# FATIGUE LIFETIME ESTIMATION IN STRUCTURES USING AMBIENT VIBRATION MEASUREMENTS

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**Abstract.** *A methodology is proposed for estimating damage accumulation due to fatigue in the entire body of a metallic structure using output-only vibration measurements from a sensor network installed at a limited number of structural locations. This method can also be seen as a tool for a life-time prognosis within SHM concepts. Available frequency domain stochastic fatigue methods based on Miner's damage rule, S-N fatigue cycle curves and Dirlik's probability distribution of the stress range are used to predict the expected fatigue accumulation of the structure in terms of the power spectral density of the stress processes. In predicting the damage and fatigue lifetime, it is assumed that the unmeasured excitations can be modeled by stationary stochastic processes. The power spectral densities of stresses at unmeasured locations are estimated from the response time history measurements available at the limited measured locations using Kalman filter and a model of the structure. The proposed formulation is demonstrated using a MDOF spring-mass chain model arising from structures that consist of members with uni-axial stress states.*

# 1  INTRODUCTION

Damage accumulation due to fatigue is an important safety-related issue in metallic structures. The linear damage accumulation law introduced by [1] is often used to evaluate fatigue damage using available methods for cycle counting per stress level in measured stress response time histories. Available S-N fatigue curves [2] are used to predict the fatigue damage at different stress levels for harmonic stress time histories. The damage accumulation predictions are based on measurements taken from a sensor network, consisting usually of strain rosettes, attached to the structure and are only applicable for the locations where measurements are available. Due to practical and economical considerations, the number of sensors placed in a structure during operation is very limited and in most cases they do not cover all critical locations. Moreover, there are locations in the structure that one cannot install sensors such as underwater locations in offshore structures (oil refinery structures, offshore wind turbines, offshore steel jackets, etc.), within the body of solid structures, and non-approachable areas of extended structures. Available fatigue prediction methods based only on measurements cannot be used to predict fatigue damage accumulation at the locations where measurements are not available. In order to infer damage due to fatigue at structural members where measurements are not available, one needs to predict the stress response time histories in these structural members using the available measurements obtained from the sensory system. In limited circumstances, such predictions can be possible if one combines the available measurements with the information obtained from a model (e.g. a finite element model) of the structure.

The methods for fatigue damage accumulation have been extended to treat the case the excitations can be represented by a stochastic vector process with known correlation characteristics. Assuming that the structure behaves linearly and the excitation is modeled by a Gaussian stochastic vector process, the stress response at any point is a stochastic process that can be completely defined using the correlation characteristics of the stochastic excitations [3]. The fatigue damage accumulation at a structural location can then be computed using the characteristics of the stochastic processes of the components of the stress sensor at such location. Methods for fatigue accumulation for Gaussian narrow band stress processes have been first introduced (e.g. [4]) and then extended to the case of wide-band Gaussian stress processes (e.g. [5]). A review and comparison of spectral methods for stochastic fatigue analysis based on wide-band Gaussian stochastic processes can be found in the work by Benasciutti and Tovo [6]. The formulations depend on the probability distribution of stress cycles corresponding to different stress levels in a stress response time history signal and the expected number of peaks per unit time of a stress process. Results for the expected fatigue damage accumulation predicted by the Miner linear rule have been presented in terms of the spectral moments of the stress process which are readily obtained from the power spectral density of the stress components involved. For the important case of wide band processes encountered often in applications, simulation-inspired Dirlik approximation [5] is widely used and is considered to be the most accurate formula for modeling the probability of stress cycles in terms of the spectral moments of the stress process. It is worth noting that the aforementioned frequency domain methods based the power spectral densities use no information available from a sensor network. Instead, their predictive accuracy depend on the assumptions employed for the stochastic excitation characteristics and the models representing structural behavior. However, these prediction fail to integrate the information provided by a network of sensors. Such sensor information is expected to update and improve the fatigue predictions, making them consistent with the available measurements.

This work addresses the problem of estimating the expected damage accumulation due to fatigue in the entire body of a structure using output-only vibration measurements at a limited

number of locations provided by a sensor network installed in the structure. The measurements may consist of response time histories such as e.g. strain, acceleration, velocity, displacement, etc. The proposed methodology is applicable for the case where the responses can be modeled by stationary processes and the measured response time histories are long enough so that they can be considered to approach stationary processes. The excitation in the structure is considered to be unknown. However, for most operational condition of structures for which the measured response is approaching a stationary process, the excitation can be considered to be a Gaussian stationary stochastic process with unknown intensity and frequency content.

The expected fatigue damage accumulation in the entire structure is obtained by integrating (a) methods for predicting strain/stress response time histories and their correlation/spectral characteristics in the entire structure from output-only measured response time histories available at limited locations in the structure, and (b) frequency domain methods for estimating fatigue damage accumulation using the spectral characteristics of the predicted strain/stress response time histories. The idea is to use prediction methods such as Kalman filter [7] methods to predict the strain/stress response time histories at various locations within structural components using the measurements available at a limited number of locations. Such predictions are restricted here to the case of excitations that can be adequately represented by Gaussian stationary stochastic processes.

This work is organized as follows. In Section 2, the frequency domain formulation for predicting damage due to fatigue in structural elements subjected to uniaxial stress state in linear structures under Gaussian stochastic excitations is reviewed. The formulation is applicable to Gaussian wide-band stress processes, often encountered in engineering applications, and it depends on the spectral moments of the power spectral densities of the stress process at the location of a structure. Section 3 presents the formulation for predicting the strain/stress response time histories and the associated power spectral densities at the corresponding locations of the structure using Kalman filter and the measured time histories at a limited number of locations in the structure. Section 4 demonstrates the effectiveness of the proposed methodology using a chain-like mass-spring MDOF structure and measured data that are simulated using white noise excitations. Conclusions are summarized in Section 5.

## 2   SPECTRAL METHOD FOR FATIGUE ESTIMATION

The Miner rule [1] is used to predict the damage accumulation due to fatigue. According to this rule, a linear damage accumulation law at a point in the structure is defined by the formula

$$D = \sum_i^m \frac{n_i}{N_i} \tag{1}$$

where $n_i$ is the number of cycles at a stress level $\sigma_i$, $N_i$ is the number of cycles required for failure at a stress level $\sigma_i$, and $m$ is the number of stress levels identified in a stress time history. Available S-N fatigue curves [2] are used to describe the number of cycles $N_i$ required for failure given a stress level $\sigma_i$. The number of cycles $n_i$ at a stress level $\sigma_i$ are usually obtained using available stress cycle count methods, provided that the stress time histories are available through measurements. Alternatively, frequency domain methods based on spectral methods can be used to predict the expected damage due to fatigue using the linear damage law (1). The methodology assumes that the power spectral density of the stress process is available. For linear systems excited by time-varying loads that can be modeled by stationary

stochastic processes, these power spectral densities can be straightforward computed using available random vibration results [3].

The methodology for estimating the fatigue accumulation at a point assuming a uni-axial stochastic stress state is next reviewed. For multi-axial stress states one can refer to the work by You and Lee [8] and Pitoiset and Preumont [9] to extend the applicability of the present methodology. Let $\sigma(t)$ be the uni-axial stress at a point in a structural element. The stress is considered to be a stationary Gaussian stochastic process. This is the case encountered in linear structures that are subjected to stationary Gaussian stochastic processes. Let $S_\sigma(\omega)$ be the power spectral density of the stationary Gaussian stochastic process $\sigma(t)$ and

$$\lambda_i = \int_{-\infty}^{\infty} |\omega|^i S_\sigma(\omega) d\omega \tag{2}$$

be the spectral moments of the stress process. Using frequency domain methods for fatigue estimation under stochastic excitations and the linear damage accumulation law (1), the expected fatigue damage accumulation for a uni-axial stochastic stress process is given by [6]

$$E[D] = \int_0^\infty \frac{n(\sigma)}{N(\sigma)} d\sigma = c^{-1} TE[P] \int_0^\infty \sigma^a p(\sigma) d\sigma \tag{3}$$

where $n(\sigma) = TE[P] p(\sigma) d\sigma$ is the number of cycles at stress levels within the stress interval $[\sigma, \sigma + d\sigma]$,

$$N(\sigma) = c\sigma^{-\alpha} \tag{4}$$

is the number of cycles for failure that correspond to a specific harmonic stress level of amplitude $\sigma$ obtained from available S-N curves [2], $E[P]$ is the expected number of peaks per unit time for the stress process, $p(\sigma)$ is the probability distribution of the stress cycles at stress levels within $[\sigma, \sigma + d\sigma]$, and $T$ the period of observation. The parameters $c$ and $\alpha$ are constants obtained from fatigue test experiments and depend on the material of the test specimen.

The expected time of failure due to fatigue (fatigue lifetime) $T_{life}$ corresponds to a critical expected damage value $E[D] = D_{cr}$ which is often set equal to unity ($D_{cr} = 1$). Using (3), the fatigue lifetime is given by

$$T_{life} = \frac{D_{cr}}{\bar{D}} \tag{5}$$

where $\bar{D}$ is the expected damage rate given by

$$\bar{D} = c^{-1} E[P] \int_0^\infty \sigma^\alpha p(\sigma) d\sigma \tag{6}$$

For Gaussian stochastic stress processes, the probability distribution of the stress cycles at stress levels within $[\sigma, \sigma + d\sigma]$ is given by the Dirlik formula [5,10-11] as

$$p(\Delta\sigma) = \frac{\dfrac{E_1}{Q} e^{-\frac{Z(\Delta\sigma)}{Q}} + \dfrac{E_2 Z(\Delta\sigma)}{R^2} e^{-\frac{Z^2(\Delta\sigma)}{2R^2}} + E_3 Z(\Delta\sigma) e^{-\frac{Z^2(\Delta\sigma)}{2}}}{2\sqrt{\lambda_0}} \tag{7}$$

where

$$Z(\Delta\sigma) = \frac{\Delta\sigma}{2\sqrt{\lambda_0}} \tag{8}$$

$\Delta\sigma = \sigma_{max} - \sigma_{min}$ is the range of the stress process, and $E_1$, $E_2$, $E_3$, $R$ and $Q$ are specific algebraic functions of the spectral moments $\lambda_0$, $\lambda_1$, $\lambda_2$, $\lambda_4$, given by

$$E_1 = \frac{2\left(x_m - \alpha_2^2\right)}{1+\alpha_2^2}, \qquad E_2 = \frac{1-\alpha_2-E_1+E_1^2}{1-R}, \qquad E_3 = 1-E_1-E_2 \tag{9}$$

$$Q = \frac{1.25\left(\alpha_2 - E_3 - E_2 R\right)}{E_1}, \qquad R = \frac{\alpha_2 - x_m - E_1^2}{1-\alpha_2-E_1+E_1^2} \tag{10}$$

$$x_m = \frac{\lambda_1}{\lambda_0}\left[\frac{\lambda_2}{\lambda_4}\right]^{\frac{1}{2}} = \alpha_1\alpha_2, \qquad \alpha_1 = \frac{\lambda_1}{\sqrt{\lambda_0\lambda_2}}, \qquad \alpha_2 = \frac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} \tag{11}$$

This is a semi-empirical probability density which is a mixture of one exponential and two Rayleigh distributions. It has been derived by fitting the shape of a rain-flow range distribution via minimizing the normalized error between the rain-flow ranges and the above density model. The spectral moments $\lambda_0, \lambda_1, \lambda_2, \lambda_4$ constitute a base for the construction of the approximate closed-form Dirlik formula for the probability density of the stress range. The Dirlik formula constitutes an extension of the Rayleigh distribution to non-narrow band processes. It is widely used for fatigue crack estimation under wide-band Gaussian stationary applied stress. Extension to non-Gaussian stress processes requiring the skewness and kurtosis of the stress process are available in the work by [12].

Using results from random vibration theory, the expected number of cycles $E[P]$ per second for a stochastic process is given by the spectral moments of the process in the form

$$E[P] = \frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} \tag{12}$$

Substituting (12) into (6) and using the fact that $p(\sigma) = p(\Delta\sigma)/2 = p(2\sigma)/2$, one obtains the expected damage rate in the form

$$\bar{D} = (4\pi c)^{-1}\sqrt{\frac{\lambda_4}{\lambda_2}}\int_0^\infty \sigma^\alpha p(2\sigma)d\sigma \tag{13}$$

Using the Dirlik formula (7) and carrying out the integration analytically, the expected damage rate simplifies to [6]

$$\bar{D} = (8\pi c)^{-1}\sqrt{\frac{\lambda_4}{\lambda_2}}\lambda_0^{\alpha/2}\left[D_1 Q^\alpha \Gamma(1+\alpha) + 2^{\alpha/2}\Gamma\left(1+\frac{\alpha}{2}\right)(D_2 \mid R \mid^\alpha + D_3)\right] \tag{14}$$

It is clear from the aforementioned formulation and equations (5) and (14) that the expected fatigue damage rate and, consequently, the fatigue accumulation during a time interval $T$ at a point in the structure depends only on the spectral moments $\lambda_i$, $i = 1, 0, 2, 4$, of the stress process $\sigma(t)$. Using the definition of the spectral moments in (2), the spectral moments and the fatigue predictions at a point of a structure eventually depend only on the power spec-

tral density $S(\omega)$ of the stress process $\sigma(t)$. The power spectral densities of the stress response processes at a point can be calculated from measurements, provided that these measurements are long enough to be considered stationary.

However, there is a limited number of points that can be instrumented in the structure. For the points where measurements are not available, one has to make predictions of the stress process and subsequently the power spectral density of the stress process, given the measurements at other locations. This issue is addressed at the next section. Once these measurements and predictions of the stresses are estimated at measured and unmeasured locations, the power spectral densities and the corresponding damage accumulation due to fatigue are obtained, using (5) and (14), everywhere in the structure. In this way, fatigue damage accumulation maps for the entire structure are constructed from limited number of vibration measurements.

# 3 RESPONSE PREDICTIONS IN THE ENTIRE STRUCTURE USING VIBRATION MEASUREMENTS

The objective of this section is to predict the stress response at all points (or DOFs) in a structure using the measurements at a limited number of locations (or DOFs). This is achieved using Kalman filter [7]. For convenience and without loss of generality, it is assumed in the analysis that sensors that are placed in the structure measure the strains.

Consider the dynamic response of a linear structural system subjected to deterministic and random excitations. Using a spatial discretization method, such as finite element analysis, the equations of motion for a linear structure are given by the following set of $n$ second order differential equations

$$M\ddot{\underline{q}}(t) + C\dot{\underline{q}}(t) + K\underline{q}(t) = L_u\underline{u}(t) + L_w\underline{w}(t) \tag{15}$$

where $\underline{q}(t) \in \mathbb{R}^{N_{out} \times 1}$ is the displacement vector, $M$, $C$ and $K \in \mathbb{R}^{n \times n}$ are respectively the mass, damping and stiffness matrices, $\underline{u}(t) \in \mathbb{R}^{N_{u,in} \times 1}$ and $\underline{w}(t) \in \mathbb{R}^{N_{w,in} \times 1}$ are the deterministic and stochastic excitation vectors applied respectively at $N_{u,in}$ and $N_{w,in}$ DOFs, and $L_u \in \mathbb{R}^{n \times N_{u,in}}$ and $L_w \in \mathbb{R}^{n \times N_{w,in}}$ are matrices comprised of zeros and ones that maps the $N_{u,in}$ and $N_{w,in}$ excited DOFs to the $n$ output DOFs. Throughout the analysis, it is assumed that the system matrices $M$, $C$ and $K$ are symmetric. Let $\underline{y}(t)$ be the vector that collects all measurements at different locations of the structure at time $t$. These measurements are collected in general from sensors such as accelerometers, strain gauges, etc.

Introducing the state vector

$$\underline{x} = \begin{bmatrix} \underline{q} \\ \dot{\underline{q}} \end{bmatrix} \tag{16}$$

the equation of motion can be written in the state space form

$$\dot{\underline{x}} = A_c\underline{x} + B_c\underline{u}(t) + G_c\underline{w}(t) \tag{17}$$

the measured output vector $\underline{y}(t)$ is given by the observation equation

$$\underline{y}(t) = H\underline{x} + D\underline{u}(t) \tag{18}$$

where $A_c$, $B_c$ and $G_c$ are the state transition and observation matrices given by

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$$A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{19}$$

$$B_c = \begin{bmatrix} 0 \\ -M^{-1}L_u \end{bmatrix} \quad \text{and} \quad G_c = \begin{bmatrix} 0 \\ -M^{-1}L_w \end{bmatrix} \tag{20}$$

respectively.

## 3.1 Kalman filter approach

Since measurements are available in digitized form, the presentation of the Kalman filter is next given in discrete time. Using the sampling rate $1/\Delta t$, the discrete-time state space model corresponding to (17) and (18) is

$$\underline{x}_k = A\underline{x}_{k-1} + B\underline{u}_{k-1} + G\underline{w}_{k-1} \tag{21}$$

$$\underline{y}_k = H\underline{x}_k + D\underline{u}_k + \underline{v}_k \tag{22}$$

where $\underline{x}_k = \underline{x}(k\Delta t)$ and $\underline{y}_k = \underline{y}(k\Delta t)$, $k = 1, \cdots, N$, are the digitized state and output vectors, and

$$A = e^{A_c \Delta t} \tag{23}$$

is the state transition matrix for the discrete formulation. The random variables $w_k$ and $v_k$ represent the stochastic excitation and the measurement noise, respectively. They are assumed to be independent, white and following normal probability distributions $p(w_k) \sim N(0, Q)$ and $p(v_k) \sim N(0, R)$, where $Q$ and $R$ are the stochastic excitation and the measurement noise covariances assumed to be constant, independent of time.

Kalman filter is used to estimate the state $\hat{\underline{x}}_k$ of the system described by (21) using the measurements in the vector $\underline{y}_k$ in (22). Specifically, in the prediction step, an apriori state estimate $\hat{\underline{x}}_k^-$ of the state vector $\underline{x}_k$ of the system is estimated from equation [7]

$$\hat{\underline{x}}_k^- = A\hat{\underline{x}}_{k-1} + B\underline{u}_{k-1} \tag{24}$$

In the correction step, the measured value $\underline{y}_k$ is used to calculate an a posteriori state estimate $\hat{\underline{x}}_k$, weighting the measured and estimated signals by the Kalman filter gain matrix $K_k$. This is formulated by the equation

$$\hat{\underline{x}}_k = \hat{\underline{x}}_k^- + K_k [\underline{y}_k - H\hat{\underline{x}}_k^- - D\underline{u}_k] \tag{25}$$

where the Kalman gain factor is given by

$$K_k = P_k H^T [H P_k H^T + R]^{-1} \tag{26}$$

and, for steady state response, the error covariance matrix $P_k \equiv P = E[\underline{e}_k^- (\underline{e}_k^-)^T]$, where $\underline{e}_k^- = \underline{x} - \hat{\underline{x}}_k^-$ is the a priori error estimate, satisfies the discrete time Riccati equation:

$$P = APA^T - APH^T (HPH^T + R)^{-1} HPA^T + GQG^T \tag{27}$$

Let $\underline{\sigma}_k$ be a vector containing the digitized stresses at various locations of the structure. Using structural mechanics theory, these stresses in the vector $\underline{\sigma}_k$ are related to the state vector through a linear transformation $\underline{\sigma}_k = \Sigma \underline{x}_k$, where $\Sigma$ is the transformation matrix that associates the state vector to the desired stresses in the entire structure. Consequently, an estimate of the stresses $\underline{\hat{\sigma}}_k$ at time $t = k\Delta t$ is related to the state vector estimate $\underline{\hat{x}}_k$ through the transformation:

$$\underline{\hat{\sigma}}_k = \Sigma \underline{\hat{x}}_k \tag{28}$$

Herein, the response prediction vector $\underline{\sigma}(t)$ is restricted to stresses at elements subjected to uni-axial stress states required in lifetime fatigue estimation as described in Section 2.

Using the definition of the cross power spectral density and the Kalman filter equations (24) and (25), the cross power spectral density $S_{\underline{\hat{\sigma}}}(\omega)$ of the stress response vector $\underline{\hat{\sigma}}_k$ is given with respect to the cross power spectral density $S_{\underline{y}}(\omega)$ of the measurement vector $\underline{y}_k$ in the form

$$S_{\underline{\hat{\sigma}}}(\omega) = \Sigma S_{\underline{x}}(\omega) \Sigma^T = \Sigma E^{-1}(j\omega) K S_{\underline{y}}(\omega) K^T E^{-T}(j\omega) \Sigma^T \tag{29}$$

where $E(j\omega)$ is the matrix given by

$$E(j\omega) = I e^{j\omega\Delta t} - (I - KH)A \tag{30}$$

and $I$ is the identity matrix. Equation (29) relates the power spectral densities of the stresses at various structural locations with the power spectral densities of the measured quantities in $\underline{y}_k$ available at the limited number of measured locations. This relation depends on the model (e.g. a finite element model) used to represent the behavior of the structure and the assumption that the excitation vector is broad-band so that the excitations can be modeled by zero-mean stationary white noise processes with spectral density described by $E[\underline{w}_k \underline{w}_l^T] = Q\delta_{kl}$, where $\delta_{kl}$ is the Kronecker delta.

It should be noted that in order to apply (27), an estimate of the zero-lag covariance $Q$ of the unknown input stochastic process has to be provided. This estimate is obtained using the measurements $\underline{y}(t)$ and the relation $Q_{yy} \equiv Q_{yy}(Q)$ between the covariance matrix $Q_{yy}$ of the measurement vector $\underline{y}(t)$ and the covariance matrix $Q$ of the excitation process. Using (22) with $D = 0$, this relation is given by

$$Q_{yy} = HQ_{xx}H^T \tag{31}$$

where $Q_{xx}$ is given by the discrete time Lyapunov equation in the form

$$A Q_{xx} A^T - Q_{xx} + GQG^T = 0 \tag{32}$$

The optimal values of the entries of the covariance matrix $Q$ can be obtained by minimizing the difference between the covariance matrix $Q_{yy} \equiv Q_{yy}(Q)$ predicted by the linear model given $Q$ and the covariance matrix $\hat{Q}_{yy} = \frac{1}{N}\sum_{\rho=1}^{N} \underline{y}_\rho \underline{y}_\rho^T$ obtained from the measurements in $\underline{y}_\rho$, $\rho = 1, \cdots, N$. That is, the optimal value $Q_{opt}$ is obtained by minimizing the objective function

$$J(Q) = tr \| Q_{yy}(Q) - \hat{Q}_{yy} \|^2 / tr \| \hat{Q}_{yy} \|^2 \tag{33}$$

with respect to the elements in $Q$. The optimal value $Q_{opt}$ of $Q$ is then substituted in (27) to completely define the Riccati equation (27). The solution $P$ of the Riccati equation is substituted in (26) in order to find $K$ which is needed in (29).

## 3.2 Estimation of power spectral densities of stresses

The cross power spectral density matrix $S_{\underline{y}}(\omega)$ of the sampled measurement vector $\underline{y}_k$, involved in (29), can be obtained using available signal processing techniques such as the Welch technique [13]. Once $S_{\underline{y}}(\omega)$ has been estimated from the measurements, equation (29) is used to estimate the cross power spectral density $S_{\hat{\sigma}}(\omega)$ of the stress response vector $\underline{\hat{\sigma}}_k$. Alternatively, the PSD $S_{\hat{\sigma}}(\omega)$ of the stress response vector $\underline{\hat{\sigma}}_k$ is obtained by using equations (24) and (25) for the Kalman filter to provide estimates $\hat{\underline{x}}_k$ of the system state vector which are then used in equations (28) to estimate the stress vector $\underline{\hat{\sigma}}_k$. Finally, available signal processing techniques such as the Welch technique are used to compute the PSD $S_{\hat{\sigma}}(\omega)$ from the sampled stress response vector $\underline{\hat{\sigma}}_k$. The length of the sampled time history should be sufficient large in order for the estimates to be accurate.

The diagonal elements $diag[S_{\hat{\sigma}}(\omega)]$ of the cross power spectral density matrix $S_{\hat{\sigma}}(\omega)$ given in (29) contain the power spectral density estimates required for fatigue predictions using equations (5) and (14).

## 4 APPLICATION

The applicability and effectiveness of the methodology is illustrated using simulated "measurements" from a simple spring mass chain-like model of a structure. The stress state at critical locations are uni-axial and the fatigue prediction methodology can be directly applied. The measurements are assumed to be strain measurements. These measurements are simulated from the model of the structure using wide-band excitation resembling white noise.

Consider the class of $N$-DOF spring mass chain-like model fixed at the two ends as shown in Figure 1. The model is used to represent a structure consisting of a series of bar and body elements as shown in Figure 2. The structure consists of $N$ bodies with the $i$-th body having mass $m_i$. The $i-1$ and the $i$ bodies are connected by elastic bar element which provide the stiffness $k_i$ to the system. The number of bar (or spring) elements of the chain model is $N+1$. The material of the bar elements is considered to be steel. For steel bar elements, the values of the fatigue constants in equation (4) are taken to be $c = 4.06 \times 10^{88}$ and $\alpha = 9.82$ for steel samples. The $i$-th bar element has length $L_i$, cross-sectional area $A_i$ and modulus of elasticity $E_i$. For simplicity, each bar element is represented by a spring element with stiffness $k_i = E_i A_i / L_i$ as shown in Figure 1. Also, the nodal mass $m_i$ in Figure 1 includes the effect of the $i$ body mass and the lumped mass arising from the bar elements connected to node $i$. The $i$ component $q_i(t)$ of the vector $\underline{q}(t)$ corresponds to the displacement of the node $i$ of the model. The system is subjected to an unmeasured excitation applied at node $e$.
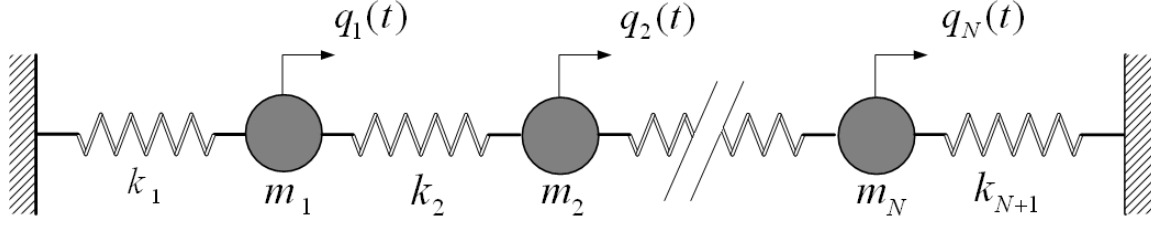
Figure 1: $N$-DOF spring mass chain-like model.
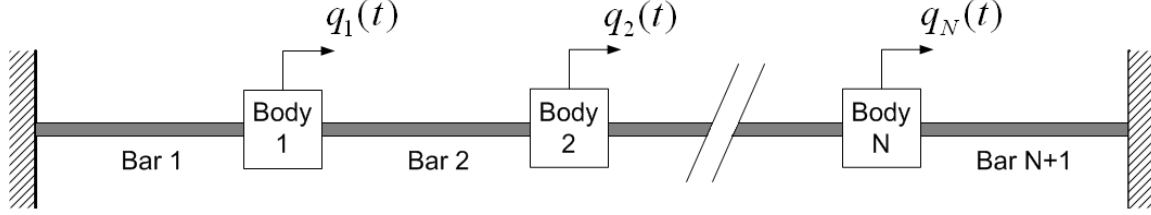


Figure 2: Structure consisting of $N$ masses and $N+1$ bar elements.

| Property Type | Property Value | Units |
|---|---|---|
| Material | Steel | |
| Modulus of Elasticity, $E$ | 2.1e11 | $N/m^2$ |
| Radius | 3.5e-3 | $m$ |
| Length | 300e-3 | $m$ |
| Stiffness $k_1 = \cdots = k_6$ | 2.7e7 | $N/m$ |
| Mass $m_1$ | 21 | $Kg$ |
| Mass $m_1$ | 15 | $Kg$ |
| Mass $m_1$ | 12 | $Kg$ |
| Mass $m_1$ | 15 | $Kg$ |
| Mass $m_1$ | 21 | $Kg$ |

Table 1: Geometric and material properties.

For demonstration purposes, the excitation vector $\underline{w}_k$ is assumed to be broadband and is modeled by Gaussian stationary white noise vector process with constant spectral density matrix. White noise provides a good approximation of the input whenever the correlation time of the input is sufficiently small compared to the system time constants. The discrete state space formulation of the equations of motion is used to simulated response time history data as well as compute estimates of the covariance responses and the power spectral density of the responses using the white noise excitation applied at node $e$. In particular, the strain and stress response time histories $\underline{\varepsilon}_k$ and $\underline{\sigma}_k$, respectively, are simulated at all bar elements using the state space formulation. The time discretization step used in simulated the sampled data is $\Delta t = 0.5 \times 10^{-3}$ seconds. The simulated strain and stress response time histories are the refer-

ence stress response time histories that are considered to be the exact stress response time histories for the excitations used. These response time histories and the corresponding power spectral densities can be used with the fatigue prediction methodology to compute the lifetime of the structure due to fatigue failure. Such predictions constitute the reference (exact) predictions against which the predictions from the proposed Kalman filter approach should be compared to for assessing the accuracy of the proposed methodology.

Let $o$ be the set that contains the bar element numbers where the strains are measured. The simulated sampled time history of the strain responses $\underline{\varepsilon}_k$ associated with the nodes or DOFs included in the set $o$ are included in the measurement vector $\underline{y}_k = \underline{\varepsilon}_k^{(o)}$. In practice, these measurements are collected using appropriate sensors such as strain gauges. Let $p$ be the set that contains the bar element numbers where the stresses will be predicted. The set $p$ is selected to be $p = \{1, \cdots, n+1\}$, i.e. it is assumed that stresses are predicted at all bar elements.

Results demonstrating the effectiveness of the proposed methodology are presented for a five degree of freedom system ($N = 5$) shown in Figure 1. The nodal masses as well as the geometric and material properties of the bars are listed in Table 1. The damping matrix $C$ in the equations of motion is chosen assuming that the system is classically damped. Rayleigh damping is assumed, i.e. $C = aM + bK$. A single excitation is considered which is applied at node $e = 5$. The strain response time histories $\underline{y}_k = \underline{\varepsilon}_k^{(o)}$ at the measured DOFs are used to predict the stress response time histories at all bar elements identified in the set $p$ using the proposed Kalman filter approach.
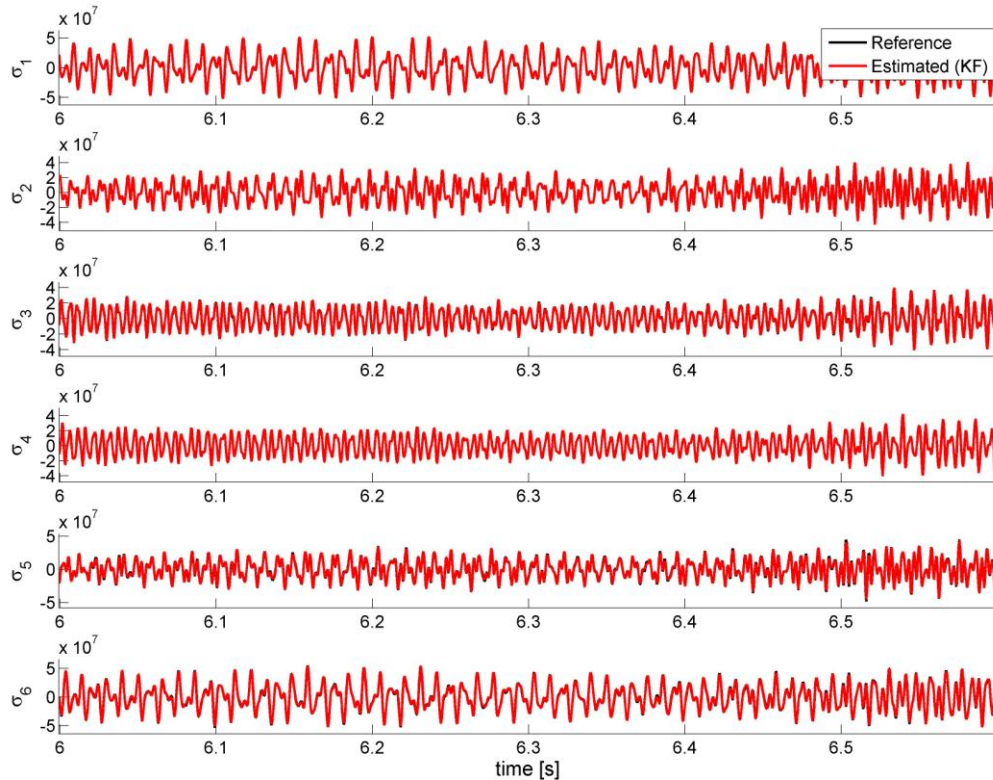


Figure 3: Comparison between reference and estimated from Kalman filter stress response time histories.

Comparison between the reference (exact) stress time histories simulated by the model and the estimated time histories from the Kalman filter are given in Figure 3 for all bar elements in the set $p$, assuming that the measured strains are at bar elements $o = \{2,4\}$. Similar comparisons for the power spectral densities of the stress time histories are shown in Figure 4. It can be seen that the estimated stress time history and the power spectral densities of the stress at the bar elements 2 and 4, where measurements are available, coincides with the corresponding reference stress time histories and stress power spectral densities simulated by the model. At the other bar elements, there is a discrepancy between the estimated and reference (exact) stress time histories or power spectral densities. These differences are due to the estimation error associated with the Kalman filter. However, it should be noted that the predictions from the Kalman filter approach are quite good, especially for the high amplitudes around the resonance peaks which mainly contribute to the fatigue process.
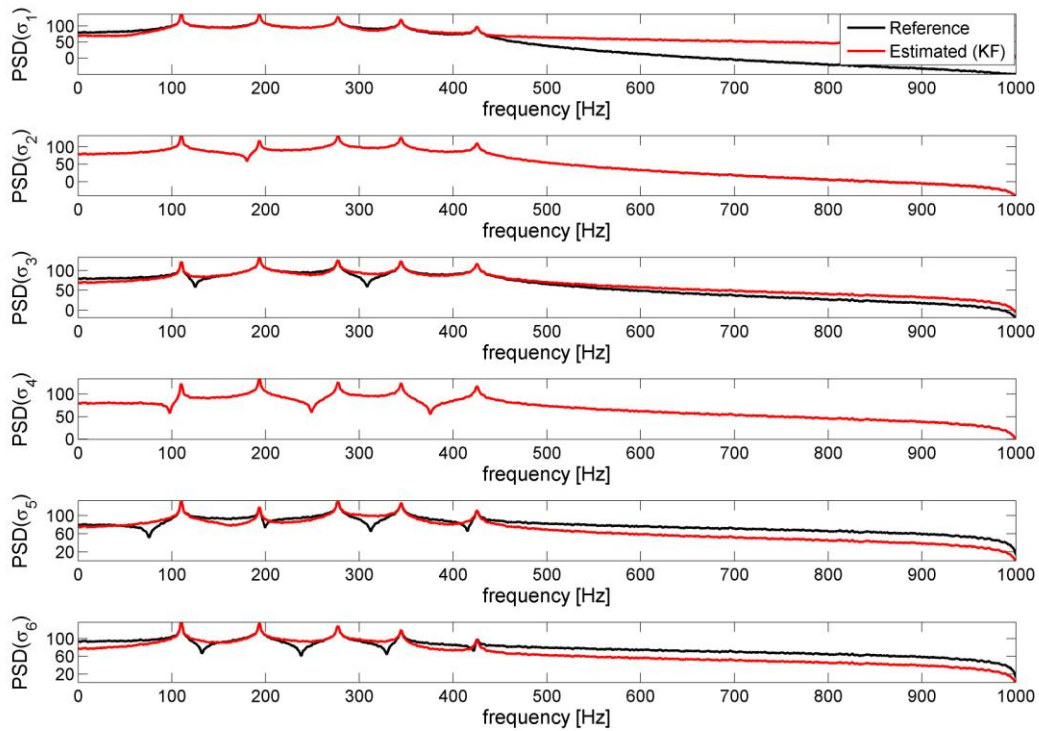


Figure 4: Comparison between reference and estimated from Kalman filter power spectral densities of the stress responses.

Lifetime predictions due to fatigue are shown in Figure 5 for all six bar (spring) elements of the structure as a function of the intensity $s_F$ of the input white noise, where $s_F$ is the standard deviation of the input. The lifetime values in the Figure are obtained using the fatigue prediction formula (5). For each bar element, there are two lifetime fatigue predictions. The first prediction is based on the reference time histories simulated by the model and it is used as the exact value against which to study the accuracy of the predictions from the proposed methodology. The second fatigue-based lifetime estimate is the one predicted by the methodology based on the use of Kalman filter approach to estimate the stress response time histories at all bar elements.
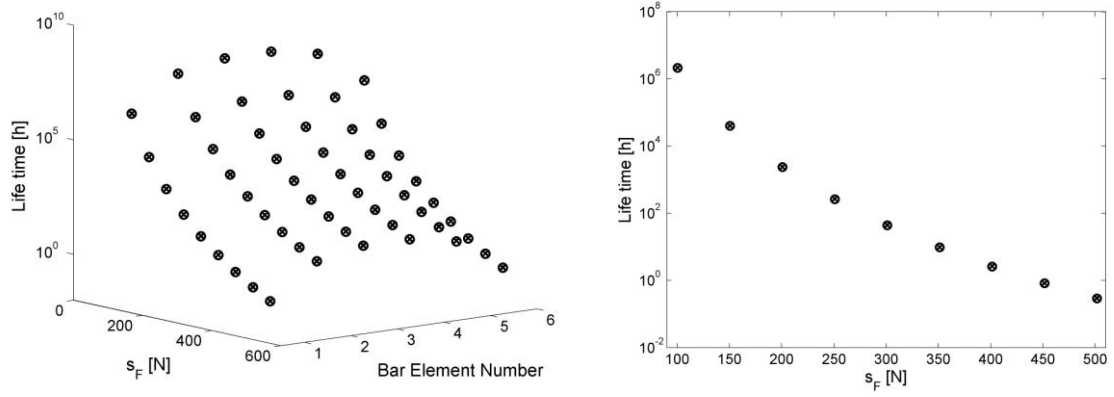
Figure 5: Comparison of reference fatigue-based lifetime estimates and the estimates predicted by the Kalman filter as a function of the excitation amplitude for the $N = 5$ DOF model.

To study the effect of the number and location of sensors (measurement points) on the accuracy of the predictions, Figures 6a and 6b presents predictions in the entire structure (all six bar elements) using two sensors placed at bar elements $o = \{2, 4\}$ and $o = \{2, 3\}$, respectively, while Figures 6c and 6d present predictions using three and four sensors placed at bar elements $o = \{1, 2, 4\}$ and $o = \{1, 2, 3, 6\}$, respectively.
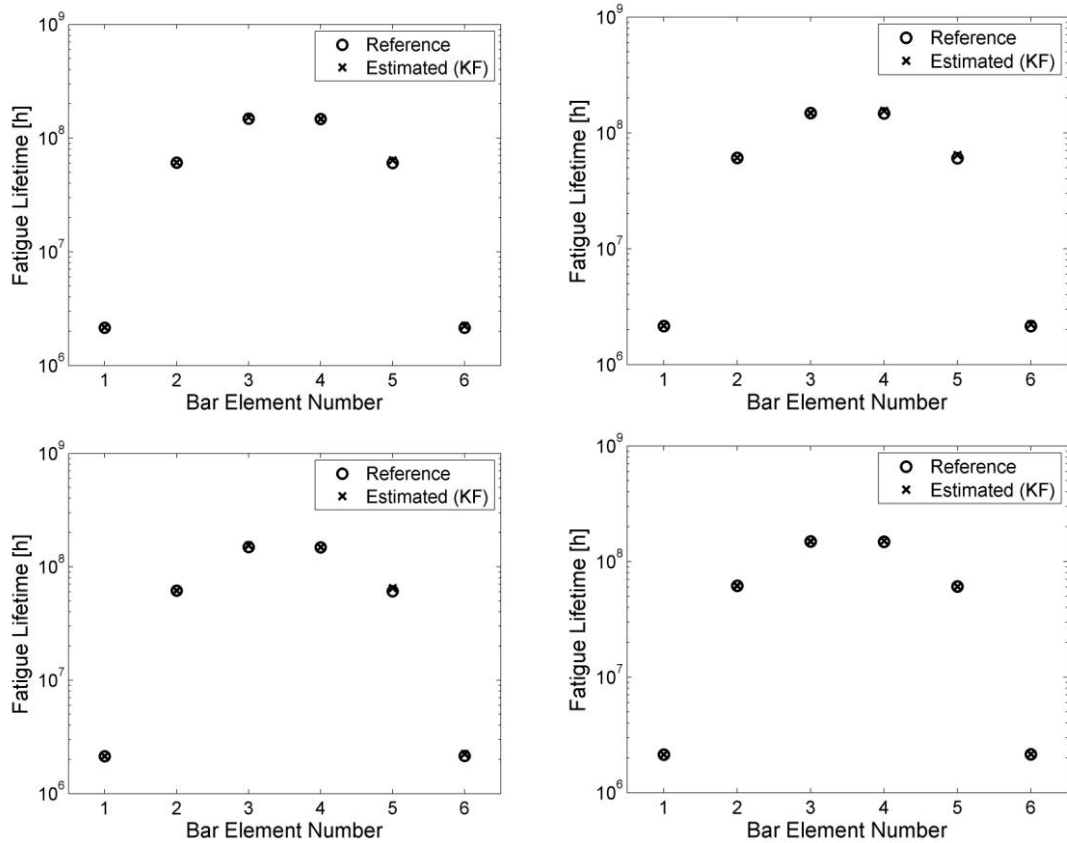


Figure 6: Comparison of reference fatigue-based lifetime estimates and the estimates predicted by the Kalman filter for the $N = 5$ DOF model.

It can be seen that the estimates based on the Kalman filter predictions are quite close to the reference fatigue values obtained from the actual simulations. Comparing the fatigue life-time predictions in Figures 6a-d, it becomes clear that the accuracy of the predictions depend on the number and location of sensors in the structure. One can improve the estimates considerably using optimal sensor location methodologies [14].

## 5 CONCLUSIONS

A methodology for estimating damage due to fatigue on the entire body of a structure using spectral methods and output only vibration measurements at a limited number of locations was presented. The fatigue predictions presented in this work are limited to structural members subjected to a uni-axial stress state. These predictions can be extended using available methods to structural members subjected to multi-axial stress state. Kalman filter is used for predicting the power spectral densities of the stresses in the entire body of the structure using the available measurements and a model of the structure. These predictions can be used to construct fatigue accumulation and lifetime prediction maps consistent with measurements provided by a monitoring system. The results from the proposed methodology are useful for designing optimal maintenance strategies for metallic structures using information for the actual condition of a structure, collected from a sensor network placed at limited number of locations.

## REFERENCES

[1] M.A. Miner, Cumulative damage in fatigue. *J Appl Mech*, **12**, A-159, 1945.

[2] B.F. Langer, Design of pressure vessels for low-cycle fatigue. *ASME J. Basic Eng*, **84**, 389–402, 1962.

[3] L.D. Lutes, S. Sarkani, *Random vibrations: analysis of structural and mechanical systems*. Butterworth-Heinemann, 2003.

[4] I., Rychlik, On the 'narrow-band' approximation for expected fatigue damage. *Probabilistic Engineering Mechanics*, **8**, 1–4, 1993.

[5] T. Dirlik, *Applications of computers to fatigue analysis*. PhD Thesis, Warwick University, 1985.

[6] D. Benasciutti, R. Tovo, Comparison of spectral methods for fatigue analysis of broadband Gaussian random processes. *Probabilistic Engineering Mechanics*, **21**, 287–299, 2006.

[7]   G.F. Franklin, J.D. Powell, M.L. Workman, *Digital control of dynamic systems.* Second Edition, Addison-Wesley, 1990.

[8]   B.R. You, S.B.  Lee, A critical review on multiaxial fatigue assessments of metals. *International Journal of Fatigue*,**18** (4), 235–244, 1996.

[9]   X. Pitoiset, A. Preumont, Spectral methods for multiaxial random fatigue analysis of metallic structures. *International Journal of Fatigue*, **22**, 541–550, 2000.

[10]  N.N. M. Bishop, F. Sherrat, A theoretical solution for the estimation of the rainflow ranges from power spectral density data. *Fatigue and Fracture of Engineering Materials and Structures*, **13** (4), 311-326, 1990.

[11]  D. Benasciutti, R. Tovo, Spectral methods for lifetime prediction under wide-band stationary random processes. *International Journal of Fatigue*, **27**, 867–877, 2005.

[12]  X. Wang, J.Q. Sun, Multi-stage regression fatigue analysis of non-Gaussian stress processes. *Journal of Sound and Vibration*, **280**, 455-465, 2005.

[13]  P.D. Welch, The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short modified periodograms. *IEEE Trans. Audio Electroacoustics*, **15**, 70-73, 1967.

[14]  C. Papadimitriou, Optimal sensor placement methodology for parametric identification of structural systems. *Journal of Sound and Vibration*, **278**(4), 923-947, 2004.