDITION
C
C
IF (NCIRC.GT.0) GJ TO 1151
DO $15 \mathrm{~J}=2, \mathrm{HY} 1$
DO $16 I=2,14 \mathrm{X} 1$
$U(I, J, 1)=3 \operatorname{In}(X(I)) * S I H(Y(J))+X(I) * * 2+Y(J) * * 2$
16 continue
15 COHTINUE
1151 COATINUE
DO $150 \mathrm{~K}=1, \mathrm{KMAX}$
DO $151 \mathrm{I}=2,11 \times 1$
$U(I, 1, K)=X(I) * * 2$
$U(I, M Y, K)=S I H(X(I)) * S I N(Y(M Y)) * E X P(-T(K))+X(I) * * 2+Y(M Y) * 2$
151
CONTINJE

```
150
CONTINUE
C
C THE THEORITICAL SOLUTION
C
    INC=1
    DO 153 K=11, KHAX,INC
    IF(K.GE.1) IHC=10
    DO 154 J=1.HY
    DO 155 I =1,i:X
    UALA(I,J,K)=SIN(X(I))*SIN(Y(J))*EXP(-T(K))+X(I)**2+Y(J)**2
155 CCNTINUE
154 CONTINUE
153 COMTINUE
C
C CALLING THE SUBROUTIME FOR AGE ALGORITHM
C
C
    CALL AGE2(DX,DY,DT,N:AY,G,U,R)
C
C
C CO:APARISON :ITH THE THEORETICAL RESIILTS
C
C
C
    INC=1
    DO 343 K=11,KMAX,INC
    IF(K.GE.1) INC=10
    DO }342\textrm{J}=1,\textrm{HY
    DO 341 I=1.14X
    ERR(I,J,K)=ABS(JANA(I,J,K)-U(I,J,K))
    IF(ABS(UANA(I,J,K)).LE.(10.O**(-5))) SO TO 3414
    PER(I,J,K)=(FRR(I,J,K)/AFSS(!JNMA (I,J,K)))*100.0
    GO TO 341
3414 PER(I,J,K)=0.0
341 CONTINUE
342 COMTINUS
343 CONTINUE
C
C
C WRITTNG THE RESJLT
C
C
    IHC=1
    DO 210 K=11, Ki1AX,INC
    IF(K.GE.1) INC=10
    WRITE(2,201) T(K)
201 FORMAT(/5X,'T:IE TIME IS=',F8.6)
    WRITE(2,202)(X(I),I=1,6)
202 FOR:1AT(/1X,'X=',9X,6(F6.3,10X))
    DO 208 J=1,MY
    WaITE(2,203) Y(J),(U(T,J,K),T=1,6)
203 FOR:MAT(1X,'Y=',F6.3.3X,6(F13.3,3X))
    WRITE (2,204)(UANA (I,J,K),I=1,6)
204 FOR:1AT(4X,'UANA=',3X,6(F13.8.3X))
```

```
        VRITE(2,350) (ERR(I,J,K),I=1,6)
350
FORHAT(3X,'ERROR=',3X,6(F10.8,6X))
        WRITE(2,5340) (PER(I,J,K),I=1,6)
5340 FOR1AT(3X,'PER=',5X,6(F10.8,6X))
208 CONTINUE
        WRITE(2,205) (X(I),I=7,11)
205 FOR'1AT(/1X,'X=',9X,5(F6.3,10X))
        DO 209 J=1,11Y
        WRITE(2,206) Y(J),(J(I,J,K),I=7,11)
206 FORMAT(1X,'Y=',F6.3,3K,5(F13.8.3X))
        WPITE(2, 207)(13AHA(I,J,K),I=7,11)
207 FORMAT(4X,'IJAMA =',3X,5(F13.8.3X))
        WRITE(2,360) (ERR(I,J,K),I=7,11)
    360 FORMAT (3X,'ERROR=',3X,5(F10.8,6X))
        WRITE(2,5341) (PER(I,J,K),I=7,11)
5341 FORMAT(3X,'PER=',5X,5(F10.?,6X))
209 CONTINUE
210 CONTINUE
        HC IRC = NC IRC+1
        IF(NCIRC.EQ.NCIR) GO TO }33
        DO 3660 J=2,MY1
        DO 361 I=?,i1X1
        U(I,J,1)=U(I,J,KMAX)
351 CONTINUE
3660 CONTINUE
        TUASE=T(KHAX)
        GO TO 320
330 CONTINUE
C
        GO TO 6789
C
C
C THAT IS THE END OF THE PROGRAM
C
C
        STOP
        EHD
C
C
C
C THE (S)AGE ROJTINE
C
C
    SUBROUTTNE AGE2(DX,DY,DT,K!AXX,G,U,R)
C
C SUBROUTINE FOR CALCULATING THE SOLUTION OF TWO DIMENSIONAL PARABOL
C IC EqUATIONS USING THE AGE ALGORITHM
C
C DX IS THE X-SPACE INTERVAL
C
C DY IS THE Y-SPACE INTERVAL
C
C NUABER OF EOTli INTERVAL ARE EVEN AND THEY ARE MX-1 AND MY-1
```

```
C
C
C
C
    REAL DX,DY,FA,FB,FC,R,U(11,11,21),DT,G(11,11,21)
    INTEGER NC
C
c
    IMTEGER MMAX,HX,NX1,4X2,NX3,MY1,MY2,HY3,K,KMAX9
C
C
C
C
:1X=1.0/DX+1.0
MY=1.0/DY+1.0
MX1=11X-1
MY1=NY-1
MX2=:1X-2
MY2=MY-2
14 X = 11X-3
HY}3=11Y-
K\AXX1=K:AAX-1
NC=1
C
C
C
C
C
C
c start calculating u jsing first group
C
41 MC= = CO+1
18 K=K+1
c
C FOR THE RLOCK FAR FROM ARY ROUNDARY
C
C
DO 20 J=2,14Y3,2
DO 19 I =2,!\3,2
FA=R*U(I-1,J,K)+(1.0-2.0*R)*U(I,J,K)+R*U(I,J-1,K)+G(I,J,K+1)*DT
FB=R*J(I+2,J,K)+(1.0-2.O*R)*!J(I+1,J,K)+R*U(I+1,J-1,K)+G(I+1,J,K+1)
1*DT
    FC=R*U(I+2,J+1,K)+(1.O-2.O*R)*U(I+1,J+1,K)+R*U(I+1,J+2,K)+G(I+1,J+
11,K+1)*DT
    FD=R*U(I-1,J+1,K)+(1,0-2.O*?)*U(I,J+1,K)+R*U(I,J+2,K)+G(I,J+1,K+1)
1*DT
    U(I,J,K+1)=((1.0+4.0*R+2.0*R*R)*FA+R*(1.0+2.0*R)*FE+2.0*R*R*FC+R*(
11.0+2.0*R)*FD)/((1.0+2.0*R)*(1.0+4.0*R))
    U(I+1,J,K+1)=( R*(1.0+2.0*R)*FA+(1.0+4.0*R+2.0*R*R)*FE+R*(1.0+2.0*
```

```
    1R)*FC+2.0*R*R*FD)/((1.0+2.0*R)*(1.0+4.0*R))
    U(I+1,J+1,K+1)=(2.0*R*R*FA+R*(1.0+2.0*R)*FE+(1.0+4.0*R+2.0*R*R)*F
    1+R*(1.0+2.0*R)*FD)/((1.0+2.0*R)*(1.0+4.0*R))
    U(I,J+1,K+1)=(R*(1.0+2.0*R)*FA+2.0*R*R*FB+R**(1.0+2.0*R)*FC+(1.0+4.
    10*R+2.0*R*R)*FD)/((1.0+2.0*R)*(1.0+4.0*R))
    CONTIMUE
    contINUE
    CaLCULATE U NEAR THE RIGHT BOUGDARY AND UPPER ROUNDARY
    DO 21 J=2,4Y3,2
    FA=R*U(MX2,J,K)+(1.0-2.O*R)*!J(MX1,J,K)+R*U(IUX1,J-1,K)+R*U(MX,J,K+
    11)+G(3X1,J,K+1)*OT
    FD=R*U(MX2,J+1,K)+(1,0-2.0*R)*II(IX1, J+1,K)+R*U(MX1,J+?,K)+R*IJ(MX,
    1 J+1,K+1)+G(MX1, J+1,K+1) #DT
    U(MX1,J,K+1)=(FA * (1.0+2.0*R)+R*FD )/(1.0+4.0*R+3.0*R*R)
    U(1X1,J+1,K+1)=(FA*R+FD *(1.0+2.0*R))/(1.0+4.0*R+3.0*R*R)
    comTINuE
    DO 22 I=2,MX3,2
    FA=P*!f(I-1,HY1,K)+(1.0-2.0*R)*U(I,HY1,K)+R*U(I,:YY2,K)+R*U(I,MY,K+1
    1)+G(I, liY 1,K+1)*DT
    FR=R*U(T+2,MY1,K)+(1.0-2.0*R)*U(I+1,MY1,K)+R*U(I+1,MY2,K)+R*!J(I+1,
1HY,K+1)+G(I+1,NY1,K+1)*DT
    U(I,GY1,K+1)=(FA*(1.0+2.0*R)+FR*R)/(1.0+4.0*R+3.0*R*R)
    U(T+1,MY1,K+1)=(FA*R+FB*(1.0+2.0*R))/(1.0+4.0*R+3.0*R*R)
    comitNuE
    CALCULATING U AT THE TOP-RIGHT CORNER
    U(MX1,MY1,K+1)=(R*U(MX2,MY1,K)+(1.O-2.O*R)*U(MX1,MY1,K)+R*U(MX1,MY
    12,K)+R*U(KK,MY1,K+1)+R*U(#MX1,MY,K+1)+G(MX1,MY1,K+1)*DT)/(1.0+2.*R)
    PROCEED TO !UEXT TIME-LEVEL IF !ECESSARY
    IF(K.EQ. MAAXT)RETURN
    IF(NC.LT.2) GO TO 41
    MC=0
    MC=NC+1
    K=K+1
C CALCULATING U ustig alternate group
calculating u at the top-right cofner
\(U(M X 1, M Y 1, K+1)=(R * U(M X 2, M Y 1, K)+(1 . \cap-2 . O * R) * U(I K X 1, M Y 1, K)+R * U(M X 1, M Y\) \(\left.12, K)+R^{*} U(1: X, M Y 1, K+1)+R * U(11 X 1, A Y, K+1)+G(A X 1, H Y 1, K+1) * D T\right) /(1.0+2 . * R)\)
PROCEED TO !UEXT TIME-LEVEL IF IECESSARY
IF (K.EQ. KAAXT)RETURN
IF (NC.LT.2) GO TO 41
\(\mathrm{NC}=0\)
\(K=K+1\)
CALCULATING U USIAG aLTERNATE GROUP
FOR THE BLOCK NEAR THE LEFT BOUNDARY
```

C
C
C
C
C
C
C
C

C

$1(1, M Y 1, K+1)+R * U(2, M Y, K+1)+(2,: 1 Y 1, K+1) * J T) /(1, C+2.0 * R)$

DO $23 \mathrm{~J}=2, \mathrm{MY} 3,2$
$F B=R * U(3, J, K)+(1,0-2.0 * R) * U(2, J, K)+R * J(2, J-1, K)+R * U(1, J, K+1)+C(2, J$
1, $K+1) * D T$
$F C=R * J(3, J+1, K)+(1,0-2,0 * R) * U(2, J+1, K)+R * J(2, J+2, K)+R * J(1, J+1, K+1)$
$1+G(2, J+1, K+1){ }^{*} D T$
$U(2, J, K+1)=((1.0+2.0 * R) * F B+R * F C) /\left(1.0+4.0 * R+3 . O^{*} R * R\right)$
$U(2, J+1, k+1)=(R * F B+(1.0+2.0 * R) * F C) /(1.0+4.0 * R+3.0 * R * R)$
continue
Calculating u at the top-left corner
Calculating u at the top buUndary
DO $24 \mathrm{I}=3,4 \times 2,2$
$F A=R * U(I-1, H Y 1, K)+(1.0-2 . O * R) * U(I, A Y 1, K)+R * I J(I, H Y 2, K)+R * I J(I, M Y, K+1$
1) $+G(I, M Y 1, K+1) * D T$
$F B=R * U(I+2, H Y 1, K)+(1.0-2.0 * R) *!(I+1, H Y 1, K)+R * U(I+1, M Y 2, K)+R * U(I+1$,
$1: 1 Y, K+1)+G(I+1, N Y 1, K+1) * D T$
$U(I, A Y 1, K+1)=(F A *(1.0+2.0 * R)+F B * R) /(1.0+4.0 * R+3.0 * R * R)$
$U(T+1, H Y 1, K+1)=\left(F A^{*} R+F B^{*}(1.0+2.0 * R)\right) /(1 . C+4 \cdot C * R+3.0 * R * R)$
continue
CALCULATING U FOR BLOCK FAR FROM THE BOUNDARY
DO $26 \mathrm{~J}=2, \mathrm{HY} 3,2$
DO $25 \mathrm{I}=3,11 \times 2,2$
$F A=R * J(I-1, J, K)+(1,0-2.0 * R) * J(I, J, K)+R * U(I, J-1, K)+G(I, J, K+1) * D T$
$F B=R * U(I+2, J, V)+\left(1,0-2 . O^{*} R\right) * U(I+1, J, K)+R * U(I+1, J-1, K)+G(I+1, J, K+1)$
1*DT
$F C=R * J(I+2, J+1, K)+(1,0-2 . O * R) * U(I+1, J+1, K)+R * U(I+1, J+2, K)+G(I+1, J+$
$11, K+1) * D T$
$F D=R * U(I-1, J+1, K)+\left(1, C-2, O^{*} R\right) * U(I, J+1, K)+R * U(I, J+2, K)+G(T, J+1, K+1)$
1*DT

$11.0+2.0 * R) * F D) /\left(\left(1.0+2.0^{*} R\right) *(1.0+4.0 * R)\right)$
$U(I+1, J, K+1)=\left(R^{*}(1.0+2.0 * R) * F A+(1.0+4.0 * R+2.0 * R * R) * F B+R *(1.0+2 . \cap * R\right.$
1)*FC+2.0*R*R*FD)/((1.0+2.0*R)*(1.0+4.0*R))
$U(I+1, J+1, N+1)=(2.0 * R * R * F A+R *(1.0+2.0 * R) * F B+(1.0+4.0 * R+2.0 * R * R) * F C$
$1+R *(1.0+2.0 * R) * F D) /((1.0+2.0 * R) *(1.0+4.0 * R))$

```
    U(I,J+1,K+1)=(R*(1.0+2.0*R)*FA+2.0*R*R*FB+R*(1.0+2.0*R)*FC+(1.0+4.
    10*R+2.0*R*R)*FD)/((1.0+2.0*R)*(1.0+4.0*R))
25 CONTIIUUE
26 CONTINUE
    IF(K.EQ.KMAX1)RETURN
    IF(NC.LT.2) GO TO 42
    NC=0
    GC TO 41
    RETURN
C
C
C THIS PRCGRAM WAS CCAPILE,TESTED AND RUN USIMG ICL 1900
C AT THE COHPUTER CENTRE,UHIVERSITY OF TECHNOLOGY,
C LOUGHBORUOGH,LEICS. LE 11 3TU.
C
C
    EID
    FIIISH
```

MASTER ALDC 1
C THIS IS THE PROGRAM FOR SOLVING THE DIFFUSION CONVECTION
C EQUATION BY THE GROUP EXPLICIT METHOD

THE EXA:APLE CONSIDERED HAVING THE DIRICHLET BOUNDARY CONDITION
The NOTATIONS ARE AS FOLLONS
C X representing the space variable
C T REPRESENTIMG THE TIME VARIABLE
C U REPRESENTING THE DEPENDENT VARIABLE U.R.T. X AND T
C U(I,J) UEANS THE VALUE OF $U$ AT POINT I*DX,J*DT WHERE
C DK AND DT ARE THE SPACE AND TIME INTERVALS RESPECTIVEL
C CK IS THE COEFFICIENT ASSOCIATED WITH THE SPATIAL
C DERIVATIVE DU/DX.
C E IS THE COEFFICIENT ASSOCIATED IITH THE SECOND DERIVATIVE
C D2U/DX2.
C $\quad \mathrm{R}$ IS THE RATIO DT/H**2
C B1 IS THE LEFT fOUNDARY CONDITION
C B2 IS THE RIGHT BOUNDARY CONDITION
C TETA IS TIE WEICHTING FACTOR
C M IS THE NUMBER OF SUBIHTERVALS FOR $X$ VARIABLE
C JMAX IS THE NUMBER OF MAXIMUM TIAE-LEVELS CONSIDERED
C
TiIS IS THE MAIN PROGRAH

FIRST THE NECESSARY DATAS AND DIMENSIONING

REAL X(11), T(101), UE (11, 101), U(11, 101), DK, DT,CK, E, R, ERR(11, 101),
 1101), JA(11,101)

Integer maxdat, ndata, ll, Jilax, ife
$\operatorname{READ}(1,707)$ llaXDAT
FORMAT(IO)
C
NDATA $=0$
600 IIDATA =NLATA +1
IF (IDATA.IT.IIAXDAT) STOP
C
C READING DATA
C
$\operatorname{READ}(1,60) \mathrm{CK}, \mathrm{E}, \mathrm{DX}, \mathrm{DT}, \mathrm{TIME}$
FORMAT (5F0.0)
C
C SPACE-MODE
C
$M=1.0 / 0 X+1.0$
DO $61 I=1,14$
$X(I)=(I-1.0) * D X$

```
61 cONTIGUE
C
C TIME-IMTERVAL
C
    JIAK=TTAE/DT+1.O
    OC S2 J=1, J'4X
    T(J)=(J-1.!j)*DT
62 COMTHHJE
C
C INITIAL COMDITION
C
    DO 53 I=2,":1
    U(I, 1)=0.0
    UA(T, 1)=0.0
    UE(I, 1)=0.0
63 CONTIMUE
C
C TIE DOUNDARY CONOITION
C
    00 04 J=1,JMX
    UA (1,J)=O.C
    U(1,J)=0.0
    UE(1,J)=0.0
    UA(i,N,J)=1.0
    !(II,J)=1.0
    UE(U,J)=1.0
    coNTINUE
C
C THE INITIAL CONDITICN AND BOHIDARY CONDITIONS
C
    D0 23 I=1,:1
    !PO(I, 1)=! (I, 1)
    U:E(I, 1)=U(I, 1)
23 COMTISIJE
    DC 24 J=1,JHAX
    UPO(1,J)=U(1,J)
    Ij:AE(1,J)=!!(1,J)
    OPO(1,J)=(1(1,J)
    UPE (i, J)=1:(!, J)
    contI:%星
C
c
    CALL AMAL(X,T,CT,UE,I,JTAX,C:, E)
C
C
    CALL ALDC(OX,DT,R,JHAX,:, E,CR,U,UPO,UTE)
    CALL CAGE2(DX,NT,R,H,JMAX,UA,CX,E)
C
C
C Coipare tiE fesults
C
    INC=1
    DO 93 J=1, JMAy, INC
    IF(J.SE.1) INC=20
```

```
    DO }37\textrm{I}=1,
    ERRPO(I,J)=UPO(I,J)-!EE(I,J)
    ERRNE(I,J)=UNE(T,J)-UE(T,J)
    ERR(I,J)=U(I,J)-UE(I,J)
    ERRIJA(I,J)=JA(I,J)-UE(I,J)
97 CONTINUE
98 continue
C
C
C PRIMT THE RESULT
    URITE(2,050) CK,E,DX,DT,TMME,R
650 FORMAT(////15X,'THE SCLUTION OF DIFFUSION-CONVECTION EQUATION USIN
    1G NEW METHOD'//10X,'CONSTANT K=',F6.4/10X,'EPSILON=',F6.4/10X,'SPA
    2CE-IMTERVAL=',F6.4/10X,'TIME-INTERVAL=',FR.5/10X,'HAXIMUM-TIME=',F
    33.5/10X,'R=',FG.4)
    IMC=1
    DO 70 J=1. JMAX,INC
    IF(J.GE.1) INC=20
    WRITE(2,6005) T(J)
6006 FORMAT(/5X,'THE TIME=',F8.5)
    WRITE (2,6007)
6007 FORMAT(/2X,'X-VALUE',3X,'-EXACT'.5X,'U-NEG DIREC',5X,'ERRNE',5X,
    1'J-PMS DIREC',5X,'ERRPO',5X,'J-MVE',5X,'ERR',5X,'U-AGE',5X,'EFRAGE
    2')
        DO 6008 I=1,H
        WRITE(2,6009) X(I),VE(I,J),JNE(I,J), ERRNE(I,J),UPO(I,J), ERRPO(I,J)
        1,U(I,J),ERR(I,J),UA(I,J),ERRUA(I,J)
600O FORMAT(2X,F6.4,5X,F7.5,5X,4(F7.5,5X,F7.5.5X))
600S CONTINUE
70 CONTINUE
    GO TO }50
    END
C
C TIE ROUTINE FCR ANALYTICAL SOLUTION
C
    SUBROUTI汭 AHAL(X,T,DT,Y,:H,JMAX,CK,E)
    FEAL EK(11), EL(11), X(11),T(101),DT,V(11,101),CK, E, SUM, ALPHA,BETA,
    TPI
    IHTEGEP :H,JMAX,M1,NN,IMC
    PI=3.141592554
    M1=1-1
    DO 12 I=2,:4
    EK(I)=(EXP(CK*X(I)/E)-1.C)/(EXP(CK/E)-1.0)
    EL(I)=EXP(CK*(X(I)-1.0)/(2.0*E))
12 CONTINUE
    I:C=1
    DO 14 J=2, JMAX,INC
    IF(J.GE.11) INC=10
    DO 13 I=2,M1
    SUI:=0.0
    N:S=0
140 NN=NN+1
    ALPHA=((-1.)**NN)*(NH*PI/((NN*PI)**2+(CK/(2.0*E))**2))
```

```
BETA=(-1.0)*(((NN*PI)**2)*E+(CK**2)/(4.0*E))
V(I, J)=SUM +ALPHA*EL(I)*SIM(NN*PI*X(I))*EXP(BETA*T(J))
IF(N%.GE.100) GO TO 130
IF(ABS(SUM-V(I,J)).LE.(10.0**(-8.0))) GO TO 130
SUM=V (I, J)
GO TO 140
130 V(I,J)=EK(I)+2.0*V(I,J)
13 CONTINUE
14 CONTINUE
RETURN
    EMD
C
C TIEE ROUTINE FOR ASSMETRIC METHOD
    SUBROUTINE ALDC(DX,DT,R, SMAX,M,E,CK,U,UPO,UNE)
C
C THE ISE OF INDIVIDUAL AGE FORHILLAE AND ITS AVERAGE
C
KEAL DX,DT,F,E,CK,E1,E2,!(11,101),UPO(11,101),UNE(11,101)
C
    INTEGER M,JMAX,M1,J,K
C
H1=N-1
C
C
        SMAX1= J\AXX-1
        R=DT/(DX**2)
        R=R/2.0
        E1=E-CK*DX/2.0
        E2=E+CK*DX/2.0
C
C
        J=0
C
C USING LEET TO RICHT DIRECTION
C
10 J=J+1
        IF(J.GE.SMAX) RETURN
C
C
        DO 20 I=2,M1
        UPO(I,J+1)=(R*E2*UPO}(I-1,J+1)+(1.0-R*E1)*UPO(I,J)+R*E1*IJPO(I+1,J)
        1/(1.0+R*E2)
20 continje
C
    DO 32 I=2,:1
    UNE (I,J+1)=UPPO(I,J+1)
32 CONTINUS
C
C USING THE RIGHT TO LEFT DIRECTIOM FORGULAE
C
J= J+1
IF(J.GE.JMAX) RETURN
C
DO 21I=2,M1
```

```
    K=14+1-I
    UNE(K,J+1)=(R*E1*UNE (K+1,J+1)+R*E2*UNE (K-1,J)+(1.0-R*E2)*UNE (K,J))
1/(1.0+R*E1)
CONTINUE
C
    DO 42 I=2,M1
    UPO(I,J+1)=IJNE (I,J+1)
    42. CONTINUE
C
C uSE OF the average
C
    DO 22 I=2,.A1
    U(I,J+1)=0.5*(UPO(I,J+1)+UNE(I,J+i))
    continue
C
    GO TO 10
    END
C
    SUBROUTINE CAGE2(DX,DT,R,M, JMAX,UA,CK,E)
C
C
    REAL X(11),UA(11, 101),UI(11,101),DX,DT,R,CK,E,E1,E2,EE1,EE2,ED1,E
    1D2,EF1,EF2, E01,E02
    INTEGER M, JHAX, J:IAX1, J,:11,112,113
C
C
    M1=11-1
    M2=4-2
    M3=M-3
    JMAX 1= MMAX-1
    F=DT/(DX**2)
    R=R/2.0
    E1=E-CK*DX/2.0
    E2=E+CK*DX/2.0
    EE1=R*E1*(1.0-R*E1)
    EE2=R*E2*(1.0-R*E2)
    ED2=1.0-(R*E2)**2
    ED1=1.C-(R*E 1)**2
    EF2=R*E2*(1.C+R*E2)
    EF1=R*E1*(1.0+R*E1)
    EO1=(R*E1)**2
    EO2=(R*E2)**2
    DETA=1.O+2.0*E*R
    DC 541 J=1,JMAX
    UI(1,J)=UA(1,J)
    UI(H,J)='JA(M,J)
541 CONTINUE
    DO 542 I=2,111
    UI(I,1)=UA(I, 1)
542 CONTINUE
    J=0
C
10 J=J+1
```

c
c using ger group

DO $20 I=2,43,2$ 1deta

1) 10 ETA
continue 1R*E1)

## USING GEL GROUP

 1(1.+ine2)DO $40 I=3, \therefore 2,2$ 12, $\mathrm{J}+1$ ))/DETA $1(\mathrm{I}+2, \mathrm{~J}+1)$ )/DETA
continue
ustng gel group

$$
\begin{aligned}
& J=J+1 \\
& \operatorname{IF}(J . G E . J M A X) \text { RETURN }
\end{aligned}
$$ 1)

DO $50 \mathrm{I}=3,42,2$ 1DETA
1)/DETA

Continue
USING GER GROUP
c
$\operatorname{UI}(I, J+1)=\left(E D 2^{*} \operatorname{JA}(I, J)+E E 1^{*} \cdot \operatorname{JA}(I+1, J)+E F 2^{* J A}(I-1, J)+E O 1^{*} U A(I+2, J)\right) /$
$\mathrm{UI}(\mathrm{I}+1, \mathrm{~J}+1)=($ EE2*UA $(\mathrm{I}, \mathrm{J})+E D 1 * \mathrm{JA}(\mathrm{I}+1, \mathrm{~J})+E 02 * \mathrm{JA}(\mathrm{I}-1, \mathrm{~J})+E F 1 * \mathrm{UA}(\mathrm{I}+2, \mathrm{~J})$
$U I(M 1, J+1)=(R * E 1 * U A(M, J+1)+(1 .-R * E 2) * U A(11, J)+R * E 2 * I J A(M 2, J)) /(1.0+$
$U A(2, J+1)=(R * E 2 * J A(1, J+1)+(1,-R * E 1) * U I(2, J+1)+R * E 1 * U I(3, J+1)) /$

UA $(I, J+1)=(E D 2 * U I(I, J+1)+E E 1 *!I(I+1, J+1)+E F 2 * U I(I-1, J+1)+E 01 *!I I(I+$
$\mathrm{UA}(\mathrm{I}+1, \mathrm{~J}+1)=(E E 2 * U I(\mathrm{I}, \mathrm{J}+1)+E D 1 * U I(I+1, \mathrm{~J}+1)+E C 2 * U I(\mathrm{I}-1, \mathrm{~J}+1)+E F 1 * U I$
$\operatorname{UI}(2, J+1)=(R * E 2 * U A(1, J+1)+(1 .-R * E 1) * J A(2, J)+R * E 1 * U A(3, J)) /(1 .+R * E 2$

$U I(I+1, J+1)=(E E 2 * U A(I, J)+E D 1 * J A(I+1, J)+E 02 * U A(I-1, J)+E F 1 * \operatorname{UA}(T+2, J)$

```
        DO 55 I=2,113,2
    UA(I,J+1)=(ED2*JII(I,J+1)+EE1*UII (I +1,J+1)+EF2*!II(I-1,J+1)+EO1*UI (I+
12, J+1))/DETA
    UA(I+1,J+1)=(EE2*UI (I,J+1)+ED1*UI(I +1,J+1)+En2*UII(I-1,J+1)+EF1*UI
1(I+?,J+1))/DETA
    CONTINJE
    UA(!11,J+1)=(R*E1*JA(M,J+1)+(1.-R*E2)*UI(M1,J+1)+R*E2*UI(M2,J+1))/
    1(1.C+R*E1)
    UA(M1,J+1)=(R*E1*UA(M,J+1)+(1.-R*E2)*UI(:11,J)+R*E2*UI (H2,J))/
1(1.0+R*E1)
    GO TO 10
```

```
C
C THIS PROGRAM WAS RUN USING ICL 1900
C
    END
****
```

```
    HASTER AGE
C
C
    PROGRAH FOR QUASILINEAR PARABOLIC EQUATICN
C
    REAL R,DX,DT,TIHE,FRRIT(21,201),FRRN(21,201),ME(21,201),N(21,201),
    1JM(21,201),T(201),X(21)
C
    IHTEGER NDATA,MAXDAT, JMAX,H, INC,ITER(20,201)
    I!TEGER ITMAX1
    I:TEGER ITER1(201)
C
    PEAD(1,4212) IT:MAX1
4212 FORAAT(IO)
    READ(1,707)MAXDAT
707 FOR:AAT([0)
C
    NDATA=0
600 :OATA =NDATA +1
    IF(HDATA.GT.\AXDAT) STOP
C
C READING DATA
C
    READ(1,60)DX,DT,TIME
60 FORMAT(3FO.0)
C
C SPACE-TODE
C
    N=1.0/DX+1.0
    DO 61 [=1,M
    X(I)=(I-1.0)*DX
61 CONTINUE
C
C TIHE-INTERVAL
C
    JHAX=TTME/DT+1.0
    DO 62 J=1, J:MAX
    T(J)=(J-1.O)*DT
62 CONTINUE
C
C CALL A!ALYTICAL SOL!JTION
C
    CALL ANAL(X,T,UE,JMAX,H)
c
C DEFIHING THE INITTAL yALIjES
C
    D0 53 I=1,M
    U(I, 1)=UE(I,1)
63 ContIHUE
C
C BOUNDARY CONDITIONS
C
    DO 64 J=2, JMAX
    U(1,J)=iJE (1,J)
    U(M,J)=JE (M,J)
```

```
64 COHTIHJE
C
C
    CALL GU{SD(DX,OT,R,I,J!AX,TTER1,U, ITMAX1)
C
C
C
C ComPanflG THE RES!LT
C
    CC=1
    DO ge J=1, JIAX, TiC
    IE(U.CE.11) I:SC=10
    DO 9? I=1,i!
    ERRIT(T,J)=U(I,J)-IE(T,J)
O? COMTTMHE
g3 CONTISUE
C PKINT TIE RESHLT
    MKITE(2,550) DX,NT,TI:AE,R
```



```
    10% AXE ALOORITHO'//10X,'SPACE-THTFRYAL=',FE.4/10X,'TTME-TITERVAL='
```



```
        IN=1
        DO 70 J=1, J:ANX, IEC
        IF(J.SE.11) INC=10
        MTITE(2,6005) T(J)
\sigma006 FORIAT(/5%,'mHE TIME=',F8.5)
        A!ITE(2,6007)
6007 FORMAT(/5X,'X-VALUE',5X,'U-EYACT',10X,'I-RITH', 10X,'E4RTT', 10X,'I-
    1:ETMOM-T"ER', 1OX,'ERRH')
        DO COC3 I=1,`
        ARITE(2,GOOO) X(I),UE(I,J),G([,J),ITERT(J),FRSIT(I,J),UH(r,J),TTER
        1(I,J), ERRY(I,J)
6009 FOBMAT(5X,F6.4,5X,F13.9.2X,F13.8,'(',55,')',5X,F10.2.5%,
    1F13.%'(',I4,')',5X,F10.2)
6002 COMTIME
70 CoMTLHJE
    GO TO 600
    E(ii)
C
C
C tias is the routide for amalymtcal solumtom
C
C
    SJBHOUTIME ANAL(X,T,JE, NAAX,:O
C
C THE A:ALYTICRL SOLUTIOA OF mAF SUSTLIMEAR PARAFOLTC [OJATIO:
C
    REAL X(21),T(201),UE(21,201),UAPP(21,201)
C
    INTEGER :1,JMX
C
C IMITIAL APPROYI'MATOA OG m:HE SOLITION
```

C
$0-J=1, \mathrm{~d} A \mathrm{X}$
DO $4 I=1,1$
$\operatorname{UAPP}\left([, J)=1.0+\left(4.2-E X P\left(2.0^{*}\left(X(I)-2 . n^{* T}(J)\right)\right)\right) /\left(4.0+2.0^{* R y P}\left(2.0^{*}(X(T\right.\right.\right.$
1)-2.0*T(d)))

4
3
c
c
C
Do $6 \mathrm{~J}=1, \mathrm{Jan}$
DO $5 I=1, \mathrm{O}$
7 UE(I,J) $=0.5+(1.5-0.5 * \operatorname{ALOG(1JAPP}(I, J)-0.5)-X(I)+2.0 * ?(J)) *(1.0-1.7 /($
12.0*(JAPP(I, J)))

$\operatorname{UAPP}(\mathrm{I}, \mathrm{j})=\mathrm{JE}(\mathrm{I}, \mathrm{d})$
IF(JAPP(I,J)-0.5) 5.5.7
comtinue
6 contrine
RETURM
EN!
C
C
$c$ THIS IS THE ROUTIME FOR AGE METHOD
c
$c$

c
c
REAL X(21), R, SX, DT, UH(21,201), UT(21,201)
C

fitecler ITH:X1
$c$

C
111:"-1
$112=: 1-2$
$11=:!-3$
J:1ax $=\mathrm{Jin} \times-1$
$\mathrm{R}=\mathrm{DT} /(\mathrm{DX**2})$
C
DO 1425 d=2. Jyax
$U[(1, J)=\ln (1, J)$

1425 comtide
C
$j=0$
c
c ustig ger group
C
$n=1$
$10 \quad N=: I C+1$
$\mathrm{J}=\mathrm{J}+1$

```
IF (J.GE.J:AX) EETURN
```

$$
\mathrm{C}
$$

            \(i C=0\)
    C

$110=10+$
$J=J+1$
IF (J.OE.JIAK) BETURN
C
C THE FEABT UNGROUFED POINT
C
D) $20 \mathrm{I}=2,133,2$

B1=n*(UH(T, J)+JH( $[-1, J))$
A1=R*(1n $(I+2, J)+U(I+1, J))$
$K=I$
$\operatorname{ITER}(K, j+1)=0$
$U I(I, j+1)=U N(I, J)$
$U T(I+1, J+1)=U H(I+1, J)$
$\operatorname{ITER}\left(K, c^{\prime}+1\right)=\operatorname{ITGR}(K, j+1)+1$
$B:=R *(U I(I+1, J+1)+J I(I, J+1))$
DETA=1.0+2.0*13:
C1 $=(1.0-21) * \operatorname{HP}(I, J)+B 1 *: 1+(I-1, J)$
$C 2=(1.0-A 1) * \operatorname{Un}(I+1, J)+A 1 \times \operatorname{Ha}(I+2, J)$
$U: I(I, J+1)=\left((1,2+B H) * C 1+D H^{*} C_{2}\right) /$ OETA
U $(I+1, J+1)=\left(1 A^{*} C 1+(1.3+B S) * C 2\right) / D E T A$

$U I(I, J+1)=U M(I, J+1)$
$U I(I+1, J+1)=1 J N(I+1, J+1)$
G) TC 25

20 contcme
C

$\operatorname{ITER}(11, j+1)=0$
$\operatorname{UI}(: 11, d+1)=\operatorname{JH}(:: 1, j)$
$\operatorname{ITER}(111, J+1)=\operatorname{ITER}(11, J+1)+1$
By=i* $(U:(i, J+1)+U Y(111, J+1))$
$\operatorname{IF}(\operatorname{ITE}(11, \mathrm{~J}+1)$. EQ. ITMAX1) क TO 15
$J-(11, J+1)=J(i, 1, d+1)$
(1) TO 35

IF (10.EQ.2) SO TO 40
© TO 10

ic $\quad$ ic 0
UDIIG GLL GROUP

IE (ABS (UM $(I, J+1)-J I(I, J+1)) \cdot G T .(10.0 * *(-6.0)))$ GJ TO 30
$\operatorname{IF}(\operatorname{ABS}(U \operatorname{Ha}(I+1, j+1)-I T([+1, j+1)) . \operatorname{LT}(10.0 * *(-6.0)))$ as T0 20

$\operatorname{IF}\left(\operatorname{ARG}(\mathrm{UY}(\mathrm{B} 1, \mathrm{~J}+1)-\mathrm{JT}(11, \mathrm{~J}+1)) . \operatorname{LT} .\left(10 . \mathrm{C}^{* *}(-\mathrm{r}, 0)\right)\right)$ GTO 15

A $1=R^{*}(3 H(3, J)+1 H(?, J))$
$\operatorname{ITER}(2, j+1)=0$

```
    gEAL U(21,201),'U(21,201),马,OY,DT, AL`,AL
    IPTEGER ITER1(201),!1,:1,N!2,\because3,J!AX, J:AX1, J,TTUAX\
    111=!-1
    11?=1-2
    43=:1-3
    J:AK1=J#AX-1
    R=OT/(DX**?)
    DO 432 J=2, J!AX
    U|(1,J)=!J(1,J)
    Ufi(:i,J)=U(i, J)
```

```
    ITER(2,J+1)=ITER(2,J+1)+1
```



```
    IF(ITER(2,J+1).EO.ITHAX1) SO IO 55
    IF(ABS(J!:(2,J+1)-i![(2,J+1)).LT.(10.0**(-6.0))) G? T0 55
    UI (2,J+1)=13:4(2,J+1)
    G) T0 50
    COHTI:HJ
    DO 65 I=3,i4,2
    B1=[*(UN(I, J)+JA: (I-1,J))
    A1=R*(0N(T+2,J)+j)(I+1,J))
    K=I
    ITER(K,J+1)=0
    UI(I,J+1)=UM(I,J)
    UI}(I+1,J+1)=|:(I+1,J
    ITER (\therefore,J+1)=ITER (K,J+1)+1
    BM=R*(UI(I+1,J+1)+UI(I,J+1))
    DETA=1.0+2.n*ET
    Cl=(1.(-d1)*|N(I,J)+E1*:!: (I-1,J)
    C?=(1.0-A1)*リN(I+1,J)+A 1*1Hi'(T+?,J)
    U*(I, J+1)=((1.I+R1)*C1+BM*C2)/DETA
    UA(T+1,J+1)=(2M*C1+(1.n+3N)*C, (NETA
    IF(ITEK(!, J+1).EQ. ITMAX1) OJ TO 65
    IF(ABS(O!(I,J+1)-UI(T,J+1)).GT.(10.0**(-6.0))) G} T0 70
    IP(ARS(UM(I+1,J+1)-!II(I+1,J+1)).LT.(10.O**(-(.0))) O) TO 65
    JI (I, J+1) =ija(I, J+1)
    UI (I+1,J+1)=U!(I+1,J+1)
    GO TO 60
    CO:Trume
    IF(AC.LT.2) 30 T0 45
    :C=0
    G) TO 10
    E*D
*
    THE ROUTI|E FOR ABSYMERICAL ALGOITEA
```



```
    LOH ORDER FORIULAE WITH THE CHAHGE OF SHEEPIMG DIRECTTOH
    contrmue
```

    \(U I(2, J+1)=!!!(2, J)\)
    ```
C
J=0
C
C positive derection
C
10 N-ic+1
    J=J+1
    IF(J.EQ.JIAX) RETURM
    ITER1(J+1)=?
C
C INITIAL MUESSES
C
    DO 433 I=2,:1
    U(I,J+1)=U(I,J)
433 CONTTHJE
430 ITER1(J+1)=\operatorname{TER1}(J+1)+1
    IF(ITER1(J+1).RQ.ITMAK1) GO TO 450
    DO 434 I=2,H
    ALI=R*(J(I+1,J)+JJ(I,J))
    AL=R*(J(I, I+1)+IJ:(I-1,J+1))
    U:I(I,J+1)=(AL*J!(I-1,J+1)+AL.**S(I+1,J)+(1.0-AL1)*!(T,J))/(1.O+!L)
434 COMTIHUE
    EPS=0.0
    DO 435 I=2,N1
    EPS=EPS+(UA(I,J+1)-! (I,J+1))
435 CONTINOE
    EPS=EPS/:12
    IF(GPS.LT.10.0**(-5)) C0 T0 440
    DO 42r, I=?,:i9
    U(I,J+1)=iNN([,J+1)
4ju comTIMUS
    Gu TO 4.9
440 COMTEN:OE
    DO 437 I=2,:11
    U(I,J+1)=!f:([,J+1)
437 codTcume
    I#(:C.E\alpha.2) Gu TO 30
    G) TO 10
C
C gegative otgection
30 :0=0
4C :
    J=J+1
    IF(J.EO.JHAX) aETURA
    ITER1(J+1)=0
C
C InItIAL MUESSES
C
    D0 452 I=?,!11
    U(I,J+1)=J(I,J)
452 COHTT:OUE
453 ITERT(J+1)=ITER1 (J+1)+1
```

```
    IF(ITER1(J+1).EQ.ITMAX1) SO TO 460
    DO454 II=2,N1
    I=1-II+1
    ALI=R*(U(I,J)+U(I-1,J))
    AL=?*(iN(I+1,J+1)+U(I,J+1))
    U!I([,j+1)=(fL*UH(I+1,J+1)+(1.0-AL1)*U(I,J)+AL{*U(I-1,J))/(1.O+1L)
454
    ONTINDE
    EPS=0.0
    D) 455 I=?,:1
    EPS=EPS+(JN([,J+1)-J(I,J+1))
455 CovmThJE
    EPS=EPS/:12
    IF(EPS.IT.10.0**(-5)) OO TO 4600
    D0 456 I=2,:11
        U([,J+1)=UH(I,J+1)
456 CONTINUE
        GO TO 453
4600 COTTITUE
    D) 437C I=2,011
        U(I,J+1)=Jil(I,J+1)
4370 COMTPWJE
        IF(4C.LT.Z) SO TO 40
        H=0
        (0) ro 10
460 vaITE(2.4624) ITMAX1.J
4624 FORMAT(//10X,'ITER EQUAL TO ',I5,'AFTER I=',I5)
C
C
C THTS FROGRAH WAS RUH UST:G ICL 1000
C
C
    E:D
    FTNISH
```

```
    MASTER BJRGER
C
C PROGRAM FOR RIRGER'S EOUATION
C
    REAL E,DX,DT,TMAE,T(21),Q,U(101,21),JN(101,21), FRR(101,21),X(101)
C
O
C
    GMTEGER MAMOAT,MMX,HDATA,MCIR,MCIPC,NTRIE,ITER(101,21) ,M1, TMC
C
C
    PI=3.141592654
    FEAD(1,707) :1AXDAT
707 FOQ4AT(IO)
C
60 NDATA=WDATA+1
    IF(:IDATA.UT.MAXDAT) STOP
C
C READIMG DATA
C
    READ(1,60) E,DX,DT,KTRUE,NOTR
60) FORMAT(3:0.0,2IO)
C
C SFACE :WDE
C
        M=1.0/0K+1.0
        R=DT/(DX**2)
        DO E1 I=1,::
        Y(I)=(T-1.0)*DX
61 COWTINUE
        XMAK=(KTRUE-1)/MCIR+1
        :11=:1
C
c
C WRIPE THE TITLE
C
    HGTE(2,200) DX,UT,K:AX,R,UTQ1E,E
200 FOR:AT(/////15X,'THE SOLUTION OE TIE RURORRS FOUATIOM BY AGE ALGOR
            1ITH1'//10X,'SPACE-IMTEPUAL=',F6.4/10X,'TIME-TITERVAL=', ER.6/10X'MA
        2XINW STEP PER CIRCULATIOH=',I5/1O%,'R=',E6.4/1OX,'TPUE TTMF STEP=
        3',IG/10X,'EPSILON=',F2.6)
            HCIRC=O
            TEASE=0
320 COMTIn!E
C
C TiE TiME IntgPyAL
            00 102 K=1,N1AX
            T(K)=TBASE+(K-1)*DT
102 covaI:DE
C
C
C
C IMITIAL COMDITION
```

```
C
C
    IR(NCIRC.GT.0) GO TO 1151
    DO 6? I=?,H1
    J:(I, 1)=3[H(PI*X([))
63 ComTrmue
1151 COLTI:UE
C
C
C BOMNAKY COMDITIONS
C
    On 6: J=1,k1AX
    Un(!:, J)=0.0
    Ua(1, 1,J)=0.0
64 ConTthue
C
C CALLTMG ALGORITH FOQ frre
C
    Catl barage(OY,P,H,G1AX,04,E,ITFR)
C
C COMPAHING THE RESULT
C
C
C britImg tiE results
    TMC=1
    DO 70 J=1, TMAX,TMC
    IF(:GIRC.GT.O.OR.J.OE.11) I!C=5
    HMITE(2,6006) T(J)
EOOG FORMAT(/5X,'TIGE TTME=',F3.5)
    WHITE(?,6007)
6007 FOR'MAT(/5X.'X-VALIE'.5X.'U-ilm,MEIC-ITEP')
    006008 = = 1, i1
    SRITE(2,5009) Y(I),0!(T,J),TRER(T,J)
6009 FOMAT(5X,F6.4,5X,F13.3,'(',rC,')')
6003 conTfuJE
70 COUTENHE
3246 :OTRC=1CIRC+1
    IP(UCIRC.EQ.DCRR) SO TO 3j"
    DO 3060 I=R,:1
    UA(I, 1)=!N(I,N4AX)
3060 ComTrive
    TBASE=T(YOAX)
    G) TO 320
C
C
330 COMTIUUE
C
C
        G) TO SOC
        E!iD
C
C
C THE ROUTIUE FOR AGE :ETHOD
```

```
C
C
C
C THE AUS MOUTISE FOR NOMLINEAR MUROER'S EOUATION
    GEAL R,E,DX,UH(101,21),U=(101,21),E1, E2, E3, E4,C1,C2,C3,C4,05,C&,C?
    REAL CE
    HMTEGER ITER(101,21), J:1AX,1,:11,:12,:3,J,K
C
C
    :11=1-1
    12=1-2
    113=11-3
    J\AX1=J:AXX-1
C
C
    J=0
C
C ISTHO OER GPOUP
C
10. :C=00+1
    J=J+1
    IF(J.GS.J:AX) RETIRN
C
C
    10 20 I=2,43,2
    K=I
    ITEH(%,J+1)=0
    UT(I, ', 1)=[JM(I,j)
    UI}(I+1,I+1)=U!U(I+1,J
25 ITRR(K,j+1)=ITER(K,J+1)+1
    E1=E-DX*(UI([,J+1)+UII(I,J))/4.0
    E2=E+EX*(UI (I,J+1)+!1:(I,J))/4.n
    EZ=F+DX*(UI(I+1,J+1)+US:(I+1,J))/4.0
```



```
    DETA=1.O+E1*?+E 3*R
    C1=1.0+R*(E3-5?)-52*E?*R*?
    C?=E1*F*(1.n-E4*R)
    C3=E3*R*(1.0-E2*R)
    C4=1.0+E1*R-E4*R-E1*E4*R*R
    C5=(1.0+E3*R)*F2*R
    Cf=C1*G4*?*R
    C7=E2*巳3*R*R
    C`=(1.C+E1*?)*E4*R
```




```
1TA
    IF(ITER(K,J+1).EQ.20) (0 TO 20
    IF}(a2S(1H(I.J+1)-1IT(I,J+1)).GT.(10.0**(-6.0))) GO TO 3.
    IF(ABS(1)N(I+1,J+1)-UI([I+1,J+1)).LT.(10.0**(-6.0))) SO TO 20
    UI(I,J+1)=SN(I,J+1)
```

```
    UI (I+1,J+1)=UM(I+1,J+1)
    CO TO 25
20
    CONIIMUE
C
C THE LIST m!SaOupED POINT
C
    TgCg(11, j+1)=0
    UE(111, J+1)=1:1!(!11,J)
35 ITCR(:1,J+1)=\operatorname{ImER}(0,1,J+1)+1
    E1= =-) Y*(!JI(:1,J+1)+U!(!4,J))/4.0
    E2=E+0)**(JT(11,J+1)+10(111,.1))/4.0
```



```
    1+R*E1)
    IF([TER(:1,J+1).E2.20) 20 T0 15
    IE(ARS(U|(i11, J+1)-JI(111,j+1)).LI.(10.0**(-6.0))) G? T0 15
    UT}(:11,J+1)=JN(111,J+1
    GO TO 35
15 IF(NC.EO.2) 30 TO 40
    GO ro 10
C
40 NC=0
C
C. MGIMG GEL GROUP
C
45 MC=:16+1
    J=J +1
    IF(J.OE.J1AX) RETURO
C
C
C
    TER}(2,j+1)=
    UZ(2,J+1)=!:M(2,j)
    IMFR(2,J+1)=ITER(2,J+1)+1
    E.T=E-ON* (U[(?,J+1)+JI(?,J))/4.0
    E?=E+M草(UT(2,J+1)+J!(?, (?))/4.?
```



```
    1E?)
    IF(ITER(2.J+1).EQ.20) GO TO 55
```



```
    UT (2,J+1)=JH(2,J+1)
    G) T0 50
55 conrinue
    DO 65 I=3,:2,2
    K=I
    IMER(K,J+1)=0
    UI(I,J+1)=0.I(I,J)
    UT(I+1,J+1)=U6(I+1,J)
60 ITER (K,J+1)=\operatorname{TTER}(%,J+1)+1
    E1=E-DX*(UI (I,J+1)+JN(I,J))/4.n
    E?=5+DY*(UT(I,J+1)+JN(T,J))/4.C
    E3=E+i)X*(UI (I+1,J+1)+1N(I + 1,J))/4.0
    Ea=E-D)X*(UI (I+1,J+1)+in(I (I 1,J) )/4.0
    DETA=1.(1+E1*R+E3*R
    Cl=1.O+R*(E3-E2)-E2*E3*?*R
```

```
    C2=E1*R*(1.0-24*R)
    C3=E_*!*(1.0-E2*?)
        C4=1.O+E1*R-E4*P-E1*E!**R*R
    C5=(1.O+E3*iN)*E2*R
    C0=E1*E4*:0*?
    C7=E2*E`*R**
    CS=(1.0+E1*隹*E4*R
```




```
    1TA
    IF(ITER(K,J+1).FQ.20) 20 T0 55
    IF(ABS(il:(I,J+1)-JI(I,J+1)).GT.(12.0**(-6.0))) (1) T0 70
    IF(ATS(JM(I+1,J+1)-!I(I+1,J+1)).LT.(12.0**(-5.0))) 50 T0 55
70 JT(I,J+1)=(0!(I,J+!)
    UI}(I+1,J+1)=|{:([+1,U+1
    GO T0 60
65 CnHTruus
    IF(%C.LT.2) %) TO 45
    HC=0
    Gu ro 10
C
C
C TUIS PROGRAM WAS EMA JING ICL ISOO
C
C
    E:D
    Er:IS:%
```

```
C THIS IS THE PQCGRAM FOR SOLVIHG THE DIFFISSIOH CONVECTIOH
C EqUATION BY THE FTUITE ELEMENT NETIOD
C
C IHVOLVIIG こ-TIME-LEVELS J AND J+1
C
C
C
C
C
C
C
C
T REPRESENTIHG THE TME VAgIARLE
J PEPRESENTING TiE DEPENDENT VAFIABLE Y.f.T. X A!D T.
C U(I,J) IEANS THE VALUE OF I! AT POINT T*&, J*DT MAESE
C H Alid DT ARE THE SFACE AHD TI'E IMTERVALS RESPECTIVELY
C CK IS TIE COEFFICIENT ASSOCIATED UITH THE SPATTAL
C DERIVATIVE DU/DX.
C E IS THE COEFEIOIEHT ASSOCIATED WTTH THE SECOMD DRRIVATIVE
C DZJ/DX2.
C R IS THE FATTO DT/U**2
C Pl IS THE LEFT POUNDARY CO!DITIOH
C R2 IS TJE RIGHT BOUHDARY CONOITION
C TETA IS THE WEIGLTIHG FACTOR
C M IS THE MUMEE OF SUBIMTERVALS FOR X vartarle
C LAST TS THE NOMEER OF HAYIHWM TH{ELEVELS COISIDEfED
C A IS THE DIACONAL ELEIENT OF THE TRIDIAGOWAL 'ATRIX
C C IS THE SUPEFOTAGONAL ELEUEHT OF THE TRIDIAOOMAL UATRIX
C B IS THE SUHDIAGODAL ELEIENT OF TIE TAIDTAGOMAL TATRTY
C F IS THE RIGHT-4AHD SIDE ELEYENT OF THE SYSTEH OF ECTATION
C THIS IS THE IAIN PPOGRA:!
C
C
C
    FIRST THE IECESSARY CATAS AMD DIGEHSIONTHG
    PEAL*3 CK, E,OT,A,PI,B2,E1,Y(11),T(201),0(11,201),
    1A(11),E(11),C(11),F(11)
    REAL*3 AF,CF,FF,BE
    REAL*3 TETA
    JHTEGSR il ,11,:12,IL,LAST
    B1=1.0
    B?=0.0
    H二19
    11=:-1
    112=14-2
    H=1./(M-1.)
    READ(1,*) CK,E,DT,TETA,LAST
    R=DT/(H**2)
    PI=3.141592554
C
C THE INITIAL CONDITION
C
C
DO 50 I=2,11
```

$$
0060 \quad[=1.11
$$

$$
X(I)=(I-1 .) * H
$$

comTIMUE
continue
THE LEFT DOMDARY COMDITION
DO $55, j=1, \operatorname{LAST}$
$U(1, U)=1.0$
comtinue
THE SPACE IHTERVAL
TUE TTAE IUTERVAL
(1) $55 \mathrm{~J}=1$, LAST
$T(J)=(J-1) * J$.
Contivue
USTIG THE LU DEGOMPOSITION
$A(2)=1.13 .+Q^{*} E^{*}(1 .+4 . * T E T A) / E$.
DO $70 \quad I=3,: 1$
$A(I)=A(2)$
$C(I)=C(2)$
COMTINJE
$A(M)=1.13 .+R^{*} E^{*}(1 .+4 . * T E T A) / 6$.
$R(!)=1 . / 6 .-R^{*} *(1 .+4 . *$ TETA $) / 5$.
U( $[, 1)=0.0$
SOLI:G THE Finite dffferevce equation

$C(2)=1.112 .-(1 .+4 . * T E T A) *(R * E / 12 .-C K * R *!/ 24$.
$E(I)=1 . / 12 .-(1 .+4 . * T E T A) *\left(R^{*} E / 12 .+C I * F_{i}^{*} / / 24.\right)$

```
    IF(A(2).EQ.0.0)CALL EXIT
    C(2)=C(2)/A(2)
    DO 75 I=3,[11
    A(I)=A(T)-3(I)*C(I-1)
    IE(A(I).EO.O.0)CALL EXIT
    C(I)=C(I)/A([)
    CONTTNiE
    A(1)=4(:1)-B(11)*C(11)
C
C
C
C
C
    AF=+1./3.-R*E*(5.-4.*TETA)/6.
    CF}=+1./12.+(5.-4.*TETA)*(R*E/12.-CK*?*:1/24.)
    BF=+1./12.+(5.-4.*TETA)*(R*E/12.+Cए*R*!/24.)
    BE=0.5*:2*E+CK*R*IF*0.25
    LAST1=LAST-1
    DO 95 J=1,LAST1
    F(2)=3E*E1+AF*J(2,j)+CF*:J(3,J)
    DO d0 I=3,:11
    F(I)=HE*(I(I-1,J)+AF*U(I,J)+CF*I)(I+1,J)
    CONTINJE
    F(A)=(DF+CF)*!(IM1,J)+AF*!(A,J)
C
C
C
C
C
        U(2,J+1)=F(2)/A(2)
        DO }85\textrm{I}=5.
        U(I,J+1)=(F(I)-B(I)*IJ(I-1,J+1))/A(I)
        CONTIMUE
        DO 90 E=1,42
        IL=1-I
        U(IL,J+1)=U([LL,J+1)-C(IL)*!J(IL+1,J+1)
OO CovT[rue
95 CONTINJE
C
C
C
C
C
C
    thITE(1,190)
100 FCRMAT(/////10X,'THE MMMERICAL SOLITION OF SERIVATIVE ROUADARY PRO
    1BLE|'//10X,'3Y THE FIMITE ELEMENT METHOD')
        WEITE(1,105) CK,E,DT,H,R
105 FORHAT(///5X,'THE RESIULS FOLLOUS'//5X,'COMSTATIT F'=',F8.4/5X,'EPSI
    1LOH=',FE.4/5X,'TEME-LEVEL=',FR.4/5X,'SPACE-INTEPYAL=',F3.4/
    25X.'R=',F12.6)
        DO 135 J=1,LAST,10
        WEITE(1,11C) T(J)
110 FORHAT(///5^,'THE TIME IS=',FR.4)
```

```
    WRITE(1,115) (X(I),I=1,6)
115 FORi{AT(//1X,'/ALJF K',3X,6(F3.4,PX))
    WRITE(1,120) (U(I,J), I=1,N)
120 FORHAT(/1X,'U-HMA', 3X,6(F13.8,3X))
    WRITE(1,125) (X(I),I=7,11)
125 FOBMAT(//1X,'VALJE X'. ЗX,5(FE.4, X))
    MRITE(1,130) (J(I,J),I=7,11)
130 EORAMT(/1X,'リ-HU4',3X,5(F13.3,3X))
135 CONTINUE
    CALL EXIT
    E!iD
C
C
C THIS PROGAAH WAS TEETED AHD RJM UST:GG PRIME H00
C at the coputer centre, matversity df tech:%ology,
C LOUGIEOROUGH,LEICESTERSHIRE,LE11 ZTU.
```

C T REPRESENTING THE TIME YAREADLE
C U REPRESENTIMG THE DEPEMDENT VARIABLE W.R.T. X AND T.
C U(I, J) IEAMS The VALUE OF II AT PUIHT I*U, J*DT GUERE
C $\quad$ A ADD DT ARE THE SPACE AOT TIME TMTERVALS RESPECTIVELY
C CR IS THE COEFFICIEAT ASSUCTATED UETH THE SPATAAL
C DEFIVATIVE DUMX.
C E IS THE COEFFICIEAT nSSOCIATED UITH THE SECO:D DERTVATIVE
C D2U/DK2.
$\mathrm{C} \quad \mathrm{R}$ IS THE RâTO DT/H**2
C B1 IS THE LEFT BCONDARY COMDITICN
C B2 IS THE RIGHT ROUNDARY GOMDITION
C TETA IS THE VEIGHTING FACTOR
C $\because 1$ IS THE numeer of subintervals for t vartanle.
$C$ H TS THE HUMER OF SUZIVTERVALS IA $X$ VAPIARLE
C THE HETHOD LEAD TO A BLOCK TYPE LIMEAR SYSTEY OF EQUATTOH
C UITH TEIDIAGONAL BLCCK FORA
C EACH block inatrix are of trioiagomal fona
C A IS THE DAGOMAL RLOCK OE TIE TRTOTACOMAL GLOSK :ATRIK
$C$ C IS THE SUPERDIAGOMAL DLOCK OF THE TRTVTAGOBAL SOC: MATRTV
C 3 IS THE SURDIAGUMAL BLOCK OE THE TRTDTAGODOL RLOCK MATRTX
C F IS THE PIGHT-HAM SIDE ELEHEHT DF THE EYSTEH OF EQUATION
C
C
C
C
c
C EPBHAX IS THE COMVERGENCE TOLERAMCE
C IS IS THE RELAYATIOH: PARATETER
C UEID IS THE FIMAL VALUE OF W JSED
C WIHC IS TUE INCREMEHT IN $I$ USED
C ITEAAX IS THE YAXIMIM ITERATION NISBES ASSHPED EOR CONVERGENCE

```
```

C

```
```

C
C
C
C START IITH THE EACH SET OF DATA
C START IITH THE EACH SET OF DATA
C
C
5 IDATA=?DATA+1
5 IDATA=?DATA+1
C
C
C
C
1 0
1 0
C
C
C
C
C
C
C
C
C
C
C
C
:11=:1-1
:11=:1-1
M?=-2
M?=-2
C
C
C
C
C
C
EO 15 J=1,i:
EO 15 J=1,i:
U(1,J)=0.0
U(1,J)=0.0
J(:, J)=0.0
J(:, J)=0.0
CONTINJE
CONTINJE
C THE HODE POTHTS SPACEITSE
C THE HODE POTHTS SPACEITSE
C
C
M 20 I=1,M
M 20 I=1,M
X(I)=(I-1.)*!!
X(I)=(I-1.)*!!
conTrwue
conTrwue
DO 25 E=1,41
DO 25 E=1,41
U(I, 1)=EYP(CK*Y(I)/(2.*E))*X(I)*(1.- - (I))
U(I, 1)=EYP(CK*Y(I)/(2.*E))*X(I)*(1.- - (I))
COHTI!NF.
COHTI!NF.
C
C
C THE HODE POLUTS TIUEITOE
C THE HODE POLUTS TIUEITOE
D0 30 J=1,`     D0 30 J=1,`
TT}(J)=(J-1.)*!T
TT}(J)=(J-1.)*!T
comiliue
comiliue
C
C
C the steady state socijTOM AT tite T
C the steady state socijTOM AT tite T
C
C
DO 35 I= 1.:1
DO 35 I= 1.:1
U(i,i)}=0.
U(i,i)}=0.
contINUE
contINUE
C
C
C TIE LIST OF ALL AEBREVIATIO:S
C TIE LIST OF ALL AEBREVIATIO:S
C

```
C
```

```
    NDATA=0
```

    NDATA=0
    C READING THE II!DIVIDUAL DATA SET
C READING THE II!DIVIDUAL DATA SET
HEAD(1,10) SY,E,DT,H,T,TTEMAX,EPSIANX,H,MEMD,ITHC
HEAD(1,10) SY,E,DT,H,T,TTEMAX,EPSIANX,H,MEMD,ITHC
FOMIAT(5FO.O,IO,UEO.0)
FOMIAT(5FO.O,IO,UEO.0)
IT:II:=ITEMAX
IT:II:=ITEMAX
THE UJ:OEP OF SPACE TITERVAL : -1
THE UJ:OEP OF SPACE TITERVAL : -1
M=1.0/H+1.0
M=1.0/H+1.0
H1=:1-1
H1=:1-1
112=1-2
112=1-2
THE MHMEEP OG THIE IITERVAL :i-1
THE MHMEEP OG THIE IITERVAL :i-1
:=T/DT+1.0
:=T/DT+1.0
TIE IHITIAL MHD BOMMDARY COMDITICN

```
    TIE IHITIAL MHD BOMMDARY COMDITICN
```

```
    R=DT/(H**2)
    B1=-4.0*E*R
    B2=2.O*E*R-CK*?*!1
    Bj=2.0*E*R+CK*R*iH
    C1=-1.0
    C2=0.0
    CZ=0.0
    A 1=1.0
    A2=0.0
    A 3 =0.0
C
C
C
C GIVING THE IMITIAL GESSES FOR U(I,J)
C
45
    ITER=[TEP+1
    ITHAX1=ITEMAX +1
    IF(ITER.LT.ITHAX1) 00 T0 05
    WRITE(2,CC) U, rTEMAX
    FOMHAT(//10X,'FOR M=',2X,FE,4,2X,'ITERATIOM :O. GREATER THAH',
        12X.16)
    O) TO 140
5
    EPS=0.0
    DO 70 J=2,41
    DO 70 I=?,!:1
    UOLD(I,J)=U(I,J)
70 contimue
C
C
C BIGUT WAMD STDE FOS TUE SECOID LIME(TIME-LEVEL)
    F(2)=-(A S*U(1,1)+A 1*U(2,1)+A2*U(3,1)+Bj*J(1,2)+C3*IJ(1, ?))
    DO 75 I=3.1:2
    F(I)=-(A3*!J(I-1,1)+A1*J(I, 1)+A2*!j(I+1,1))
75 COHTENUE
    F(11)=-(&3*J(12,1)+A1*|(11,1)+A2*!J(%,1)+R2*)(N,2)+C2*U(1, 3))
C
C ADDING IITH THE MATRIX C :MLTIPLY BY YOLD(I, 3)
    F(2)=F(2)-C1*:IOLD(2,3)-C2*UOLD(3,3)
    DU 30 I = 3,N2
    F(I)=F(I)-C3*UOLD(I-1,3)-C1*:JGLD(I, 3)-C2*1OLD(T+1,3)
```

    CONTINIE
    F(!1)=F(11)-C2*UOLD(12,3)-C1*)
    C
C FIHD U(I,J) FOR J=2
C
CALL TRID(31, E2, R3,F,1,2,:1)
DO 85 T=2,:1
U(T, 2)=!*!J(I, 2)+(1.0-V)*11OLD(I, 2)
EPS=EPS+APS(U(I,2)-!OLD(I,2))
CONGINUE
C
C RIGHT HAHO STDE FOR THIRN TO -2 LEVEL
DO 110 J=3.:?
F(?)=-(A3*U(1,J-1)+5\mp@subsup{3}{}{*}J(1,J)+C}\mp@subsup{3}{}{*}J(1,J+1)
DO 50 I=3,1%
F(I)=0.0
90 COMTMUS
F(N1)=-(A2*(J(N,J-1)+B2*U(N,J)+C2*iJ(!,J+1))
C
C ADDIUG NITH THE MATRIX A iHLTTPLY BY U(I,J-1)
F(2)=F(2)-1 1*!J(2,J-1)-n2*|(2,J-1)
DO 95 I=3,i.2
F(I)=F(I)-A 3*! ([-1,J-1)-A1*U(T,J-1)-A2*:J(T+1,J-1)
CONTIMJE
F(id1)=F(N1)-A3*!(N2,J-1)-i:1*iJ(|1,J-1)
C
C AMNIMG IITH THE MATRIY C WULTPLY BY U(T, J+1)
F(2)=F(2)-6,1*!OLD(2,J+1)-C2*IOOLO(3,I+1)
DO 100 I=3,12
F(I)=F(I)-C3*UOLD(I-1,J+1)-C1*:JOLD(I,J+1)-C2*10LD(I+1,J+1)
1OO CONTINEE

```

```

C
C ETHD J(I,J) FOR J=3 TO M?
C
C
CALL TRIN(R1,122,U3,F,U,J,:N)
DO 105 I=2,:11
J(I,J)=,*|(I,J)+(1.0-I)*i!OLD(I,J)
EPS=EPS+ABS(II(I,J)-UOLD(T,J))
105 CODTPME
110 SONTISUE
C :IGHT HA:D STDE FOR (H-1)ST. LEVEL
C
F(2)=-(13*J(1,12)+13*!J(1,:11)+C3*U(1,11)+C1*!1(2,!')+C2*!J(3,!1))
DO 115 I=3,42

```

```

115 COMTT:UE

```

```

C
C ADOIYG NITH THE MATRIX A MILTIPLY BY U(I,:|-2)
C
F(2)=F(2)-A1*1J(2,:2)-12*!J(3,12)
DO 120 I=3,112

```
```

        F(I)=F(I)-A 3*J(I-1,H2)-A1*U(I,H2)-A2*U(I+1,H2)
    120 COMIMJE
F(111)=F(!11)-A3*J(N2,142)-A1*U(N1,!!2)
C
C EIND IJ(I,J) FOR J=M1
C
C
CALL T:ID(E1,?2,B3,F,\,:1,!:)
DO 125 I=2,:1
U(I,!11)=|*U(I,!1)+(1.0-4)*IOLD(I,:11)
EPS=EFB+ABS(U(T,11)-JCLD(T,:1))
CONTINUS
C
C TESTING THE COHVERGEHCE
C
IF(EPS .GT.EPSMAX) IC TO 55
C IF COHVEPGE GRITE THE RESULT
C
IF(ITER.GE.TTHIH) OO TO 135
ITH:TH=TTER
DO 130 J=1,4
DC 130 [=1,:
UUEN(I,J)=U(I,J)
130 COHTIUNE
135 GRITE(2,175) |
WRITE(2,170) ITER
140 IF(W.LT.NEID) GO TO 45
DO 105 J=1,11
Gnime(2,145)TT(J)
145 FORMAT(///JX,'THE TIME TS=',F3.4)
WIITE(2,155)
DO 16n I= 1, \
MRITE(2,150) X(I),NHEM(I,J)
150 FGR:4T(5X,F8.4,5又,F13.8)
155 FOR:AT(/10X,'《',13X,'J')
100 CoNTPMUE
165 CONTIME
170 FGFMAT(//5%,'CONVERGFICE COMDITION FULEILLED APTEF',?X,IG,?X,'ITER
1ATIOHS')
175 FORMAT(//'IME RESULT EOQ OMEMA=',FQ.4)
|DATA=:DDATA+1
IF(HDATA.LE.2) Oी TO 5
C
C THAT IS THE END OE TIE PROGRAY
STOP
E:D
C
C
THF SUBROUTEAE FOR SOLVIMG TRIDIAGONAL SYSTEU OF EOHATION
Su3ROUTI:,E TPID(B1,D2,E3,F,'1,TIMF,H:
REAL E1,E2,33,F(11),BA(11),EE(11),FC(11),1)(11,201)
IMTEGER TIIE,!1,N,H2

```
```

    N1=N-1
    N2=N-2
    C E!B=TITAOO:AL
C BA=SUSDIACOMAL
C SC=SUFERDEAGONAL
C
DU 50 I=2,H1
BP(I)=131
BA(I)=B3
BC(I)=6?
ConTIHUE
BA(1)=0.0
BC(il1)=0.0
C
C LU UECO:POSITION
PC(?)=BC(2)/R3(2)
DO 51 I=3,.42
DP(I)=BC(I)-BA(I)*RC(I-1)
IF(DR(I).EQ.G.0) RETURN
3C(I)=BC(I)/BB(I)
51 CONTIMUE
BR(:1)=RP(91)-RA(11)*PC(122)
C
C RACK SURSTITUTICN A:D SOLUTTCH
J(2,TTHE)=F(2)/PB(2)
DO 52 I=3.N1
U(I,TI:E)=(F(I)-BA(I)*J([-1,TIME))/3B(I)
contimje
D0 53 I=2,1%?
IL=I-I
U(IL,TM\&E)=0(IL,TTME)-BC(IL)*|(IL+1,TIME)
52 CONTIHUE
RETUSN
EMD
C
THIS PROGRAM :IAS COMPILE,TESTED AGD RUN WSIHG ICL 1.OO
AT FUE COMPUTER CEHTRE,UIIVERSITY OF TECHWOLOGY,
LOUGHBOROUGH,LEICSTERSHIRE,LE 11 3TH.
FI!TS4

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```

C
C PROGRAM FOR 3-POINT GROUP EXPLICIT :METHOD
C
C THE HETHOD TESTED UITH THE PARABOLIC EQUATION UITH
C DIRICHLET BOUNDARY CONDITION
C
C
C
C
C
C
C
C
C
C
C
C
5
C
C
C
C
C
C
C
C
READ(1,*) DX,DT,TI4E
M=1/DX+1
H1=4-1
DEFIUING THE POINTS ALOHG THE X-AXIS
DO 10 I= 1,H
X(I) =(I-1.)*DX
CONTINUE
SMAX IS THE HAXIMUM TIME-LEVEL
T IS THE TIME VARIABLE
NAAX=TIHE/DT+1
DO 15 J=1, J!1AX
T(J)=(J-1.O)*DT
CONTINUE
C
C
C
C
THE X-DOMAIH IS ASSU!EED TO BE DIVIDED INTO M+1 S!JAINTERVALS
WhERE :1 IS THE INTEGER HULTIPLICITY OF 3
HENCE THE NUHBER OF IHTERNAL HHKMOWN POINTS ARE I|
MULTIPLICITY OF 3
dEFIIING THE TYPE OF VARIAELES OR CONSTAMTS
IIPLICIT REAL (A-H,O-Z)
DIMEUSION X(21),T(201),01(21,201)
READING THE NU:IBER OF SET OF DATAS
READ(1.*) :1AXDAT
NOATA =0
UDATA =!DATA +1
IF(NDATA.GT.MAXDAT) CALL EXIT
ImPLEAENTATION TO EACH DATA
READING TIIE DATAS
DX IS THE SPACE INTERVAL AND DT IS THE TIME INTERVAL
TIME IS THE HAXIMUM TIME CONSIDERED
THE INITIAL CONDITION FOR THE DEPENDENT VARIABLE U1
DO 20 I=2,111

```
\(\mathrm{U} 1(\mathrm{I}, 1)=4.0 * X(I) *(1.0-X(I))\)

C

CONTINUE

THE BOUNDARY CONDITIONS

DO \(25 \mathrm{~J}=1, \mathrm{~J}: 4 \mathrm{AX}\)
\(U 1(1, J)=0.0\)
\(\mathrm{U} 1(\mathrm{M}, \mathrm{J})=0.0\)
continue

CALLING THE ROUTINE FOR 3-POIUT GROUP EXPLICIT HETHOD

CALL GE3PB(DX,DT,R,JMAX,M,U1)

WRITING THE HEADING
\(R\) IS THE RATIO DT/(DX**2)
\(\operatorname{WRITE}(1,30)\) DX, DT,TIME,R
FCRMAT(/////10X,'SPACE INTERVAL=',2X,FR.4/10X.'TIME INTERVAL=', 12X,F8.6/10X,'TIME=',2X,F10.5/10X, 'R=',2X,F8.5)

WRITING THE RESULTS
\(I N C=1\)
DO \(50 \mathrm{~J}=1\), JMAX, INC
IF (J.GE.11) INC=10
WfITE (1, 35) T(J)
FORAAT (//5X,'THE TME=', FB.6)
WRITE \((1,40)(X(I), I=1,6)\)
FORMAT (5X,6(F6.4,5X))
\(\operatorname{WRITE}(1,45)(U 1(I, J), I=1,6)\)
FORMAT(5X, (F (F8.6.3X))
\(\operatorname{VRITE}(1,50)(X(I), I=7,11)\)
FORMAT (5X,5(F6.4,5X))
WRITE (1,55) (111(I,J), I=7,11)
FORMAT (5X,5(F8.6.3X))
CONTINUE
GO TO 5
END

SUBROUTINE FOR THE 3-POINT GE :IETHOD

SUBROUTINE CE 3 PB(DX, DT,R,JMAX,M,U1)
```

C
C DEFINING THE VARIABLES AND CONSTANTS
C
C
IHPLICIT REAL (A-H,O-Z)
DIMENSION U1(21,201)
M1=M-1
JMAX1=JMAXX-1
R=DT/(DX**2)
M3=14-3
M2=1-2
C
C
C
C
C
E1=4.0*R-28.0*R*R+8.0
E2=-12.0*R**3+8.0*R*R+20.0*R
E3=20.0*R*R-4.0*R
E4=3.0*R*(R+1.0)
E5=4.0*(R+1.0)*(2.0+2.0*R-3.0*R*R)
ES=3.0*R*(2.0*R*R+10.0*R+4.0)
ET=6.0*R*R*(R-1.0)
E8=6.0*R*R*(R+1.0)
DETA=3.0*(3.0*R+1.0)*(R+1.0)
C
C
C
C
C
J=0
C
C
C
C
C
C
C
65J=J +1
IF(J.GE.JMAX) RETURN
DO 70 I=3,112,3
U1(I-1,J+1)=(E1*U1(I-1,J)+E2*:J1(I,J)+E3*J 1(I+1,J)+E6*UTTIT-2,J)
1+E7*U1(I+2,J))/DETA
U1(I,J+1)=(E4*(U1(I-1,J)+U1(I+1,J))+E5*U1(I,J)+E8*(J1(I-2,J)
1+U1(I+2,J)))/DETA
U1(I+1,J+1)=(E3*U1(I-1,J)+E2*U1(I,J)+E1*U1(I+1,J)+E7*:J1(I-R,J)
1+EG*U1(I+2,J))/DETA
70 CONTINUE
C
C GO TO NEXT LEVEL IF MECESSARY
GO TO 65
END
C

```
```


[^0]:    TABLE (4.11.3)

