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# Normal modes of the Indian elephant bell

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The geometrical structure of the Indian elephant bell is presented and the requirements on its normal modes from group representation theory are described. These are in good agreement with the results of a finite-element model (FEM) for a specific 16-tine case. The spectrum consists of a sequence of families of modes lying on saturation curves, completely different from those of conventional bells. Physical explanations for the occurrence of these families are presented in terms of the tines behaving as a closed loop of coupled cantilevers with constraints from the dome. Each family is found to consist of modes in one of two specific sequences of symmetry types. Experimental measurements of the modes of this same 16-tine bell, using electronic speckle pattern interferometry (ESPI), have been made and are compared with the FEM predictions. Although the interpretation of the interferograms is difficult in all but the simpler cases, agreement in terms of frequencies is surprisingly good for the first few family sequences. The ESPI study also showed up numerous harmonics and subharmonics of true normal modes, showing the system to be rather nonlinear and making comparisons with the FEM results tricky.

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I. INTRODUCTION

Beautiful multicolored Benaresware elephant bells were often on sale as souvenirs in the Indian pavilions at international exhibitions in the middle decades of the last century. They were regularly to be seen on display in United Kingdom living rooms at that time and have since become collectors items. In Fig. 1 we show a typical medium-sized example. Relatively crude, uncolored, brass bells of similar general design are also commercially available.

In the present paper we report the results of a study of the normal modes of these elephant bells using finiteelement modeling (FEM), group representation theory and electronic speckle pattern interferometry (ESPI). The only previously reported studies of elephant bells, of which we are aware, are preliminary reports of the present work<sup>1,2</sup> and one by Brailsford in 1944.<sup>3</sup> His results are of little help in understanding these bells.

# **II. BELL GEOMETRY**

All Indian elephant bells have the same basic design. They consist of a more-or-less hemispherical dome from whose rim descend roughly identical and equally spaced tines, which have an inward curvature. There is a cast-in handle on the top of the dome. The bell is rung by means of a metal ball attached to the underside of the domes center by a wire. This ball strikes the hemisphere close to the tops of the tines, where they attach to the dome. The number of tines can be even or odd and, in general, the larger the overall size of the bell the greater is this number. We have found examples with almost every value from 9 up to 19. We have been able to assemble a total of 10 different sizes of the type of bell shown in Fig. 1, which, being characterized by an extra band of metal where the tines join the dome, we call banded Benaresware bells. In Fig. 2 we have taken the diameter of this extra band as a measure of the overall size of the bell and plotted it against the number of tines for these ten cases. There is a convincing linear relationship between the two variables. Other types of elephant bell do not fit well onto this line. We have, so far, been unable to find sufficient numbers of any other types to test whether or not they obey similar relationships.

Pages: 2288-2294

# **III. THEORETICAL CONSIDERATIONS**

#### A. The unperturbed bell

With bells in general it is convenient to use cylindrical polar coordinates with the z-direction chosen to lie along the symmetry axis. We refer to displacements from equilibrium in the  $(r, \theta, z)$  directions as (u, v, w) and follow the usual convention of defining nodes to be points of zero displacement in the radial direction. If the elephant bell did not have inter-time gaps it would just be another convex bell with symmetry group  $C_{\infty v}$  and subject to the same consequences.<sup>4</sup> The vibrational patterns of Indian elephant bells are similar to those of trumpet bells,<sup>5</sup> bells,<sup>6</sup> and gongs.<sup>7</sup> Thus, the normal modes would occur in degenerate pairs with modal

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FIG. 1. (Color online) Sixteen-tine Benareseware elephant bell.

functions varying like  $\sin(m\theta)$  and  $\cos(m\theta)$  and having nodal patterns consisting of *m* equally spaced diameters and *n* circles parallel to the rim. The diameters of one member would lie exactly mid-way between those of its partner. Cases with zero diameters would be the exception as singlets. The number pairs (m, n) could be used to identify the modes.

#### B. Extensional and inextensional modes

The study of bells<sup>8</sup> and other structures with axial symmetry<sup>9</sup> leads one to expect that the lowest frequency modes will involve inextensional distortions of the bell. In other words, if one takes a section through the bell at fixed z, the resulting ring will contain a neutral circle whose total length remains unchanged throughout the cycle. This means that the radial and transverse components of the motion are related by

$$u + \frac{\partial v}{\partial \theta} = 0. \tag{1}$$

Thus, using the  $\theta$  part of the modal functions from Sec. III A we may write  $u = mA \sin(m\theta)$  and  $v = A \cos(m\theta)$ where A is an arbitrary constant. As m increases, the modes will have radial components whose amplitudes become increasingly larger than their transverse ones. One would expect modes of this type to continue to occur as one goes to higher frequencies, but to be supplemented by others satisfying the complementary extensibility condition



FIG. 2. (Color online) Diameter vs tine number for banded Benaresware bells.

$$v + \frac{\partial u}{\partial \theta} = 0. \tag{2}$$

This latter condition results in modes whose transverse components are *m* times their radial.

#### C. Perturbed bell approach

For generality we assume that the elephant bell has k tines. The set of k inter-tine gaps can be considered as a large perturbation, with symmetry group  $C_{kv}$ , on a basic convex bell. Because some, but not all, of the symmetries of  $C_{\infty v}$  have been removed, some, but not necessarily all, of the doublets will have become split. Perrin has shown<sup>10</sup> that, under these circumstances, most doublets do not split unless k is rather small, although their common frequency will change. In fact splitting occurs only when m/k is an integer or a half-integer. So, for example, in a 16-tine bell such as that shown in Fig. 1 the only doublets to split have m = 8, 16, etc.

If we now regard the elephant bell as a composite system with symmetry group  $C_{kv}$ , then its normal modes must all behave like symmetry types of this group. Because of the completeness requirement, every one of these types must appear in the spectrum. From the character table for this group (see Ref. 10, p. 308), when k = 16 there are four singlet types denoted as  $A_1, A_2, B_1$ , and  $B_2$ , whereas two types of doublets  $E_{m(1)}$  and  $E_{m(2)}$  are each restricted to m in the range 1–7. If k is odd then the character table is simpler, containing no classes of types  $B_1$  or  $B_2$ . The modes forming a doublet with m = 8 in the full bell now split into  $B_1 \oplus B_2$ . Although every mode of type  $B_2$  a singlet anti-twister, they will both still appear to have m = 8 on the dome.

#### D. Coupled cantilever approach

Alternatively one could regard the elephant bell as a collection of k cantilevers coupled together in a closed loop via the dome. A similar, but much simpler, system with the same symmetry group, and thus modes of the same symmetry types, is a collection of particles at the vertices of a k-fold regular polygon connected in pairs by identical springs along the polygons sides and constrained to lie in its plane. In this case it is possible to construct the forms of the normal modes in detail from symmetry arguments alone.<sup>11</sup> Because these constructions do not involve the details of the inter-particle forces (just requiring them all to be identical), we can reasonably expect that the elephant bell modes will be broadly similar in forms in planes of fixed z in the region of the cantilevers. A full set of results for the 16-tine case are shown in Fig. 3. Only one member of each doublet pair is included. In each diagram the arrows show the directions of particle motion.

#### E. Polygon spectrum

Perrin has shown that,<sup>12</sup> if the inter-particle forces in the polygon are in the form of identical simple springs then, using group representation methods alone, one can calculate

J. Acoust. Soc. Am., Vol. 131, No. 3, March 2012



FIG. 3. Singlet modes for 16-gon as predicted by group theory.

the actual frequencies of the modes. When plotted they form a very distinctive saturation curve as shown in Fig. 4 for the 16-particle case. This type of curve is, as we shall see, rather similar to those appearing in the spectrum of the 16-tine elephant bell.

# **IV. FINITE ELEMENT MODEL**

# A. Construction of the model

An elephant bell is much harder to model than a conventional one because it lacks complete axial symmetry. It does, however, have a high level of symmetry which can be exploited. To construct a unit cell one can take a vertical segment bounded by two planes defined by fixed values of  $\theta$ . These should be chosen such that one touches the left-hand side of a typical tine at its widest point and the other touches its right-hand neighbor at the corresponding point. The unit cell will thus contain one tine and one gap plus the segment of the dome joining them to its pole. If one first makes a FEM of this unit cell, then one for the whole bell can be generated by copying it (k - 1) times while rotating about the symmetry axis through angles of  $2\pi/k$ . To produce the unit cell careful measurements were





2290 J. Acoust. Soc. Am., Vol. 131, No. 3, March 2012



FIG. 5. (Color online) FEM predictions of frequency (Hz) vs m.

made of the dimensions and profile of a 16-tine bell. The tines and gaps were each measured separately and averages taken. The FEM was then produced with the LUSAS package using thin shell elements chosen to preserve the shape of the outside of the bell and having appropriate thicknesses. The effect of the handle was modeled by constraining the bell to be fixed at its edge. This also excluded rigid body modes from the calculations. Values for density, Young's modulus and Poisson's ratio for brass were taken from the packages library.



FIG. 6. One member of each doublet pair of symmetry type  $E_{m(1)}$  for m = 1 - 7.

Perrin et al.: Normal modes of the Indian elephant bell

#### **B.** Finite element results

The FEM was used to calculate the frequencies and display the modal forms for all the modes it could find up to about 6 kHz. Results are listed in Table I. In most cases it was easy to establish the value of *m* by looking at the behavior of the dome. However, this did become more difficult as *m* increased because evanescence set in steadily further down the dome.<sup>13</sup> The modes with 0 > m > 8 were all in degenerate pairs, as expected, with interlocking nodal diameters and frequencies differing by never more than 0.25 Hz. Modes with m = 0 were all singlets. Those with m = 8 were also singlets, as anticipated in Sec. III A, and no modes were found with m > 8.

In Fig. 5 we show the predicted spectrum up to about 6 kHz. Instead of families with fixed *n* rising steadily and indefinitely with *m*, as in a normal bell, we see family curves reaching limiting values at m = 8 in much the same way as in Fig. 6. This clearly suggests that the modes are being driven largely by inter-tine coupling. It was easy to see that all the members of the lowest family had n = 0, as one would expect. The second family all appeared to have n = 1 with the circle about two-thirds of the way down the tines, although variations in the direction of vibration of the times as one moved around the bell made it hard to be sure if these

were true nodes in the radial direction. The third family also appeared to have n = 1, which we refer to as n = 1# in Table I. The fourth had n = 2 with one circle about halfway down and the other near the tine top. The FEM predictions of modal forms for the first two families are shown in the upcoming Figs. 7 and 8.

# C. Allocation of symmetry types

Although the allocation of a particular FEM predicted mode, or doublet pair of modes, to a specific symmetry type depends primarily on its value of m, there remain at least two possible choices in each case. To distinguish between these one needs to consider the ways in which the mode transforms under the various symmetry operations of the group. However, in practice, it is possible to short-cut this process by comparing the FEM forms with those discussed in Sec. III B for the coupled point masses. It was fairly easy to do this for the first few saturation curve families just by looking at the overall forms of the motions. The FEM animation facility was particularly helpful in this process. The results are incorporated into Table I. The first four singlet modes predicted by FEM are shown in Fig. 9. For ease of comparison the singlet types of the 16-gon are repeated in Fig. 9. The doublets in the first family are



FIG. 7. FEM predictions of doublet modes of the first family (grayscale: Black = min displacement to white = max displacement).



J. Acoust. Soc. Am., Vol. 131, No. 3, March 2012



FIG. 9. FEM and group theory predictions for the first four singlet modes (grayscale: Black = min displacement to white = max displacement).

shown in Fig. 7. Comparison with Fig. 4 shows that they are all symmetry types  $E_{m(1)}$ . Likewise, by comparison with Fig. 10, the doublets in the second family, shown in Fig. 8, are all of symmetry type  $E_{m(2)}$ . To summarize, the first family, with n = 0, consists of types  $E_{m(1)}$  plus  $B_1$  (antibreather), whereas the second, with n = 1, contained  $E_{m(2)}$ plus  $B_2$  (anti-twister). The  $A_1$  (breather) and  $A_2$  (twister) modes did not fit convincingly into any family. It was clear that, in both of these families, the tines were essentially behaving like cantilevers in their fundamental modes but, due to the domes coupling generating a composite system, moving in directions that varied in a systematic way as one moved around the bell. The third family contained a repeat of the symmetry types in the first, but now with n = 1, whereas the fourth was a repeat of the second but with n = 2.

Using the animation facility on the FEM package it was possible to study the modes in some detail. Looking at the members of the first (n = 0) family it was evident that, while for modes for m = 0, 1, and 8 the motions were exactly as predicted for the coupled masses, there were some small deviations for other m values. If one breaks the motion down into radial and transverse components the radial parts becomes increasingly dominant over the transverse as *m* increases until, at m = 8, it is purely radial. For the second (n = 1) family the situation is reversed, with the transverse components becoming increasingly dominant over the radial until, at m = 8, it is purely transverse. These are precisely what are required by the inextensibility and extensibility conditions in Eqs. (1) and (2), respectively. Thus, the  $E_{m(1)}$  and  $E_{m(2)}$  modes are, respectively, inextensional and extensional. The same results apply to the pairs of higher families.



FIG. 10. One member of each doublet pair of symmetry type  $E_{m(2)}$  for m = 1 - 7.

# **V. EXPERIMENTS**

In order to compare the FEM predictions with the modes of an actual 16-tine bell, the one used as the basis for the model was studied using ESPI,<sup>14</sup> a common technique utilized for visualization of the vibrational modes of musical instruments.<sup>5,15</sup> The ESPI system used is described in detail in Ref. 16 and operates by imaging out-of-plane vibrations by digitally subtracting a speckle pattern interferogram of an object illuminated by coherent radiation before the object begins to vibrate, from one imaged subsequent to its movement. The system used was constructed from discrete components on a vibration isolated table that was inside an anechoic chamber. The laser used was a diode-pumped frequency doubled NdYVO<sub>4</sub> with an output of 532 nm. It was mounted on a vibration-isolated optical table inside an anechoic chamber in order to minimize the ambient noise. The light entered the chamber through a small hole in the wall. All the data acquisition and analysis was performed by a computer located outside of the chamber.

The vibrations of the bell were driven by a piezoelectric disk mounted to either the hemispheric cap or the tines using putty. The piezoelectric driver was connected to a high quality function generator that produced a sine wave with frequency



FIG. 11. ESPI images of selected modes.

accuracy and precision exceeding 0.1 Hz. Additionally, the frequency of the function generator was scanned while observing the ESPI images in real time to ensure that no normal modes were neglected. Examples of the ESPI images of normal modes obtained, first while viewing the bell from vertically above the dome and second from a direction normal to the symmetry axis are shown in Fig. 11.

The interpretation of fringe patterns in interferograms of vibrating three-dimensional objects is famously difficult. A large number of resonant frequencies were detected but it was possible to fully identify only a few of them. In such cases the degeneracy was always as expected from group theory. The doublets were always split so, where required, we quote the higher value. The lowest modes positively identified were a (2,0) pair at 753 Hz and a (3,0) pair at 1086 Hz. The FEM results were therefore scaled to bring the lowest (2,0) pair into agreement with experiment. This was legitimate because of the use of standardized values for material constants and of the averaging of the geometry in the FEM model. Comparing the FEM and ESPI results in frequency sequence, as included in Table I, then gave a good match for the first few families provided one ignored what were evidently subharmonics and harmonics of true modes. The presence of these is clear evidence for the system being non-linear.<sup>17</sup> It should be noted that the measured frequency of the (1,0) pair in the first family was only achieved by identifying their half sub-harmonics at 361.5 Hz. In the case of the third family agreement was again good provided one omitted a whole sequence of resonances which were exact half sub-harmonics of the fourth family. Overall the agreement of frequencies is then good for the first three families.

#### **VI. CONCLUSIONS**

The agreement between the FEM model and experiment is impressive, although more sophisticated methods for identifying the nodal patterns via interferometry would give greater certainty to the type allocations. The success of symmetry arguments in describing the overall nature of the spectrum gives us confidence that what is happening is indeed that the tines are behaving like cantilevers coupling via the dome of the bell and constrained by it to be of extensional or inextensional types. A deeper study of the non-linear behavior of the bell might be instructive.

# **APPENDIX**

Please see Table I for a comparison of finite element and ESPI results.

TABLE I. Comparison of FEM and ESPI results.

Scaled FEM frequency (Hz)	т	Degeneracy (S or D)	Symmetry type	n	Experimental frequency (Hz)
730	1	D	$E_{1(1)}$	0	723
753	2	D	$E_{2(1)}$	0	753
1085	3	D	$E_{3(1)}$	0	1086
1184	4	D	$E_{4(1)}$	0	1183
1209	0	S	$A_1$ (breather)	0	1212
1226	5	D	$E_{5(1)}$	0	1232
1247	6	D	$E_{6(1)}$	0	1248
1257	7	D	$E_{7(1)}$	0	1251
1260	8	S	$B_1$ (anti-breather)	0	1263
1321	1	D	$E_{1(2)}$	1	1299
1640	2	D	$E_{2(2)}$	1	1607
2392	3	D	$E_{3(2)}$	1	2403
2891	0	S	$A_2$ (twister)	1	2905
2972	4	D	$E_{4(2)}$	1	2981
3238	5	D	$E_{5(2)}$	1	3236
3354	6	D	$E_{6(2)}$	1	3318
3403	7	D	$E_{7(2)}$	1	
3417	8	S	$B_2$ (anti-twister)	1	
3562	0	S	$A_1$ (longitudinal)	0	3671
4083	1	D	$E_{1(1)}$	1#	4056
4251	0	S	$A_2$ (twister)	2	4235
4366	2	D	$E_{2(1)}$	1#	4315
4818	3	D	$E_{3(1)}$	1#	4791
5423	4	D	$E_{4(1)}$	1#	5403
5566	0	S	$A_1$ (breather)	2	5531
5799	1	D	$E_{1(2)}$	2	
5890	5	D	$E_{5(1)}$	1#	5955
6158	6	D	$E_{6(1)}$	1#	6132
6291	7	D	$E_{7(1)}$	1#	6233
6382	8	D	$B_1$ (anti-breather)	1#	

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J. Acoust. Soc. Am., Vol. 131, No. 3, March 2012

Perrin et al.: Normal modes of the Indian elephant bell 2293

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