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## Persistent entanglement in the classical limit

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**Abstract.** The apparent difficulty in recovering classical nonlinear dynamics and chaos from standard quantum mechanics has been the subject of a great deal of interest over the last 20 years. For open quantum systems—those coupled to a dissipative environment and/or a measurement device—it has been demonstrated that chaotic-like behaviour can be recovered in the appropriate classical limit. In this paper, we investigate the entanglement generated between two nonlinear oscillators, coupled to each other and to their environment. Entanglement—the inability to factorize coupled quantum systems into their constituent parts—is one of the defining features of quantum mechanics. Indeed, it underpins many of the recent developments in quantum technologies. Here, we show that the entanglement characteristics of two ‘classical’ states (chaotic and periodic solutions) differ significantly in the classical limit. In particular, we show that significant levels of entanglement are preserved only in the chaotic-like solutions.

The correspondence principle is one of the fundamental building blocks of modern physics. Stated simply, the principle insists that any new theory should contain the existing (experimentally verified) theories in some appropriate limit or approximation. Without this correspondence, it would not be possible to build new physical theories without undermining previous theory and experiment. When quantum mechanics was originally proposed, the natural expression of the correspondence principle could be stated as ‘If a quantum system has a classical analogue, expectation values of operators behave, in the limit  $\hbar \rightarrow 0$ , like the corresponding classical

quantities' [1]. Whilst this works for many systems, problems sometimes arise when dealing with nonlinear systems [2]. More generally, there were difficulties in obtaining nonlinear classical equations from the linear Schrödinger equation. This is a significant problem and a great deal of work has been done to analyse the way in which classical nonlinearities express themselves in isolated (Hamiltonian) systems. In this paper, we deal with an associated problem, namely nonlinear behaviour of quantum systems that are coupled to an environment (open quantum systems).

Three main approaches exist for dealing with open quantum systems. The first approach, based on work by Feynman and Vernon [3], treats the environment as a source of dissipation and decoherence and then calculates the average evolution of the quantum system by integrating the environmental degrees of freedom. This provides a good method for large ensembles of quantum systems, but the average evolution does not reflect all types of nonlinear behaviour, such as chaos. The second approach is to examine the evolution of the probability distribution of the quantum system in phase space, having averaged out the environment [4]. In this case, it can be shown that the quantum evolution of the probabilities is very similar to the evolution of the corresponding classical system, which is also coupled to a noisy environment. The third approach, often called quantum trajectories [5]–[10], is the one adopted here. This method uses continuous weak measurements on the environment and results in modified quantum evolution, which can be described by a stochastic Schrödinger equation. This equation reduces to the previous cases when averaged over an ensemble, but provides different trajectories for individual systems. The type of trajectories depend on the type of measurement that is performed on the environment [9]. Different trajectories are often referred to as unravellings of the Master equation, which would describe the average quantum evolution. For chaotic systems, quantum trajectories show a behaviour that is very similar to their classical counterparts in the correspondence limit  $\hbar \rightarrow 0$  [11]–[13]. This paper demonstrates that the chaotic-like quantum dynamics have other properties that do not reduce to the classical limit when  $\hbar$  becomes small. We show that the entanglement between two coupled chaotic oscillators can be significant even when the trajectories are essentially classical.

In this paper we have chosen to consider, as a convenient example, two coupled Duffing oscillators that are described by quantum state diffusion (QSD) [6, 7]. Duffing oscillators were chosen because they are a standard example of classical chaos, describing a driven oscillator that contains a third order nonlinearity. In addition, single Duffing oscillators have been studied extensively using quantum trajectory models [12, 13]. The QSD unravelling corresponds to a unit-efficiency heterodyne measurement (or ambi-quadrature homodyne detection) on the environmental degrees of freedom [9]. Similar chaotic-like dynamics can be obtained from the Duffing oscillator using most unravellings, and we consider another unravelling (quantum jumps, corresponding to photon number detection on the environment) further on in this paper, and we show that similar entanglement properties can be obtained. In QSD, the evolution of the state vector  $|\psi\rangle$  is given by the increment (Itô equation) [6, 7]

$$|d\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle dt + \sum_j \left[ \langle L_j^\dagger \rangle L_j - \frac{1}{2}L_j^\dagger L_j - \frac{1}{2}\langle L_j^\dagger \rangle \langle L_j \rangle \right] |\psi\rangle dt + \sum_j [L_j - \langle L_j \rangle] |\psi\rangle d\xi, \quad (1)$$

where the Linblad operators  $L_i$  represent coupling to the environmental degrees of freedom,  $dt$  is the time increment and  $d\xi$  are complex Weiner increments such that  $\overline{d\xi^2} = \overline{d\xi} = 0$  and  $\overline{d\xi d\xi^*} = dt$  [6, 7]. The first term on the right-hand side of (1) deals with the Schrödinger evolution

of the system while the second (drift) and third (fluctuation) terms describe the decoherence effects of the environment on the evolution of the state vector. To solve these equations can be a computationally demanding problem and approaching the classical limit generally requires the use of many basis states and extremely long run times.

In the classical limit, QSD for one Duffing oscillator reproduces the chaotic attractors (Poincaré sections) of the classical equations [12, 13]. In this paper, we extend this analysis and consider two identical, coupled Duffing oscillators. The Hamiltonian (total energy) for each oscillator is given by

$$H_i = \frac{1}{2}p_i^2 + \frac{\beta^2}{4}q_i^4 - \frac{1}{2}q_i^2 + \frac{g_i}{\beta}\cos(t)q_i + \frac{\Gamma_i}{2}(q_i p_i + p_i q_i), \quad (2)$$

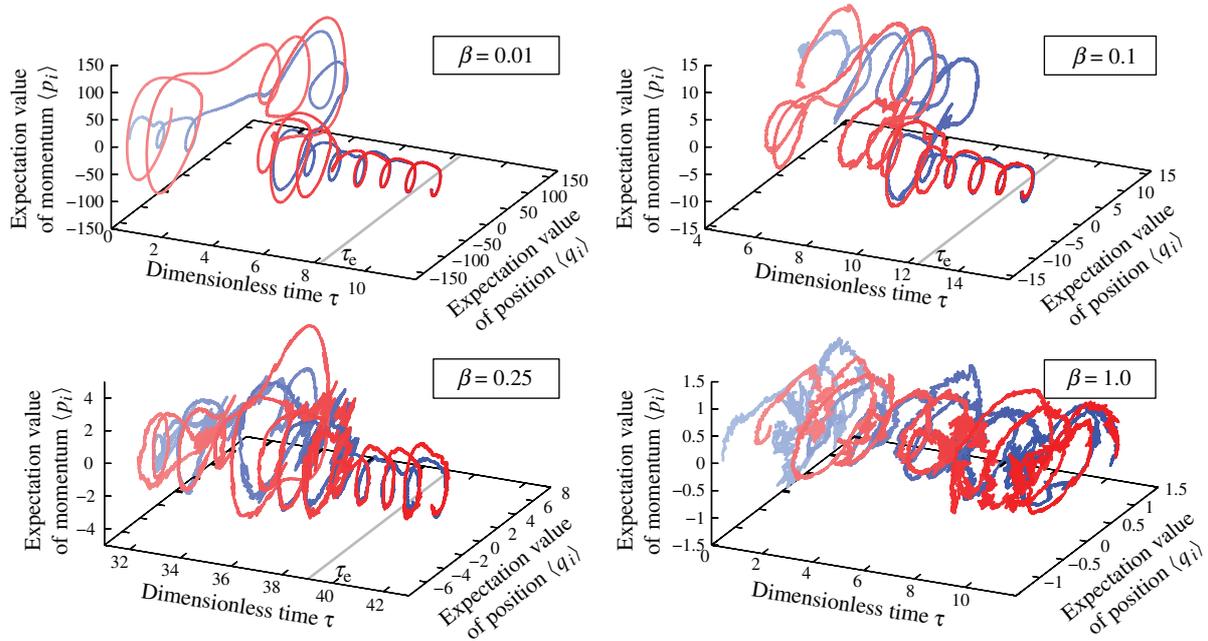
where  $q_i$  and  $p_i$  are the position and momentum operators for each oscillator. Here, the Linblad operators are simply  $L_i = \sqrt{2\Gamma_i}a_i$  (for  $i = 1, 2$ ), where  $a_i$  is the oscillator lowering operator. Following [12, 13], we set the parameters for the drive and the damping to be  $g_i = 0.3$  and  $\Gamma_i = 0.125$  respectively. The Hamiltonian for the coupled system then takes the form

$$H = H_1 + H_2 + \mu q_1 q_2, \quad (3)$$

where  $\mu = 0.2$  is a measure of the strength of the coupling between the two oscillators.

Now, if we are to consider the correspondence principle in more detail, we see that  $\hbar \rightarrow 0$  is not the only interpretation. By this we mean that, mathematically, it is equivalent to consider  $\hbar$  remaining fixed (as indeed it does) and scale the Hamiltonian so that the relative motion of the expectation values of the observables becomes large compared with the minimum area ( $\hbar/2$ ) in the phase space. In this paper, this is achieved by the introduction of a scaling parameter  $\beta$ , where the system behaves more classically as  $\beta$  tends to zero from its maximum value of one. The choice of  $g_i$ ,  $\Gamma_i$  and the various values of  $\beta$  come from the previous work on the QSD Duffing oscillator and dissipative quantum chaos [12, 13].

We solve the equation of motion for the state vector (1) using the Hamiltonian (3) starting with the initial states centred on points taken from the Poincaré section of the uncoupled classical oscillators. Explicitly, this initial state is a tensor product of coherent states for which the expectation values in position and momentum are centred in  $q$ - $p$  phase plane at points chosen from the classical Poincaré section. In figure 1, we show the evolution of the expectation values of position and momentum for the two coupled oscillators (in blue and red) as a function of time for some selected values of  $\beta$ . From these figures it is apparent that the initial phase space trajectory of the coupled system is, for low  $\beta$ —the classical limit, very similar to that of the classical Duffing oscillator. In this small- $\beta$  limit, the uncertainties in position and momentum are a very small fraction (typically a few percent) of the oscillations, indicating that the dynamics are essentially classical. As  $\beta$  increases to unity, this structure becomes harder to resolve and finally in the fully quantum limit ( $\beta = 1$ ), it is indiscernible. However, it is clear, for  $\beta = 0.01$ ,  $0.1$  and  $0.25$ , that after a certain time period, the chaotic behaviour of the coupled oscillators gives way to regular, almost periodic motion. The oscillators can be said to have entrained, i.e. their nonlinear oscillations are stabilized and synchronized by the coupling. We note for future reference that for  $\beta = 1$ , it is impossible to determine the presence of a chaotic or periodic attractor. Also, it is not possible to determine whether or not there is any level of entrainment between these oscillators. This is because, in the quantum limit, the quantum noise is large enough to drive the system away from periodic attractors and to swamp the fractal structure found in the chaotic classical attractor.



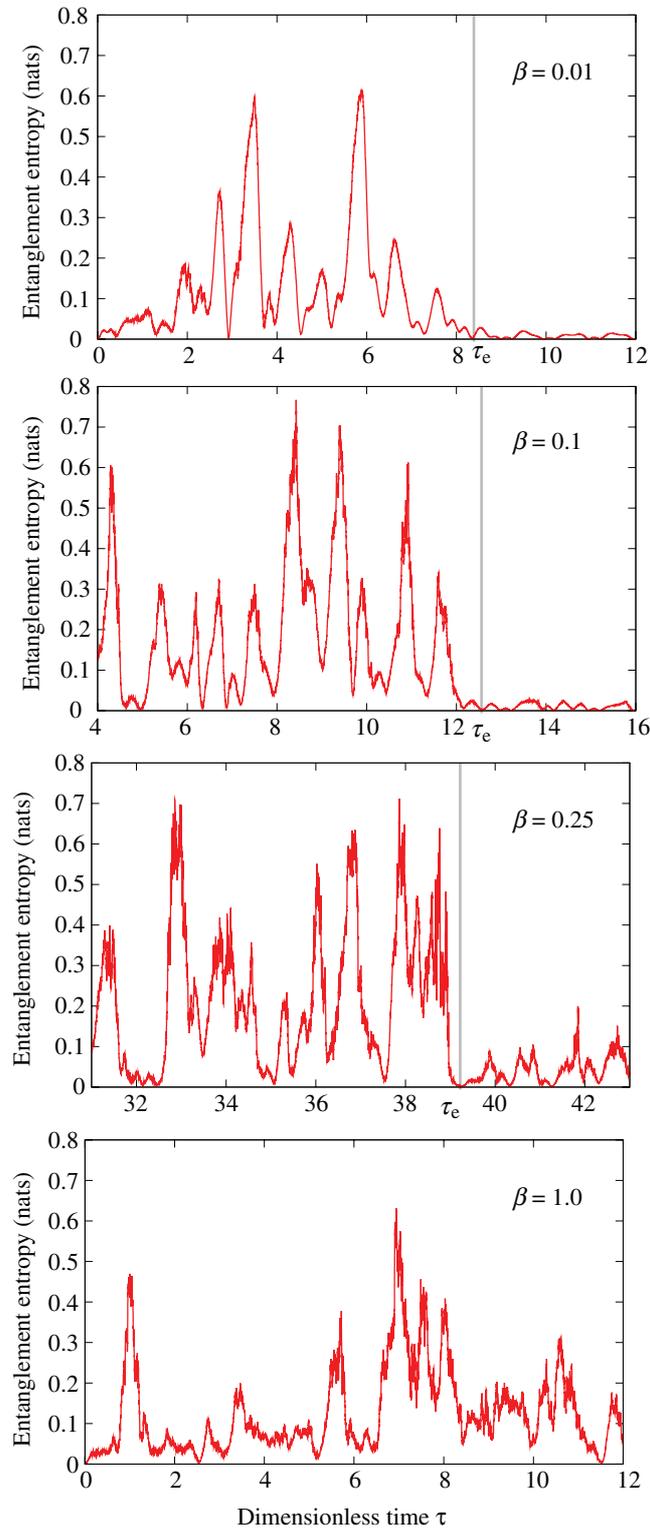
**Figure 1.** The dynamics of the expectation values of position and momentum as a function of time (normalized to drive periods) for  $\beta = 0.01$ ,  $0.1$ ,  $0.25$  and  $1.0$ . The dynamics have been taken over the same duration and displayed so that, with the exception  $\beta = 1.0$ , they approximately align at the time ( $\tau_e$ ) at which entrainment occurs.

We note that, when the two oscillators are entrained, the classical correlations between the oscillators are at a maximum (the oscillations are, after all, fairly well synchronized). It is natural, therefore, to ask what level of quantum correlation is present within this system and how the chaotic and periodic phases of motion affect this correlation. We measure this quantum correlation by computing the entropy of entanglement for the system [14], which is simply the von Neumann entropy [15] of the reduced density operator  $\rho_i$  for either of the oscillators, i.e.

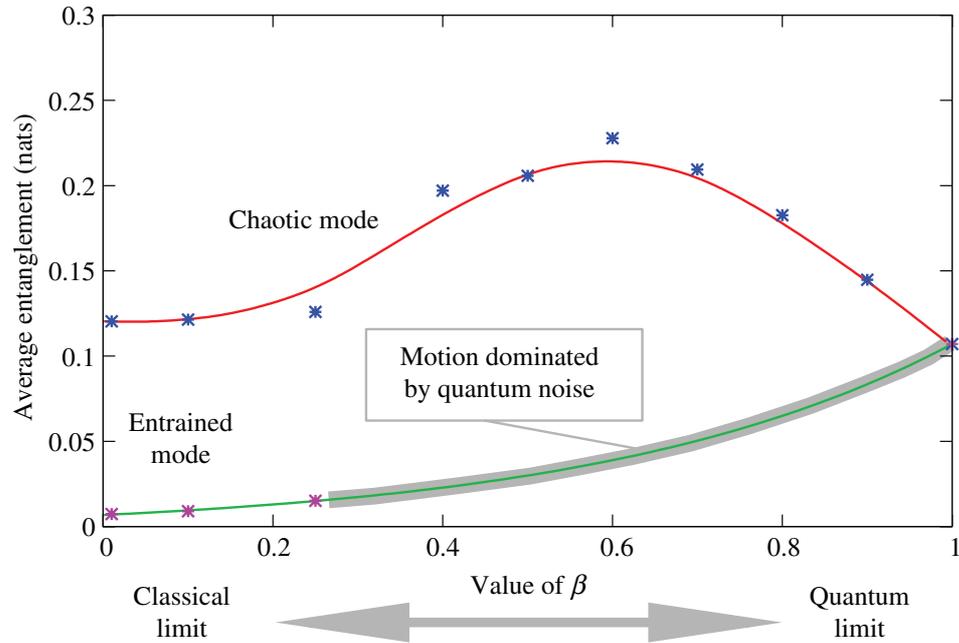
$$S(\rho_i) = -\text{Tr}[\rho_i \ln \rho_i].$$

As QSD models the evolution of a pure state,  $S(\rho) = 0$  at all times. Since entanglement is a purely quantum mechanical phenomenon, one might expect that when a quantum system approaches the classical limit the two oscillators would not be strongly entangled.

In figure 2 we show the entropy of entanglement as a function of time for the four different values of  $\beta$  as used in figure 1, over the same corresponding time intervals. By comparing figure 2 with figure 1, we can see that its entanglement remains relatively large as long as the system undergoes chaotic motion. It is only when the system becomes entrained that the entanglement drops when approaching the classical limit. As we have already said, the uncertainties in the  $\beta = 0.01$  case are very small when compared to the system's evolution in phase space. In this case, the system is well localized in phase space. The correspondence principle would seem to imply that these objects are behaving classically even though the entanglement (in the chaotic phase) does not seem to be diminishing. Once the oscillators entrain, the system becomes approximately



**Figure 2.** Entropy of entanglement as a function of time (normalized to drive periods) for  $\beta = 0.01$ , 0.1, 0.25 and 1.0. By comparing with figure 1, we can see that there is a sudden drop in entropy of entanglement after the time  $\tau_e$  at which the oscillators have entrained.



**Figure 3.** Mean entropy of entanglement as a function of  $\beta$  for the chaotic-like and periodic (entrained) states. Here, we see that the entropy of entanglement for the system in the chaotic state does not vanish as  $\beta$  approaches the classical regime.

separable and the entropy of entanglement falls as  $\beta$  reduces. These results would indicate that quantum oscillators operating in an apparently classical regime can generate significant levels of entanglement, but only in chaotic-like states. It is worth mentioning that the close connection between chaos in quantum systems and entanglement has been noted by Wang *et al* for Hamiltonian systems [16]; moreover they have recently considered the behaviour in the classical limit [17]. However, we note that the system discussed in [17] is considered in a non-chaotic regime.

Each solution computed using QSD corresponds to one realization or experiment. We obtain a more general picture if we compute the average entanglement as a function of  $\beta$  over many experiments: the mean entropy of entanglement [16], rather than the entanglement of the average (i.e. mixed) state. Clearly, the most useful quantity for quantum information processing is the entanglement of the average (mixed) state but it has already been shown that mixed state evolution is non-chaotic [11, 19]. The average entanglement of the pure state, which we calculate here, has been shown to be associated with chaotic behaviour [17] and may indicate some properties of the underlying dynamics. We show the results of such a computation in figure 3. The number of trajectories used to compute the average entropy of entanglement was determined on a point-by-point basis. We note that the time at which chaotic motion ceases and entrained motion starts, is different for each trajectory. Hence, determining these average values was a labour intensive process. Nevertheless, a sufficient number of trajectories were averaged, for each point, so as to ensure that our results had settled to  $<1\%$  or so.

The mean entanglement in the chaotic state does not appear to vanish as  $\beta \rightarrow 0$  and, possibly more surprisingly, its maximum is not at  $\beta = 1$ . However, we also see that when this system entrains (at smaller  $\beta$  values), the entanglement is rapidly suppressed. Figure 3 also shows lines

indicating the best fit to the mean data points. In order to guide the eye in the case of the entrained state, we have fitted an exponential curve to our data. This includes the quantum limit ( $\beta = 1$ ), even though there is no evidence for an entrained state for this  $\beta$ -value.

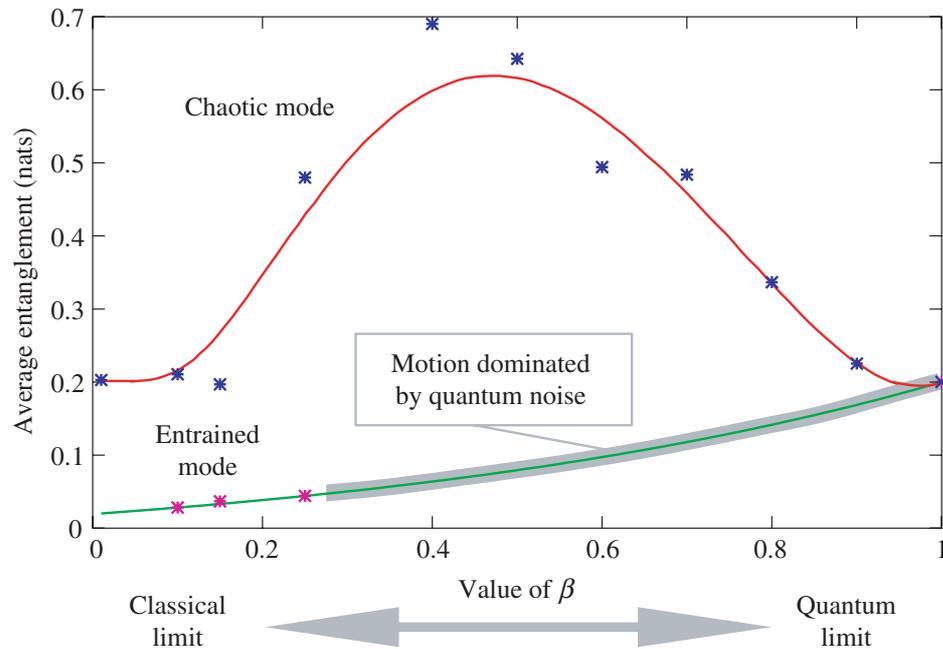
For the preceding results, we have used the QSD unravelling of the master equation. This unravelling makes assumptions about the underlying measurement process which is applied to the environment: unit efficiency heterodyne detection for QSD. However, studies have shown that the level of entanglement is dependent upon which unravelling is used [18]. It is natural then to ask whether the properties of the entanglement that we have described above are particular to the QSD unravelling. To consider this, we choose another unravelling—the quantum jumps unravelling [5, 8]. This model is very different from QSD as it is based on a discontinuous photon counting measurement process, rather than a diffusive continuous evolution. However, it has already been shown that this rather different unravelling can produce, in the classical limit, the expected chaotic dynamics of the Duffing oscillator [19]. It seems natural, therefore, to use the quantum jumps unravelling as a second test unravelling for our example system. In this model, the pure-state stochastic evolution equation for quantum jumps is given by

$$|d\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle dt - \frac{1}{2} \sum_j [L_j^\dagger L_j - \langle L_j^\dagger L_j \rangle] |\psi\rangle dt + \sum_j \left[ \frac{L_j}{\sqrt{\langle L_j^\dagger L_j \rangle}} - 1 \right] |\psi\rangle dN_j, \quad (4)$$

where  $dN_j$  is a Poissonian noise process such that  $dN_j dN_k = \delta_{jk} dN_j$ ,  $dN_j dt = 0$  and  $\overline{dN_j} = \langle L_j^\dagger L_j \rangle dt$ , i.e. jumps occur randomly at a rate that is determined by  $\langle L_j^\dagger L_j \rangle$ .

As we can see from figure 4, the quantum jumps unravelling produces entanglement that persists in the classical limit so long as the oscillators remain in the chaotic-like solution. Once again, if the oscillators are seen to entrain, then the entropy of entanglement drops significantly (in figure 3 we have fitted the results to an exponential function to guide the eye). The dependence of the entropy of entanglement on  $\beta$  is similar in form for both unravellings (one continuous and one discontinuous). We find, as might be expected [18], that the entropy of entanglement for quantum jumps differs from that of QSD. However, even though the average level differs between unravellings, the behaviour of the average entanglement, both chaotic-like and non-chaotic, as a function of  $\beta$  is very similar. We note that the data points shown in figure 4 exhibit a slightly more erratic nature than those shown in figure 3. The explanation for this lies in the fact that the entanglement entropies obtained for the evolution of the system under the quantum jumps unravelling take much more time to settle to a good average than for the QSD unravelling. Without very long computational times (e.g. several months), we cannot achieve much more significant accuracy with the computational power currently available to us.

In this paper, we have considered the classical limit of two coupled nonlinear quantum oscillators. The analysis is based on standard examples from classical chaos and open quantum systems. While these results are indicative of a possible general effect, the results presented in this paper do not establish whether or not this is indeed the case. In order to fully understand the implications of these results, a more detailed analysis will be required. This study should include both linear and nonlinear systems in order to understand exactly which classes of oscillators behave in this manner. We have shown that when these coupled Duffing oscillators enter the almost periodic entrained state, the entanglement falls rapidly as the system approaches the classical regime, as expected. Conversely, we have also shown that these oscillators can remain significantly entangled even when the dynamics of the system appear to be classical, although



**Figure 4.** The calculation of figure 3 using quantum jumps instead of QSD. Again we show the mean entropy of entanglement as a function of  $\beta$  for the chaotic-like and periodic (entrained) states. As with QSD we see that when using quantum jumps the entropy of entanglement for the system in the chaotic state does not vanish as  $\beta$  approaches the classical regime.

only when in the chaotic state. Moreover, by using both the QSD and quantum jumps models, we have shown that these results are not unique to a particular unravelling of the master equation. We find the results presented in this paper are surprising, since the quantum correlations represented by the nonzero entanglement do not have classical counterparts yet they persist in the classical (correspondence) limit. Some understanding can be gained if we consider that in order to obtain the classical limit it is sufficient that the state of the system be localized relative to a large classical action. However, in order to guarantee that the entanglement between the two oscillators can be eliminated from the system, we require that the localization of the wavefunction, in the tensor product space, be of the order of  $\hbar$ . Hence, although entanglement has no equivalent in classical mechanics, satisfying the classical limit does not imply that entanglement will be removed.

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