This item was submitted to Loughborough's Research Repository by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

## Condensed phase of converting boson-fermion mixtures

PLEASE CITE THE PUBLISHED VERSION

LICENCE

CC BY-NC-ND 4.0

## REPOSITORY RECORD

Alexandrov, A.S.. 2019. "Condensed Phase of Converting Boson-fermion Mixtures". figshare. https://hdl.handle.net/2134/1169.

# Condensed phase of converting boson-fermion mixtures 

A. S. Alexandrov<br>Department of Physics, Loughborough University, Loughborough LE11 3TU, United Kingdom


#### Abstract

Theory of a condensed state of hybridised bosons and fermions is developed. Normal and anomalous Green's functions are obtained diagrammatically and analytically using the Hamiltonian of the boson-fermion model (BFM). A pairing of bosons analogous to the Cooper pairing of fermions is found. There are three coupled condensates in the model, described by the off-diagonal singleparticle boson, pair-fermion and pair-boson fields. The Gor'kov expansion in the strength of the order parameter near the transition yields no linear homogeneous term in the Ginzburg-Landau equation, and the infinite Levanyuk-Ginzburg parameter, $G i=\infty$, which indicates that previous mean-field discussions of BFM are flawed.


PACS numbers: $74.20 . \mathrm{Mn}, 71.10 .-\mathrm{w}, 74.25 . \mathrm{Bt}, 03.75 . \mathrm{Fi}$

A two component model of negative $U$ centers coupled with the Fermi sea of itinerant fermions was originally employed to study superconductivity in disordered metal-semiconductor alloys [1, 2]. Later on it was applied more generally to describe pairing electron processes with localisation-delocalisation [3], and to the polaron-bipolaron crossover problem in the intermediate electron-phonon coupling regime [4]. The model attracted more attention in connection with exotic (5), and high-temperature superconductors 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. When the attractive potential $U$ is large, the model is reduced to localised bosons spontaneously decaying into itinerant electrons and vice versa, different from a non-converting mixture of mobile charged bosons and fermions [19, 20]. More recently this boson-fermion model has been adopted for a description of superfluidity of atomic fermions scattered into bound (molecular) states 21].

Most studies of BFM below its transition into a lowtemperature condensed phase applied a mean-field approximation (MFA), replacing zero-momentum boson operators by c-numbers and neglecting the boson selfenergy in the density sum rule $[5,7,8,10,11,17,18,21]$. When the bare boson energy is well above the chemical potential, the BCS ground state was found with bosons being only virtually excited [7, [8]. MFA led to a conclusion that BFM exhibits features compatible with BCS characteristics 10], and describes a crossover from the BCS-like to local pair behaviour 18]. The transition was found more mean-field-like than the usual Bose condensation, i.e. characterized by a relatively small value of $G i$ (11).

At the same time our study of BFM 12] beyond MFA revealed a crucial effect of the boson self-energy on the normal state boson spectral function and the transition temperature $T_{c}$. Ref. 12] proved that the Cooper pairing of fermions via virtual bosonic states is impossible in anydimensional BFM. It occurs only simultaneously with the Bose-Einstein condensation of real bosons in 3D BFM [13, 14], and vanishes in 2D BFM due to the absence of the Bose-Einstein condensation in two dimensions 13]. The origin of this simultaneous condensation lies in a softening of the boson mode at $T=T_{c}$ caused by its hy-
bridization with fermions. The energy of zero-momentum bosons is renormalized down to zero at $T=T_{c}$, no matter how weak the boson-fermion coupling and how large the bare boson energy are. Bosons look like overdamped diffusive modes, rather than quasiparticles in the long-wave limit 12, 13, 15] contrary to the conclusion of Ref. (9] that there is 'the onset of coherent free-particle-like motion of the bosons' in this limit. One can expect that the boson self-energy should qualitatively modify the whole condensed phase of 3D BFM below $T_{c}$.

In this Letter a closed set of equations for fermion and boson Green's functions (GFs) is derived taking into account self-energy effects in the condensed state of 3D BFM. There exist a boson pair condensate along with the fermion Cooper pair and the single-particle boson condensate in the model. Remarkably, the Gor'kov expansion [22] of GFs in the strength of the order parameter yields a zero linear term at any temperature below $T_{c}$, and $G i=\infty$. It shows that the transition is not a meanfield second order transition, and there is no crossover from the BCS-like to a local pair behaviour at any values of the parameters of BFM.

The Hamiltonian of BFM in an external magnetic field $\mathbf{B}=\nabla \times \mathbf{A}$ is defined as

$$
\begin{align*}
H & =\int d \mathbf{r} \sum_{s} \psi_{s}^{\dagger}(\mathbf{r}) \hat{h}(\mathbf{r}) \psi_{s}(\mathbf{r})+g\left[\phi(\mathbf{r}) \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r})+H . c .\right] \\
& +E_{0} \phi^{\dagger}(\mathbf{r}) \phi(\mathbf{r}), \tag{1}
\end{align*}
$$

where $\psi_{s}(\mathbf{r})$ and $\phi(\mathbf{r})$ are fermionic and bosonic fields, $s=\uparrow, \downarrow$ is the spin, $E_{0}$ is the bare energy of bosons with respect to their chemical potential $2 \mu, \hat{h}(\mathbf{r})=-[\nabla+$ $i e \mathbf{A}(\mathbf{r})]^{2} /(2 m)-\mu$ is the fermion kinetic energy operator , and $g$ is the hybridization interaction converting a boson into two fermions and vice versa. Here and further I take $\hbar=c=k_{B}=1$, and the volume of the system $V=1$.

The Matsubara field operators, $Q=$ $\exp (H \tau) Q(\mathbf{r}) \exp (-H \tau), \bar{Q}=\exp (H \tau) Q^{\dagger}(\mathbf{r}) \exp (-H \tau)$ $\left(Q \equiv \psi_{s}, \phi\right)$ evolve with the imaginary time $-1 / T \leqslant \tau \leqslant 1 / T$ as

$$
\begin{align*}
-\frac{\partial \psi_{\uparrow}(\mathbf{r}, \tau)}{\partial \tau} & =\hat{h}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}, \tau)+g \phi(\mathbf{r}, \tau) \bar{\psi}_{\downarrow}(\mathbf{r}, \tau)  \tag{2}\\
\frac{\partial \bar{\psi}_{\downarrow}(\mathbf{r}, \tau)}{\partial \tau} & =\hat{h}^{*}(\mathbf{r}) \bar{\psi}_{\downarrow}(\mathbf{r}, \tau)-g \bar{\phi}(\mathbf{r}, \tau) \psi_{\uparrow}(\mathbf{r}, \tau)  \tag{3}\\
-\frac{\partial \phi(\mathbf{r}, \tau)}{\partial \tau} & =E_{0} \phi(\mathbf{r}, \tau)+g \psi_{\downarrow}(\mathbf{r}, \tau) \psi_{\uparrow}(\mathbf{r}, \tau) \tag{4}
\end{align*}
$$

The theory of the condensed state can be formulated with the normal and anomalous fermion GFs [22], $\mathcal{G}\left(\mathbf{r}, \mathbf{r}^{\prime}, \tau\right)=-\left\langle T_{\tau} \psi_{s}(\mathbf{r}, \tau) \bar{\psi}_{s}\left(\mathbf{r}^{\prime}, 0\right)\right\rangle, \mathcal{F}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}, \tau\right)=$ $\left\langle T_{\tau} \bar{\psi}_{\downarrow}(\mathbf{r}, \tau) \bar{\psi}_{\uparrow}\left(\mathbf{r}^{\prime}, 0\right)\right\rangle$, respectively, where the operation $T_{\tau}$ performs the time ordering. Fermionic and bosonic fields condense simalteneously [12]. Following Bogoliubov [23] the bosonic condensate is described by separating a large matrix element $\dot{\sim}_{0}(\mathbf{r})$ in $\phi(\mathbf{r}, \tau)$ as a number, while the remaining part $\tilde{\phi}(\underset{\sim}{\mathbf{r}}, \tau)$ describes a supracondensate field, $\phi(\mathbf{r}, \tau)=\phi_{0}(\mathbf{r})+\tilde{\phi}(\mathbf{r}, \tau)$. Then using Eq.(4) one obtains

$$
\begin{equation*}
g \phi_{0}(\mathbf{r})=\Delta(\mathbf{r}) \equiv-\frac{g^{2}}{E_{0}} \mathcal{F}(\mathbf{r}, \mathbf{r}, 0+) \tag{5}
\end{equation*}
$$

where $\mathcal{F}\left(\mathbf{r}, \mathbf{r}^{\prime}, \tau\right)=\left\langle T_{\tau} \psi_{\downarrow}(\mathbf{r}, \tau) \psi_{\uparrow}\left(\mathbf{r}^{\prime}, 0\right)\right\rangle$. The equations for GFs are obtained by using Eqs $(2,3,4)$ and the diagrammatic technique 24] in the framework of the noncrossing approximation [25], as shown in Fig. 1 and Fig.2.

An important novel feature of BFM is a pairing of supracondensate bosons, generated by their hybridization with the fermionic condensate, as follows from the last diagram in Fig.2. Hence, one has to introduce an anomalous supracondensate boson GF, $\mathcal{B}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}, \tau\right)=\left\langle T_{\tau} \overline{\tilde{\phi}}(\mathbf{r}, \tau) \overline{\tilde{\phi}}\left(\mathbf{r}^{\prime}, 0\right)\right\rangle$ along with a normal boson GF, $\mathcal{D}\left(\mathbf{r}, \mathbf{r}^{\prime}, \tau\right)=-\left\langle T_{\tau} \tilde{\phi}(\mathbf{r}, \tau) \overline{\tilde{\phi}}\left(\mathbf{r}^{\prime}, 0\right)\right\rangle$. The diagrams, Fig. 1 and Fig.2, are transformed into analytical equations for time Fourier-components of the fermion GFs with the Matsubara frequencies $\omega=\pi T(2 n+1)(n=0, \pm 1, \pm 2, \ldots)$ as

$$
\begin{align*}
& {[i \omega-\hat{h}(\mathbf{r})] \mathcal{G}_{\omega}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)=\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)-\Delta(\mathbf{r}) \mathcal{F}_{\omega}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) } \\
- & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{G}_{-\omega^{\prime}}(\mathbf{x}, \mathbf{r}) \mathcal{D}_{\omega-\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{G}_{\omega}\left(\mathbf{x}, \mathbf{r}^{\prime}\right) \\
- & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{F}_{\omega^{\prime}}^{+}(\mathbf{r}, \mathbf{x}) \mathcal{B}_{\omega+\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{F}_{\omega}^{+}\left(\mathbf{x}, \mathbf{r}^{\prime}\right)  \tag{6}\\
& {\left[-i \omega-\hat{h}^{*}(\mathbf{r})\right] \mathcal{F}_{\omega}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)=\Delta^{*}(\mathbf{r}) \mathcal{G}_{\omega}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) } \\
- & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{G}_{\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{D}_{\omega^{\prime}-\omega}(\mathbf{x}, \mathbf{r}) \mathcal{F}_{\omega}^{+}\left(\mathbf{x}, \mathbf{r}^{\prime}\right) \\
+ & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{F}_{-\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{B}_{-\omega-\omega^{\prime}}^{+}(\mathbf{r}, \mathbf{x}) \mathcal{G}_{\omega}\left(\mathbf{x}, \mathbf{r}^{\prime}\right) \tag{7}
\end{align*}
$$

and,

$$
\begin{aligned}
& \left(i \Omega-E_{0}\right) \mathcal{D}_{\Omega}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)=\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \\
- & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{G}_{\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{G}_{\Omega-\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{D}_{\Omega}\left(\mathbf{x}, \mathbf{r}^{\prime}\right)
\end{aligned}
$$

\$


FIG. 1: Diagrams for the normal and anomalous fermion GFs. Vertexes (dots) correspond to the hybridization interaction $g$, and zig-zag arrows represent the single-particle Bose condensate $\phi_{0}$.

$$
\begin{align*}
- & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{F}_{\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{F}_{\Omega-\omega^{\prime}}(\mathbf{r}, \mathbf{x}) \mathcal{B}_{\Omega}^{+}\left(\mathbf{x}, \mathbf{r}^{\prime}\right)  \tag{8}\\
& \left(-i \Omega-E_{0}\right) \mathcal{B}_{\Omega}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)= \\
& g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{F}_{-\omega^{\prime}}^{+}(\mathbf{r}, \mathbf{x}) \mathcal{F}_{-\Omega+\omega^{\prime}}^{+}(\mathbf{r}, \mathbf{x}) \mathcal{D}_{\Omega}\left(\mathbf{x}, \mathbf{r}^{\prime}\right) \\
- & g^{2} T \sum_{\omega^{\prime}} \int d \mathbf{x} \mathcal{G}_{-\omega^{\prime}}(\mathbf{x}, \mathbf{r}) \mathcal{G}_{\omega^{\prime}-\Omega}(\mathbf{x}, \mathbf{r}) \mathcal{B}_{\Omega}^{+}\left(\mathbf{x}, \mathbf{r}^{\prime}\right) \tag{9}
\end{align*}
$$

for the boson GFs with $\Omega=2 \pi T n$, and $\mathcal{B}\left(\mathbf{r}, \mathbf{r}^{\prime}, \tau\right)=$ $\left\langle T_{\tau} \tilde{\phi}(\mathbf{r}, \tau) \tilde{\phi}\left(\mathbf{r}^{\prime}, 0\right)\right\rangle$

These equations can be formally solved in the homogeneous case without the external field, $\mathbf{A}=0$. Transforming into the momentum space yields GFs' time-space Fourier components as

$$
\begin{align*}
\mathcal{G}(\mathbf{k}, \omega) & =-\frac{i \tilde{\omega}^{*}+\xi_{\mathbf{k}}}{\left|i \tilde{\omega}-\xi_{\mathbf{k}}\right|^{2}+|\tilde{\Delta}(\mathbf{k}, \omega)|^{2}}  \tag{10}\\
\mathcal{F}^{+}(\mathbf{k}, \omega) & =\frac{\tilde{\Delta}^{*}(\mathbf{k}, \omega)}{\left|i \tilde{\omega}-\xi_{\mathbf{k}}\right|^{2}+|\tilde{\Delta}(\mathbf{k}, \omega)|^{2}} \tag{11}
\end{align*}
$$

and

$$
\begin{align*}
\mathcal{D}(\mathbf{q}, \omega) & =-\frac{i \tilde{\Omega}^{*}+E_{0}}{\left|i \tilde{\Omega}-E_{0}\right|^{2}+|\Gamma(\mathbf{q}, \Omega)|^{2}}  \tag{12}\\
\mathcal{B}^{+}(\mathbf{q}, \omega) & =\frac{\Gamma^{*}(\mathbf{q}, \Omega)}{\left|i \tilde{\Omega}-E_{0}\right|^{2}+|\Gamma(\mathbf{q}, \Omega)|^{2}} \tag{13}
\end{align*}
$$

where $\tilde{\omega} \equiv \omega+i \Sigma_{f}(\mathbf{k}, \omega), \tilde{\Omega} \equiv \Omega+i \Sigma_{b}(\mathbf{q}, \Omega)$. The fermionic order parameter, renormalised with respect to


FIG. 2: Diagrams for the supracondensate boson GFs. The Cooper-pairing of fermions leads to the Cooper-pair-like boson condensate, described by the boson anomalous $\mathrm{GF}, \mathcal{B}^{+}$.
the mean-field $\Delta$ due to the formation of the boson-pair condensate, is given by
$\tilde{\Delta}(\mathbf{k}, \omega)=\Delta+g^{2} T \sum_{\omega^{\prime}} \int \frac{d \mathbf{q}}{2 \pi^{3}} \mathcal{F}^{+}\left(\mathbf{k}-\mathbf{q}, \omega^{\prime}\right) \mathcal{B}\left(\mathbf{q}, \omega+\omega^{\prime}\right)$,
and the boson-pair order parameter, generated by the hybridization with the fermion Cooper pairs, is

$$
\begin{equation*}
\Gamma(\mathbf{q}, \Omega)=g^{2} T \sum_{\omega^{\prime}} \int \frac{d \mathbf{k}}{2 \pi^{3}} \mathcal{F}\left(\mathbf{k}, \omega^{\prime}\right) \mathcal{F}\left(\mathbf{q}-\mathbf{k}, \Omega-\omega^{\prime}\right) \tag{15}
\end{equation*}
$$

Hence, there are three coupled condensates in the model described by the off-diagonal fields $g \phi_{0}, \tilde{\Delta}$, and $\Gamma$, rather than two, as in MFA. At low temperatures all of them have about the same magnitude, as the fermion and boson self-energies,

$$
\begin{gather*}
\Sigma_{f}(\mathbf{k}, \omega)=-g^{2} T \sum_{\omega^{\prime}} \int \frac{d \mathbf{q}}{2 \pi^{3}} \mathcal{G}\left(\mathbf{q}-\mathbf{k},-\omega^{\prime}\right) \mathcal{D}\left(\mathbf{q}, \omega-\omega^{\prime}\right),  \tag{16}\\
\Sigma_{b}(\mathbf{q}, \Omega)=-g^{2} T \sum_{\omega^{\prime}} \int \frac{d \mathbf{q}}{2 \pi^{3}} \mathcal{G}\left(\mathbf{k}, \omega^{\prime}\right) \mathcal{G}\left(\mathbf{q}-\mathbf{k}, \Omega-\omega^{\prime}\right), \tag{17}
\end{gather*}
$$

respectively.
On the other hand, when the temperature is close to $T_{c}$ (i.e. $T_{c}-T \ll T_{c}$ ), the boson pair condensate is weak compared with the single-particle boson and the Cooper pair condensates. Since $\Gamma$, Eq.(15) is of the second order in $\Delta, \Gamma \propto \Delta^{2}$, the anomalous boson GF can be neglected in this temperature range, where $\Delta$ is small. The fermion self-energy, Eq.(16) is a regular function of
$\omega$ and $\mathbf{k}$, so that it can be absorbed in the renormalized fermion band dispersion. Then the fermion normal and anomalous GFs, Eqs. $(10,11)$ look like the familiar GFs of the BCS theory, and one can apply the Gor'kov expansion [22] in powers of $\Delta(\mathbf{r})$ to describe the condensed phase of BFM in the magnetic field near the transition. Using Eq.(7) and Eq.(5) one obtains to the terms linear in $\Delta$

$$
\begin{equation*}
\Delta^{*}(\mathbf{r})=\frac{g^{2}}{E_{0}} T \sum_{\omega_{n}} \int d \mathbf{x} \mathcal{G}_{-\omega_{n}}^{(n)}(\mathbf{x}, \mathbf{r}) \Delta^{*}(\mathbf{x}) \mathcal{G}_{\omega_{n}}^{(n)}(\mathbf{x}, \mathbf{r}) \tag{18}
\end{equation*}
$$

The spatial variations of the vector potential are small near the transition. If $\mathbf{A}(\mathbf{r})$ varies slowly, the normal state $\mathrm{GF}, \mathcal{G}_{\omega}^{(n)}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)$ differs from the zero-field normal state GF, $\mathcal{G}_{\omega}^{(0)}\left(\mathbf{r}-\mathbf{r}^{\prime}\right)$ only by a phase 22] $\mathcal{G}_{\omega}^{(n)}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)=$ $\exp \left[-i e \mathbf{A}(\mathbf{r}) \cdot\left(\mathbf{r}-\mathbf{r}^{\prime}\right)\right] \mathcal{G}_{\omega}^{(\mathbf{0})}\left(\mathbf{r}-\mathbf{r}^{\prime}\right)$. Expanding all quantities near the point $\mathbf{x}=\mathbf{r}$ in Eq.(18) up to the second order in $\mathbf{x}-\mathbf{r}$ inclusive, one obtains the linearised equation for the fermionic order parameter as

$$
\begin{equation*}
\gamma[\nabla-2 i e \mathbf{A}(\mathbf{r})]^{2} \Delta(\mathbf{r})=\alpha \Delta(\mathbf{r}) \tag{19}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=1+\frac{\Sigma_{b}(0,0)}{E_{0}} \approx 1-\frac{g^{2} N(0)}{E_{0}} \ln \frac{\mu}{T} \tag{20}
\end{equation*}
$$

and $\gamma \approx 7 \zeta(3) v_{F}^{2} g^{2} N(0) /\left(48 \pi^{2} T^{2} E_{0}\right)$. Here $v_{F}$ is the Fermi velocity, and $N(0)$ is the (renormalized) fermion density of states at the Fermi level.

In the framework of MFA one takes the bare boson energy in Eq.(20) as a temperature independent parameter, $E_{0}=g^{2} N(0) \ln \left(\mu / T_{c}\right)$ 11], or determines it from the conservation of the total number of particles (the density sum-rule) neglecting the boson self-energy $\Sigma_{b}(\mathbf{q}, \Omega)$ [8, 17, 18, 21]). Then Eq.(19) looks like the conventional mean-field Ginzburg-Landau (GL) equation with a negative $\alpha \propto T-T_{c}$, a relatively small fluctuation region $G i$ [11], and a finite $H_{c 2}(T)$ [17]. As a result one concludes that the phase transition is almost the conventional BCSlike transition, at least at $E_{0} \gg T_{c}$ 7, 11], and BFM, Eq.(1) describes the crossover from the BCS-like to local pair behaviour by tuning the parameters 18]. This conclusion is incorrect. The main problem with MFA stems from the density sum-rule, which determines the chemical potential of the system and consequently the bare boson energy $E_{0}(T)$ as a function of temperature,

$$
\begin{equation*}
-T \sum_{\Omega_{n}} e^{i \Omega_{n} \tau} \int \frac{d \mathbf{q}}{2 \pi^{3}} D\left(\mathbf{q}, \Omega_{n}\right)=n_{b}-\int d \mathbf{r}\left|\phi_{0}(\mathbf{r})\right|^{2} \tag{21}
\end{equation*}
$$

Here $\tau=+0, n_{b}=n-n_{f}$ is the number of bosons, $n$ is the total number of particles, and $n_{f}$ is the number of fermions. The term of the sum in Eq.(21) with $\Omega_{n}=0$ is given by the integral

$$
\begin{equation*}
T \int \frac{d \mathbf{q}}{2 \pi^{3}} \frac{1}{E_{0}+\Sigma_{b}(\mathbf{q}, 0)} \tag{22}
\end{equation*}
$$

where $\Sigma_{b}(\mathbf{q}, 0)=\Sigma_{b}(0,0)+q^{2} / 2 M^{*}$ for a small $\mathbf{q}$ is calculated using Eq.(17) with the normal state fermion GF 12] (here $M^{*}$ is a constant). The integral converges, if and only if $E_{0} \geqslant-\Sigma_{b}(0,0)$. This exact result means that $\alpha \geqslant 0$ at any temperature. In fact, this coefficient is strictly zero in the Bose-condensed state, because $\mu_{b}=-\left[E_{0}+\Sigma_{b}(0,0)\right]$ corresponds to the boson chemical potential relative to the lower edge of the boson energy spectrum. More generally, $\mu_{b}=0$ corresponds to the appearance of the Goldstone-Bogoliubov mode due to a broken symmetry below $T_{c}$. As a result, the GL coefficient $\alpha(T)$, is zero at any temperature below $T_{c}, \alpha(T) \equiv 0$, and not only at $T_{c}$ in the exact theory of BFM. On the other hand, MFA violates the density sum-rule, predicting the wrong negative $\alpha(T)$ below $T_{c}$.

There are a few important physical consequences. Since $\alpha(T)=0$, the Levanyuk-Ginzburg parameter [26] is infinite. It means that the phase transition is never a BCS-like second-order phase transition even at large $E_{0}$ and small $g$. In fact, the transition is driven by the BoseEinstein condensation of real bosons with $\mathbf{q}=0$, which occur due to the complete softening of their spectrum at $T_{c}$ in 3D BFM. Remarkably, the conventional upper critical field, determined as the field, where a non-trivial
solution of the linearised Gor'kov equation (19) occurs, is zero in BFM, $H_{c 2}(T)=0$. It is not a finite $H_{c 2}(T)$ found in Ref. 17] using MFA. In the homogeneous case $\Delta(T)$ should be determined from Eq.(21) rather than from the BCS-like equation (5), which is actually an identity 12], since $E_{0}=-\Sigma_{b}(0,0)$ below $T_{c}$. To get an insight into the magnetic properties of the condensed phase one has to solve Eqs. (6-9, 14-17) and Eq.(21) keeping the nonlinear terms. Even at temperatures well below $T_{c}$ the condensed state is fundamentally different from the MFA ground state, because of the pairing of bosons. The latter is similar to the Cooper-like pairing of supracondensate ${ }^{4} \mathrm{He}$ atoms 27], proposed as an explanation of the small density of the single-particle Bose condensate in superfluid Helium-4. The pair-boson condensate should significantly modify the thermodynamic properties of the condensed BFM compared with the MFA predictions. The common wisdom that at weak coupling the bosonfermion model is adequately described by the BCS theory, is therefore negated by our theory.

I highly appreciate enlightening discussions with A.F. Andreev, L.P. Gor'kov, V.V. Kabanov, A.I. Larkin, and A.P. Levanyuk, and support by the Leverhulme Trust (UK) via Grant F/00261/H.
[1] E. Simanek, Solid State Commun. 32, 731 (1979).
[2] C.S. Ting, D.N. Talwar and K.L. Ngai, Phys. Rev. Lett. 45, 1213 (1980).
[3] S.P. Ionov, Izv. AN SSSR Fiz 49, 310 (1985).
[4] J. Ranninger and S. Robaszkiewicz, Physica B (Amsterdam) 53, 468 (1985). It was shown that this interpretation of the polaron-bipolaron crossover stemmed from a misunderstanding of the polaron dynamics in the Holstein model (Y.A. Firsov, V.V. Kabanov, E.K. Kudinov, and A.S. Alexandrov, Phys. Rev. B 59, 12132 (1999); A.S. Alexandrov, ibid 61, 12315 (2000)).
[5] S. Robaszkiewicz, R. Micnas, and J. Ranninger, Phys. Rev. B 36, 180 (1987).
[6] G.M. Eliashberg, Pis'ma Zh. Eksp. Teor. Fiz. (Prilozh.) 46, 94 (1987) (JETP Lett. Suppl. 46, S81 (1987)).
[7] R. Friedberg and T.D. Lee, Phys. Rev. B40, 6745 (1989); R. Friedberg, T.D. Lee, and H.C. Ren, Phys.Rev. B42, 4122 (1990).
[8] R. Micnas, J. Ranninger, and S. Robaszkiewicz, Rev. Mod. Phys. 62, 113 (1990); J. Ranninger and J.M. Robin, Physica C235, 279 (1995).
[9] J. Ranninger, J.M. Robin, and M. Eschrig, Phys. Rev. Lett. 74, 4027 (1995).
[10] T. Kostyrko and J. Ranninger, Phys. Rev. B 54, 13105 (1996).
[11] V.B. Geshkenbein, L.B. Ioffe, and A.I Larkin, Phys. Rev.B 55, 3173 (1997).
[12] A. S. Alexandrov, J. Phys.: Condens. Matter 8, 6923 (1996).
[13] A.S. Alexandrov, Physica C274, 237 (1997); ibid 316, 239 (1999).
[14] R. Friedberg, H. C. Ren and O. Tchernyshyov, J. Phys.: Condens. Matter 10, 3089 (1998); A. S. Alexandrov, ibid, 3093.
[15] I. Grosu, C. Blaga, and M. Crisan, J. Supercond. 13, 459 (2000).
[16] L. P. Gor'kov, J. Supercond. 13, 765 (2000)
[17] T. Domański, Phys. Rev. B66, 134512 (2002); T. Domański, M. M. Maśka and M. Mierzejewski, Phys. Rev. B 67, 134507 (2003).
[18] R. Micnas, S. Robaszkiewicz, and A. Bussmann-Holder, in Highlights in condensed matter physics (eds. A. Avella et al., AIP Conference Proceedings, Melville, New York ) 695, 230 (2003) and references therein.
[19] A.S. Aleksandrov (Alexandrov) and A.B. Khmelinin, Fiz. Tverd. Tela (Leningrad) 28, 3403 (1986) (Sov. Phys. Solid State 28, 1915 (1986)).
[20] A.F. Andreev, Pis'ma Zh. Eksp. Teor. Fiz. 79, 100 (2004).
[21] M.L. Chiofalo, S.J.J.M.F. Kokkelmans, J.N. Milstein, and M.J. Holland, Phys. Rev. Lett. 88, 090402 (2002).
[22] L.P. Gor’kov, Zh. Eksp. Teor. Fiz. 34, 735 (1958) (Sov. Phys.-JETP 7, 505 (1958)); ibid 36, 1918 (1959) (Sov. Phys.-JETP 9, 1364 (1958))
[23] N.N. Bogoliubov, Izv. Acad. Sci. (USSR) 11, 77 (1947).
[24] A.A. Abrikosov, L.P. Gor'kov, and I.E. Dzyaloshiskii, Methods of Quantum Field Theory in Statistical Physics (Prentice-Hall, Englewood Cliffs, N.J., (1963)).
[25] The crossing diagrams are small as $g / \mu \ll 1$ 12]. Taking them as well as hard-core and other dynamic interactions into account change none of our conclusions.
[26] L.D. Landau and E.M. Lifshitz, Statistical Physics, 3rd Edition, Part 1 (eds. E.M. Lifshitz and L.P. Pitaevskii, Pergamon Press, Oxford- New York (1980)), page 476.
[27] E.A. Pashitskii, S.V. Mashkevich, and S.I. Vilchynskyy, Phis. Rev. Lett. 89, 075301 (2002).

