Optimal Design of Civil Engineering Structures Using Optimality Criteria Methods

by

Young-Shik Chung, BS, MSc

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S U M M A R Y

This report presents an Optimality Criteria method for the optimal design of civil engineering structures subject to multiple behavioural constraints on element stresses and nodal displacements and also to constraints on design variables.

The method makes use of a first order approximation for both deflection and stress constraints instead of the zero order approximation based on the concept of Fully Stressed Design used for stress constraints by the majority of Optimality Criteria approaches. The better approximation for stress constraints, introduced by considering the stress components as linear combinations of the generalized displacements, removes the difficulties arising from the use of stress ratios which, particularly for a well-known 10-bar planar truss, leads in many cases to a wrong design.

The method is also used to design continuous beams with tapered elements. The beam has rectangular or I-shape sections. The depth of sections at nodal points is chosen for the design variable since the depth of the tapered elements is continuously varying. The maximum bending stress at any section of the beam can be expressed by a linear combination of the rotational displacements of the section concerned and another section adjacent to it at an infinitesimally small distance away and thus the proper approximation for bending stresses is always possible.

A redesign algorithm is derived from the Kuhn-Tucker necessary conditions for optimality and the Newton-Raphson method is used to solve the system of nonlinear constraint equations. When applied to various

trusses and continuous beams it proves accurate and efficient, probably due to its mathematical rigour and the proper approximation for all kinds of constraints. The method can also solve the problems with nonlinear objective functions and thus enables us to obtain minimum cost designs as well as minimum weight designs. For civil engineering structures, which are of great variety in their types of element and predominant constraints, the method presented in this report shows much promise.

To my parents, wife and children

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NOTATION

F	Cost objective function
W	Weight objective function
$\underset{\sim}{X}$	Design Vector in general terms
A_i	Design variable i in truss problems
a_s	Size of member s in truss problems
D_i	Design variable i in beam problems
$d_t(x)$	Depth of beam element t varying linearly
ρ_s	Mass density of member s
L_s	Length of member s
E_s	Elastic modulus of member s
B_t	Breadth of beam element t with rectangular section
A_{ft}	Flange area of beam element t with I-section
ρ_t	Mass density of beam element t
L_t	Length of beam element t
E_t	Elastic modulus of beam element t
g_k	The kth constraint function
u_k	Deflection component k
σ_j	Stress of member j or at node j
l	Number of design variables
m	Number of stress constraints
n	Number of deflection constraints
λ_k	Lagrange multiplier associated with kth deflection constraint
λ_{n+j}	Lagrange multiplier associated with jth stress constraint
$\overline{g_k}, \overline{u_k}, \overline{\sigma_j}$	Prescribed values of behavioural constraints
$\underline{A_i}, \underline{D_i}$	Minimum size restrictions on design variables
F_s	Axial force of member s due to actual loads
$F_s^{(k)}$	Axial force of member s due to a unit load applied at the node and in the direction associated with the kth deflection component

$F_s^{(j)}$	Axial force of member s due to two unit loads, each applied at one of the nodes connecting member s in the direction of member s
$M_t(x)$	Bending moment distribution over beam element t due to actual loads
$M_t(x)^{(k)}$	Bending moment distribution over beam element t due to a unit load applied at the node associated with U_k
$M_t(x)^{(j)}$	Bending moment distribution over element t due to two couples associated with σ_j
h_i	Derivative of objective function with respect to design variable i
a_{tk}	Coefficient to evaluate approximate deflections in beam problems
b_{tj}	Coefficient to evaluate approximate stresses in beam problems
c_{ik}	Coefficient to evaluate approximate deflection gradients in beam problems, and approximate deflections and deflection gradients in truss problems
d_{ij}	Coefficient to evaluate approximate stress gradients in beam problems, and approximate stresses and stress gradients in truss problems
$\{T_{ti}\}$	Transformation matrix to obtain depths of nodes from design values in beam problems
r_{ti}	Ratio of the depth of node t to design value i
r_t	Ratio of the depth of node t+1 to that of node t
$G1$	Set of active design variables
$G2$	Set of passive design variables
U	Set of active deflection constraints
S	Set of active stress constraints

1. INTRODUCTION

The desire for structural optimization might date back to the dawn of human civilization. The development of structural forms, such as arches, domes, beams, slabs, ect., was inevitably of a very slow evolutionary process, but probably in an optimal manner. In fact they have been in use for hundreds of years or even thousands of years, and it seems that no radical changes of basic structural forms have yet appeared in spite of recent striking developments in theory and technologies.

Nevertheless, the revolution in calculation brought about by the computer together with improvements in other techniques has made a new wave of innovative designs possible. It enables the designer to focus more on the physical reality rather than a mathematical abstraction, largely by providing extensive capability of structural analysis by now. Whereas structural analysis can be carried out with reasonable accuracy solely by the computer, it can hardly be said that structural design is also the subject the computer can yield satisfactory results without human intervention and/or excessive simplification. Probably human judgement should always play a more important role in any kind of engineering design. The designer need not and must not hand over whole responsibility to the computer, but he may wish to rely on the capabilities of the computer to such extent as to make the most appropriate configuration or proportions of the structure be selected under the necessary conditions of selection which he is liable to feed into the computer.

The emergence of mathematical programming techniques enabled a wide range of optimization problems in engineering to be solved rigourously. Among them are the problems of optimal design of structures, which have a significant amount of quantifiable portions. The combining of computer oriented structural analysis techniques with mathematical programming methods led to the development of automated procedures for iterative redesign directed towards an optimum design. In this approach structural design is idealized as a problem of mathematical extremization of pre-defined merit in a solution space constrained by prescribed quantities such as stress and deflection limits. Since the problem of structural design usually involves a large number of quantifiable solution variables and response values, its subproblem consisting of those quantities may be of great importance, and undoubtedly the mathematical solution to the subproblem contributes to the whole solution process to quite a meaningful extent. Any automated procedure, if it can solve the subproblem mathematically or numerically and thus give an optimum design, will be of much greater help to the designer than providing merely capability of structural analysis. Since it decides the design values set by the designer without requiring human intervention, the designer can put aside numerical manipulations, concentrate on innovative designs and even use trial and error without worrying the burden of repeated calculations.

It seems, however, that such optimum design procedures as to be applicable universally and confidently to practical structures have not yet appeared although enormous development has been achieved during the past two decades. As the structures being designed become larger and more complex, the solution process by mathematical programming techniques confronts serious difficulties and becomes increasingly inefficient and inaccurate. In the solution space with greatly increased dimension, the elaborate mathematical transformations for determining search directions and step sizes become not only time-consuming but often erroneous. To offset some of these difficulties optimality criteria approaches were proposed, but these also confront difficulties in the presence of multiple constraints.

Behind the development of optimum design methods the aircraft and aerospace structures have provided a strong driving force. Their designs should be directed towards an obvious and urgent objective - minimizing the weight without compromising structural integrity. Naturally light weight and high strength materials are used. The structural form will be of truss-type, if applicable, to increase the stiffness of the overall structure. Therefore the problems of this category may be those of finding the minimum weight designs of trusses where only a few deflection constraints are likely to be eminent. The optimality criteria methods are particularly suitable for these problems and have been used successfully.

The design of civil engineering structures, however, puts forward different aspects. They may be assembled with bar elements, bending elements or both types of elements. Stress constraints will be of greater importance in many cases, and many of them will become equally restrictive constraints. The objective is undoubtedly cost-minimization, but the cost of a structure is by no means such a physical quantity as the weight which can be defined and stated in a definite form. Moreover, reducing the costs for materials may cause increases of other costs, and the relative importance among various costs may vary from problem to problem. For the problems of this nature mathematical programming techniques may be suitable in the light of their generality, but they will soon become inefficient as the size of problems increases. The optimality criteria methods, on the other hand, may suit large-scale problems, but their problem-dependent nature and lack of mathematical rigour make it difficult to apply them directly to civil engineering structures.

Among the various optimality criteria approaches, the method developed by Taig and Kerr of British Aircraft Corporation has ability to solve rigourously problems with multiple constraints making use of the Newton-Raphson method. The work described in this thesis attempts firstly to improve the method substantially in both respects of reliability and efficiency, secondly to extend the scope of problems it can tackle to such extent as to include structures with bending elements and problems where stress constraints are rather restrictive, and consequently to provide a basis for further developments leading to practical use of optimization methods for civil engineering structures.

The central feature of the optimization process is the solution of the optimality criteria and constraint equations for the Lagrange multipliers. Since the constraint equations in structural optimization problems are highly involved nonlinear equations, it is not at all an easy task to find their solution which satisfies the optimality conditions. Moreover, it is always possible to encounter many local minima, which are hardly recognizable because the behaviour of the constraint surfaces in the design space is not yet fully understood. Therefore finding the optimum solution to any problem, even within the context of the quantifiable aspects, seems still remote from materialization. When the author started this work, he came across a quotation,

" The optimist proclaims that we live in the best of all possible worlds; and the pessimist fears this is true. " – J.B. Cabell, in the book, " Methods of Optimization ", by G.R. Walsh, Wiley, 1975, and felt that he did not need to fear the true optimum could be found by the mathematical and/or numerical optimization methods.

In Chapter 2 a comparison is made between mathematical programming techniques and optimality criteria methods by listing advantages and disadvantages of both classes. A review on various optimality criteria methods appears in Chapter 3. Chapter 4 describes the purpose and scope of this work and delineate the problems treated in this work. Chapter 5 is devoted to a detailed description of the method developed in this work. It starts with depicting some aspects of the problem and proceeds with a step-by-step description of the method. In Chapter 6 examples of stress limited trusses are taken, and an interesting feature concerning optimality of fully stressed designs of trusses is explored. A range of truss problems widely appearing in the literature are solved and their results are compared favourably with those obtained by other methods in Chapter 7. The trusses are subjected to both deflection and stress constraints. The adverse effect from using stress ratios to resize overstressed members is demonstrated also in this chapter. A number of beam examples are treated in Chapter 8. They are 2 to 5 span continuous beams with varying sections and subjected to both deflection and stress constraints. Chapter 9 discusses some difficulties of this method encountered throughout this work and suggests possible further developments to counter the difficulties and also to extend the scope of problems to which the method can be applied. Appendix "A" provides a guide for the user and notes for the programmer of the TRUSS-program developed in this work. Appendix "B" is for the BEAM-program.

2. MATHEMATICAL PROGRAMMING TECHNIQUES and OPTIMALITY CRITERIA METHODS

The first attempt at coupling finite element analysis and nonlinear mathematical programming to create automated optimum design capabilities for elastic structural systems was by Schmit [1]. He presented some three-bar truss results different from and apparently lighter than fully stressed designs. Naturally the contrary-to-intuition results brought attention to the potential flaw in the basic premise of the simultaneous failure mode method, which was a prevailing approach at the time. Since then various mathematical programming techniques such as Sequence of Linear Programs (SLP), Sequence of Unconstrained Minimizations Techniques (SUMT) and Methods of Feasible Directions have been used to tackle structural design problems combining with computer aided structural analysis methods.

The structural optimization problem viewed as a nonlinear mathematical programming problem will have the form of

$$\left.\begin{array}{ll} \text{to minimize} & W(\underset{\sim}{x}) \\ \text{subject to} & g_k(\underset{\sim}{x}) - \bar{g}_k \leq 0 , \\ & k = 1, \cdots, m , \end{array}\right\} \cdots (2.1)$$

where

$$\underset{\sim}{x} = \{ x_1, x_2, \quad \ldots , x_\ell \}^T$$

Since the constraint equations in the majority of the problems are highly involved nonlinear equations and hardly explicit, the solution to the problem has to be found in an iterative way. The usual approach therefore is that of determining successive moves from a trial design as shown below.

$$\underset{\sim}{X}^{(r+1)} = \underset{\sim}{X}^{(r)} + \eta^{(r)} \underset{\sim}{d}^{(r)} \quad \ldots\ldots\ldots \quad (2.2)$$

Using the information obtainable from a trial design, $\underset{\sim}{X}^{(r)}$, the solution algorithm decides a direction vector, $\underset{\sim}{d}^{(r)}$, and a step size, $\eta^{(r)}$, such that the resulting design, $\underset{\sim}{X}^{(r+1)}$, is an improved one. The improved design may not be the optimum and thus it is used as the trial design of this step to improve the design further.

Various mathematical programming techniques may adopt different strategies in deciding the direction vector and the step size but they all have the following features in common.

a) Any design problem, whether it is the design of a structural system or individual elements, can be formulated as a mathematical programming problem.

b) The behavioural characteristics of the optimum design need not be presumed, rather they emerge as a consequence of the design procedure.

c) A wide variety of constraints on structural behaviour including stress, displacement, buckling, dynamic and thermal response can be dealt with.

d) The objective function is not necessarily restricted to representing a specific merit such as the weight of the structure, it may have any complicated form as long as it is differentiable.

The mathematical programming techniques are therefore rather general and can be used as a "black-box" optimizer if a proper algorithm is provided.

However, difficulties arise when the problem involves a great number of design variables. Since the way of search is direct, many entries constituting the direction vector, $\underset{\sim}{d}^{(r)}$, in Equ. (2.2) may be erroneous and thus the convergence to the optimum becomes painfully slow as the number of design variables increases. For this reason these approaches are not very successful and the practical use of them has been restricted to problems of a moderate size in spite of their problem-independent nature. Indeed, a grim assessment of them appeared in 1971 (in Ref. 2 and also in Ref. 3). The decade 1960-1970 was characterized as a "period of triumph and tragedy for the technology of structural optimization", and it was also suggested that the mathematical programming approach to structural optimization was little more than "an interesting research toy".

Leaving the aforementioned approaches, using purely numerical search based on the mathematical form of the problem, another class of approaches, called optimality criteria methods, emerged in the late 1960's. These are the approaches to find the optimum design of a structure in an indirect manner making use of the nature peculiar only to the optimum structure. Whereas the mathematical programming techniques stick to the mathematical form and the local behaviour of the objective functions and the constraint equations, the new methods consider the physical behaviour of the structure implied in the mathematical form and aim at reaching the optimum design by solving a system of nonlinear equations obtained by applying the Kuhn-Tucker necessary conditions. These methods also adopt an iterative method but find the next design from the information obtained by analysing the current design rather than by deciding on a move from the current design.

The following recurrence relation is used in most optimality criteria methods.

$$x_i^{(r+1)} = C_i^{(r)} x_i^{(r)} \quad \ldots\ldots\ldots \quad (2.3)$$

The correction factor, $C_i^{(r)}$, in Equ. (2.3) should be determined for each design variable. Seemingly, the set of correction factors may become erroneous, as the direction vector in Equ. (2.2), when the problem involves too many design variables. But this is not the case with the optimality criteria methods. Rather, difficulties are encountered when there are many behavioural constraints. The correction factor for the ith design variable is determined from

$$C_i^{(r)} = \left(\sum_{k=1}^{m} \lambda_k \frac{-\frac{\partial g_k}{\partial x_i}}{\frac{\partial W}{\partial x_i}} \right)^{\frac{1}{N}} \quad \ldots\ldots \quad (2.4)$$

where

λ_k ; Lagrange multiplier,

N ; relaxation parameter.

The values of the Lagrange multipliers are determined such that the design resulting from them satisfies Equu. (2.5).

$$\lambda_k (g_k(\underset{\sim}{X}) - \bar{g}_k) = 0 , \quad k = 1, 2, \cdots, m \quad \ldots (2.5)$$

The redesign process by Equ. (2.3) is repeated until

$$\sum_{k=1}^{m} \lambda_k \frac{-\frac{\partial g_k}{\partial x_i}}{\frac{\partial W}{\partial x_i}} = 1 \quad \ldots\ldots\ldots \quad (2.6)$$

holds for all design variables except those controlled by the side constraints. Equ. (2.5) and Equ. (2.6) are the optimality criteria derived from the Kuhn-Tucker necessary conditions.

The derivatives in Equ. (2.4) are obtainable directly from the results of structural analysis. Determining the Lagrange multipliers, on the other hand, not only calls for a significant amount of computing but sometimes confronts difficulties particularly when the problem involves many behavioural constraints. The number of design variables, however, does not affect the ease and stability of determining the Lagrange multipliers and the correction factors. Since the same set of Lagrange multipliers are used for all design variables, it is straightforward to determine the correction factors for any number of design variables once the set of Lagrange multipliers are determined.

The features of the optimality criteria methods with some references to those of mathematical programming techniques are listed below.

a) The optimality criteria methods usually tackles the design problem of a structural system, leaving that of individual elements to the mathematical programming techniques.

b) The methods are still efficient even for large-scale problems while the mathematical programming techniques suffer from numerical difficulties arising from the increased number of design variables.

c) It is necessary to explore the behavioural characteristics of the optimum design to develop an optimality criteria method. Therefore the method so developed must be problem-dependent.

d) The existence of multiple constraints, particularly of different types, presents difficulties and diminishes the admirable efficiency of the methods.

e) The majority of the problems tackled to date by the optimality criteria methods have a specific class of objective functions representing the weight of the structure.

There are other classes of approaches falling into the category of mathematical programming. Among them geometric programming has been successfully employed for the civil engineering structures such as reinforced concrete beams. Templeman and Winterbottom [4] demonstrated that many problems arising in optimum structural design could be formulated in such a way as to be easily and rapidly solved using geometric programming, and that geometric programming was particularly suitable for the design of many different types of bending elements and so general that it could be programmed as a standard package of subroutines.

It was also shown by Templeman [5] and Morris [6] that geometric programming could also be used for the optimum design of structural systems such as the truss-type structures solved previously by various methods. Nevertheless, it appears that the method also confronts difficulties as the number of design variables and/or behavioural constraints increases.

3. REVIEW ON VARIOUS OPTIMALITY CRITERIA METHODS

In developing an optimality criteria method for a particular class of problems, the first task is to establish the optimality criteria relevant to the problem since the optimality criteria methods are problem-dependent. Based on the principle of minimum potential energy, Prager [7) 8) 9) 10)] developed optimality criteria for such structures as beams, sandwich plates and trusses subject to a single behavioural constraint or multiple constraints. VenKayya and co-workers [11) 12) 13)] derived a strain energy criterion, also based on the principle of minimum potential energy, and coupled it with a search procedure to find the optimal design. When multiple constraints are present, a Lagrangian approach is used and the optimality criteria become similar to the Kuhn-Tucker necessary conditions, from which many authors in the 1970's derived directly the optimality criteria mainly for trusses subjected to static loading.

The analytical treatments by Prager [7-10)] will be hard for practical use, although they provide a deeper insight into the analytical nature of the optimality criteria. The strain energy criterion by Venkayya et al [11-13)] raises a little doubt whether it can alway yield the true optimum. For a stress limited truss the requirement "the ratio of the strain energy in the element to its energy capacity should be the same throughout the structure" in Ref. 12 can be replaced by "all elements should be equally stressed" when the truss is made of one material. Therefore this criterion will lead to a fully stressed design, which may not be the optimum. A test on a three-bar truss built with different materials was made in Ref. 14 and proved the resulting design was not optimum.

The second phase of developing an optimality criteria method is that of devising an algorithm to force the design to satisfy the established optimality criteria. When the problem is subject to only a single behavioural constraint the algorithm will be straightforward. Determining the Lagrange multiplier and the correction factor for each design variable in Equ. (2.4) is simply a matter of scaling such that the resulting Lagrange multiplier satisfies the constraint in an equality sense. The major difficulties in this phase stem from the presence of multiple behavioural constraints. It is not at all an easy task to find whether a constraint is active and, if so, what the contribution of the constraint is to the overall requirement. Many authors, who established essentially the same optimality criteria, adopted different schemes to tackle these problems. A number of optimality criteria methods dealing with truss problems subject to multiple behavioural constraints are to be outlined under separate headings bearing the authors' names.

Prior to describing the solution algorithms, the optimality criteria and recurrence relations which most methods share are presented. The k^{th} component of the generalized displacements, U_k, is expressed as

$$U_k = \sum_{i=1}^{I} \frac{F_i F_i^{(k)} L_i}{A_i E_i} \qquad \text{. . . . . (3.1)}$$

where A_i, E_i, L_i represent area, elastic modulus and length of member i, and $F_i, F_i^{(k)}$ represent axial forces of member i due to actual and corresponding virtual loads. Throughout a redesign iteration, U_k and its derivatives with respect to the design variables are evaluated from Equ. (3.2) and (3.3) assuming that F_i and $F_i^{(k)}$ remain unchanged.

$$u_k = \sum_{i=1}^{l} \frac{c_{ik}}{A_i} \quad \cdots\cdots\cdots \quad (3.2)$$

where

$$c_{ik} = \frac{F_i F_i^{(k)} L_i}{E_i} \quad ; \quad \text{constant}$$

$$\frac{\partial u_k}{\partial A_i} = -\frac{c_{ik}}{A_i^2} \quad \cdots\cdots\cdots \quad (3.3)$$

Then the optimality criteria, Equ. (2.6) and Equ. (2.5), and the recurrence relation, Equ. (2.3) and (2.4), for the truss problems will appear as follows.

$$\sum_{k=1}^{m} \lambda_k \frac{c_{ik}}{\rho_i L_i A_i^2} = 1 \quad \cdots\cdots\cdots \quad (3.4)$$

$$\lambda_k (u_k - \bar{u}_k) = 0 \quad , \quad k = 1, \cdots, m \quad \cdots \quad (3.5)$$

$$A_i^{(r+1)} = C_i^{(r)} \cdot A_i^{(r)} \quad \cdots\cdots\cdots \quad (3.6)$$

$$C_i^{(r)} = \left[\sum_{k=1}^{m} \lambda_k \frac{c_{ik}}{\rho_i L_i A_i^2} \right]^{\frac{1}{N}} \quad \cdots\cdots \quad (3.7)$$

In most cases, the design values, A_i , are determined, repeatedly in a redesign iteration, from the Lagrange multipliers using the following relation.

$$A_i = \left[\sum_{k=1}^{m} \lambda_k \frac{c_{ik}}{\rho_i L_i} \right]^{\frac{1}{N}} \quad \cdots\cdots \quad (3.8)$$

3.1 Gellatly and Berke [15)]

This method deals with the deflection constraints in Equ. (3.7) separately. From the results of structural analysis and the virtual unit load method, the derivatives of deflection constraints with respect to design variables are calculated. Combinations of new design values are computed from Equ. (3.5) and (3.8) assuming that only one constraint takes part in Equ. (3.8) for each combination. Then the largest area is selected for each member from all the combinations. The areas so generated are compared with those based upon stress ratios or minimum sizes and the larger values are selected for each member. This evaluation of areas is made again for each deflection constraint, but in this case those members critically designed in the preceding step by stress limits or minimum sizes or by a deflection constraint other than that being currently considered are kept at their previous values and considered as passive. This cycle is repeated until no transfer occurs between the members designed by deflections and by stresses or minimum sizes. The resulting design is then reanalysed and scaled until critical. Throughout the redesign process, Equ. (3.8) for each design variable and each constraint equation will be evaluated several times, assuming that $C_{ik}'s$ remain unchanged. In this case the axial forces of each member due to actual and virtual loads are assumed constant.

In this method it is not necessary to decide the set of active constraints in advance. If only one active constraint emerges the method will work, but difficulties arise if there are more active constraints. Let us assume that two deflection constraints emerged and determined the values of active design variables at the end of a redesign procedure. Then the design variables would be divided into two groups, each governed by one of the two constraints. Therefore it cannot be said that the two constraints make the set of active constraints in a strict sense. For this reason the method may not be accurate or efficient when the problem is subject to many behavioural constraints.

3.2 Venkayys, Khot and Berke [16)]

This method also uses the virtual unit load to derive the derivatives of deflection constraints. For the truss problem, the ratio of the derivative of constraint k to that of the objective function appearing in Equ. (2.4) and Equ. (2.6) happens to be the same as the virtual strain energy density per unit mass of member i when the virtual unit load is associated with constraint k.

This method therefore uses the term, virtual strain energy density, instead of the ratio of the derivatives in Equ. (2.4) and (2.6) and states "the optimum structure for a specified displacement is the one in which the virtual strain energy density per unit mass is the same for all its elements". In the presence of multiple constraints the optimality condition becomes "the virtual strain energy densities of a member associated with all the constraints, each multiplied by a weighting parameter constant for all members, add up to unity". The weighting parameter stands for the Langrange multiplier and this condition is exactly the same as that of Equ. (3.4)

The iterative algorithm proposed in this method determines the values of the Lagrange multipliers very simply as shown below.

$$\lambda_k = \frac{\sum_{i=1}^{\ell} \rho_i L_i A_i}{\sum_{i=1} \frac{F_i F_i^{(k)} L_i}{A_i E_i}} \quad \ldots\ldots \quad (3.9)$$

The Lagrange multiplier associated with constraint k is set to the ratio of the total weight to the amount of the k^{th} displacement. Therefore the less restricitive the constraint is, the greater the associated Lagrange multiplier becomes.

This method is very simple but gives rise to adverse situations because the multipliers associated with inactive constraints should vanish but do not. When this method was applied to the three-bar truss in Ref. 14 the weights of the resulting designs were ever increasing.

3.3 Berke and Khot [17)]

This method proposed a simple iterative scheme to determine the values of the Lagrange multipliers. Firstly initial values of all Lagrange multipliers are obtained considering all constraints separately. With these $\lambda's$, $A_i's$ are calculated from Equ. (3.8) and used for evaluating u_k from Equ. (3.2). If u_k so evaluated satisfy Equ. (3.5) for all k the latest values of $\lambda's$ are accepted as the final values of the current redesign iteration. Otherwise, they are updated using the following relation.

$$\lambda_k^{(r+1)} = \left(\frac{u_k'^{(r)}}{\bar{u}_k'} \right)^2 \cdot \lambda_k^{(r)} \quad \ldots \quad (3.10)$$

The prime in Equ. (3.10) means that those terms corresponding to passive variables in Equ. (3.2) are deducted from the evaluated

$u_k^{(\nu)}$ and the prescribed $\bar{u}_k$.

This simple formula assumes that λ_k effects only the satisfaction of the k^{th} constraint. At the outset and during the iteration,the multipliers are treated separately but interrelated indirectly since all the multipliers participate in evaluating the deflection values. This approach also has the effect of eliminating inactive constraints and showed reasonably good behaviour when applied to the three-bar truss in Ref. 14.

Later, Gellatly et al [18] reported that the method showed rather high sensitivity to the initial values of λ's and low rate of convergence when applied to a small problem involving only two behavioural constraints. They also suggested that it might be most effectively used in combination with other solution techniques. The unstableness of the method even for such a small problem suggests that it may not be appropriate for large-scale problems in spite of its simplicity.

3.4 Kiusalaas [19] and Rizzi [20] [21]

In the foregoing methods the relaxation parameter N has been 2 , which is a moderate value for the truss problems. In this method a relaxation parameter of different type was used and resulted in the following relation being introduced into the recurrence relation, Equ. (2.3).

$$C_i^{(\nu)} = \alpha - (1 - \alpha) \sum_{k=1}^{m} \lambda_k \frac{\frac{\partial g_k}{\partial x_i}}{\frac{\partial W}{\partial x_i}} \quad \ldots . (3.11)$$

Here α is a scalar relaxation parameter that ranges in value from zero to unity, and is adjusted so as to improve convergence.

Khot et al [22] made a comparison of the two types of recurrence relation, exponential and linear, and presented the following relation existing between the relaxation parameters.

$$\alpha = \left[1 - \frac{1}{N} \right] \quad \ldots\ldots\ldots\ldots\ldots \quad (3.12)$$

Therefore if $N = 2$ and $\alpha = 0.5$, Equ. (2.4) and Equ. (3.11) yield the same value for the correction factor. A further comparison was made by Arora [23] between the step size $\eta^{(r)}$ in Equ. (2.2), when the gradient projection method was used, and the scalar relaxation parameter α as follows.

$$\eta^{(r)} = 1 - \alpha \quad \ldots\ldots\ldots\ldots\ldots \quad (3.13)$$

It appears therefore that the matter is not the type of the parameter but the value assigned to it.

The Lagrange multipliers are chosen in such a way that the resulting design satisfies all the constraints currently considered active in an equality sense within a first order approximation. In other words the resulting Lagrange multipliers move the design to the intersection of hyperplanes, each tangent to one constraint surface at the current design. For this purpose it is necessary to form a system of linear equations and solve it for λ's. If some of λ's turn negative, they are deemed to be associated with inactive constraints and new λ's are found with the remaining active constraints.

Naturally the relaxation parameter α takes part in the linear equations and plays an important role in determining the values of the Lagrange multipliers. Therefore it seems that the success of this method is sensitive to some extent to the value of α whereas the parameter N in other method does not affect the values of the Lagrange multipliers.

3.5 Dobbs and Nelson [24) 25)]

In this method an auxilliary function $\vartheta(\underset{\sim}{\lambda})$ is formed such that

$$\vartheta(\underset{\sim}{\lambda}) = \sum_{i=1}^{l} [1 - I_i]^2 \quad \ldots\ldots\ldots (3.14)$$

where

$$I_i = -\sum_{k=1}^{m} \frac{\lambda_k \frac{\partial g_k}{\partial x_i}}{\frac{\partial W}{\partial x_i}} \quad \ldots\ldots\ldots (3.15)$$

and is minimized by solving the set of equations

$$\frac{\partial \vartheta(\underset{\sim}{\lambda})}{\partial \lambda_k} = 0 , \quad k = 1, 2, \cdots, m \quad \ldots\ldots\ldots (3.16)$$

for the Lagrange multipliers λ_k . If the set of multipliers so obtained satisfies the optimality criteria, Equ. (2.6), the values of I_i will be unity for all i and $\vartheta(\underset{\sim}{\lambda})$ will be zero. If not, a new design is obtained from I_i and Equ. (2.3) and (2.4). In this case the method restricts the values of I_i within a certain limit as follows.

$$1 - \Delta \leq I_i \leq 1 + \Delta \quad \ldots\ldots\ldots (3.17)$$

Any I_i less than $1-\Delta$ or greater than $1+\Delta$ are set to the limit value.

If there are any design variables less than their minima, they are deleted from Equ. (3.14) and a new design is sought. This process is repeated until no design variable is found less than its minimum size. At the outset of the process active or near active constraints take part in Equ. (3.15), but on completion of the process some Lagrange multipliers will be negative. If this happens, the whole process should be repeated after deleting those constraints associated with negative Lagrange multipliers from Equ. (3.15).

This method was applied to the beam examples in Ref. 14. It was found that the success of the method was sensitive to the value of Δ and moreover the appropriate value of Δ varied from problem to problem.

3.6 Khan, Willmert and Thornton [26)27)]

This rather simple method involves only one active constraint. The most restrictive deflection constraint is considered active and the stress constraints are treated as side constraints using stress ratio as the other methods. Therefore determining the Lagrange multipliers is just a matter of scaling. This method uses a relaxation parameter to control the rate of convergence and stability ranging

$$0.001 \leq \frac{1}{N} \leq 0.2$$

as appropriate reportedly.

The value of the parameter used in this method is rather small comparing with $N = 2$ generally adopted in various optimality criteria methods and thus results in a small step size. In general, several constraints are active at the optimum. Therefore, the method dealing with only one active constraint at a time will seldom find the exact solution. The results of sample problems reported were not exact sotutions.

3.7 Taig and Kerr [28)]

This method is a rigourous approach to tackle suitably problems involving multiple constraints. The optimality criteria equations, Equ. (3.4) and (3.5), are solved for the Lagrange multipliers using the Newton-Raphson method. The number of optimality criteria equations is $\ell+m$ (number of design variables+number of constraints) whereas the number of unknowns is m. However, Equ. (3.4) makes A_i's and λ's interrelated and gives values of A_i's from λ's. Therefore the problem is to solve a system of nonlinear constraint equations in the space spanned by λ's. Since strict equalities are not observed for the inactive constraints they should be excluded from the system of the equations. The passive design variables should also be excluded since they are not determined by Equ. (3.4).

In spite of its mathematical rigour, the method confronts a number of difficulties stemming from how to discriminate active/inactive constraints and the appearance of negative Lagrange multipliers during the Newton-Raphson iterations. This thesis describes work done to improve the method in many respects. The improvements will be presented in chapter 5 in full detail. In addition the use of the Newton-Raphson method was extended to problems with stress limits as behavioural constraints and beam problems.

3.8 Sander, Fleury and Geradin [29)30)31)32)]

In these works, a proper linearization of the stress constraints was introduced by considering the stress components as linear combinations of the generalized displacements. The optimality criteria approaches were related to the dual statement of the problem as an auxiliary maximization problem in the Lagrange multipliers. The problem in its primal form was solved in terms of the reciprocal variables. The use of the reciprocal variables made the deflection and stress constraints of the truss problems linear at the expense of making the objective function nonlinear.

3.9 Applications to Bending Elements

The method by Taig and Kerr [28)] was applied to the design of continuous beams in Ref. 14. An optimality criterion for continous beams subject to multiple deflection constraints was derived and the numerical solution for the problems was based on the Newton-Raphson method. Although some difficulties were encountered the method showed promising from the viewpoint of accuracy and rapid convergence.

Armand and Lodier [33)] derived an optimality criterion for finite element structural representations using constant-moment plate-bending triangular elements. Only single displacement constraint was involved in the solution process and stress limits were treated as side constraints.

Gorzynski and Thornton [34)] presented a design method for trusses and frames based on a recursion formular similar to that given by

Venkayya [13)] but with the requirement that the energy ratios of all members at convergence be the same eliminated. Instead, the energy ratio of each member was allowed to become as large as possible. The energy ratio was defined as the ratio of the actual strain energy of a member to the strain energy capacity of the member, which was taken to be the strain energy that would be in the member when the entire cross section was stressed to the yield point. The ratio was also referred to as the "efficiency" of the member, which made the efficiency of the overall structure when summed over all the members. The solution algorithm was therefore the maximization of the efficiency of the structure. The method looks attractive, but it is not obvious if the method yields true optima.

4. THE PURPOSE OF THIS WORK

The various optimality criteria methods, as outlined in the preceding chapter, generally concern truss problems with predominant deflection constraints, frequently encountered in the design of aircraft structures. Stress constraints are usually treated as side constraints. Therefore the number of stress constraints does not affect the stability and efficiency of the methods. They merely replace the minimum size restrictions.

In designing structures for stress constraints, the fully stressed design approach has been used for reasons of simplicity. When deflection constraints are present this approach is no longer applicable, and therefore the optimality criteria approaches become more useful and reliable design methods. These approaches are particularly good at designing structures for deflection constraints because deflection constraints are seldom active except those at top nodes of a cantilever-type truss or at midspans and thus only a few active constraints have to be dealt with. If the stress constraints are to be treated as behavioural constraints, a large number of active stress constraints will disturb the solution process. For this reason the majority of the optimality criteria methods put aside the stress constraints while the optimality criteria equations are solved, and later take into account the stress limits using the stress ratio method.

The purpose of this work is first of all to devise a solution scheme to cope with difficulties arising from multiple constraints as well as to yield exact solutions. It was felt that for this purpose the method presented by Taig and Kerr[28)] was appropriate since it took advantage of the splendid Newton-Raphson method. Nevertheless, it was found that the method also had drawbacks, as was usually the case, and a number of modifications were necessary.

The next objective is to extend the scope of problems to be dealt with by the new method. Stress limits are treated as behavioural constraints and thus take part in the solution process of the optimality criteria equations. This of cause gives rise to a large number of active constraints, and thus requires a more powerful solution scheme. Then the three basic constraints - deflection constraints, stress constraints and minimum size restrictions - are treated properly, all within a first order approximation. Other types of constraints, such as stability and dynamic response, may be of potentially greater importance, but they are generally and generically related to the three basic constraints and excluded from the scope of this work.

A further and rather important extension in the context of civil engineering is to apply the method to the bending elements. In this work, however, the type of problems is restricted to the design of continuous beams.

The behaviour of bending elements is certainly different from that of bar elements. The response quantities are dependent not only on the cross sectional area but also on the shape of the section. Therefore some characteristics of the section, qualitative or quantitative, need to be predetermined. This work concerns two types of sections, rectangular section and I-section. The breadth of the rectangular section is predetermined, whereas the depth is allowed to vary and thus makes the design variable. For the I-section, the depth is the design variable, and the cross sectional areas of both flanges, upper and lower, are predetermined.

The stress and elongation of a bar element are both inversely proportional to the cross sectional area, the design variable. Therefore, it is possible to express the stress of a member as a linear combination of the generalized displacements with constant coefficients and thus to obtain the stress gradients in the same way as the deflection gradients. However, the bending stress at the extreme fibres of a rectangular beam section is inversely proportional to the depth squared whereas the flexural flexibility of the section is inversely proportional to the depth cubed. The coefficients therefore are not constant but linear functions of the depths when the stress at the extreme fibres is expressed as a linear combination of the generalized rotational displacements with them. This also applies to the I-section beams. This fact makes the expressions of the stress gradients more complicated and the solution process more difficult.

In this work, the depths at nodes are taken as the design variables. As a consequence of this the beam element will have a tapered configuration, which enables continuity of structure at the element boundaries. As there can be many elements in a span the profile view of the beam will appear continuously varying as commonly seen in civil engineering structures such as bridges.

Employing tapered elements as well as taking stress constraints into account in the solution process creates a number of difficulties not only in establishing an appropriate optimality criterion but devising an algorithm to solve the equations rigourously. It seems that there has been no published method to tackle such a problem. A survey paper by Haftka and Prasad [35] indicated only the use of fully stressed design approach for the stress constraints. Prasad and Haftka [36] derived a formula to obtain the derivatives of the stresses of plate finite elements, but it was assumed that the stress-displacement relation is independent of the design variables. The use of tapered elements is hardly found in the literature. In some analytical approaches for simple structures such as circular disks, tapered shapes were dealt with. Miller and Moll [37] presented an automatic design scheme for tapered member gabled frames using a modified interior penalty function approach. Venkateswara Rao [38] proposed an optimality criterion approach using tapered finite elements, but it was applied to a simple problem, optimization of a cooling fin with a temperature constraint.

In summary, the purpose of this work is to develop an optimality criteria method to solve the design problems of structures built with either bar or beam elements subject to the three basic constraints, deflection, stress and minimum size. The stress constraints should be treated as behavioural constraints. The method should be stable and reliable based on the mathematical rigour. The method may require more computing time per iteration than other methods, but largely the stability should bring the method back to efficiency. It is the ultimate objective to make the optimality criteria method efficient, reliable and problem-independent, such that it can handle all kinds of elements and constraints, and eventually applicable to practical civil engineering structures.

5. THE OPTIMALITY CRITERIA METHOD DEVELOPED

The optimality criteria method developed in this thesis tackles the optimal design of structures falling into two types. These are planar or space trusses assembled with bar members and continuous beams assembled with tapered elements, both subject to deflection and stress constraints. Due to the different structural behaviour the equations and formulae adopted in developing the optimality criteria method for one type are different from those for the other. Prior to the step-by-step description of the method some aspects of the problems are depicted.

The truss-type structures are those appearing widely in the literature and have no particular conditions imposed in this thesis. Their minimum-weight designs are sought. The bar members take axial forces and their stiffnesses are strictly proportional to their cross-sectional areas. Besides such behavioural constraints as deflection of nodes and/or stresses in bars, the trusses are subject to side constraints of minimum size restriction and design variable linking. The problem therefore is the minimization of the weight of a truss expressed as a linear function of a set of design variables A_i , the cross-sectional areas of individual or groups of bars and can be represented mathematically as:-

minimize $$W = \sum_{i=1}^{\ell} h_i A_i$$

subject to

$$u_k - \overline{u}_k \leq 0 \;, \quad k = 1, \cdots, n$$

$$\sigma_j - \overline{\sigma}_j \leq 0 \;, \quad j = 1, \cdots, m$$

$$\underline{A_i} - A_i \leq 0 , \quad i = 1, \cdots, \ell$$

$$h_i = \sum_{s \in I_i} \rho_s L_s$$

$$a_s = A_i \quad for \quad s \in I_i$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5.1)$$

where

A_i ; the ith design variable,

u_k ; the kth deflection component,

σ_j ; stress of the jth member,

$\overline{u_k}, \overline{\sigma_j}, \underline{A_i}$; prescribed values,

ρ_s ; mass density of member s ,

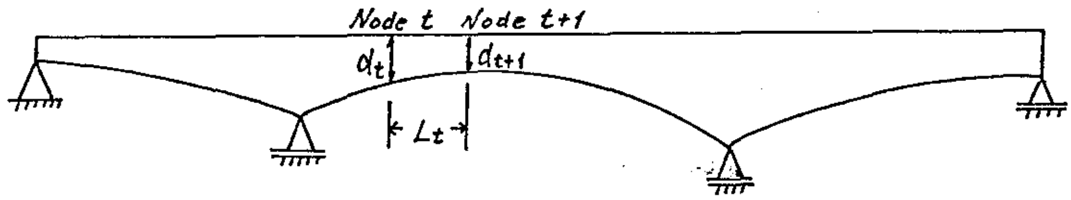
L_s ; length of member s ,

a_s ; cross sectional area of member s ,

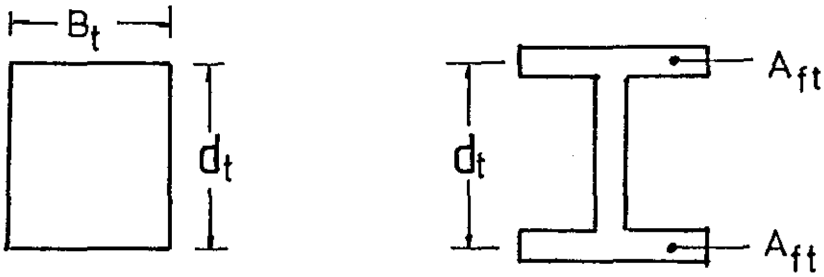
I_i ; set of member No.'s associated with design variable A_i .

The categories of design variables other than the cross-sectional area of each member, such as the topology and geometry of the structure and material properties, are assumed predetermined. For simplicity the above mathematical expression involves only one load case, but the extension for multiple load cases presents no difficulties. u_k and σ_j in Equ. (5.1) represent only the magnitudes of the corresponding deflection and stress. The number of design variables, bar members and deflection components are represented by ℓ, m and n , and unless otherwise stated throughout this thesis suffix i takes the values $1,\ldots,\ell$, j takes $1,\ldots,m$ and k takes $1,\ldots,n$.

The beam-type structures are those such as is shown in Fig. 1. The beam may be of rectangular section with predetermined breadth or I-section whose flanges have a predetermined cross sectional area. The depths at the nodes are allowed to vary and thus make a set of design variables. Therefore any element of the beam has linearly varying depths. It is also assumed that the loads are applied only at the nodes.



a) Profile View



b) Rectangular Section c) I-section

Fig. 1 A Typical Design of Beam

A deflection constraint may be imposed on any node, but one node per span would be reasonable. The stress at any of the nodes of a element makes a stress constraint. An element, say element t in Fig. 2, has linearly varying depths and is subject to linearly varying

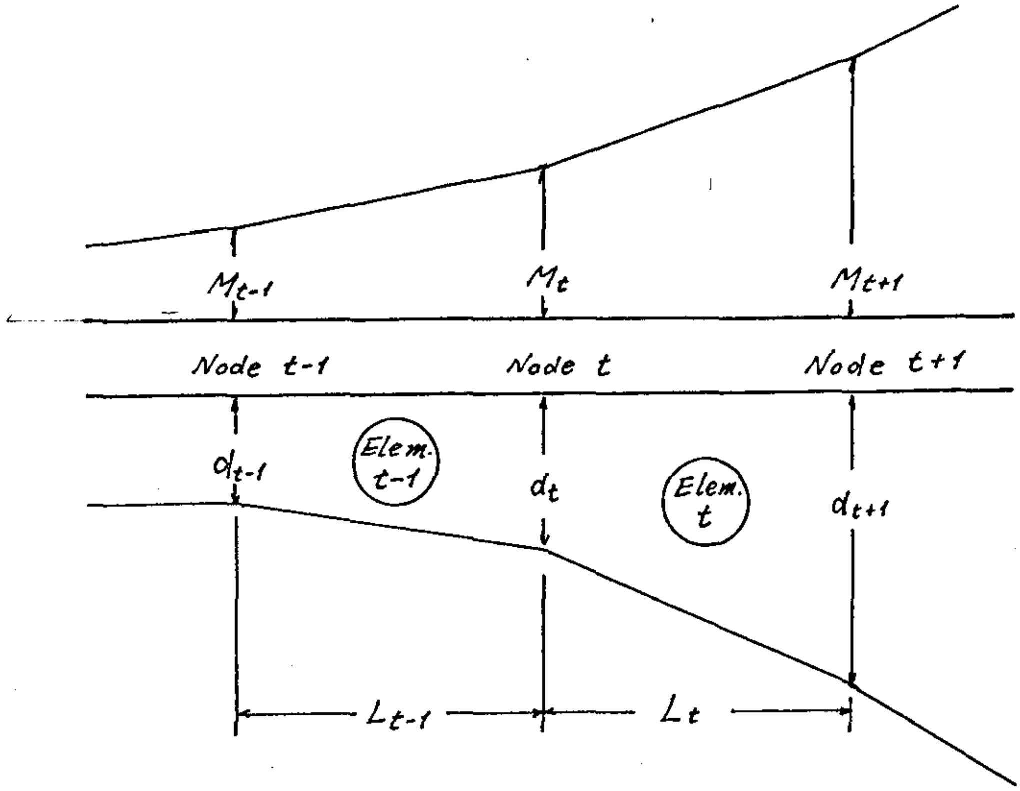


Fig. 2 A Typical Beam Element

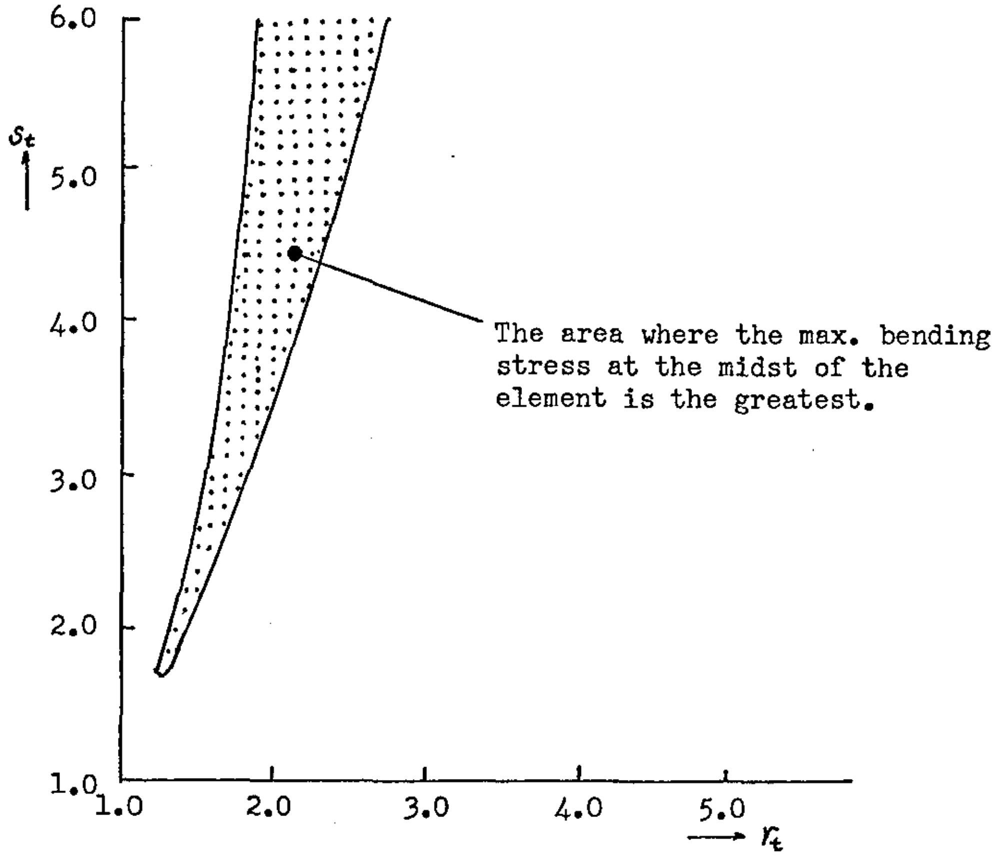


Fig. 3 Relationship between the maximum bending stresses at various points

bending moments. If we let r_t be the ratio of the depth at node $t+1$ to the depth at node t and s_t be that for the bending moments as expressed in Equ. (5.2),

$$\left.\begin{aligned} r_t &= d_{t+1} \,/\, d_t \\ s_t &= M_{t+1} \,/\, M_t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.2)$$

where d_t : depth at node t,

M_t : bending moment at node t,

we establish a relationship between the maximum bending stresses at node t, $t+1$ and at the midst of element t denoted by σ_t, σ_{t+1} and σ_{mt} respectively, as follows:-

$$\sigma_t : \sigma_{mt} : \sigma_{t+1} = 1 : \frac{2(1+s_t)}{(1+r_t)^2} : \frac{s_t}{r_t^2} \quad \ldots\ldots (5.3)$$

for the rectangular section, and

$$\sigma_t : \sigma_{mt} : \sigma_{t+1} = 1 : \frac{1+s_t}{1+r_t} : \frac{s_t}{r_t} \quad \ldots\ldots (5.4)$$

for the I-section. The shaded area of Fig. 3 representing those combinations of r_t and s_t in which σ_{mt} is greater than any of σ_t and σ_{t+1} for the rectangular section shows that there is only a little chance for σ_{mt} to be greater than σ_t and σ_{t+1} except for those elements close to a point where the sign of bending moment changes and thus unlikely to be subject to big bending moments. For the I-section, σ_{mt} can never be greater than any of σ_t and σ_{t+1}, and therefore taking into account the maximum bending stresses at nodes will be reasonable.

The objective function of this problem takes the form of Equ. (5.5).

$$F = C + \sum_{t=1}^{m} \alpha_t d_t^{\beta} \quad \cdots\cdots\cdots\cdots \quad (5.5)$$

where C, α_t and β are constants, and F may represent the cost of the beam if appropriate constants are chosen. If we let the constants have the following values,

$$\left.\begin{aligned} C &= 0.0 \\ \beta &= 1.0 \\ \alpha_t &= \frac{1}{2}(\rho_{t-1} B_{t-1} L_{t-1} + \rho_t B_t L_t),\ t=1,\cdots,m \\ \rho_o B_o L_o &= \rho_m B_m L_m = 0.0 \end{aligned}\right\} \cdots (5.6)$$

where ρ_t : mass density of element t,

B_t : breadth of element t,

L_t : length of element t.

F represents the total weight of the rectangular section beam or that of the I-section beam provided that C represents the weight of flanges and B_t represents the breadth of the web of element t. Multiplying α_t by appropriate values and assigning β some value possibly less than 1.0 will allow Equ. (5.5) to represent the cost of the beam. In the case of the I-section beam, the second term of Equ. (5.5) represents the cost of the web including stiffeners while C represents the cost of the flanges.

Equ. (5.7) is the mathematical expression of the constraints,

$$\left.\begin{aligned} u_k - \overline{u}_k &\leq 0, \quad k = 1,2,\cdots,n \\ \sigma_j - \overline{\sigma}_j &\leq 0, \quad j = 1,2,\cdots,m \\ \underline{d}_t - d_t &\leq 0, \quad t = 1,2,\cdots,m \end{aligned}\right\} \cdots\cdots \quad (5.7)$$

which is in implicit terms the same as that for the truss problem, but the way of design variable linking is different. A group of nodes may have the same depth. Some other group, however, will contain those nodes whose depths are not the same but instead interpolated by two design variables. The design variable linking for the beams is refered to as 'design variable linking by ratio', and will be explained later. For the beam problems, n represents the number of deflection constraints, m represents the number of nodes, stress constraints and design variables. Thus the number of elements is $m-1$. When design variable linking is employed l represents the number of design variables instead.

5.1 Constraints and Their Derivatives

The scope of most optimization techniques is the minimization (or maximization) of differentiable merit functions subject to constraints on the design variables in which the constraint functions are also differentiable. It is also the case with the techniques for structural optimization and all the foregoing methods require the use of derivatives of the constraint function and of the objective function with respect to the design variables. The objective function may well be of the form of Equ. (5.1) or Equ. (5.5) and the derivation of its derivatives is straightforward. However, the constraint equations arising in the structural optimization problem hardly have explicit expressions in terms of the design variables and thus there is no way to determine their derivatives but by numerical approaches.

Since the calculation of the derivatives takes a significant part of the total computing effort, it is important to carry out the task as efficiently as possible. The Virtual Load method, based on the principle of virtual work, has most been used particularly for the optimality criteria methods. An approach based on the concept of the design space was first suggested by Fox [39][40]. Another approach, called State Space method, has been developed by Haug and Arora [41][42], and Arora and Haug [43] made an analysis of the various methods mentioned above. Recently Johnson [44] presented a general expression for the derivation of design sensitivities via the flexibility method.

In this work the Virtual Load method was used because:-

a) as far as the constraint equation has the form of Equ. (5.1) or Equ. (5.7) it makes no difference whichever method is used,

b) this method allows us to selectively determine the derivatives of the constraints considered as active in a particular redesign process,

c) the optimality criteria method of this thesis requires not only the derivatives of the constraints but the explicit expressions of the constraints in terms of the design variables, although they are of an approximation, which can be given only by the Virtual Load method,

d) it is even more desirable for the beam problems since the beams treated in this thesis can be analyzed more efficiently by the force method.

5.1.1 Deflection Constraints

The Virtual Load method makes it possible to obtain explicit expressions for the constraints in the vicinity of the current design and thus to express their derivatives with respect to the design variables. The deflection component, u_k, is now expressed in the form of the work done by a unit virtual load associated with u_k.

$$u_k = \sum_{s=1}^{m} \frac{F_s \, F_s^{(k)} \, L_s}{a_s \, E_s} \qquad \qquad (5.8)$$

for trusses,

$$u_k = \sum_{t=1}^{m-1} \int_0^{L_t} \frac{12 \cdot M_t(x) \cdot M_t(x)^{(k)}}{E_t \, B_t \, d_t(x)^3} \, dx \qquad \qquad (5.9)$$

for rectangular section beams, and

$$u_k = \sum_{t=1}^{m-1} \int_0^{L_t} \frac{2 \cdot M_t(x) \cdot M_t(x)^{(k)}}{E_t \, A_{ft} \, d_t(x)^2} \, dx \qquad \qquad (5.10)$$

for I-section beams where

F_s : axial force of member s due to actual loads,

$F_s^{(k)}$: axial force of member s due to a unit load applied at the node and in the direction associated with the kth deflection component,

E_s : elastic modulus of member s,

$M_t(x)$: bending moment distribution over element t due to actual loads,

$M_t(x)^{(k)}$: bending moment distribution over element t due to a unit load applied at the node associated with u_k,

E_t : elastic modulus of element t,

$d_t(x)$: depth of element t varying linearly,

A_{ft} : cross-sectional area of a flange of element t.

If we let

$$C_{sk}' = \frac{F_s F_s^{(k)} L_s}{E_s} \quad \ldots\ldots\ldots\ldots\ldots \quad (5.11)$$

and

$$C_{ik} = \sum_{s \in I_i} C_{sk}' \quad \ldots\ldots\ldots\ldots\ldots \quad (5.12)$$

to include all members controlled by the same design variable A_i, then Equ. (5.8) becomes

$$U_k = \sum_{i=1}^{l} \frac{C_{ik}}{A_i} \quad \ldots\ldots\ldots\ldots\ldots \quad (5.13)$$

Therefore

$$\frac{\partial U_k}{\partial A_i} = - \frac{C_{ik}}{A_i^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (5.14)$$

assuming that the forces F_s and $F_s^{(k)}$ are independent of A_i and thus C_{ik} remains constant. Equ. (5.13) and Equ. (5.14) together with the assumption of the constant C_{ik} allow us to determine readily the magnitude of a deflection component and its derivative with respect to any design variable whenever the design changes. This remains valid until a redesign iteration finishes and the structure with the new design is analyzed and thus new C_{ik}'s are calculated.

For the beam problems, such formulations as Equ. (5.13) and Equ. (5.14) call for more complicated process. Not only the depth of an element but the bending moment distributed over the element vary continuously. Moreover, the depth at any point of an element has to be decided by two design varibles. An important assumption, however, enabled the depths at nodes to be the design variables. It is that the ratio of the depth at a node to the depth of its adjacent node, i.e r_t in Equ. (5.2), remains unchanged throughout a redesign iteration. Based upon this assumption and Simpson's rule Equ. (5.9) can be modified as follows.

$$U_k = \cdots\cdots + \frac{2L_{t-1}}{E_{t-1}B_{t-1}}\left[\frac{f_{t-1}^{(k)}}{d_{t-1}^3} + \frac{4f_{m\,t-1}^{(k)}}{d_{m\,t-1}^3} + \frac{f_t^{(k)}}{d_t^3}\right]$$

$$+ \frac{2L_t}{E_tB_t}\left[\frac{f_t^{(k)}}{d_t^3} + \frac{4f_{mt}^{(k)}}{d_{mt}^3} + \frac{f_{t+1}^{(k)}}{d_{t+1}^3}\right] + \cdots\cdots$$

$$= \sum_{t=1}^{m-1} \frac{2L_t}{E_tB_t}\left[\frac{f_t^{(k)}}{d_t^3} + \frac{16\,f_{mt}^{(k)}}{(1+r_t)^3 d_t^3} + \frac{16\,f_{mt}^{(k)}}{(1+\frac{1}{r_t})^3 d_{t+1}^3} + \frac{f_{t+1}^{(k)}}{d_{t+1}^3}\right]$$

$$= \sum_{t=1}^{m} \frac{1}{d_t^3}\left\{\frac{2L_{t-1}}{E_{t-1}B_{t-1}}\left[\frac{16f_{m\,t-1}^{(k)}}{(1+\frac{1}{r_{t-1}})^3} + f_t^{(k)}\right] + \frac{2L_t}{E_tB_t}\left[f_t^{(k)} + \frac{16\,f_{mt}^{(k)}}{(1+r_t)^3}\right]\right\}$$

$$= \sum_{t=1}^{m} \frac{a_{tk}}{d_t^3} \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (5.15)$$

where

$$a_{tk} = \frac{2L_{t-1}}{E_{t-1}B_{t-1}}\left[\frac{16f_{m\,t-1}^{(k)}}{(1+\frac{1}{r_{t-1}})^3} + f_t^{(k)}\right] + \frac{2L_t}{E_tB_t}\left[f_t^{(k)} + \frac{16f_{mt}^{(k)}}{(1+r_t)^3}\right] \cdots \quad (5.16)$$

$$L_0 = L_m = 0.0$$

$$r_t = d_{t+1} / d_t$$

$$d_{mt} = \frac{1}{2}(d_t + d_{t+1}) = \frac{1}{2}(1 + r_t)\, d_t$$

$$d_{m\,t-1} = \frac{1}{2}(d_t + d_{t-1}) = \frac{1}{2}\left(1 + \frac{1}{r_{t-1}}\right) d_t$$

$f_t^{(k)}$; the value of $M_t(x) \cdot M_t(x)^{(k)}$ at node t,

$f_{t+1}^{(k)}$; the value of $M_t(x) \cdot M_t(x)^{(k)}$ at node $t+1$,

$f_{mt}^{(k)}$; the value of $M_t(x) \cdot M_t(x)^{(k)}$ at the midst of element t.

As is the case with the truss problems, it is assumed that $M_t(x)$ and $M_t(x)^{(k)}$ are independent of d_t, and so are $f_t^{(k)}$, $f_{t+1}^{(k)}$ and $f_{mt}^{(k)}$. As r_t and r_{t-1} are assumed unchanged, a_{tk}'s remain constrant untill a redesign process finishes. The derivative of u_k have the form of Equ. (5.17),

$$\frac{\partial u_k}{\partial d_t} = -3\,\frac{C_{tk}'}{d_t^{4}} \quad \cdots\cdots \quad (5.17)$$

$$C_{tk}' = \frac{2L_{t-1}}{E_{t-1}B_{t-1}}\left[\frac{32 f_{mt-1}^{(k)}}{(1+\frac{1}{r_{t-1}})^4} + f_t^{(k)}\right] + \frac{2L_t}{E_t B_t}\left[f_t^{(k)} + \frac{32 f_{mt}^{(k)}}{(1+r_t)^4}\right] \cdot (5.18)$$

taking into account the existence of d_{mt} and d_{mt-1} in Equ. (5.15) which are governed partly by the design variable d_t.

Equ. (5.19) - (5.22) are the deflection constraints and their derivatives for I-section beams derived in the same way as for the rectangular section beams.

$$u_k = \sum_{t=1}^{m} \frac{a_{tk}}{d_t^{2}} \quad \cdots\cdots \quad (5.19)$$

$$a_{tk} = \frac{L_{t-1}}{3\cdot E_{t-1}A_{ft-1}}\left[\frac{8 f_{mt-1}^{(k)}}{(1+\frac{1}{r_{t-1}})^2} + f_t^{(k)}\right] + \frac{L_t}{3\cdot E_t A_{ft}}\left[f_t^{(k)} + \frac{8 f_{mt}^{(k)}}{(1+r_t)^2}\right] \cdots (5.20)$$

$$\frac{\partial u_k}{\partial d_t} = -2\,\frac{C_{tk}'}{d_t^{3}} \quad \cdots\cdots \quad (5.21)$$

$$C_{tk}' = \frac{L_{t-1}}{3E_{t-1}A_{ft-1}}\left[\frac{16 f_{mt-1}^{(k)}}{(1+\frac{1}{r_{t-1}})^3} + f_t^{(k)}\right] + \frac{L_t}{3E_t A_{ft}}\left[f_t^{(k)} + \frac{16 f_{mt}^{(k)}}{(1+r_t)^3}\right] \cdot (5.22)$$

5.1.2 Stress Constraints

The stress of any member of a truss can be obtained from generalized displacements and the stress-displacement relation existing in the structural system. For the stress of a particular member, say member j in Fig. 4, to be expressed in terms of virtual work we simply employ a pair of unit virtual loads to obtain the relative displacement of the nodes connecting the member to the overall system.

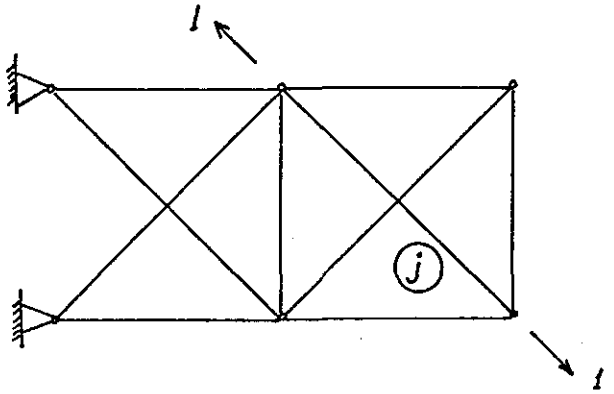


Fig. 4 Virtual Loads to express the Stress of Member j in terms of Virtual Work.

The member stress σ_j can then be found by virtual work:

$$\sigma_j = \sum_{s=1}^{m} \frac{F_s F_s^{(j)} L_s}{a_s E_s} \cdot \frac{E_j}{L_j} \quad \ldots\ldots\ldots \quad (5.23)$$

where $F_s^{(j)}$ is the axial force of member s due to the two unit loads as shown in Fig. 4. If we let

$$d_{sj}' = \frac{F_s F_s^{(j)} L_s}{E_s} \cdot \frac{E_j}{L_j}$$

and

$$d_{ij} = \sum_{s \in I_i} d_{sj}'$$

then

$$\sigma_j = \sum_{i=1}^{\ell} \frac{d_{ij}}{A_i} \quad \ldots\ldots\ldots \quad (5.24)$$

Therefore

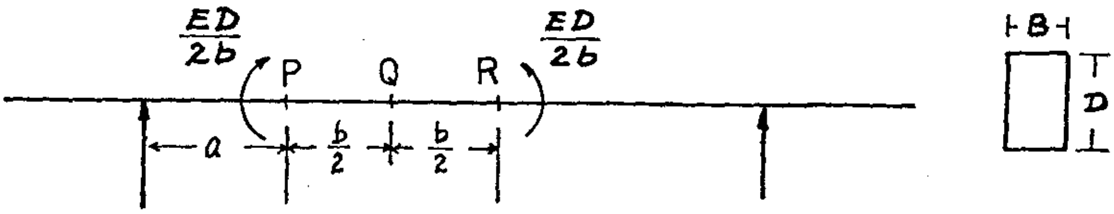
$$\frac{\partial \sigma_j}{\partial A_i} = - \frac{d_{ij}}{A_i^2} \quad \ldots\ldots\ldots \quad (5.25)$$

Equ. (5.24) and Equ. (5.25) have been derived in the same way as Equ. (5.13) and Equ. (5.14) were done and the same assumption is applied. In fact, any member stress of a truss is a linear combination of generalized displacements with a set of coefficients independent of the member sizes.

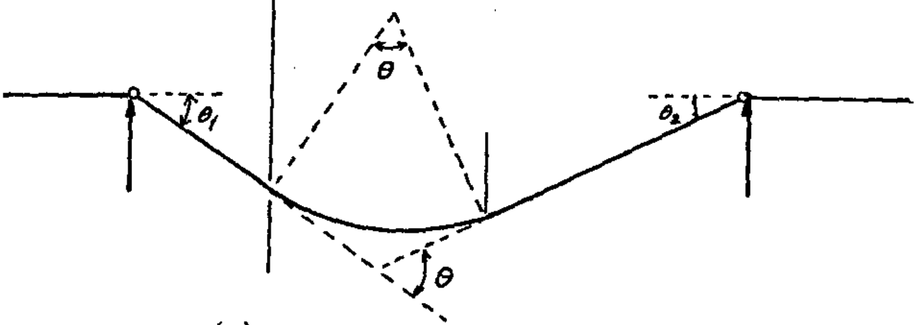
This fact applies also to the beam problems, but the different nature of the problem presents some difficulties. The coefficients used for converting displacements into stresses are no longer independent of the design variables. Obtaining stresses at nodal points gives rise to even more difficulties. But the following procedure makes the calculation simple and efficient and the result accurate.

If we are to find the maximum bending stress at point Q of Fig. 5-a, we may employ a pair of virtual loads applied at point P and R respectively as shown in the figure, provided that the beam segment P-R has a constant section. The magnitude of the virtual loads, $\frac{ED}{2b}$, is obtainable from the slope-deflection equations of the beam segment. Fig. 5-b shows the deformed shape of the beam due to the virtual loads after introducing hinges at the supports. Since the beam is analysed by the force method and the bending moments at the nodes due to the virtual loads have to be calculated, it is necessary to calculate θ, and then θ_1 and θ_2. If the beam segment P-R has a constant section θ can be calculated as follow.

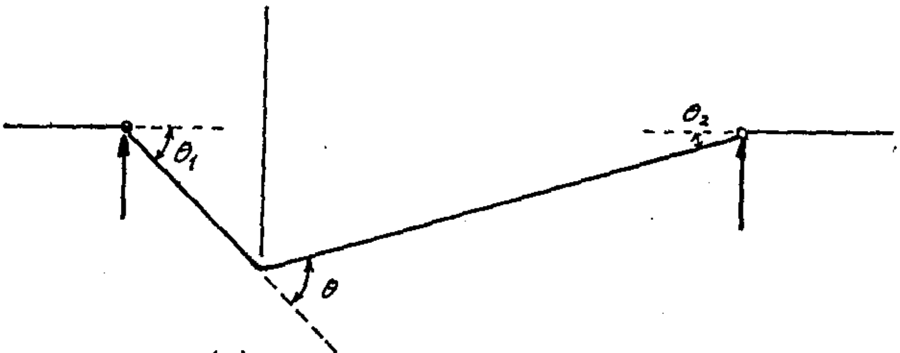
$$\theta = \frac{\frac{ED}{2b} \cdot b}{EI} = \frac{6}{BD^2}$$



(a) Virtual Loads to express the maximum bending stress at Q in terms of virtual work



(b) Deformation of the simply supported beam due to the Virtual Loads



(c) Deformation of the beam after R approached to P

Fig. 5 Virtual Loads to express the Stress Constraint at point P in terms of virtual work.

where

B ; breadth of the section,

D ; depth of the section,

E ; elastic modulus of the segment.

Let point R approach infinitesimally to point P. Then Q approaches to P and the value of b approaches to zero, but the value of θ dose not change and remains as a finite value. Fig. 5-c shows the deformed shape when point R has approached to point P, and in this case the requirement of the segment P-R having a constant section is by all means met. The slope deflections θ_1 and θ_2 can be calculated from θ easily.

Let P be at node j and $M_t(x)^{(j)}$ be the virtual bending moments so calculated and distributed over element t. Then the maximum bending stress at node j can be found by

$$\sigma_j = \sum_{t=1}^{m-1} \int_0^{L_t} \frac{12\, M_t(x)\, M_t(x)^{(j)}}{E_t\, B_t\, d_t(x)^3}\, dx$$

$$= \sum_{t=1}^{m} \frac{b_{tj}}{d_t^{\,3}} \qquad \dots\dots\dots\dots\dots\dots \quad (5.26)$$

$$b_{tj} = \frac{2L_{t-1}}{E_{t-1}B_{t-1}}\left[\frac{16 f_{m\,t-1}^{(j)}}{(1+\frac{1}{r_{t-1}})^3} + f_t^{(j)}\right] + \frac{2L_t}{E_t B_t}\left[f_t^{(j)} + \frac{16 f_{mt}^{(j)}}{(1+r_t)^3}\right] \cdots (5.27)$$

in the same way as for the deflection constraint.

The derivation of the derivatives, however, should be made differently. The magnitude of the virtual loads depends on the depth at node j and so do $f_t^{(j)}$, $f_{mt}^{(j)}$ and $f_{m\,t-1}^{(j)}$. Therefore the derivatives should be expressed as follows.

$$\frac{\partial \sigma_j}{\partial d_t} = -3 \, \frac{d_{tj}'}{d_t^4} + \delta_{tj} \frac{\sigma_j}{d_j} \quad \cdots\cdots \quad (5.28)$$

$$d_{tj}' = \frac{2L_{t-1}}{E_{t-1}B_{t-1}}\left[\frac{32 f_{m\,t-1}^{(j)}}{(1+\frac{1}{r_{t-1}})^4} + f_t^{(j)}\right] + \frac{2L_t}{E_t B_t}\left[f_t^{(j)} + \frac{32 f_{mt}^{(j)}}{(1+r_t)^4}\right] \cdot (5.29)$$

where $\delta_{tj} = 1$ if $t = j$, 0 otherwise.

Also for the I-section beams we use the same virtual loads, but the bending stress of the flanges at node j, σ_j , and its derivatives with respect to the design variables read as follows.

$$\sigma_j = \sum_{t=1}^{m} \frac{b_{tj}}{d_t^2} \quad \cdots\cdots \quad (5.30)$$

$$b_{tj} = \frac{L_{t-1}}{3E_{t-1}A_{ft-1}}\left[\frac{8 f_{m\,t-1}^{(j)}}{(1+\frac{1}{r_{t-1}})^2} + f_t^{(j)}\right] + \frac{L_t}{3E_t A_{ft}}\left[f_t^{(j)} + \frac{8 f_{mt}^{(j)}}{(1+r_t)^2}\right] \cdot (5.31)$$

$$\frac{\partial \sigma_j}{\partial d_t} = -2 \, \frac{d_{tj}'}{d_t^3} + \delta_{tj} \frac{\sigma_j}{d_j} \quad \cdots\cdots \quad (5.32)$$

$$d_{tj}' = \frac{L_{t-1}}{3E_{t-1}A_{ft-1}}\left[\frac{16 f_{m\,t-1}^{(j)}}{(1+\frac{1}{r_{t-1}})^3} + f_t^{(j)}\right] + \frac{L_t}{3E_t A_{ft}}\left[f_t^{(j)} + \frac{16 f_{mt}^{(j)}}{(1+r_t)^3}\right] \cdot (5.33)$$

5.2 Design Variable Linking by Ratio

Design variable linking generally stands for assigning one value to a number of design variables, and thus results in a reduction of the number of independent design variables, improved efficiency in designing and ease of construction. This ordinary way of design variable linking was employed for the truss problems as shown in Equ. (5.1) but will not show such usefulness for the beam problems concerned. It diminishes the advantage of using tapered elements.

The way of design variable linking for the beam problems is illustrated in Fig. 6. The depths at node 1, 4, 6, 9, are the design variables 'a', 'b', 'c', 'd', respectively. But the depths at node 2 and 3 are decided by interpolating the values of design variable 'a' and 'b', and so on. The beam elements are divided into a number of groups each of which contains several consecutive elements. The element belonging to a group have the same rate of tapering. A transformation matrix, $\{T_{ti}\}$, is now defined and from the matrix and the values of the design variables we obthain the depths at nodes as shown in Equ. (5.34).

$$\{d_t\} = \{T_{ti}\}\{D_i\} \quad \cdots\cdots\cdots\cdots \quad (5.34)$$

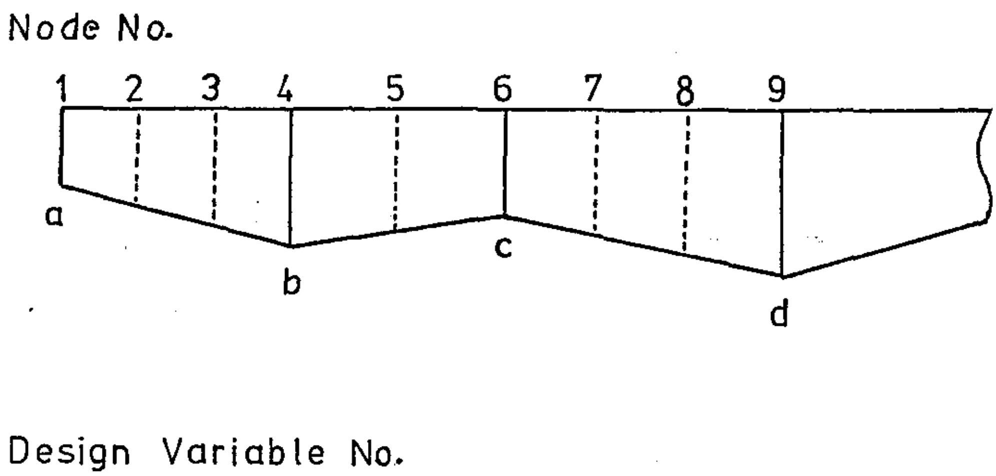


Fig. 6 Node No. and Design Variable No.

Let the objective function, Equ. (5.5), have the form of Equ. (5.35) assuming that β is unity,

$$F = C + \sum_{t=1}^{m} \alpha_t d_t \quad \ldots \ldots \quad (5.35)$$

and α_t have the form as in Equ. (5.6). Then the objective function, representing the weight of the beam, can be expressed in terms of design variables as follows.

$$\begin{aligned} F &= C + \sum_{t=1}^{m} \alpha_t \sum_{i=1}^{\ell} T_{ti} D_i \\ &= C + \sum_{i=1}^{\ell} \left[\sum_{t=1}^{m} \alpha_t T_{ti} \right] D_i \\ &= C + \sum_{i=1}^{\ell} h_i D_i \quad \ldots \ldots \quad (5.36) \end{aligned}$$

where ℓ is the number of design variables. Let h_i be multiplied by some value and β take some value possibly less than unity. Then the objective function expressed by

$$F \doteq C + \sum_{i=1}^{\ell} h_i D_i^{\beta} \quad \ldots \ldots \quad (5.37)$$

may represent the cost of the beam. The meaningfulness of this cost objective function has not been investigated in this work.

The derivatives of constraints with respect to design variables are derived from Equ. (5.17) and Equ. (5.28) using the transformation matrix.

$$\frac{\partial u_k}{\partial D_i} = \sum_{t=1}^{m} \frac{\partial u_k}{\partial d_t} \cdot \frac{\partial d_t}{\partial D_i}$$

$$= \sum_{t=1}^{m} T_{ti} \frac{-3\, c_{tk}'}{d_t^{\,4}}$$

$$= -\, 3 \left[\sum_{t=1}^{m} T_{ti} \cdot \frac{c_{tk}'}{r_{ti}^{\,4}} \right] \frac{1}{D_i^{\,4}}$$

$$= -\, 3 \,\frac{c_{ik}}{D_i^{\,4}} \qquad \ldots\ldots\ldots\ldots\ldots \quad (5.38)$$

$$c_{ik} = \sum_{t=1}^{m} T_{ti} \frac{c_{tk}'}{r_{ti}^{\,4}} \qquad \ldots\ldots\ldots\ldots\ldots \quad (5.39)$$

$$\frac{\partial \sigma_j}{\partial D_i} = \sum_{t=1}^{m} \frac{\partial \sigma_j}{\partial d_t} \cdot \frac{\partial d_t}{\partial D_i}$$

$$= \sum_{t=1}^{m} T_{ti} \left[\frac{-3\, d_{tj}'}{d_t^{\,4}} + \delta_{tj} \frac{6\, M_j}{B_j\, d_t^{\,3}} \right]$$

$$= -\, 3 \,\frac{d_{ij}}{D_i^{\,4}} + \delta_{K_j} \frac{6\, T_{ii} M_j}{B_j\, r_{ji}^{\,3}} \cdot \frac{1}{D_j^{\,3}} \qquad \ldots\ldots \quad (5.40)$$

$$d_{ij} = \sum_{t=1}^{m} \frac{T_{ti}\, d_{tj}'}{r_{ti}^{\,4}} \qquad \ldots\ldots\ldots\ldots\ldots \quad (5.41)$$

where

$$r_{ti} = \frac{d_t}{D_i}$$

$$K_i = \{ t : T_{ti} \neq 0 \}$$

$$\delta_{K_ij} = 1, \text{ if } j \in K_i ,$$

$$\quad 0, \text{ otherwise.}$$

The ratio r_{ti} is assumed constant throughout a redesign iteration. This assumption, together with those in the preceding section leads to constant c_{ik} and d_{ij} throughout a redesign iteration.
The derivatives of constraints for I-section beams are derived from Equ. (5.21) and Equ. (5.32) in the same way.

$$\frac{\partial u_k}{\partial D_i} = -2\,\frac{c_{ik}}{D_i^{3}} \quad \cdots\cdots\cdots\cdots\cdots \quad (5.42)$$

$$c_{ik} = \sum_{t=1}^{m} \frac{T_{ti}\, c_{tk}'}{r_{ti}^{3}} \quad \cdots\cdots\cdots \quad (5.43)$$

$$\frac{\partial \sigma_j}{\partial D_i} = -2\,\frac{d_{ij}}{D_i^{3}} + \delta_{ij}\,\frac{T_{ti} M_j}{A_{fj}\, r_{ji}^{2}}\,\frac{1}{D_j^{2}} \quad \cdots \quad (5.44)$$

$$d_{ij} = \sum_{t=1}^{m} \frac{T_{ti}\, d_{tj}'}{r_{ti}^{3}} \quad \cdots\cdots\cdots\cdots\cdots \quad (5.45)$$

Most problems treated in this thesis are subject to such design variable linking as is illustrated in Fig. 7 rather than in Fig. 6. Each span has three groups of elements. The elements of the second group are made to have equal depths by further linking the two design variables governing their depths. The foregoing equations are still valid even after this further design variable linking provided that the two corresponding columns of the transformation matrix are merged into one column to correspond with the new design variable.

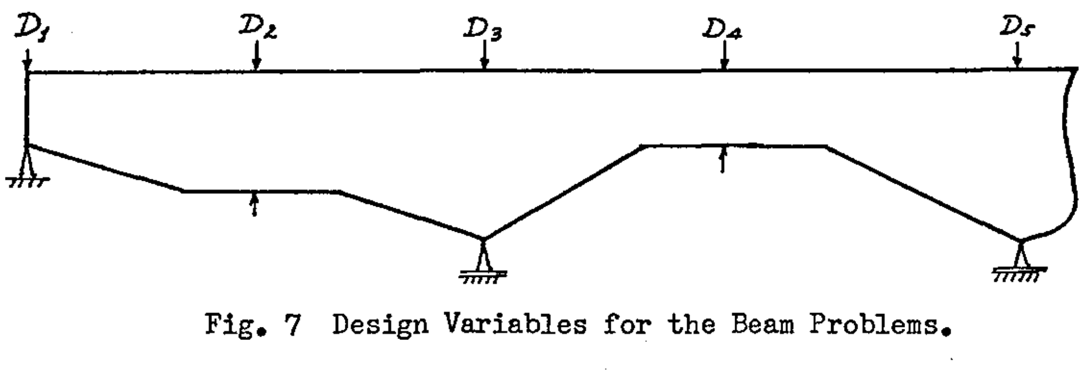


Fig. 7 Design Variables for the Beam Problems.

5.3 The Optimality Criteria

The optimality criteria, upon which the proposed method is based, are to be derived from the Kuhn-Tucker necessary conditions expressed as:-

$$\frac{\partial W}{\partial A_i} + \sum_k \lambda_k \frac{\partial u_k}{\partial A_i} + \sum_j \lambda_{n+j} \frac{\partial \sigma_j}{\partial A_i} - \gamma_i = 0 \quad \cdots (5.46)$$

$$\gamma_i \; (\underline{A_i} - A_i) = 0 \quad \cdots\cdots\cdots\cdots (5.47)$$

for the truss problems,

$$\frac{\partial F}{\partial D_i} + \sum_k \lambda_k \frac{\partial u_k}{\partial D_i} + \sum_j \lambda_{n+j} \frac{\partial \sigma_j}{\partial D_i} - \gamma_i = 0 \quad \cdots (5.48)$$

$$\gamma_i \; (\underline{D_i} - D_i) = 0 \quad \cdots\cdots\cdots\cdots (5.49)$$

for the beam problems, and

$$\lambda_k \; (u_k - \bar{u}_k) = 0 \quad \cdots\cdots\cdots\cdots (5.50)$$

$$\lambda_{n+j} \; (\sigma_j - \bar{\sigma}_j) = 0 \quad \cdots\cdots\cdots\cdots (5.51)$$

$$\lambda_k , \lambda_{n+j} , \gamma_i \geq 0 \quad \cdots\cdots\cdots\cdots (5.52)$$

for both problems, where λ_k , λ_{n+j} and γ_i are Lagrange multipliers.

Prior to formulating the optimality criteria the design variables and constraints are classified by their roles during the redesign process. The design variables are divided into two groups. Group 1 contains those design variables whose values are greater than their specified minimum values. These are the 'active' variables and their associated Lagrange multipliers, γ_i , will be zero. Group 2 contains the remaining design variables, the 'passive' variables, whose values are set to the specified minima and whose associated Lagrange multipliers, γ_i , may be greater than or equal to zero. The constraints are also divided into two groups, active and inactive ones. The Lagrange multipliers, λ_k or λ_{n+j} associated with active constraints may be greater than or equal to zero, and those associated with inactive constraints will be zero. Some index sets concerning these classifications are made as follows to be used in the forthcoming equations.

$$G1 = \{ i \; : \; A_i > \underline{A_i} \quad \text{or} \quad D_i > \underline{D_i} \},$$

set of group 1 design variables;

$$G2 = \{ i \; : \; A_i = \underline{A_i} \quad \text{or} \quad D_i = \underline{D_i} \},$$

set of group 2 design variables;

$$U = \{ k \; : \; u_k = \bar{u}_k \},$$

set of active deflection constraints;

$$S = \{ j \; : \; \sigma_j = \bar{\sigma}_j \}$$

set of active stress constraints.

The optimality criteria for the truss problems can be derived directly from the Kuhn-Tucker necessary conditions, Equ. (5.46), and Equ. (5.1), (5.14) and (5.25) as follows.

$$\frac{1}{A_i^2 h_i}\left[\sum_{k\in U}\lambda_k c_{ik} + \sum_{j\in S}\lambda_{n+j} d_{ij}\right] + \frac{\gamma_i}{h_i} = 1 \quad \cdots (5.53)$$

Equ. (5.53) excludes those terms associated with inactive constraints without loss of generality because the associated Lagrange multipliers vanish. Since the last term of Equ. (5.53) is nonnegative and, if i belongs to Group 1, can be excluded by the same argument, the optimality criteria, completed by the rest of the Kuhn-Tucker necessary conditions, can better be expressed as follows.

$$\frac{1}{A_i^2 h_i}\left[\sum_{k\in U}\lambda_k c_{ik} + \sum_{j\in S}\lambda_{n+j} d_{ij}\right] = 1 \quad , i \in G1 \quad \cdot\cdot (5.54)$$

$$\frac{1}{A_i^2 h_i}\left[\sum_{k\in U}\lambda_k c_{ik} + \sum_{j\in S}\lambda_{n+j} d_{ij}\right] \leq 1 \quad , i \in G2 \quad \cdot\cdot (5.55)$$

$$\left.\begin{array}{l} \lambda_k\,(\,u_k - \bar{u}_k\,) = 0 \\ \lambda_{n+j}\,(\,\sigma_j - \bar{\sigma}_j\,) = 0 \\ \gamma_i\,(\,\underline{A}_i - A_i\,) = 0 \\ \lambda_k,\ \lambda_{n+j},\ \gamma_i \geq 0 \end{array}\right\} \quad \cdots\cdots\cdots (5.56)$$

Physically, any term of Equ. (5.54)

$$\lambda_k \frac{c_{ik}}{A_i^2 h_i} \qquad \text{or} \qquad \lambda_{n+j}\frac{d_{ij}}{A_i^2 h_i}$$

represents the sum of the product of some non-negative coefficient by the virtual strain energy density of each member associated with design variable A_i under the virtual loads concerned with the active constraint k or j . If there is no design variable linking, it simply represents the virtual strain energy density of the member multiplied by the non-negative coefficient, the Lagrange multiplier. It can therefore be stated that Equ. (5.54) for each design variable is a linear combination with non-negative coefficients of virtual strain energy densities, each of which is concerned with an active constraint, and that at the optimum the linear combination with the same non-negative coefficients for any of the design variables, is equal to unity for Group 1 design variables and less than or equal to unity for Group 2.

The optimality criteria for the beam problems come from the Kuhn-Tucker necessary conditions, Equ. (5.48), and Equ. (5.37), (5.38), (5.40), (5.42) and (5.44).

$$\frac{1}{D_i^{(3+\beta)}} \cdot \frac{3}{\beta h_i} \cdot \left[\sum_{k \in U} \lambda_k c_{ik} + \sum_{j \in S} \lambda_{n+j} d_{ij} \right] - \frac{1}{D_i^{(2+\beta)}} \frac{1}{\beta h_i} \sum_{j \in S} \delta_{K_i j} \frac{6 \lambda_{n+j} T_{ji} M_j}{B_j r_{ji}^3} = 1 , \quad i \in G1 \qquad \ldots\ldots\ldots\ldots\ldots (5.57)$$

$$\frac{1}{D_i^{(3+\beta)}} \cdot \frac{3}{\beta h_i} \cdot \left[\sum_{k \in U} \lambda_k c_{ik} + \sum_{j \in S} \lambda_{n+j} d_{ij} \right] - \frac{1}{D_i^{(2+\beta)}} \frac{1}{\beta h_i} \sum_{j \in S} \delta_{K_i j} \frac{6 \lambda_{n+j} T_{ji} M_j}{B_j r_{ji}^3} \leqslant 1 , \quad i \in G2 \qquad \ldots\ldots\ldots\ldots\ldots (5.58)$$

for the rectangular section beams,

$$\frac{1}{D_i^{(2+\beta)}}\cdot\frac{2}{\beta h_i}\left[\sum_{k\in U}\lambda_k c_{ik} + \sum_{j\in S}\lambda_{n+j} d_{ij}\right] - \frac{1}{D_i^{(1+\beta)}}\frac{1}{\beta h_i}\sum_{j\in S}\delta_{k_i j}\frac{\lambda_{n+j} T_{ji} M_j}{A_{ij}\, r_{ji}^2} = 1 \;, \quad i \in G1 \qquad \ldots\ldots\ldots\ldots\ldots \;(5.59)$$

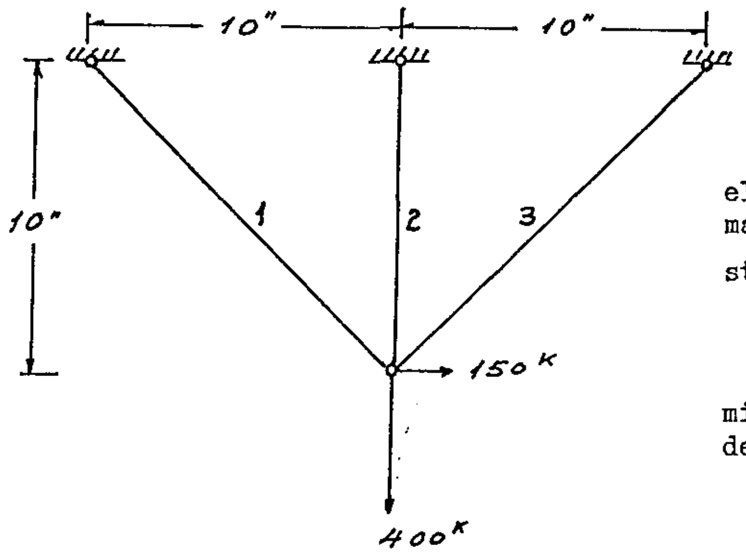
$$\frac{1}{D_i^{(2+\beta)}}\cdot\frac{2}{\beta h_i}\cdot\left[\sum_{k\in U}\lambda_k c_{ik} + \sum_{j\in S}\lambda_{n+j} d_{ij}\right] - \frac{1}{D_i^{(1+\beta)}}\frac{1}{\beta h_i}\sum_{j\in S}\delta_{k_i j}\frac{\lambda_{n+j} T_{ji} M_j}{A_{ij}\, r_{ji}^2} \leq 1 \;, \quad i \in G2 \qquad \ldots\ldots\ldots\ldots\ldots \;(5.60)$$

for the I-section beams, and for both beams completed by

$$\left.\begin{array}{lcl} \lambda_k \;(\, u_k - \bar{u}_k \,) & = & 0 \\ \lambda_{n+j} \;(\, \sigma_j - \bar{\sigma}_j \,) & = & 0 \\ \gamma_i \;(\, \underline{D}_i - D_i \,) & = & 0 \\ \lambda_k ,\; \lambda_{n+j} ,\; \gamma_i & \geq & 0 \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(5.61)$$

In the foregoing optimality criteria and constraint equations, u_k and σ_j represent only the magnitudes of the corresponding deflection and stress. This provision has been made in order to keep the Lagrange multiplier positive even though its associated constraint is a negative deflection or a negative stress (compressive stress of a bar or negative bending moment at a node), and in consequence such values as of a_{tk} , b_{tj} , c_{ik} , d_{ij} and M_j should be multiplied by -1 if the corresponding constraint, u_k or σ_j is negative.

Three-bar truss, as shown in Fig. 8, has been taken as an illustrative example. Table-1 shows the final design and its response quantities such as member stresses, stress gradients and Lagrange



Material Data

elastic modulus ; 10^4 Ksi
mass density ; 0.1 pci

stress limits
member 1 ; $\pm$ 50 Ksi
member 2 ; $\pm$ 55 Ksi
member 3 ; $\pm$ 10 Ksi

minimum size ; 0.1 in.^{2}
deflection limits ; none

Fig. 8 Three-Bar Truss Problem

Table-1 Final Design of 3-Bar Truss

Member (i)	Area (A_i)	Stress (σ_i)	d_{ij}/A_i^2 $j=1$	d_{ij}/A_i^2 $j=2$	γ_i	*	h_i
1	4.2527	50.0	11.50	0.3478	0	1.4143	1.4142
2	4.5326	55.0	0.2705	11.78	0	1.0003	1.0000
3	0.1000	5.0	-1.115	1.479	1.427	1.4141	1.4142
			λ_1 0.1205	λ_2 0.08215	$\lambda_3=0$		

* left hand side of Equ. (5.53) times h_i

multipliers, and that the optimality critera, Equ. (5.54)-(5.56), are all satisfied.

5.4 Redesign Algorithm

Having established the optimality criteria, it is now necessary to devise a redesign algorithm which will force the design to satisfy the optimality criteria. The key feature of the redesign process of this thesis is the use of the Newton-Raphson method to solve the system of nonlinear constraint equations for the Lagrange multipliers. This approach was first presented by Taig and Kerr [28] but no improvement has yet been reported in the literature. The method of this thesis achieved a number of improvements. These will be explained wherever appropriate.

At the outset of the redesign process, such values as of a_{tk}, b_{tj}, c_{ik}, d_{ij} and μ_j are determined from the initial design and used to find a new design hopefully satisfying the optimality criteria. While the design changes they also change, and in addition they can be evaluated only numerically by means of a structural reanalysis. Therefore, to find the optimum design, there is no alternative to using an iterative method. A new design can be found from Equ. (5.54), (5.57) or (5.59) assuming that a_{tk} , b_{tj} , c_{ik} , d_{ij} and μ_j remain unchanged until the redesign iteration finishes. Then the structure of the new design is analysed, new values of the coefficients are determined and the next redesign iteration starts. The resulting set of design values after a redesign iteration are compared with the set current at entry to the iteration and accepted when changes in the objective function or of individual variables are below an acceptable tolerance.

There are a group of values to be determined in each redesign iteration. They are the Lagrange multipliers and the only information not obtainable from the results of structural analysis. Therefore, the main task of a redesign iteration is that of determining the values of the Lagrange multipliers.

One pass through the redesign process is illustrated in Fig. 9 and is referred to as one iteration. Before entering the Newton-Raphson process it is necessary to find which constraints are active since we need not consider inactive constraints and then to calculate such values as of c_{ik} , d_{ij} , etc. for the active constraints and all design variables. The Newton-Raphson process starts by estimating the Lagrange multipliers and proceeds in an iterative way. The process may be interrupted by the appearance of negative Lagrange multipliers. Therefore it is sometimes unavoidable to discard some of the active constraints and get the Newton-Raphson process to start again. So far, all design variables are deemed to be active, i.e. in Group 1, but upon completion of the process some of them may be found below their minimum values. If this happens, the variables below their minima must be set to the minima and another round of the Newton-Raphson process starts including only the remaining active design variables. A detailed description of the redesign process follows.

5.4.1 Finding active constraints

In the first redesign iteration, the most critical constraint(s) is the only active constraint. Whilst the redesign proceeds iteration by iteration, however, the set of active constraints gradually expands by taking more constraints if they are more restrictive than any of those considered active in the preceding iteration. In other

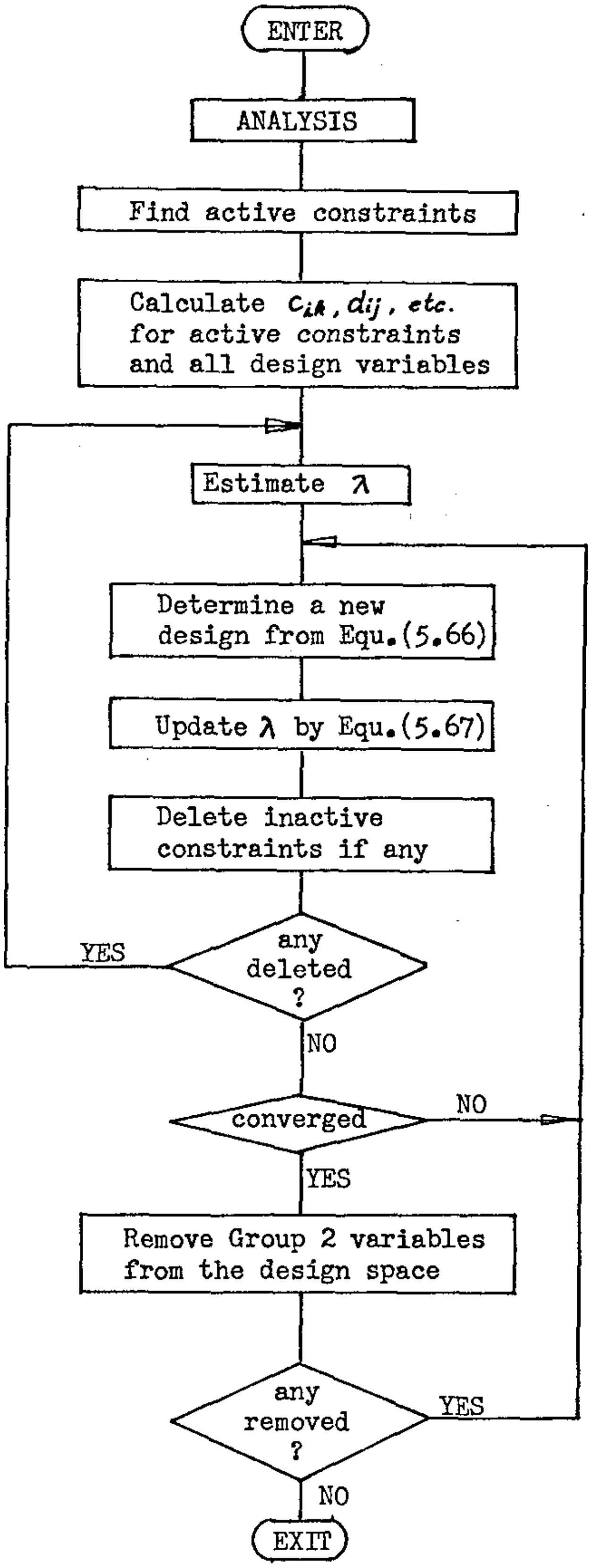


Fig. 9 Flow Diagram of the Redesign Process

words the number of active constraints grows up to a certain redesign iteration and thereafter the set of active constraints becomes fixed. This approach shows efficiency but has a fallacy due to the absence of some active constraints in the earlier redesign iterations. What the fallacy is, how to get rid of it and the advantage of this strategy will be discussed later.

5.4.2 Estimating Lagrange multipliers

The Newton-Raphson method is used to solve the active constraint equations in Equ. (5.56) and (5.61) for the Lagrange multipliers, λ's , associated with the deflection and stress constraints. The first task therefore is estimating initial values for λ's . These should be as accurate as possible: otherwise the Newton-Raphson procedure will be disturbed.

We first assume that the contribution of each term in Equ. (5.54) to the overall value is the same and makes unity altogether. Then each Lagrange multiplier is estimated in turn such that the associated constraint eqution is satisfied by the estimated value. Let n_a be the number of active constraints and λ_p be the Lagrange multiplier associated with the pth deflection constraint. We obtain the following equations from Equ. (5.54), (5.56) and (5.13).

$$\frac{1}{A_i^2 h_i} \lambda_p C_{ip} = \frac{1}{n_a} \qquad \cdots\cdots\cdots \text{(a)}$$

$$u_p = \sum_i \frac{C_{ip}}{A_i} = \bar{u}_p \qquad \cdots\cdots \text{(b)}$$

We now modify Equ. (b) such that

$$\sum_{+i} \frac{c_{ip}}{A_i} = \bar{u}_p - \sum_{-i} \frac{c_{ip}}{A_i} \quad . . . . \quad (c)$$

where $\sum_{+i}$ means summation over i for which $c_{ip} > 0$,

$\sum_{-i}$ means summation over i for which $c_{ip} \leqslant 0$.

Equ. (a) is introduced only into the left hand side of Equ. (c) to avoid a negative argument with a non-integer exponent. Then λ_p is obtainable from the new equation as shown in Equ. (5.62).

$$\lambda_p = \frac{1}{n_a} \left[\frac{\sum_{+i} (c_{ip} h_i)^{\frac{1}{2}}}{\bar{u}_p - \sum_{-i} c_{ip}/A_i} \right]^2 \quad . . . . \quad (5.62)$$

For the qth stress constraint, the associated Lagrange multiplier will be estimated as follows.

$$\lambda_q = \frac{1}{n_a} \left[\frac{\sum_{+i} (d_{iq} h_i)^{\frac{1}{2}}}{\bar{\sigma}_q - \sum_{-i} d_{iq}/A_i} \right]^2 \quad . . . . \quad (5.63)$$

We leave the beam problems for the time being. Equations and formulae applicable to the beam problems will appear at the end of this section.

5.4.3 Improving Larange multipliers by the Newton-Raphson method

The estimatied Lagrange multipliers are now introduced into the constraint equations and examined if they satisfy all the active constraints in the equality sense. The approximate constraint equations,

$$u_k - \bar{u}_k = \sum_i \frac{c_{ik}}{A_i} - \bar{u}_k = 0 \quad \cdots\cdots\cdots \quad (5.64)$$

$$\sigma_j - \bar{\sigma}_j = \sum_i \frac{d_{ij}}{A_i} - \bar{\sigma}_j = 0 \quad \cdots\cdots\cdots \quad (5.65)$$

are not expressed in terms of the Lagrange multipliers but the design variables. Therefore we first determine the design variables from Equ. (5.54) with the Lagrange multipliers, and introduce them into the above constraint equations.

Equ. (5.54) is re-formed to determine the design variables as

$$A_i = \left[\sum_{k\in U} \frac{\lambda_k c_{ik}}{h_i} + \sum_{j\in S} \frac{\lambda_{n+j}\, d_{ij}}{h_i}\right]^{\frac{1}{2}} \quad \cdots\cdots \quad (5.66)$$

and the value of A_i so determined is used to evaluate Equ. (5.64) and (5.65). The values of A_i given by Equ. (5.66) are kept for the time being even if they are less than the minimum value. For some design variable, however, the value of the expression in the bracket may be negative. If this happens the design variable is given the minimum value, excluded when updating the Lagrange multipliers and re-calculated in the next iteration.

The values of A_i determined above do not really make a design at this point. Their role is in fact a set of intermediate parameters which make it possible to evaluate Equ. (5.64) and (5.65) and determine the Newton direction with the current values of the Lagrange multipliers. However, if all the forthcoming requirements are met the values will make the design of the current redesign iteration.

The design values obtained above may or may not satisfy the constraint equations, Equ. (5.64)-(5.65). If not, the Lagrange multipliers are updated aiming at a better satisfaction to the constraint equations by the improved design values calculated from the new Lagrange multipliers. This task is done by the Newton-Raphson method and the following relation is used.

$$\underset{\sim}{\lambda}^{(1)} = \underset{\sim}{\lambda}^{(0)} - \left[\begin{array}{c:c} X_{11} & X_{12} \\ \hdashline X_{21} & X_{22} \end{array}\right]^{-1} \left[\begin{array}{c} u_k(\underset{\sim}{\lambda}^{(0)}) - \bar{u}_k \\ \hdashline \sigma_j(\underset{\sim}{\lambda}^{(0)}) - \bar{\sigma}_j \end{array}\right] \cdots \quad (5.67)$$

$$\left.\begin{aligned} X_{11} &= \left\{\frac{\partial u_k}{\partial \lambda_p}\right\} = \left\{-\frac{1}{2}\sum_{i\in G1}\frac{c_{ik}c_{ip}}{A_i^3 h_i}\right\}_{\underset{\sim}{\lambda}=\underset{\sim}{\lambda}^{(0)}} \\ X_{22} &= \left\{\frac{\partial \sigma_j}{\partial \lambda_{n+g}}\right\} = \left\{-\frac{1}{2}\sum_{i\in G1}\frac{d_{ij}d_{ig}}{A_i^3 h_i}\right\}_{\underset{\sim}{\lambda}=\underset{\sim}{\lambda}^{(0)}} \\ X_{12} &= \left\{\frac{\partial u_k}{\partial \lambda_{n+g}}\right\} = \left\{-\frac{1}{2}\sum_{i\in G1}\frac{c_{ik}d_{ig}}{A_i^3 h_i}\right\}_{\underset{\sim}{\lambda}=\underset{\sim}{\lambda}^{(0)}} \\ X_{21} &= X_{12}^T \end{aligned}\right\} \cdots\cdots \quad (5.68)$$

$$\underset{\sim}{\lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_{m+n}\} \cdots\cdots\cdots \quad (5.69)$$

$$p = 1, 2, \cdots, n$$

$$g = 1, 2, \cdots, m$$

In evaluating each entry of the Jacobian matrix, the summation is only over the Group 1 design variables since the rest, Group 2 design variables, have been given their minimum values, and therefore they are not sensitive to the Lagrange multipliers.

If the updated set of Lagrange multipliers, $\underset{\sim}{\lambda}^{(1)}$, yield design values satisfying the constraint equations, Equ. (5.64)-(5.65), the Newton-Raphson process finishes. But probably there remain some design variables below their minimum values. If a particular constraint is found to be inactive we should delete the corresponding rows and columns from the matrices in Equ. (5.67) and (5.69) and set the associate Lagrange multiplier to zero. During the Newton-Raphson process some λ's may turn negative. If this happens we should either consider the corresponding constraints inactive or do some remedial measures. Therefore it is more than desirable to know the set of active constraints in advance since it reduces the order of Jacobian matrix remarkably and will make the Newton-Raphson process more stable and efficient.

5.4.4 Deleting constraints relating to negative Lagrange multipliers

The appearance of negative Lagrange multipliers during the Newton-Raphson process creates difficulties since the multipliers are not allowed to have negative values. A negative Lagrange multiplier indicates that its associated constraint selected as active is not active and thus should be deleted from the set of active constraints.

It was found that as the Newton-Raphson process proceeded, successive values of each multiplier did not smoothly converge to final values but showed a good deal of "noise" over the trend values. This led to the difficulty of clearly distinguishing between those multipliers which were definitely converging to a negative value, and therefore were to be eliminated as relating to inactive constraints, and those converging to small positive values. Fig 10 illustrates the two possible sequences. The solution process converged rapidly once the correct set of active constraints were identified, but premature elimination of an ultimately active constraint on the first occasion that the associated multiplier went negative caused instability and "looping" in which a constraint continued to flip between active to inactive states. Damping of the Newton-Raphson process did not solve this problem but the successful method finally adopted was simply to allow any multiplier which went negative, one more chance before elimination. Its value was set to zero for the purpose of determining the new design from Equ. (5.66) and a new value of the multiplier calculated from Equ. (5.67). A multiplier which went negative twice in successive iterations was deemed to be associated with an inactive constraint which was then eliminated.

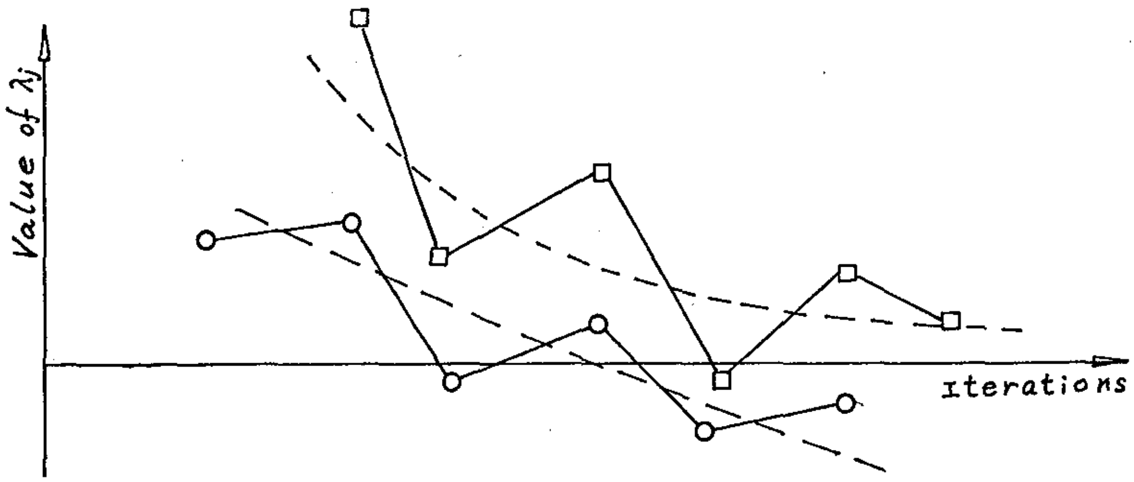


Fig. 10 Progress of Lagrange multipliers during Newton-Raphson iterations.

If more than one multiplier went negative, all of them were set to zero and the Newton-Raphson process continued. If there was at least one multiplier which came back positive, the Newton-Raphson method was allowed to proceed. Table-2 shows the history of the process when Example-1 of Ref. 14 was solved. The example was the same as Ex.B-5 appearing later, except that its elements were not tapered and stress constraints were treated as side constraints. At the second step of Table-2 two multipliers went negative and were set to zero. One of them came back positive at the next step and as the Newton-Raphson process proceeded further the rest also came back positive. Eventually the process converged as shown and the final values satisfied all the constraints exactly. An explanation of the above is given in Fig. 11, two-dimensional space spanned by λ_2 and λ_3, although the behaviour of the multipliers are not clearly known. Heavily overestimated λ_2 at P_1 could have caused the negative λ_2 at P_2 but P_2 was made to move to P_3 by setting λ_2 to zero and thereafter the process converged to the true solution.

Table-3 shows another case where the multiplier was allowed to stay even though it turned negative successively. λ_1 was always negative and therefore set to zero at every step. The process did converge but the final values did not satisfy any of the active constraints. It appears that the process converged to S' in Fig. 12 which satisfied neither of the constraints, and that the true solution is S at which $\mathcal{U}_2(\underset{\sim}{\lambda})=0$ and $\lambda_1=0$. Therefore it is a reasonable measure to deem a constraint inactive when its associated multiplier turns negative twice in succession. Another adverse situation happened as shown in Table-4. Successively setting the negative multipliers to zero gave rise to divergence. Therefore if any multiplier set to zero causes

Table-2 Iteration History of Newton-Raphson Process
- successful case

Iter.	Values of Lagrange Multipliers			
	λ_1	λ_2	λ_3	λ_4
1	9,988	12,690	7,338	8,166
2	0	0	3,658	5,955
3	0	3,920	4,906	6,720
4	135	6,163	6,422	6,879
5	752	7,416	7,637	6,998
6	1,707	7,069	8,294	6,880
7	2,243	6,611	8,647	6,775
8	2,308	6,556	8,705	6,770
9	2,309	6,556	8,706	6,770

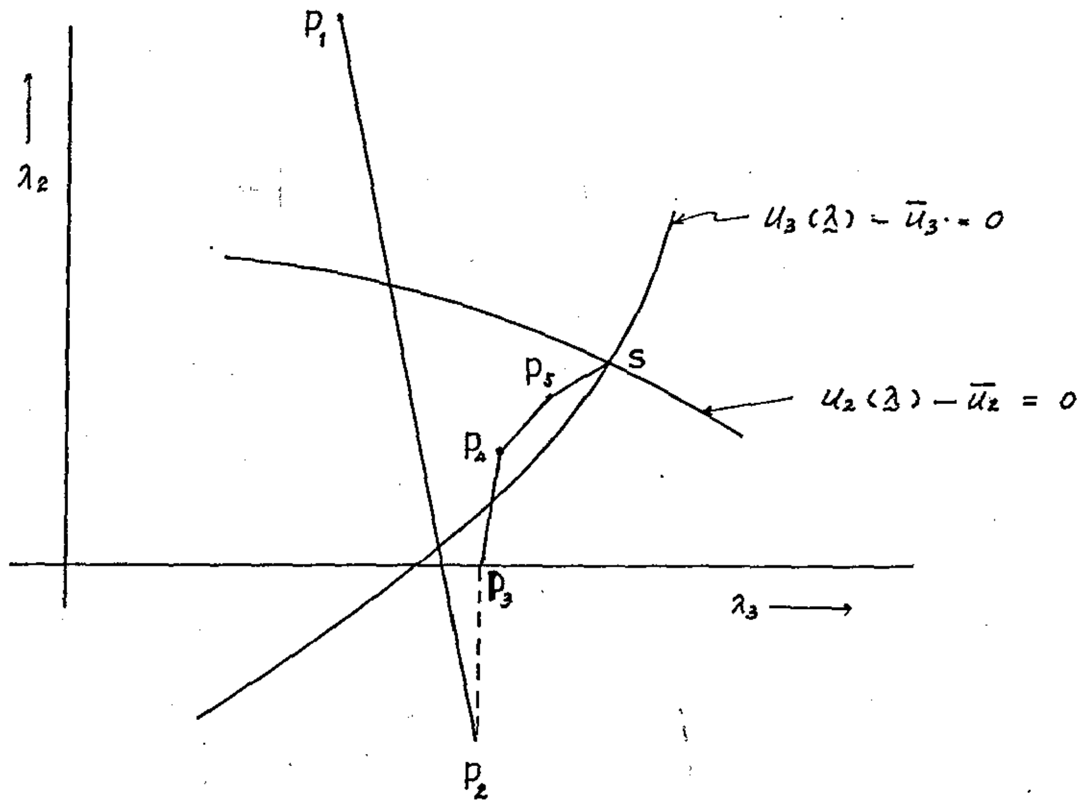


Fig. 11 Newton-Raphson process
- successful case

Table-3 Iteration History of Newton-Raphson Process
- unsuccessful case

Iter.	Values of Lagrange Multipliers			
	λ_1	λ_2	λ_3	λ_4
1	2,309	6,556	8,706	6,770
2	0	11,590	5,964	5,203
3	0	11,530	6,050	5,433
4	0	11,600	6,066	5,431
5	0	11,590	6,063	5,432
6	0	11,590	6,063	5,432

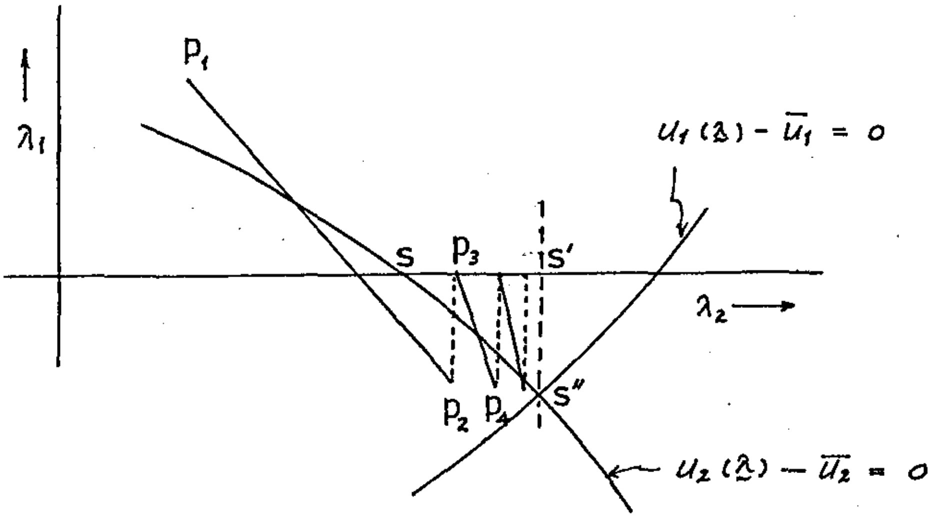


Fig. 12 Newton-Raphson process
- unsuccessful case

Table-4 Iteration History of Newton-Raphson Process
- unsuccessful case

Iter	of Lagrange multipliers				
	λ_1	λ_2	λ_3	λ_4	λ_5
1	1,885	7,073	8,496	5,982	1,011
2	3,723	1,572	9,752	21,020	0
3	12,910	0	0	21,800	0
4	0	850,300	82,180	0	0
5	0	0	0	25,920	$2\text{x}10^{6}$
6	0	$1\text{x}10^{10}$	0	$2\text{x}10^{8}$	0
7	0	0	$4\text{x}10^{14}$	$1\text{x}10^{14}$	$4\text{x}10^{12}$
8	0	$2\text{x}10^{23}$	0	0	0
9	0	0	$3\text{x}10^{37}$	$9\text{x}10^{36}$	0
10	0	$2\text{x}10^{68}$	0	0	0

other multipliers to turn negative, as λ_5 of Table-4, it should be deemed to be associated with an inactive constraint which has to be eliminated.

The strategies of finding active constraints and deleting inactive constraints explained above achieved substantial improvements on Taig's method [28]. Taig's method deletes inactive constraints one by one whenever negative multipliers appear. If a number of multipliers turn negative the method picks up one of them according to their magnitudes and deletes its associated constraint. This method therefore could delete a wrong constraint and require more Newton-Raphson iterations until the set of active constraints is fixed. In addition, this method considers all the constraints active at the outset of each redesign iteration. This costs many Newton-Raphson iterations with high order Jacobian matrices in every redesign iteration to get the set of active constraints fixed.

For the purpose of assessing the improvements a measure of efficiency was taken as follows.

$$\mathcal{M} = \sum_i N_i^2 * I_i$$

where N_i : order of Jacobian matrix,

I_i : number of Newton-Raphson iterations where Jacobian matrices of order Ni were solved.

When the beam example taken in this section was solved by Taig's method and the method of this thesis, the values of $\mathcal{M}$ were 6,225 and 1,155 respectively. For this example, the method was five times as efficient as Taig's.

5.4.5 Use of stress ratio

However, the strategy of finding active constraints has a fallacy as mentioned in Section 5.4.1. If a member stress σ_x is not included in the set of active constraints the member size A_x when calculated from Equ. (5.66), may be underestimated because the predominant term containing d_{xx} is absent, and if σ_x is in fact active this underestimation makes the design biased against the optimum design.

Although we show later that the concept of fully-stressed-designs can lead to non-optima, it was felt that for members such as those just described, the concept might lead towards more realistic and unbiased designs. We therefore introduced an alternative redesign method for this group of members.

A second value for the member size A_x was calculated from the member stress σ_x and its permitted value $\bar{\sigma}_x$ as:-

$$A_x^{s} = A_x^{(0)} \frac{\sigma_x}{\bar{\sigma}_x}$$

i.e. making the member fully stressed.

If A_x^s was larger than the value from Equ. (5.66), then A_x^s was used for the next design stage and the design variable removed from the active set, Group 1. The new set was called Group 3.

In summary, if A_i^* is the value given by Equ. (5.66) and $\underline{A_i}$ is the minimum size, then the new value A_i is given as below.

$A_i = A_i^*$; all active variables - Group 1

$A_i = \underline{A_i}$; minimum size - Group 2

$A_i = A_i^s$; stress ratio size - Group 3

This process proved very successful in avoiding biased designs in the initial stages. Variables assigned to Group 3 went to Group 1 or Group 2 before the design process terminated.

Table-5 shows active constraints and Group No.'s of the design variables in each iteration when the 3-bar truss was designed by the method of this thesis. In the first iteration the stress of member 1 was the only active constraint and design variable 2 and 3 were of Group 3. In iteration 2 design variable 2 went to Group 1 while the associated member stress became active. Design variable 3 stayed in Group 3 up to iteration 5, but eventually went to Group 2 in iteration 6.

5.4.6 Changing active into passive design variables.

As defined in Section 5.3, an active design variable is one contained in Group 1 and has a greater value than its minimum value. A passive design variable may be contained either in Group 2 or in Group 3. The variables contained in Group 3 are passive in nature and treated in the same way as those of Group 2 during the redesign process.

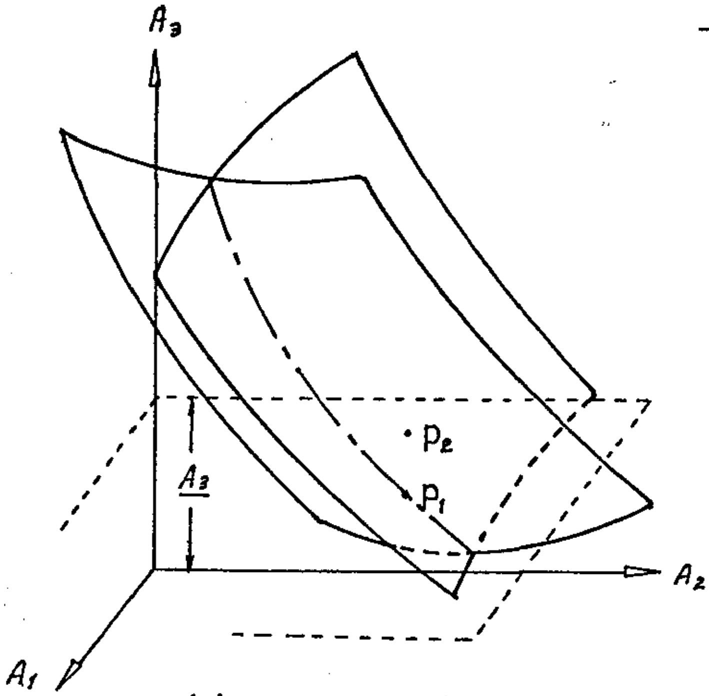
Table-5 Active Constraints and Group No.'s of 3-Bar Truss.

Iteration No.	Active Constraints		Group No. of Design Variables		
			1	2	3
1	1		1	3	3
2	1	2	1	1	3
3	1	2	1	1	3
4	1	2	1	1	3
5	1	2	1	1	3
6	1	2	1	1	2

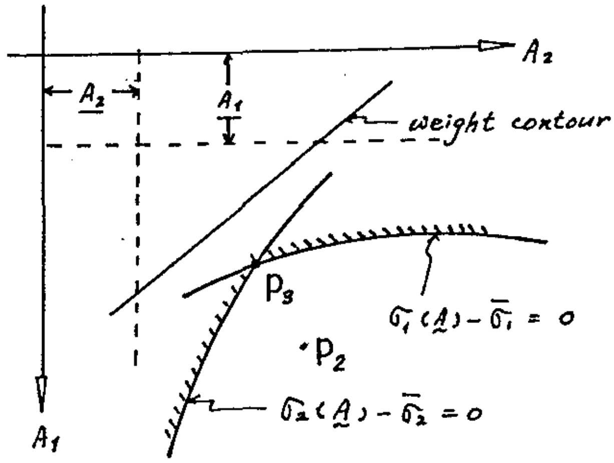
Upon completion of a round of the Newton-Raphson process those design variables, given their values by Equ. (5.66) and thus considered active so far, are not necessarily above their minimum values or the values determined by the stress ratios where these apply. If all the active variables have values big enough to stand in Group 1 the redesign iteration finishes, otherwise the redesign iteration requires another round of the Newton-Raphson process. Before the new round starts, some of the active variables below their minimum values or the values by stress ratios are removed to Group 2 or Group 3 and made passive. Then the new Newton-Raphson process is carried out in the subspace, of the original design space, spanned by the active variable coordinates. Since the passive variables are given fixed values, they no longer have a part in the redesign process.

Fig. 13 shows how the design of 3-bar truss behaved in the successive rounds of the Newton-Raphson process. The first round in redesign iteration 6 found design P_1 in the 3-dimensional design space, in which the area of member 3 was below the minimum. Therefore a new design, represented by P_2 in both figures, was generated by giving the variable of member 3 its minimum. Since the design P_2 was not the optimum, the next round was carried out in the 2-dimensional design space and found the optimum design.

This way of treating passive variables is another important improvement on Taig's method. In Taig's method, any variable for which Equ. (5.66) defines a value below the minimum is set to the minimum immediately. But this approach sometimes presented serious numerical difficulties when beam problems were solved. The beam problems had



(a) 3-dimensional space



(b) 2-dimensional space

Fig. 13 Design Space Map of 3-Bar Truss Problem

Table-6 Two Designs of Example-1 of Ref. 14.

Element No.	Design Values	
	Design 1	Design 2
.		
10	52.30	55.09
11	37.21	40.39
12	42.26	38.34
13	54.49	48.59
14	62.80	56.32
15	61.61	54.85
.		
Total Volume	75322	74429

Design 1 ; solved by Taig's method
Design 2 ; solved by the method of this thesis.

much relaxed minimum size restrictions, i.e. large minimum sizes, compared with the truss problems usually appearing in the literature. Therefore many variables were set to the minima and this shift of the design within a Newton-Raphson iteration was big enough to form a loop. Setting these variables to their minima had the consequence of changing the Lagrange multipliers, and thus the resulting Newton direction was that at a point different from the current set of the Lagrange multipliers. This shift of the design values therefore often caused a loop to form. The length of the loop was usually 2 but sometimes reached 30 when two or three active constraints were involved. Keeping the value of Equ. (5.66) even if it was below the minimum removed the problem of the loop.

Another interesting result is that the two approaches of Taig and this thesis resulted in different designs for the beam example of the preceding section. Table-6 shows the total volumes of the designs and the values of some variables which were given apparently different values. They are seemingly two different local minima, but were not examined closely to explore their nature. However, the method of this thesis resulted in the same design, i.e. design 2, even when the design given by Taig's method was used as the initial design.

5.4.7 Terminating a redesign iteration

A round of the Newton-Raphson process is terminated when changes of individual values of the Lagrange multipliers or the residual of the constraint equations are below an acceptable tolerance. Having terminated the round the design variables are examined to see if any of them should be removed to Group 2 or Group 3. If there are none the current redesign iteration finishes.

If there are such variables the next round of the Newton-Raphson process starts having removed them to Group 2 or Group 3 and taking the solution of the last round as its starting values. This approach was quite helpful since it took advantage of the characteristics of the Newton-Raphson method, stable and very fast if used with good estimates.

The method developed in this work follows the line of mathematical rigour at the cost of ease of computing. In particular the way of treating passive variables calls for more Newton-Raphson iterations. However we can reduce computing effort substantially by taking better estimates of the Lagrange multipliers as explained above. In addition further improvement was achieved by giving the Lagrange multipliers, as the starting values of the current iteration, the final values of the first round of the preceding iteration when the set of active constraints did not change.

5.4.8 Differences for the beam problems

Although the same approach can be used for the beam problems, there are a number of differences due to the different structural behaviour of the bending elements from that of the bar elements. Equations and formulae applicable to the beam problems are shown below.

The estimation of λ's for the beam problems will be made based on the same concept, but further assumptions are yet required. Those terms not including c_{ik} or d_{ij} in Equ. (5.57) and (5.59) are neglected and the objective function is assumed linear. When design variable linking is employed, the following relations are also assumed to hold.

$$\sum_{t=1}^{m} \frac{a_{tk}}{d_t^3} = \sum_{i=1}^{\ell} \frac{\sum_{t=1}^{m} T_{ti}\, a_{tk}}{D_i^3} = \sum_{i=1}^{\ell} \frac{y_{ik}}{D_i^3}$$

$$\sum_{t=1}^{m} \frac{b_{tj}}{d_t^3} = \sum_{i=1}^{\ell} \frac{\sum_{t=1}^{m} T_{ti}\, b_{tj}}{D_i^3} = \sum_{i=1}^{\ell} \frac{z_{ij}}{D_i^3}$$

$$\sum_{t=1}^{m} \frac{a_{tk}}{d_t^2} = \sum_{i=1}^{\ell} \frac{\sum_{t=1}^{m} T_{ti}\, a_{tk}}{D_i^2} = \sum_{i=1}^{\ell} \frac{y_{ik}}{D_i^2}$$

$$\sum_{t=1}^{m} \frac{b_{tj}}{d_t^2} = \sum_{i=1}^{\ell} \frac{\sum_{t=1}^{m} T_{ti}\, b_{tj}}{D_i^2} = \sum_{i=1}^{\ell} \frac{z_{ij}}{D_i^2}$$

Then the Lagrange multipliers are estimated from

$$\lambda_p = \frac{1}{3n_a} \left[\frac{\sum_{+i} h_i^{\frac{3}{4}} y_{ip} / c_{ip}^{\frac{3}{4}}}{\bar{u}_p - \sum_{-i} y_{ip}/D_i^3} \right]^{\frac{4}{3}} \quad \ldots\ldots\ldots \quad (5.70)$$

$$\lambda_g = \frac{1}{3n_a} \left[\frac{\sum_{+i} h_i^{\frac{3}{4}} z_{ig} / d_{ig}^{\frac{3}{4}}}{\bar{\sigma}_g - \sum_{-i} z_{ig}/D_i^3} \right]^{\frac{4}{3}} \quad \ldots\ldots\ldots \quad (5.71)$$

for the rectangular section beam, and

$$\lambda_p = \frac{1}{2n_a} \left[\frac{\sum_{+i} h_i^{\frac{2}{3}} y_{ip} / c_{ip}^{\frac{2}{3}}}{\bar{u}_p - \sum_{-i} y_{ip}/D_i^2} \right]^{\frac{3}{2}} \quad \ldots\ldots\ldots \quad (5.72)$$

$$\lambda_g = \frac{1}{2n_a} \left[\frac{\sum_{+i} h_i^{\frac{2}{3}} z_{ig} / d_{ig}^{\frac{2}{3}}}{\bar{\sigma}_g - \sum_{-i} z_{ig}/D_i^2} \right]^{\frac{3}{2}} \quad \ldots\ldots\ldots \quad (5.73)$$

for the I-section beam.

To determine the design variables for the rectangular beam from the Lagrange multipliers, Equ. (5.57) is re-formed as:

$$\left.\begin{aligned} D_i^{(3+\beta)} - v_i D_i - \omega_i &= 0 \\ v_i &= \frac{1}{\beta h_i} \sum_{j \in S} \delta_{Kij} \frac{6 \lambda_{n+j} T_{ji} M_j}{B_j r_{ji}^{3}} \\ \omega_i &= \frac{3}{\beta h_i} \left[\sum_{k \in U} \lambda_k c_{ik} + \sum_{j \in S} \lambda_{n+j} d_{ij} \right] \end{aligned}\right\} \cdots (5.74)$$

The roots of Equ. (5.74) may be obtained in an iterative way, but their nature should be considered here. If v_i is equal to zero, i.e. no active stress constraint is found at the nodes governed, fully or partly, by the ith design variable, the solution is straightforward. A unique positive root will be found if ω_i is positive, no root will be available otherwise. If v_i is positive, there is no definite way to explore the existence or uniqueness of the roots. If we assume that the objective function is linear, the first of Equ. (5.74) becomes

$$D_i^{4} + v_i D_i - \omega_i = 0$$

and we know from the Descartes' sign rule that a unique positive real root exists if ω_i is positive, and otherwise there is no positive real root. It appears that Equ. (5.74) also follows this rule since β is usually given a value not far from unity.

From the above discussion and the fact that v_i is always nonnegative, we can conclude that the existence of a unique positive real root of Equ. (5.74) depends solely on the sign of ω_i. The design variables with nonpositive ω_i-value are therefore given the minimum values while the rest have the roots of Equ. (5.74). This method also applies to the I-section beam but the equation whose roots are sought. Equ. (5.75), which came from Equ. (5.59), is used for the I-section beam.

$$\left.\begin{aligned} & D_i^{(2+\beta)} + V_i D_i - W_i = 0 \\ & V_i = \frac{1}{\beta h_i} \sum_{j \in S} \delta_{K_{ij}} \frac{\lambda_{n+j} T_{ji} M_j}{A_{fj} r_{ji}^2} \\ & W_i = \frac{2}{\beta h_i} \left[\sum_{k \in U} \lambda_k c_{ik} + \sum_{j \in S} \lambda_{n+j} d_{ij} \right] \end{aligned}\right\} \cdots\cdots (5.75)$$

The approximate constraint equations

$$u_k - \bar{u}_k = \sum_{t=1}^{m} \frac{a_{tk}}{d_t^3} - \bar{u}_k = 0 \quad \cdots\cdots (5.76)$$

$$\sigma_j - \bar{\sigma}_j = \sum_{t=1}^{m} \frac{b_{tj}}{d_t^3} - \bar{\sigma}_j = 0 \quad \cdots\cdots (5.77)$$

for the rectangular section beam, and

$$u_k - \bar{u}_k = \sum_{t=1}^{m} \frac{a_{tk}}{d_t^2} - \bar{u}_k = 0 \quad \cdots\cdots (5.78)$$

$$\sigma_j - \bar{\sigma}_j = \sum_{t=1}^{m} \frac{b_{tj}}{d_t^2} - \bar{\sigma}_j = 0 \quad \cdots\cdots (5.79)$$

for the I-section beam, are expressed in terms of the depths at nodes. Therefore it is necessary to determine the depths from the design values using the transformation matrix, $\{T_{ti}\}$, defined in Equ. (5.34). Then the equations (5.76) - (5.79) will be evaluated using the depths so obtained.

The entries of the Jacobian matrix in Equ. (5.67) are obtained as follows instead of using Equ. (5.68). For the rectangular beam we use

$$X_{11} = \left\{ \frac{\partial u_k}{\partial \lambda_p} \right\} = \left\{ \sum_{i \in G1} \frac{-3\,C_{ik}}{D_i^{4}} \; \frac{3\,C_{ip} / \beta h_i}{(3+\beta)\,D_i^{(2+\beta)} + v_i} \right\}$$

$$X_{22} = \left\{ \frac{\partial \sigma_j}{\partial \lambda_{n+q}} \right\} = \left\{ \sum_{i \in G1} \left[\frac{-3\,d_{ij}}{D_i^{4}} + \delta_{kij} \frac{6\,T_{ji}\,M_j}{B_j\, r_{ji}^{3}\, D_j^{3}} \right] \right.$$

$$\left. * \frac{3 \dfrac{d_{iq}}{\beta h_i} - \delta_{kiq} \dfrac{6\,T_{qi}\,M_q\,D_i}{\beta h_i\, B_q\, r_{qi}^{3}}}{(3+\beta)\,D_i^{(2+\beta)} + v_i} \right\}$$

$$X_{12} = \left\{ \frac{\partial u_k}{\partial \lambda_{n+q}} \right\} = \left\{ \sum_{i \in G1} \frac{-3\,C_{ik}}{D_i^{4}} \; \frac{3 \dfrac{d_{iq}}{\beta h_i} - \delta_{kiq} \dfrac{6\,T_{qi}\,M_q\,D_i}{\beta h_i\, B_q\, r_{qi}^{3}}}{(3+\beta)\,D_i^{(2+\beta)} + v_i} \right\}$$

$$X_{21} = \left\{ \frac{\partial \sigma_j}{\partial \lambda_p} \right\} = \left\{ \sum_{i \in G1} \left[\frac{-3\,d_{ij}}{D_i^{4}} + \frac{\delta_{kij}\, 6\,T_{ji}\,M_j}{B_j\, r_{ji}^{3}\, D_j^{3}} \right] * \frac{3\,C_{ip} / \beta h_i}{(3+\beta)\,D_i^{(2+\beta)} + v_i} \right\}$$

$$v_i = \frac{1}{\beta h_i} \sum_{j \in S} \delta_{kij} \frac{6\,\lambda_{n+j}\,T_{ji}\,M_j}{B_j\, r_{ji}^{3}}$$

.... (5.80) and for the I-section beam

$$X_{11} = \left\{\frac{\partial u_k}{\partial \lambda_p}\right\} = \left\{ \sum_{i \in G1} \frac{-2c_{ik}}{D_i^3} \cdot \frac{2\, c_{ip}/\beta h_i}{(2+\beta)\, D_i^{(1+\beta)} + v_i} \right\}$$

$$X_{22} = \left\{\frac{\partial \sigma_j}{\partial \lambda_{n+g}}\right\} = \left\{ \sum_{i \in G1} \left[\frac{-2d_{ij}}{D_i^3} + \delta_{kj}\frac{T_{ji}\, M_j}{A_{fj}\, r_{ji}^2\, D_j^2}\right] \right.$$

$$\left. * \frac{2\dfrac{d_{ig}}{\beta h_i} - \delta_{kg}\dfrac{T_{gi}\, M_g\, D_i}{\beta h_i\, A_{fg}\, r_{gi}^2}}{(2+\beta)\, D_i^{(1+\beta)} + v_i} \right\}$$

$$X_{12} = \left\{\frac{\partial u_k}{\partial \lambda_{n+g}}\right\} = \left\{ \sum_{i \in G1} \frac{-2c_{ik}}{D_i^3} \cdot \frac{2\dfrac{d_{ig}}{\beta h_i} - \delta_{kg}\dfrac{T_{gi}\, M_g\, D_i}{\beta h_i\, A_{fg}\, r_{gi}^2}}{(2+\beta)\, D_i^{(1+\beta)} + v_i} \right\}$$

$$X_{21} = \left\{\frac{\partial \sigma_j}{\partial \lambda_p}\right\} = \left\{ \sum_{i \in G1} \left[\frac{-2d_{ij}}{D_i^2} + \frac{\delta_{kj}\, T_{ji}\, M_j}{A_{fj}\, r_{ji}^2\, D_j^2}\right] \frac{2\, c_{ip}/\beta h_i}{(2+\beta)\, D^{(1+\beta)} + v_i} \right\}$$

$$v_i = \frac{1}{\beta h_i} \sum_{j \in S} \delta_{kj} \frac{\lambda_{n+j}\, T_{ji}\, M_j}{A_{fj}\, r_{ji}^2}$$

.... (5.81)

5.4.9 Summary of differences between the methods of Taig and this thesis

The differences between the methods of Taig and this thesis are listed in Fig. 14. The improvements achieved can be summarized in two respects. Firstly, the strategies of finding and deleting constraints have been changed entirely. These prevented the constraints from filpping between active to inactive states and improved the efficiency substantially. Secondly, postponing the removal of Group 2 variables until the completion of a Newton-Raphson process, i.e. Stage 9, eliminated the formation of loops which is a fatal drawback of the Newton-Raphson method.

Stage of redesign process	Differences
1. Find active constraints.	Taig's method does not have this stage. It considers all the constraints active.
2. Calculate c_{ik}, d_{ij} for active constraints.	Taig's method calculates c_{ik} for all the deflection constraints, but does not calculate d_{ij} since it treats stress limits as side constraints.
3. Estimate λ's	the same.
4. Determine a new design.	Taig's method removes variables below their minima to Group 2 at this stage.
5. Update λ's	the same.
6. Delete inactive constraints if any.	When some negative λ's appear, Taig's method deletes one of them each time. The method of this thesis may delete several constraints at a time or may not delete any depending on the history of the λ's .
7. If any deleted, GO TO 3.	the same.
8. If converged,	In Taig's method, EXIT. In this method, GO TO 9.
9. Remove Group 2 variables.	Taig's method does not have this stage. This operation is carried out at stage 4 instead.
10. If any removed, GO TO 4. EXIT otherwise.	Taig's method does not have this stage.

Fig. 14 Differences between the methods of Taig and this thesis.

5.5 Optimality Test

Another application of the optimality criteria is to test if a given design is at a local minimum. Having analysed the structure, we can find which constraints are active/inactive, which design variables are active/passive, and from Equ. (5.56) which Lagrange multipliers should be greater than/equal to zero for the optimality of the design. Then the system of simultaneous linear equations, Equ. (5.53), will be solved for the Lagrange multipliers.

For the sake of convenience and geometrical explanation, the equations of Equ. (5.53) are rearranged and expressed in matrix form.

$$\left[\begin{array}{c:c:c} \left(\dfrac{C_{ik}}{A_i^2}\right)_{i\in G1} & \left(\dfrac{d_{ij}}{A_i^2}\right)_{i\in G1} & O \\ \hdashline \left(\dfrac{C_{ik}}{A_i^2}\right)_{i\in G2} & \left(\dfrac{d_{ij}}{A_i^2}\right)_{i\in G2} & I \end{array}\right] \left[\begin{array}{c} \lambda_k \\ \hdashline \lambda_{n+j} \\ \hdashline \gamma_i \end{array}\right] = \left[\begin{array}{c} \left(f_i\right)_{i\in G1} \\ \hdashline \left(f_i\right)_{i\in G2} \end{array}\right] \quad \cdots\cdots \quad (5.82)$$

Let

$$\underset{\sim}{N_1} = \left[\begin{array}{c:c} \left(\dfrac{C_{ik}}{A_i^2}\right)_{i\in G1} & \left(\dfrac{d_{ij}}{A_i^2}\right)_{i\in G1} \end{array}\right]_{\substack{k\in U \\ j\in S}}$$

$$\underset{\sim}{N_2} = \left[\begin{array}{c:c} \left(\dfrac{C_{ik}}{A_i^2}\right)_{i\in G2} & \left(\dfrac{d_{ij}}{A_i^2}\right)_{i\in G2} \end{array}\right]_{\substack{k\in U \\ j\in S}}$$

$$\underset{\sim}{\lambda} = \left[\begin{array}{c} \lambda_k \\ \hdashline \lambda_{n+j} \end{array}\right]_{\substack{k\in U \\ j\in S}} = \{\lambda_1, \lambda_2, \cdots, \lambda_{na}\}^T$$

$$\underset{\sim}{\gamma} = \left[\ \gamma_i\ \right]_{i\in G2}$$

$$\underset{\sim}{B_1} = \left[\, h_i \, \right]_{i \in G1}$$

$$\underset{\sim}{B_2} = \left[\, h_i \, \right]_{i \in G2}$$

Then Equ. (5.82) becomes

$$\left[\begin{array}{c:c} \underset{\sim}{N_1} & O \\ \hdashline \underset{\sim}{N_2} & I \end{array} \right] \left[\begin{array}{c} \underset{\sim}{\lambda} \\ \hdashline \underset{\sim}{\gamma} \end{array} \right] = \left[\begin{array}{c} \underset{\sim}{B_1} \\ \hdashline \underset{\sim}{B_2} \end{array} \right] \quad \cdots\cdots\cdots\cdots \quad (5.83)$$

and we first solve

$$\underset{\sim}{N_1} \, \underset{\sim}{\lambda} = \underset{\sim}{B_1} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5.84)$$

for $\underset{\sim}{\lambda}$ and with the solution we find $\underset{\sim}{\gamma}$ from

$$\underset{\sim}{\gamma} = \underset{\sim}{B_2} - \underset{\sim}{N_2} \, \underset{\sim}{\lambda} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (5.85)$$

The matrix $\underset{\sim}{N_1}$ in Equ. (5.84) may not be a square matrix but rather a rectangular matrix with a greater number of rows than of columns. To set up an approach for finding $\underset{\sim}{\lambda}$ we define the residual vector for Equ. (5.84), form the square of its length, and then look at the conditions for its length to be a minimum. The square of its length is expressed by

$$L(\underset{\sim}{\lambda}) = (\underset{\sim}{N_1} \underset{\sim}{\lambda} - \underset{\sim}{B_1})^T (\underset{\sim}{N_1} \underset{\sim}{\lambda} - \underset{\sim}{B_1})$$

$$= \underset{\sim}{\lambda}^T \underset{\sim}{N_1}^T \underset{\sim}{N_1} \underset{\sim}{\lambda} - 2 \underset{\sim}{\lambda}^T \underset{\sim}{N_1}^T \underset{\sim}{B_1} + \underset{\sim}{B_1}^T \underset{\sim}{B_1} \quad \cdots\cdots \quad (5.86)$$

and the stationary conditions are

$$\underset{\sim}{N_1}^T \underset{\sim}{N_1} \underset{\sim}{\lambda} - \underset{\sim}{N_1}^T \underset{\sim}{B_1} = 0 \quad \cdots\cdots\cdots\cdots \quad (5.87)$$

The square matrix $\underset{\sim}{N_1}^T \underset{\sim}{N_1}$ is nonsingular provided that the column vectors of $\underset{\sim}{N_1}$ are linearly independent and, if so, we can find $\underset{\sim}{\lambda}$ from

$$\underset{\sim}{\lambda} = (\underset{\sim}{N_1}^T \underset{\sim}{N_1})^{-1} \underset{\sim}{N_1}^T \underset{\sim}{B_1} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (5.88)$$

minimizing $L(\underset{\sim}{\lambda})$ but not necessarily satisfying Equ. (5.84).

By substituting Equ. (5.88) into Equ. (5.84) we obtain

$$P \equiv [\, I - \underset{\sim}{N_1} (\underset{\sim}{N_1}^T \underset{\sim}{N_1})^{-1} \underset{\sim}{N_1}^T]\, B_1 = O \quad \cdots \quad (5.89)$$

If Equ. (5.89) holds, $\underset{\sim}{\lambda}$ obtained from Equ. (5.88) is really the solution of Equ. (5.84) and we can proceed to Equ. (5.85) and find $\underset{\sim}{\gamma}$. Then the optimality will be ensured if no negative entry exists in either of $\underset{\sim}{\lambda}$ or $\underset{\sim}{\gamma}$.

Geometrically a column vector of $\underset{\sim}{N_1}$ is the projection of a negative constraint gradient onto the subspace spanned by the active design variable coordinates, and $\underset{\sim}{B_1}$ is that of the cost gradient. Solving Equ. (5.84) for $\underset{\sim}{\lambda}$ is therefore determining the set of coefficients with which the cost gradient is considered to be a linear combination of the constraint gradients, each multiplied by -1. If the cost gradient does not lie in the space generated by the constraint gradients, Equ. (5.84) has no solution, vector P defined in Equ. (5.89) exists, and the optimality is disproved. The existence of vector P , the projection of the cost gradient onto the intersection of all the hyperplanes perpendicular to the constraint gradients, suggests that there are better designs lying along P with the same set of active constraints as that of the current design. When P vanishes but a negative Lagrange multiplier, λ or γ , appears there are better designs with a different set of active constraints.

6. STRESS LIMITED TRUSSES

A number of stress limited truss problems are solved in section 6.1. They are 25-bar, 55-bar, 72-bar, and 124-bar trusses and their solutions are seldom found in the literature. It is demonstrated how rigourously and rapidly the method described in the preceding chapter solved the problems where many active constraints are present. In section 6.2, the nature of the fully stressed designs for a stress limited truss is investigated thoroughly in connection with the optimality of those designs.

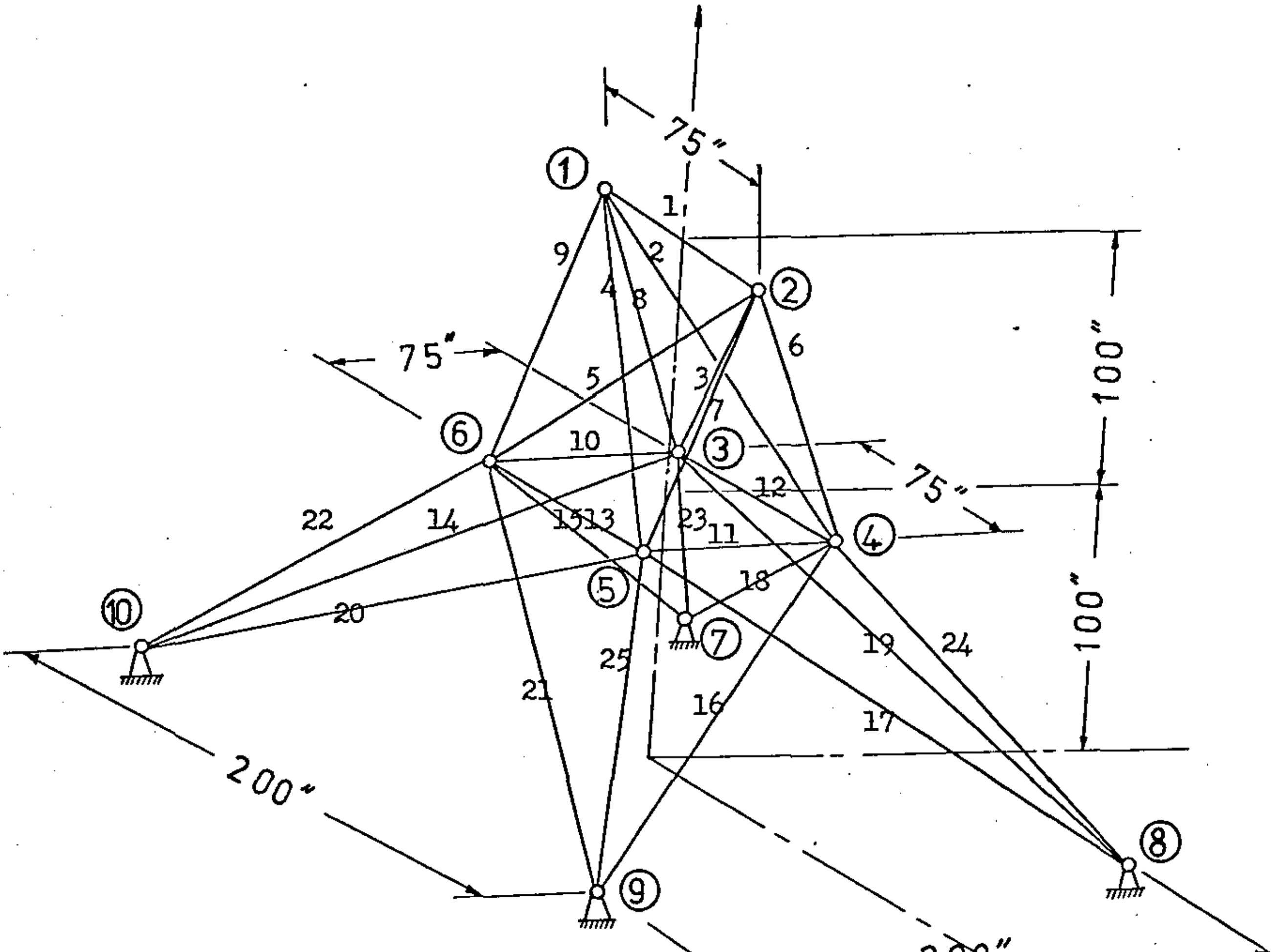
6.1 Examples

Ex.T-1 25-Bar Truss Case I

This problem, shown in Fig. 15, is the same as the 25-bar space truss frequently appearing in the literature, but in Case I stress limits for each group of members are the only behavioural constraints. The optimum design shown in Table-7 is not a fully stressed design. Neither of the members 12 and 13 associated with the fifth design variable is fully stressed despite the design variable being in Group 1. Their stresses are only 18% and 21% of the permitted value respectively. This problem was also solved by Dobbs and Nelson[24] and a design weighing 351.4 LB was obtained after 4 iterations.

Ex.T-2 25-Bar Truss Case II

This is the same example as Ex.T-1, but no design variable linking is employed and the compressive stress limit is set to -35,000 psi for all members instead of those in Fig. 15. The final design after



Material Data

$E = 10^7$ psi

$\rho \doteq 0.1$ pci

min. size ; 0.01 in^2

Stress Limits (psi)

Members	Tensile	Compressive
1	40,000	-35,090
2- 5	40,000	-11,590
6- 9	40,000	-17,300
10-11	40,000	-35,090
12-13	40,000	-35,090
14-17	40,000	- 6,760
18-21	40,000	- 6,960
22-25	40,000	-11,080

Applied Loads (Kips)

Node	X-Force	Y-Force	Z-Force
Load Case 1			
1	1.0	10.0	-5.0
2	-	10.0	-5.0
3	0.5	-	-
6	0.5	-	-
Load Case 2			
1	-	20.0	-5.0
2	-	-20.0	-5.0

Fig. 15 25-Bar Truss

18 redesign iterations is shown in Table-8, and at this design the value of Equ. (5.54), the optimality criteria equation, for each of twenty Group 1 variables is exactly 1.0000.

Table-7 Designs of 25-Bar Truss

Member	Case I	Case III
1	0.01	0.01
2-5	1.2441	1.9845
6-9	1.1182	2.9973
10-11	0.01	0.01
12-13	0.1052	0.01
14-17	0.5519	0.6841
18-21	1.6501	1.6773
22-25	1.3010	2.6609
Iteration	13	8
Weight	343.524	545.168

Table-8 25-Bar Truss Case II

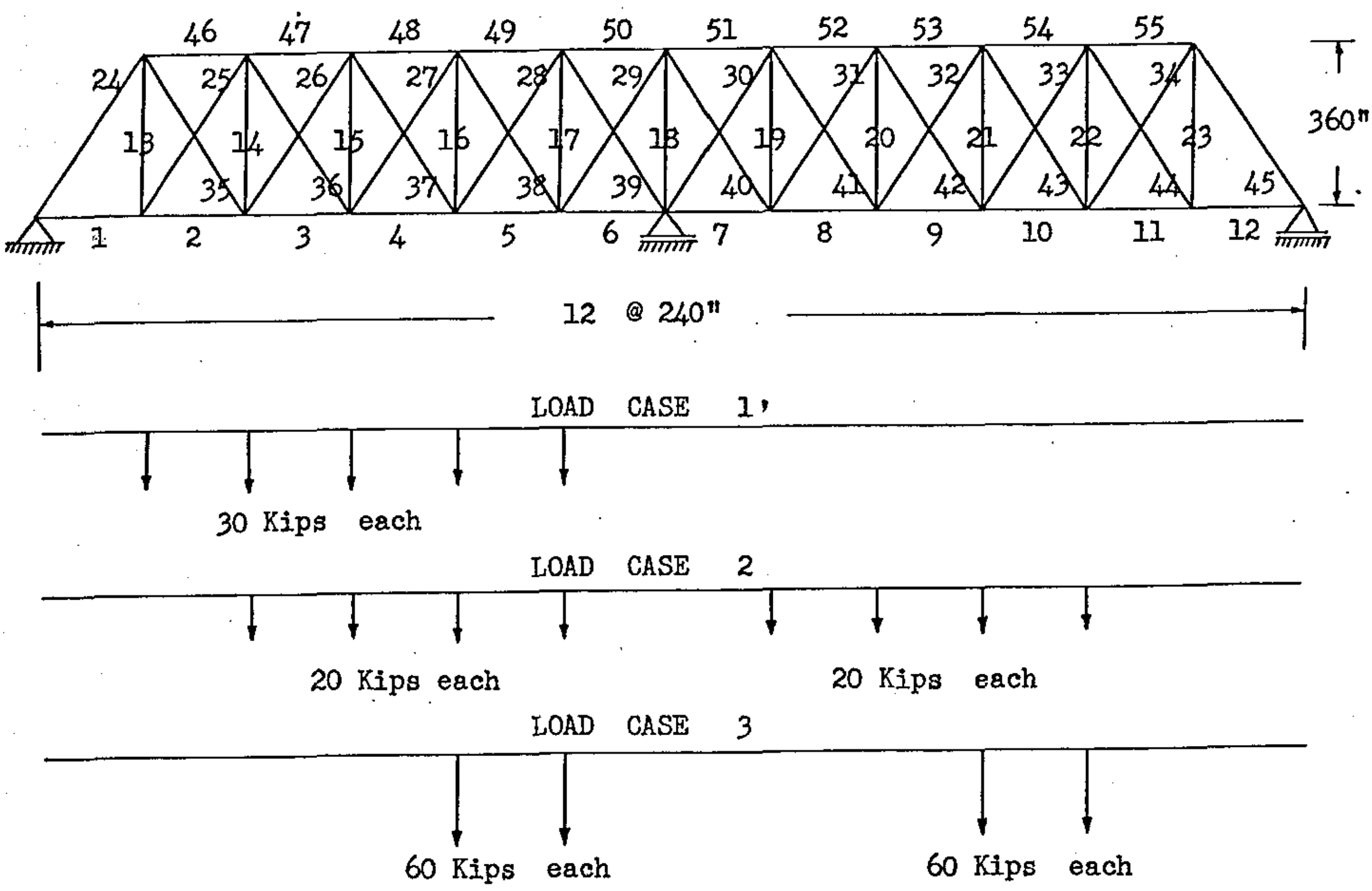
Member	Area	Member	Area	Member	Area
1	0.01	10	0.01	19	0.317
2	0.448	11	0.01	20	0.353
3	0.339	12	0.01	21	0.192
4	0.400	13	0.01	22	0.256
5	0.379	14	0.096	23	0.356
6	0.368	15	0.061	24	0.444
7	0.580	16	0.112	25	0.242
8	0.523	17	0.060		
9	0.318	18	0.224		
Weight					82.978

Ex.T-3. 55-Bar Truss

This example, illustrated in Fig. 16, is to show the ability of the method presented in this thesis to solve large-scale systems of simultaneous non-linear equations. The final design in Table-9 is exact to the extent of 1.0000 for all of the 35 design variables, including 35 active stress constraints, and achieved after only 11 iterations. The design has no Group 2 variables, the stress of a member associated with each design variable is active, and thus there are 35 active constraints. The optimality criteria equations, Equ. (5.54), and the constraint equations, Equ. (5.56), totaling 70 equations are all satisfied in an equality sense as exactly as 1.0000.

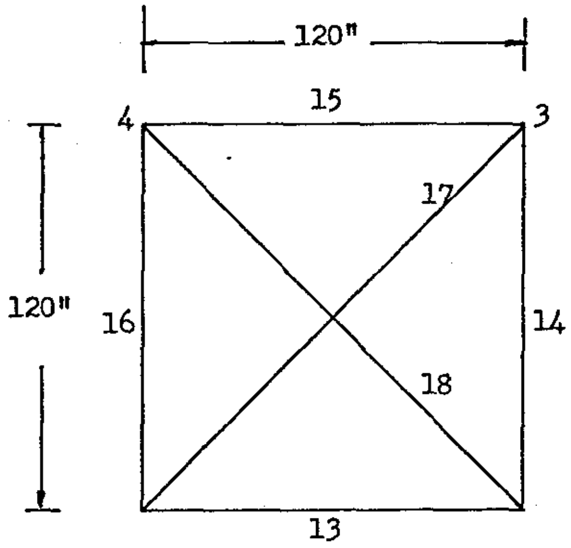
Ex.T-4 72-Bar Truss Case I

The truss, illustrated in Fig. 17, is another example solved by many researchers but mainly subjected to stress and deflection limits. The problem when subject only to stress limits was solved by Dobbs and Nelson [24)] but the result does not seem meaningful. Among sixteen design variables only four were of Group 1 and in addition very close to their minimum values. In Ex.T-4 the applied loads are ten times the magnitudes of those commonly used in the literature and deflection constraints are neglected. Table-10 shows the final design including ten Group 1 variables and it was obtained only after 6 iterations. This problem shows a good behaviour in terms of quick convergence.



Material Data	Stress Limits	Deflection Limits
$E = 3 \times 10^7$ psi	+20 Ksi	none
$\rho = 0.28$ pci	or	
min. size ; 0.1 in^2	-15 Ksi	

Fig. 16 55-Bar Truss



Material Data

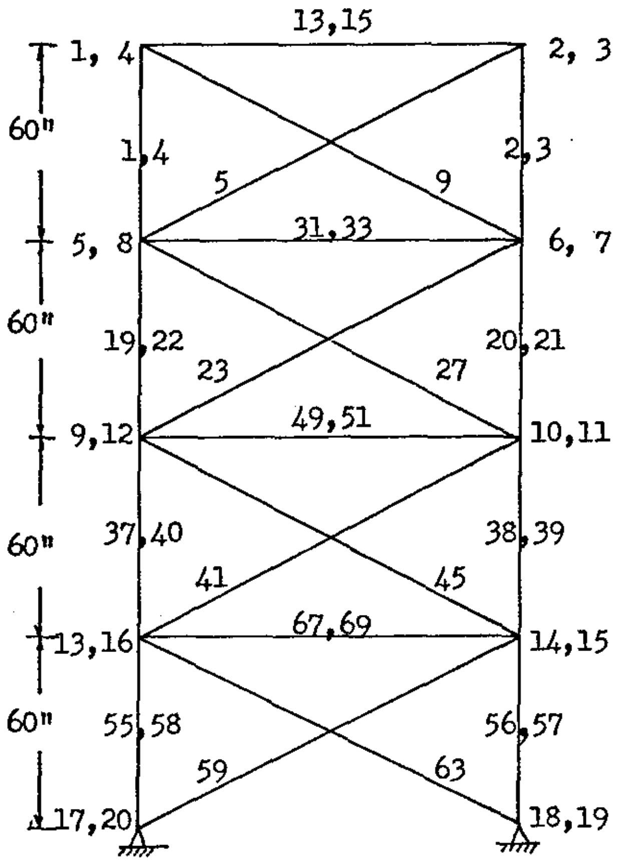
$E = 10^7$ psi

$\rho = 0.1$ pci

stress limit ; ± 25 Ksi

min. size ; 0.1 in^2

Applied Loads (Kips)



Case I

Node	X-Force	Y-Force	Z-Force
load case 1			
1	50.0	50.0	-50.0
load case 2			
1	-	-	-50.0
2	-	-	-50.0
3	-	-	-50.0
4	-	-	-50.0

Case II

Node	X-Force	Y-Force	Z-Force
load case 1			
1	5.0	5.0	-5.0
load case 2			
1	-	-	-5.0
2	-	-	-5.0
3	-	-	-5.0
4	-	-	-5.0

Fig. 17 72-Bar Truss

Table-9 Design of 55-Bar Truss

Member	Area	O.C.*	Member	Area	O.C.*
1	3.0051	1.0000	30,40	2.8690	1.0000
2	3.4616	1.0000	31,41	2.7787	1.0000
3	4.8156	1.0000	32,42	2.9514	1.0000
4	4.4472	1.0000	33,43	0.3510	1.0000
5	2.6440	1.0000	34,44	2.8679	1.0000
6	1.0267	1.0000	45	3.2380	1.0000
7	1.6334	1.0000	46	3.1638	1.0000
8	1.1206	1.0000	47	3.6543	1.0000
9	2.8802	1.0000	48	3.4362	1.0000
10	4.5948	1.0000	49	2.2941	1.0000
11	3.1986	1.0000	50	2.5137	1.0000
12	2.3948	1.0000	51	3.4825	1.0000
13-23	2.9287	1.0000	52	1.4712	1.0000
24	4.0618	1.0000	53	2.4118	1.0000
25,35	2.1887	1.0000	54	3.5342	1.0000
26,36	0.5399	1.0000	55	2.9894	1.0000
27,37	1.0197	1.0000			
28,38	2.7165	1.0000			
29,39	3.9627	1.0000			

No. of Iteration ; 11
Total Weight ; 13834.1
No. of Active Constraints; 35
* O.C. ; Values of Equ. (5.54)

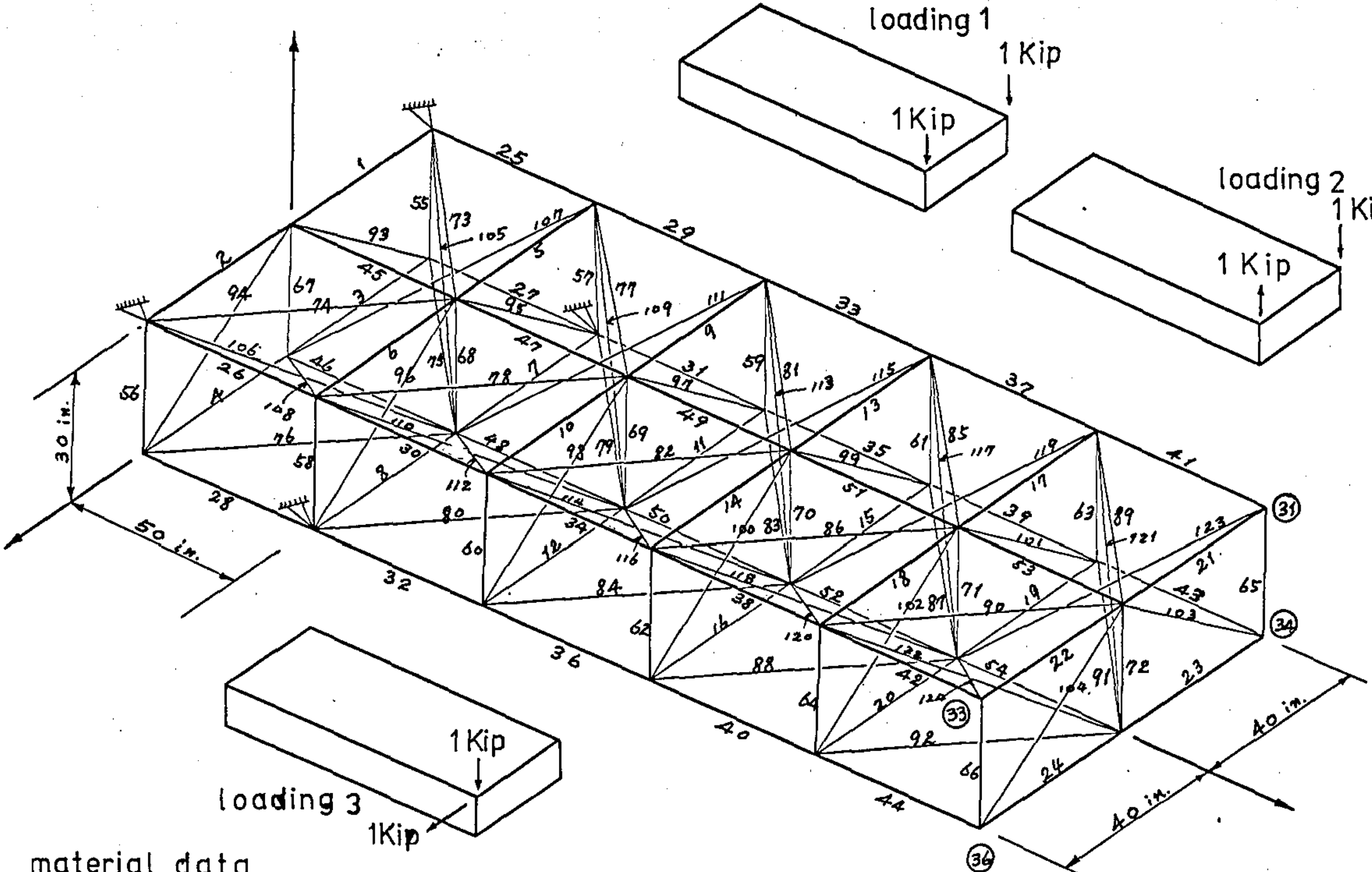
Table-10 Designs of 72-Bar Truss

Member	Case I	Case II
1-4	1.9543	0.1565
5-12	0.8591	0.5453
13-16	0.6292	0.4130
17-18	0.8520	0.5664
19-22	1.9923	0.5232
23-30	0.7245	0.5172
31-34	0.1	0.1
35-36	0.1	0.1
37-40	2.0990	1.2689
41-48	0.6626	0.5117
49-52	0.1	0.1
53-54	0.1	0.1
55-58	2.9667	1.8863
59-66	0.6396	0.5124
67-70	0.1	0.1
71-72	0.1	0.1
Iteration	6	4
Weight	609.721	379.622

Ex.T-5 124-Bar Truss Case I

This truss, which was solved by Sheu [45] for deflection limits, is illustrated in Fig. 18. Ex.T-5 is a variation of the original problem, where no displacement limit is present but instead stress limits of 10 Ksi are imposed on all members. Interestingly the optimum design obtained after 13 redesign iterations is not a fully stressed design. Member 144 is the most stressed among those associated with the 44^{th} design variable which belongs to Group 1, but its stress is 86% of the permitted stress.

The redesign process, however, failed to get rid of all Group 3 variables. The stress of member 122 was included in the set of active constraints at the outset of the redesign iteration but deleted during the Newton-Raphson process, and therefore remained in Group 3. But the optimality of the design was confirmed by the optimality test made on completion of the redesign process. The problem was also solved by the stress ratio method, which gave a fully stressed design. Whereas the optimum design had 18 active stress constraints, this design had 19 to include the stress constraint of member 144. The optimality test for the design resulted in a negative Lagrange multiplier associated with the stress of member 144 and disproved its optimality, although it was quite close to the optimum design. The two designs are shown in Table-11.



material data

$E = 10^7$ psi

$\rho = 0.1$ pci

min. size $= 0.1$ in^2

Fig. 18 124-Bar Truss

Table-11 124-Bar Truss Case I

Member	Optimum Design	FSD	Member	Optimum Design	FSD
5- 8	0.111	0.111	81- 84	0.214	0.214
9-12	0.111	0.112	85- 88	0.214	0.214
13-16	0.111	0.113	89- 92	0.214	0.214
17-20	0.114	0.114	109-110	0.171	0.174
21-24	0.163	0.163	113-114	0.154	0.119
25-28	0.608	0.608	115-116	0.107	0.118
29-32	0.522	0.521	117-118	0.158	0.162
33-36	0.371	0.378	121-122	0.136	0.136
37-40	0.196	0.193	123-124	0.119	0.120
77-80	0.272	0.279			
Total Weight				107.269	107.308

Other member sizes are all at their minima. (0.1)

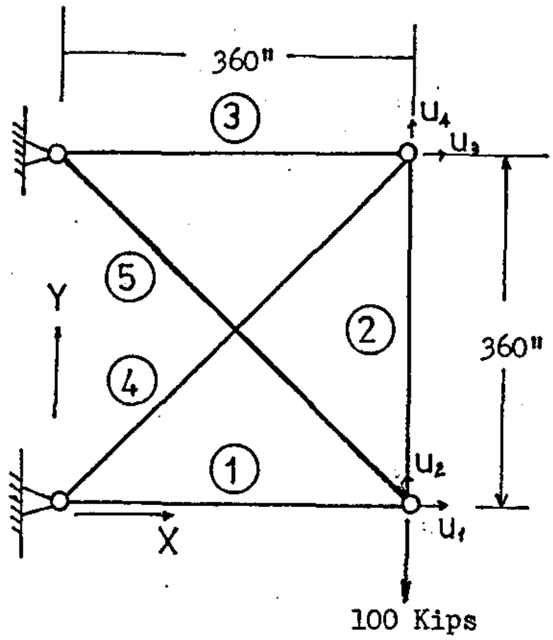
6.2 Optimality of Fully Stressed Designs

The stress ratio redesign algorithm based on the concept of fully stressing has been of great appeal to the engineer owing to its simplicity. It gives a fully stressed design (FSD), which in many cases is the optimum when the structure is subjected only to stress limits and built with one material.

A simple and rather artificial example, Ex.T-6 the 5-bar truss in Fig. 19, shows interesting features of FSD's. Table-12 lists three typical FSD's among innumerable FSD's of the problem. Design 1 and 3 of table-12 are the two extremes, and Design 2 is that obtained after a single stress ratio redesign from a uniform design. The stress-displacement relation in Equ. (6.1) shows that the five member stresses are not independent, and thus we can express the stress of member 5 in terms of other member stresses as in Equ. (6.2).

Table-12 FSD's of 5-Bar Truss

Member (i)	Design 1			Design 2	Design 3			Stress for all Designs
	A_i	λ_i	γ_i	A_i	A_i	λ_i	γ_i	
1	3.9000	3.120	-	2.2310	0.1061	0.	-	-25.00
2	0.1	2.520	30.00	1.7690	3.8939	5.76	-	25.00
3	0.1	2.520	30.00	1.7690	3.8939	5.76	-	25.00
4	0.1414	5.280	-	2.5017	5.5069	11.52	-	-25.00
5	3.6771	0.	-	2.1034	0.1	0.	-63.64	37.50
Total Weight	342.			442.14	569.64			



Material Data

$E = 10^7$ psi

$\rho = 0.1$ pci

Stress Limits;

± 25 Ksi for Member 1-4,

± 37.5 Ksi for Member 5.

Minimum Size ; $0.1\ in^2$

Fig. 19 5-Bar Truss

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \frac{10^7}{360} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad \cdots\cdots\cdots\cdots \quad (6.1)$$

$$\sigma_5 = \frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2 + \frac{1}{2}\sigma_3 - \sigma_4 \quad \cdots\cdots\cdots \quad (6.2)$$

Since the stress limits for each member were chosen deliberately such that

$$\bar{\sigma}_5 = \frac{1}{2}\bar{\sigma}_1 + \frac{1}{2}\bar{\sigma} + \frac{1}{2}\bar{\sigma}_3 - \bar{\sigma}_4$$

the stress ratio redesign immediately brought the design to an FSD but to different FSD's whenever different trial designs were used.

The Lagrange multipliers for the designs of Table-12 cannot be determined directly due to the functional dependency existing between the stress constraints. Those for Design 1 and 3 were obtained assuming, without loss of validity, that $\lambda_5 = 0$ and $\lambda_2 = \lambda_3$. Design 1 is the optimum as is seen from the nonnegative values of the Lagrange multipliers. For Design 3 the negative value of γ_5 disproves the optimality.

The stress ratio method simply forces each member fully stressed without looking at the optimality and shows such a fallacy as demonstrated above. Besides the fact that an FSD is not necessarily the optimum the fallacy, which can be encountered for any structural system since the relation in Equ. (6.1) exists in most cases, diminishes the value of the stress ratio method.

In many cases of stress limited trusses for which an FSD is the optimum, the design makes the truss degenerate to a statically determinate truss when the minimum size restriction is infinitesimally small. The stress of each member in these designs is active if the member belongs to Group 1 and is otherwise inactive. If we further assume that the stress limits of tension and compression are the same in magnitude and that the truss is made of one material, we can find interesting characteristics of the Lagrange multipliers. We obtain various optimum designs in accordance with various minimum size restrictions. The Lagrange multiplier associated with the stress limit of each Group 1 member is constant regardless of what the design is and in fact the same as that of its degenerate truss as long as the set of Group 1 members remains unchanged. Table-13 shows these characteristics of the well-known 10-bar truss Case I-a, Ex.T-7, appearing in Fig. 20. Assuming that there is no design variable linking the proof of this assertion is as follows.

Let

$\rho, E, \bar{\sigma}$; mass density, elastic modulus, stress limit of any member,

$F_i^{(D)}$; axial force of member i of the degenerate truss,

$\sigma_i = F_i / A_i$: stress of member i ,

$\sigma_i^{(j)} = F_i^{(j)} / A_i$: stress of member i under the unit virtual load applied to both ends and in the direction of member j ,

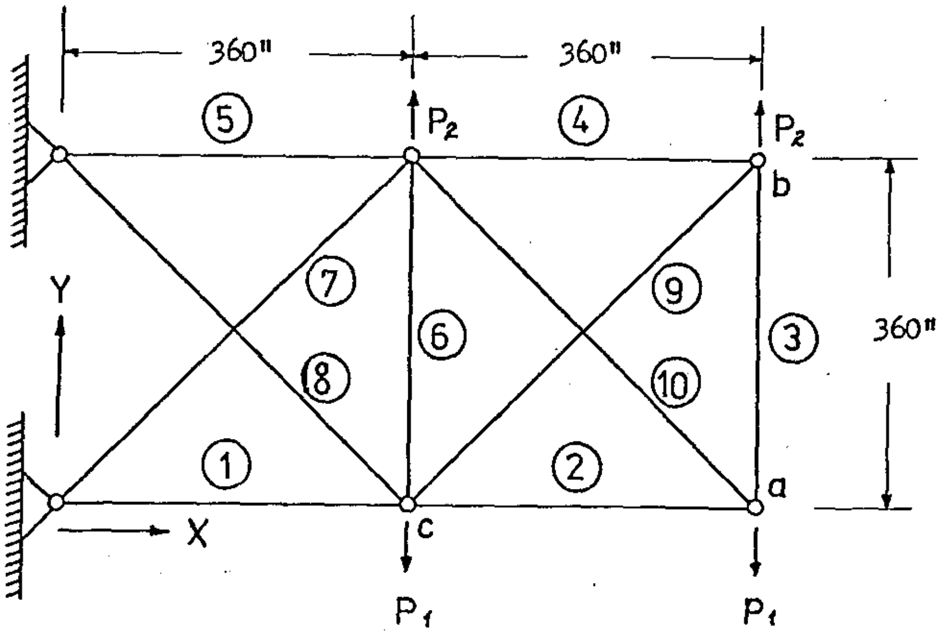
$\lambda_j^{(D)}$: λ_j of the degenerate truss.

Material Data

Elastic Modulus ; $E = 10^7$ psi

Mass Density ; $\rho = 0.1$ pci

Stress Limits ; ± 25 Ksi for Member 1-9, varying for Member 10.



Design Conditions in Various Problems

	Case I-a	Case I-b	Case I-c	Case II	Case III
Stress Limit of Member 10 (Ksi)	± 25	± 50	± 25	± 25	± 25
min. size (in^2)	varying	varying	0.1	0.1	0.1
Deflection Limit (in)	none	none	2.0	2.0	2.0
Loads P_1 (Kips)	100	100	100	150	100
Loads P_2 (Kips)	0	0	0	50	50

Fig. 20 10-Bar Truss

Table-13 Designs of 10-Bar Truss Case I-a

Member (i)	Member sizes of various designs				The same for all designs			
	1	2	3	4	λ_i	γ_i	O.C.	Stress
1*	8	8.0062	8.0624	8.6214	11.52	-	1.000	-25.00
2*	4	3.9938	3.9379	3.3787	5.76	-	1.000	-25.00
3	-	0.01	0.1	1.0	-	22.1	0.386	15.53
4	-	0.01	0.1	1.0	-	22.1	0.386	15.53
5*	8	7.9938	7.9381	7.3788	11.52	-	1.000	25.00
6	-	0.01	0.1	1.0	-	36	0.	0.
7*	5.6569	5.6481	5.5690	4.7782	11.52	-	1.000	-25.00
8*	5.6569	5.6657	5.7448	6.5356	11.52	-	1.000	25.00
9	-	0.01	0.1	1.0	-	11.6	0.772	-21.97
10*	5.6569	5.6481	5.5690	4.7782	11.52	-	1.000	25.00
Total Weight	1584.0	1584.92	1593.20	1675.82				

Design 1 : Determinate

Design 2 : min. size = 0.01

Design 3 : min. size = 0.10

Design 4 : min. size = 1.00

O.C. : the value of Equ. (5.54)

* : Group 1 members

Since the degenerate truss is statically determinate, it is easily found that

$$\lambda_j^{(D)} = \frac{\rho F_j^{(D)} L_j}{\bar{\sigma}^2} \qquad \cdots\cdots\cdots \qquad (6.3)$$

Let $\{A_i^*\}$ be an optimum design subject to a certain minimum size restriction. Then Equ. (6.4) must hold.

$$-\sum_{j\in G1} \lambda_j \frac{\partial \sigma_j}{\partial A_i^*} = \sum_{j\in G1} \lambda_j \frac{L_i}{L_j} \sigma_i \sigma_i^{(j)} = \rho L_i \quad , \text{ for } i \in G1 \quad \cdots\cdots (6.4)$$

If we introduce Equ. (6.5), derived from the reciprocal theorem, into (6.4) we obtain Equ. (6.6).

$$\sigma_i^{(j)} = \frac{L_j}{L_i} \sigma_j^{(i)} \quad \cdots\cdots\cdots (6.5)$$

$$\sum_{j\in G1} \lambda_j \sigma_i \sigma_j^{(i)} = \rho L_i, \quad \text{for} \quad i \in G1 \quad \cdots\cdots\cdots (6.6)$$

The elongation of member i, δ_i, due to the actual loads can be obtained by

$$\delta_i = \sum_{j\in G1} \frac{F_j \sigma_j^{(i)} L_j}{E} + \sum_{j\in G2} \frac{F_j \sigma_j^{(i)} L_j}{E}$$

$$= \sum_{j\in G1} \frac{F_j \sigma_j^{(i)} L_j}{E} + U2 \quad \cdots\cdots\cdots (6.7)$$

where $U2$ is the summation of the virtual strain energies over Group 2 members. If we substructure the truss into S1 consisting of Group 1 members and S2 consisting of Group 2 members, $U2$ is the same as the work done by the nodal forces due to the actual loads through the boundary nodes (from S1 to S2) as they ride along the displacements due to the virtual loads. The addition of $U2$ to the first term of Equ. (6.7) has the same effect as the subtraction of the work done by the boundary nodal forces from S2 to S1 and also as the removal of the forces applied to S1 from S2 through the boundary nodes. Since S1 is the same structure as the degenerate truss, Equ. (6.7) becomes

$$\delta_i = \sum_{j \in G1} \frac{F_j^{(D)} \sigma_j^{(i)} L_j}{E} \quad , \quad for\ all\ i \quad \cdots\cdots\cdots (6.8)$$

Multiplying both sides of Equ. (6.8) by $\frac{\rho E}{\sigma_i}$ results in Equ. (6.9).

$$\sum_{j \in G1} \frac{\rho F_j^{(D)} L_j}{\sigma_i} \sigma_j^{(i)} = \delta_i \frac{\rho E}{\sigma_i} = \rho L_i \ , \ for\ all\ i \quad \cdots (6.9)$$

By equating Equ. (6.6) and (6.9) for all $i \in G1$ we obtain

$$\sum_{j \in G1} \sigma_j^{(i)} \lambda_j = \sum_{j \in G1} \sigma_j^{(i)} \lambda_j^{(D)} \ , \ for\ i \in G1, \cdots\cdots (6.10)$$

and in matrix form,

$$\{ \sigma_j^{(i)} \} \{ \lambda_j \} = \{ \sigma_j^{(i)} \} \{ \lambda_j^{(D)} \}, \quad \cdots\cdots\cdots \quad (6.11)$$

$$\{ \sigma_j^{(i)} \} \{ \lambda_j - \lambda_j^{(D)} \} = \{ 0 \}. \quad \cdots\cdots\cdots\cdots \quad (6.12)$$

The matrix $\{ \sigma_j^{(i)} \}$ in Equ. (6.11) is nonsingular because Equ. (6.11) has a unique solution for $\{ \lambda_j \}$ and from Equ. (6.12) we obtain

$$\{ \lambda_j - \lambda_j^{(D)} \} = \{ 0 \} \quad \cdots\cdots\cdots\cdots (6.13)$$

which proves the assertion.

For Group 2 members Equ. (6.14) replaces Equ. (6.6)

$$\sum_{j \in G1} \lambda_j \sigma_i \sigma_j^{(i)} + \gamma_i = \rho L_i \quad \cdots\cdots\cdots (6.14)$$

From Equ. (6.9) and (6.14) we can find a simple expression to determine the γ's .

$$\gamma_i = \rho L_i - \frac{\rho \sigma_i}{\bar{\sigma}^2} \sum_{j \in G1} F_j^{(D)} L_j \sigma_j^{(i)}$$

$$= \rho L_i \left[1 - \left(\frac{\sigma_i}{\bar{\sigma}}\right)^2 \right] \quad \cdots\cdots\cdots \quad (6.15)$$

It can be concluded that for the particular problems mentioned above the Lagrange multipliers, both λ's and γ's , and the stress of each member are not sensitive at all to the change of the minimum size restrictions. Since the Lagrange multiplier represents the sensitivity of the objective function to the associated constraint, it is quite easy to determine the change of total weight due to the change of stress limit. If, for instance, we increase the stress limit of 10-bar truss Case I-a to 30, the decrease of total weight will be

$$\rho \sum_{j \in G1} F_j^{(D)} L_j \int_{25}^{30} \frac{d\bar{\sigma}}{\bar{\sigma}^2} = \rho \sum_{j \in G1} F_j^{(D)} L_j \left(\frac{1}{25} - \frac{1}{30} \right)$$

With 10-bar truss Case I-b, Ex.T-8, Berke[17)] demonstrated that an FSD is not necessarily the optimum design. Table-14 shows two FSD's and two optimum designs according to the different minimum size restrictions. The optimality of the FSD's is disproved by the negative Lagrange multiplier. It is also noted that the two optimum designs, neither of which is a fully stressed design, have different λ's .

Table-14 Optimum Designs and FSD's of 10-Bar Truss Case I-b

Member (i)	Design 1		Design 2		Design 1,2	
	Area	λ_i	Area	λ_i	γ_i	Stress
1	8.1002	14.16	9.0000	14.88	-	-25.00
2	3.9001	3.12	3.0000	2.40	-	-25.00
3	0.1	2.52	1.0	2.16	30	25.00
4	0.1	2.52	1.0	2.16	30	25.00
5	7.9002	8.88	7.0001	8.16	-	25.00
6	0.1	-	1.0	-	36	0.
7	5.5156	6.24	4.2427	4.80	-	-25.00
8	5.7984	16.80	7.0711	18.24	-	25.00
9	0.1414	5.28	1.4142	6.72	-	-25.00
10	3.6771	-	2.8285	-	-	37.50
Total Weight	1497.64		1584.01			
	Design 3		Design 4		Design 3,4	
Member (i)	Area	λ_i	Area	λ_i	γ_i	Stress
1	11.8940	17.28	10.9450	17.28	-	-25.00
2	0.1061	0.	1.0605	0.	-	-25.00
3	3.8940	5.76	2.9409	5.76	-	25.00
4	3.8940	5.76	2.9410	5.76	-	25.00
5	4.1061	5.76	5.0608	5.76	-	25.00
6	0.1	-	1.0	-	36	0.
7	0.1500	0.	1.4981	0.	-	-25.00
8	11.1638	23.04	9.8197	23.04	-	25.00
9	5.5069	11.52	4.1591	11.52	-	-25.00
10	0.1	0.	1.0	0.	-63.64	37.50
Total Weight	1725.26		1701.03			

Design 1 ; Optimum Design, min. size = 0.1

Design 2 ; Optimum Design, min. size = 1.0

Design 3 ; FSD, min. size = 0.1

Design 4 ; FSD, min. size = 1.0

7. TRUSSES WITH DEFLECTION AND STRESS CONSTRAINTS.

Firstly zero and first order approximations to stress gradients in truss problems with deflection and stress constraints are discussed in section 7.1. How the use of a zero order approximation can affect the resulting design, is presented. A number of truss problems frequently appearing in the literature are solved and their results are compared with the results obtained by other methods in section 7.2. Among them the well known 10-bar truss shows interesting features.

7.1 Approximation to Stress Gradients

For the prolems involving deflection constraints a fully stressed design or an evenly stressed design, obtained from the stress ratio method, is seldom the optimum design. The optimality criteria methods are therefore preferably used for the problems falling into this category, but many of them treat the stress limits as side constraints by using the stress ratio based on the concept of fully stressing. Having found a new design from the optimality criteria, including only the deflection constraints, the methods resize the overstressed members using the stress ratio. The use of stress ratios for stress limits may improve the efficiency of the redesign but may give a wrong design.

The stress gradients obtained from Equ. (5.25), (5.40) or (5.44) with constant d_{ij} and M_j are of first order approximation to the true gradients in the whole design space, and exact at the current design. On the other hand, the approximation from the stress ratio is merely the coordinate vector of the design variable with which the member concerned is associated, and therefore referred to as zero order approximation. Sander and Fleury [29] presented a graphical comparison of the two kinds of approximation, and it is shown in Fig. 21.

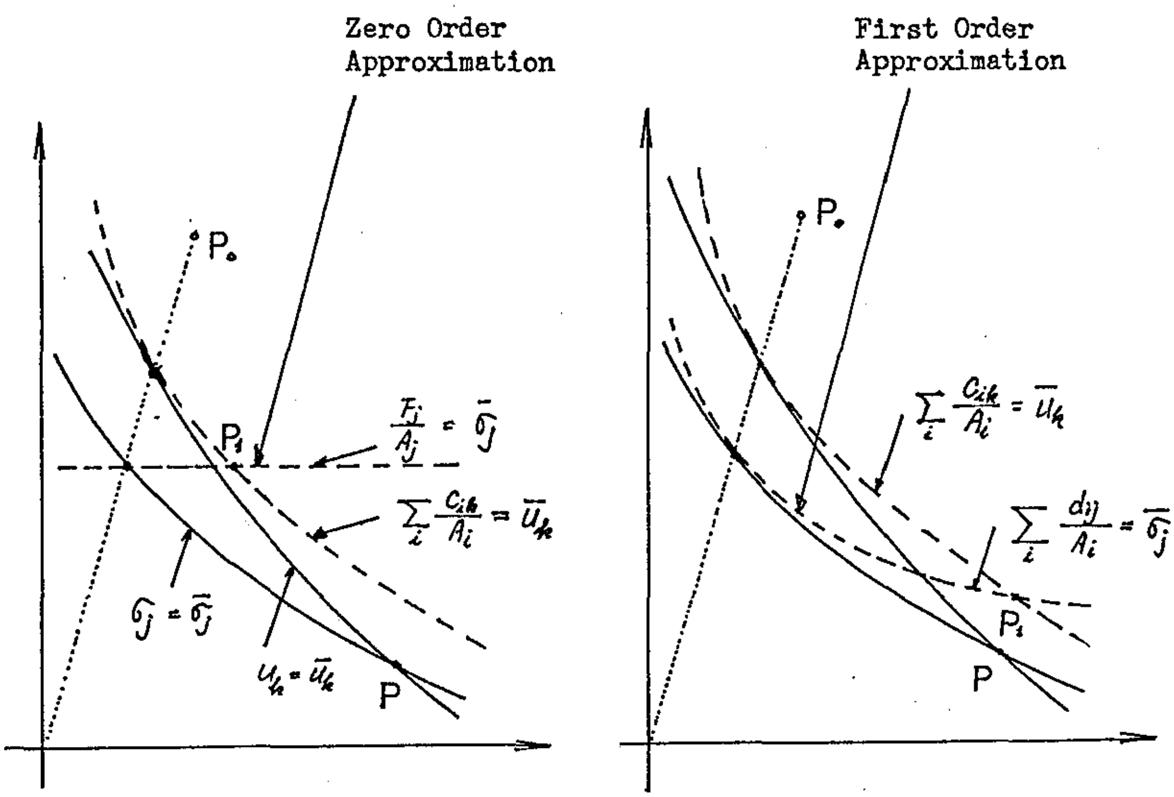


Fig. 21 Comparison of Approximations of Zero Order and First Order to True Stress Gradient.

The problems with a fewer number of active constraints than that of design variables may have a good number of designs which all satisfy the same set of active constraints in an equality sense. In this case the crude approximation to the stress gradients made by the stress ratio method can lead to a design with the same set of active constraints as the optimum design but different from it, and with very slow convergence.

Table-15 shows two quite different designs of 10-bar truss

Case III, Ex.T-9, which were obtained by the method of this thesis and the method using stress ratios respectively. In both designs, the vertical deflection at node a and the stresses of member 3 and 6 were the active constraints. It is interesting to note that the significantly heavier non-optimum design was due almost solely to the overestimated size of member 6 and at the optimum member 6 was kept at the minimum as well as fully stressed. Amazingly, the increase of the weight of member 6 by 29.6 pounds called for the increase of total weight by 290.6 pounds. As a matter of fact, the value of γ_6 at the optimum is 500, and therefore the total weight will increase by 500 units as the size of member 6 increases by one unit, although its own weight increases by only 36 units of weight. Two optimum designs in which $\underline{A_6}$ = 0.1 and $\underline{A_6}$ = 0.2 were imposed respectively are compared in Table-16 to demonstrate the usefulness of the values of the Lagrange multipliers. Each represents the sensitivity of the total weight to the corresponding constraint.

Table-15 Designs of 10-Bar Truss Case III

Member	Design 1	Design 2
1	13.7636	12.7996
2	6.2471	7.1239
3	1.9732	1.9702
4	0.1	0.1
5	7.6011	11.3854
6	0.1	0.9220
7	1.3662	5.9279
8	9.8195	6.5315
9	0.1	0.1
10	8.8347	10.0747
Weight	2096.62	2387.18

Design 1 ; by the method of this thesis
Design 2 ; by the method using stress ratio

Table-16 Change of Total Weight due to different Minimum Sizes on A_6

Member	Design 1 $A_6 = 0.1$	Design 2 $A_6 = 0.2$	Changes of Area	Changes of Weight
1	13.7636	13.6168	-0.1468	-5.2848
2	6.2471	6.4189	0.1718	6.1848
3	1.9732	1.9725	-0.0007	-0.025
4	0.1	0.1	0.	0.
5	7.6011	8.2235	0.6224	22.4064
6	0.1	0.2	0.1	3.6
7	1.3662	2.0862	0.7164	36.4731
8	9.8196	9.2311	-0.5885	-29.9615
9	0.1	0.1	0.	0.
10	8.8347	9.0777	0.2430	12.3715
Total Weight	2096.62	2142.39		45.77
γ_6	500	425		

Change of total weight calculated from γ_6
$= 0.1x(500+425)/2 = 46.25$

The two active constraints associated with member 6, its stress and the minimum size have different effects on the total weight. A more flexible member 6 would allow the size of members 1 and 8 to decrease as they become more stressed. So member 6 was fully stressed at the optimum. But its axial force has an adverse effect on the total weight. Greater axial force of member 6 makes node a deflect more. Thus the size of member 6 was kept at the minimum to reduce its axial force. Since the method using stress ratio decides the size of member 6 by its stress ratio without considering the effect of its axial force, it happened to give a bigger size to member 6 and resulted in the heavier non-optimum design.

The optimality test on the non-optimum design yielded a non-zero vector **P** defined in Eq. (5.89). The entries of the vector were zero except that corresponding to member 6. Therefore the test disproves the optimality and suggests that there exists a better design along the negative coordinate direction associated with member 6 as was expected.

7.2 Examples

Ex.T-10 10-Bar Truss Case I-c

This problem is one of the most frequently used examples in the literature and shows a number of interesting features. A number of results obtained by various methods are compared in Table-17.

The method of this thesis and Fleury et al [30], both using a first order approximation to the stress constraint, gave the best solution within a reasonable number of redesign iterations. In the solution, member 6 is not only at its minimum but also fully stressed as was the case of 10-bar truss Case III in the preceding section. It appears that the numerical instability and slow convergence encountered in the first solution of Taig et al [28] was largely due to the behaviour of member 6. The optimality criteria involving deflection constraints forced the size of member 6 to decrease in the earlier iterations. As the design approached to the optimum, the stress of member 6 increased and the stress ratio made the member bigger. In consequence the size of member 6 oscillated.

The second solution of Taig et al [28] was obtained by fixing member 3 at its minimum bearing in mind that the vertical deflections

Table-17 Comparison of Various Designs for 10-Bar Truss Case I-c

Member	Designs obtained by Gellatly (15)	Venkayya (13)	Taig-1 (28)	Taig-2 (28)	Schmit (46)	Berke (17)	Rizzi (21)
1	20.027	23.408	22.57	23.94	24.260	23.536	23.934
2	15.598	14.904	15.43	14.73	14.260	14.915	14.733
3	0.242	0.101	0.57	0.10	0.100	0.527	0.100
4	0.100	0.128	0.10	0.10	0.100	0.100	0.100
5	31.352	30.416	31.98	30.73	33.432	30.860	30.731
6	0.138	0.101	0.58	0.10	0.100	0.100	0.100
7	22.206	21.084	22.76	20.95	20.740	21.231	20.954
8	8.347	8.696	6.44	8.54	8.338	7.477	8.542
9	0.100	0.186	0.10	0.10	0.100	0.100	0.100
10	22.059	21.077	21.82	20.84	19.690	21.092	20.836
Total Weight	5112.13	5084.9	5167	5077	5089.0	5061.86	5076.66
No. of Iter.	18	25	32	8	23	28	11

Member	Designs obtained by Dobbs (25)	Taleb-Agha (47)	Khan (27)	Fleury (30)	Arora (23)		This Thesis
1	23.290	25.586	24.169	23.20	23.274		23.204
2	15.428	14.808	14.805	15.22	15.286		15.219
3	0.210	0.100	0.406	0.55	0.557		0.547
4	0.100	0.100	0.100	0.10	0.100		0.100
5	30.500	26.778	30.980	30.53	30.031		30.528
6	0.100	0.343	0.100	0.10	0.100		0.100
7	20.980	20.485	21.046	21.04	21.198		21.040
8	7.649	8.036	7.547	7.46	7.468		7.458
9	0.100	0.100	0.100	0.10	0.100		0.100
10	21.818	23.099	20.937	21.52	21.618		21.523
Total Weight	5080.0	5070.8*	5066.98	5060.85	5061.65		5060.87
No. of Iter.	15	24**	18	14	13		11

* Deflections at nodes a and b were violated.
If scaled, 5169.74.

** Number of Analyses.

at nodes a and b should reach the limit at the same time for the design to be optimum. This design, also obtained by Rizzi [21], was found to be another local minimum with a set of active constraints different from that of the design obtained by the method of this thesis. Table-18 shows these designs and other designs obtained when the problem was subject to different sets of constraints. Design 3 was obtained when the stress constraints were relieved expecting the same set of active constraints, deflections at nodes a and b, as in the Rizzi's solution. However the method found different active constraints, deflections at nodes b and c, and resulted in a better design as shown in Table-18. In this design the stress of member 6 reached 35.5 Ksi, which made the deflection at node c at the limit. Another trial was made assuming that the deflection constraints were imposed only on nodes a and b, and resulted in Design 4 in which the deflection at b was the only active constraint.

In all designs, Design 1, 3 and 4, deflections at nodes a and b used to make the set of active constraints one time in the iteration history but eventually deflection at node a was deleted from the set as shown in Table-19. It appears that the valley where the local minimum obtained by Rizzi [21], Design 2 in Table-18 and Fig. 22, resides is narrow and thus the redesign process seldom goes into the valley. But it could not get out of the valley when Design 2 was used as the starting values. It is also notable that the design converged very quickly to the optimum once the right set of active constraints were identified. For Design 1, 3, 4 and 5, only two or three redesign iterations were sufficient. Fig. 22 depicts an imaginary design space map of the problem and locates the designs in Table-18 on the map.

Table-18 Designs of 10-Bar Truss Case I-c and its Variations

Member	Design 1	Design 2	Design 3	Design 4	Design 5
1	23.204	23.934	22.433	22.107	18.998
2	15.219	14.733	15.259	15.461	12.000
3	0.547	0.1	0.961	1.434	0.1
4	0.1	0.1	0.1	0.1	0.1
5	30.528	30.731	30.913	31.377	24.384
6	0.1	0.1	0.1	0.1	0.1
7	21.040	20.954	21.839	22.306	16.625
8	7.458	8.542	5.803	4.102	6.835
9	0.1	0.1	0.1	0.1	0.1
10	21.523	20.836	21.579	21.865	16.970
Total Weight	5060.87	5076.66	5022.55	5003.51	4068.00
Active Constraints	node b member 6	node a, b	node b, c	node b	node a, b member 6

Design 1 ; Design of this thesis.

Design 2 ; Design of Rizzi(21) and Taig(28).

Design 3 ; 2.0 inch deflection limits on all nodes, no stress limits.

Design 4 ; 2.0 inch deflection limits on nodes a and b, no stress limits.

Design 5 ; 2.5 inch deflection limits on all nodes, ± 25 Ksi stress limits on all members.

Table-19 Sets of Active Constraints in Ex.T-10

Iteration	Design 1	Design 3	Design 4
– 7	node a	node a	node a
8	node a,b	node a,b	node a,b
9	node a,b member 6	node a,b	node a,b
10	node b member 6	node b	node b
11 –	node b member 6	node b,c	node b

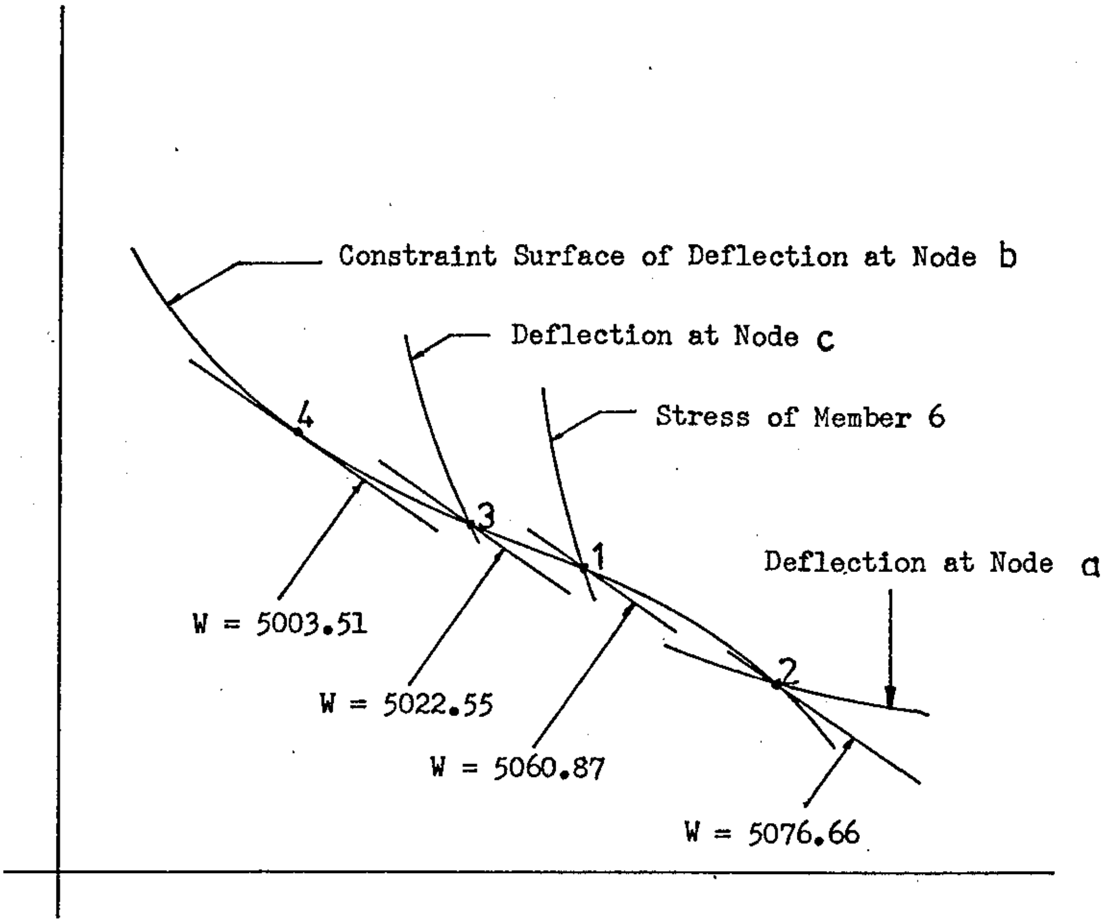


Fig. 22 Design Space Map for 10-Bar Truss Case I-c

Design 5 in Table-18 was obtained when the problem was subject to the deflection constraints of 2.5 inch imposed on all nodes and the stress limits of 25 Ksi on all members. In this design member 3 was at the minimum and both deflections at nodes a and b were active as in Design 2 but accompanying stress of member 6.

Ex.T-11 10-Bar Truss Case II

This problem has also been a popular example. The result given by the method of this thesis is compared favourably with other results in Table-20. Member 6 shows a particular behaviour also in this problem. When the problem was solved by the method using optimality criteria and stress ratios, the design was 7.5% heavier than the design obtained in this thesis.

Table-20 Comparison of Various Designs for 10-Bar Truss Case II

Member	Designs obtained by						
	Venkayya (13)	Schmit (46)	Dobbs (25)	Taleb-Agha(47)	Khan (27)	Rizzi (21)	This Thesis
1	25.419	23.346	27.233	19.767	26.541	25.291	25.285
2	14.327	13.654	16.653	14.404	13.219	14.374	14.375
3	3.144	1.970	2.024	1.969	4.835	1.970	1.970
4	0.363	0.100	0.100	0.100	0.100	0.100	0.100
5	25.190	24.290	25.813	23.130	24.716	23.533	23.531
6	0.417	0.100	0.100	0.205	0.108	0.100	0.100
7	14.612	12.544	14.218	11.281	13.775	12.825	12.828
8	12.083	12.670	12.776	12.534	12.664	12.389	12.391
9	0.513	0.100	0.100	0.100	0.100	0.100	0.100
10	20.261	21.971	22.137	25.320	18.438	20.328	20.329
Total Weight	4895.60	4691.8	5059.7	4651.2*	4792.52	4676.92	4676.93
No. of Iter.	12	22	12	23**	9	12	9

*Deflection at node a was violated.
If scaled, 4861.51.

**Number of analyses.

Ex.T-12 25-Bar Truss Case III

This problem is the same as Ex.T-1, 25-bar truss Case I, shown in Fig. 15 except that deflection limits of 0.35 inch are imposed on the nodes. The final design shown in Table-7 is one of the best among those appearing in the literature. The deflection constraints at the top nodes in y-direction and the stress constraint of member 20 were active. The method using stress ratios as well as optimality criteria yielded almost the same design and the optimality test proved its optimality.

Ex.T-13 72-Bar Truss Case II

This problem is illustrated in Fig. 17 and the same as Ex.T-4, 72-bar truss Case I, except the magnitudes of the applied loads and the deflection constraints. The final design shown in Table-10 is also one of the best presented so far in terms of both accuracy and efficiency. For this problem and 25-bar truss Case III, comparisons between the results of this thesis and those presented in the literature are prepared in Table-21 in terms of the total weight achieved and the number of iterations required.

The method using stress ratios as well as optimality criteria yielded a design close to the optimum design, but the optimality test disproved its optimality. Both designs involved the same set of active constraints, the deflections at node 1 in x- and y-direction and the stress of the member connecting nodes 1 and 5.

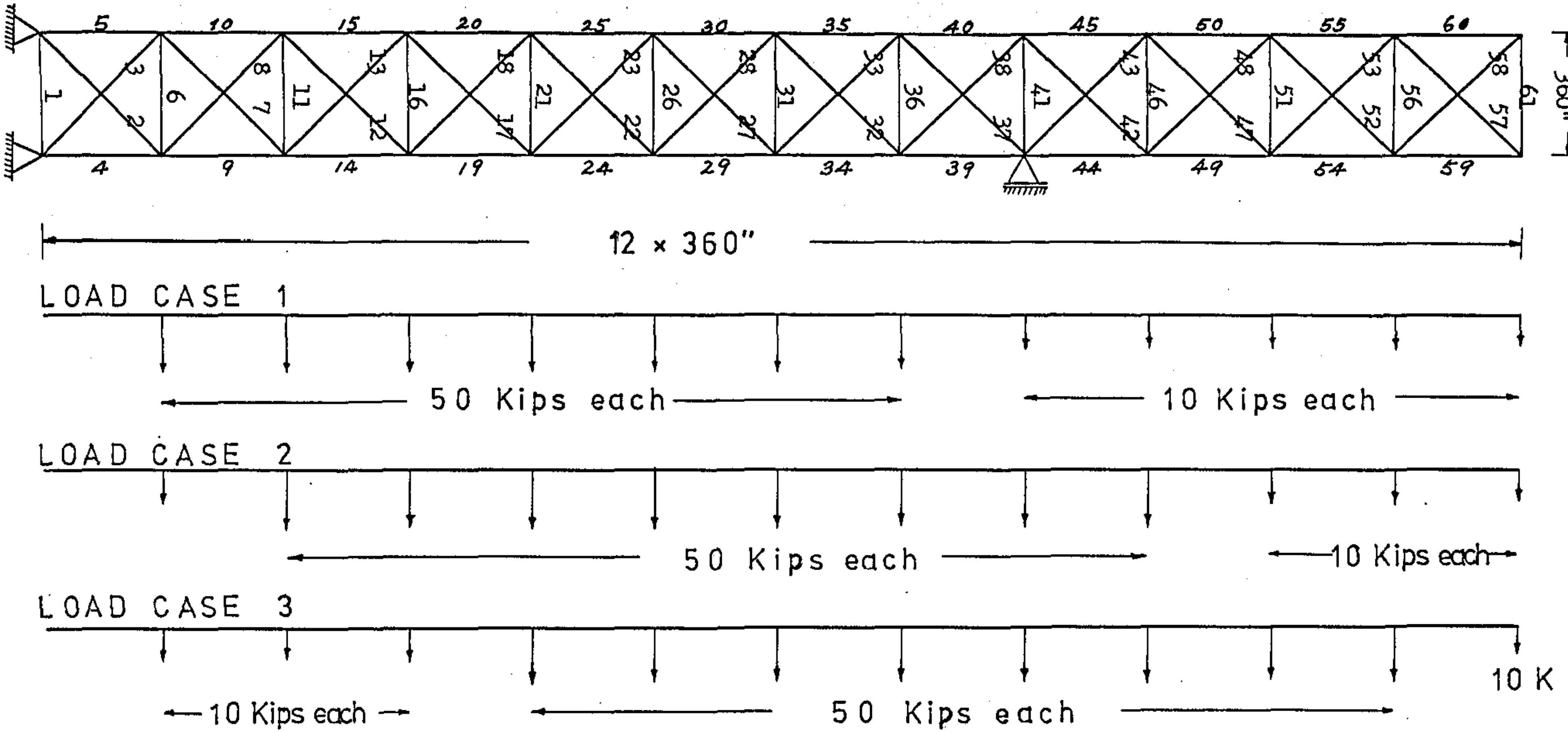
Table-21 Comparison of the Designs for 25- and 72-Bar Trusses

Method	25-Bar Truss		72-Bar Truss	
	Iter.	Weight	Iter.	Weight
Gellatly (15)	7	545.36	8	395.97
Venkayya (13)	6	545.49	11	381.2
Venkayya (16)	-	-	4	381.1
Templeman (4)	7	545.32	-	-
Taig (28)	-	-	5	379.6
Schmit (46)	15	545.23	21	388.6
Terai (48)	17	551.6	-	-
Berke (17)	-	-	3	379.67
Rizzi (21)	10	545.163	-	-
Dobbs (25)	10	553.4	-	-
Fleury (30)	6	545.23	5	379.66
Khan (27)	8	553.94	9	387.67
This Thesis	7	545.166	3	379.622

Ex.T-14 61-Bar Truss Case I

This problem is illustrated in Fig. 23 and its two different designs are shown in Table-22. Design 1 was obtained from the method of this thesis and Design 2 came from the method using stress ratios in addition to optimality criteria. This example also demonstrates that the use of stress ratios can lead to a wrong design.

The set of active constraints in Design 1 contained one deflection and fourteen stress constraints. But Design 2 involved two more stress constraints, the stresses of members 15 and 21.



Material Data		CASE I	CASE II
$E = 3 \times 10^7$ psi	Deflection Limit :	*6" on upper nodes*	*6" on lower nodes*
$\rho = 0.28$ pci	Minimum Size :	0,1 in²	0,5 in²
Stress Limit : 25 Ksi			support at node 2 removed.

Fig. 23 61-Bar Truss

Table-22 Designs of 61-Bar Truss

Design Variable Number	Member	Case I			Case II	
		Design 1		Design 2		
		Area	O.C.*	Area	Area	O.C.*
1	1	0.1000	0.0000	0.1000	3.1996	1.0000
2	2,3	6.0857	0.9994	6.5718	4.6678	1.0000
3	4,5	9.5085	0.9969	11.4292	3.3019	1.0000
4	6	2.0363	1.0058	0.1000	0.5000	0.1672
5	7,8	4.7602	0.9988	5.2384	4.0259	1.0000
6	9,10	2.3986	0.9890	4.0888	9.3471	1.0000
7	11	1.1739	1.0084	0.1000	0.5000	0.0568
8	12,13	3.4407	0.9992	3.7623	2.4132	1.0000
9	14,15	4.0197	1.0055	2.2759	12.6249	1.0000
10	16	0.8955	1.0007	0.1000	0.5000	-0.6927
11	17,18	1.9227	0.9987	2.3483	0.9016	1.0002
12	19,20	6.4998	1.0027	5.6735	15.2378	1.0000
13	21	0.6953	0.9984	0.2302	1.3866	0.9995
14	22,23	1.7396	1.0007	1.7108	2.7072	1.0001
15	24,25	7.8024	1.0017	7.0298	14.7195	1.0000
16	26	0.1433	0.9934	0.1173	0.5000	-0.5014
17	27,28	3.0376	1.0002	2.9866	3.8742	1.0000
18	29,30	7.0304	1.0010	6.4689	11.6142	1.0000
19	31	0.1000	0.0725	0.1000	0.5000	0.0248
20	32,33	4.3921	1.0001	4.3919	4.9687	1.0000
21	34,35	10.3228	1.0001	10.2873	10.9712	1.0000
22	36	0.3865	1.0004	0.3555	0.5000	0.3592
23	37,28	3.9815	1.0001	4.9281	5.7434	1.0000
24	39,40	17.4357	1.0000	17.8287	14.9705	1.0000
25	41	6.7906	0.9999	6.7509	6.7875	1.0000
26	42,43	5.1457	1.0000	5.2499	4.6023	1.0000
27	44,45	17.3211	1.0000	17.7011	14.8852	1.0000
28	46	0.4191	1.0007	0.4064	0.5000	0.3052
29	47,48	4.2314	1.0000	4.3188	3.9475	1.0000
30	49,50	10.1439	1.0000	10.3660	8.7233	1.0000
31	51	0.1000	-0.0795	0.1000	0.5000	0.0237
32	52,53	3.1766	1.0000	3.2437	2.6768	1.0000
33	54,55	4.4586	1.0000	4.5558	3.8129	1.0000
34	56	1.0151	0.9999	1.0433	0.5000	-0.7453
35	57,58	1.0240	1.0000	1.0423	1.1770	1.0000
36	59,60	0.7242	1.0000	0.7372	0.8324	1.0000
37	61	0.1000	-11.9673	0.1000	0.5000	0.5289
No. of Iterations		18		18	8	
Total Weight		33623		34032	37943	

* value of Equ. (5.54)

Ex.T-15 61-Bar Truss Case II

This problem is also illustrated in Fig. 23 and the final design is shown in Table-22. The set of active constraints of the design contained two deflection and nine stress constraints. The values of Equ. (5.54), the optimality criteria equation, for the twenty-seven Group 1 design variables were all 1.0000 except three when evaluated after 8 redesign iterations.

Ex.T-16 124-Bar Truss Case II

The 124-bar truss problem solved by Sheu[45] is illustrated in Fig. 18. Table-23 provides comparisons between the designs by Sheu[45] and this thesis and the iteration histories. It is noted that the redesign process of this thesis was faster and resulted in a little different design.

Ex.T-17 200-Bar Truss

The 200-bar truss problem shown in Fig. 24 was first solved by Venkayya et al[12] and later by Arora et al[42], Venkayya[49], Fleury[31] and Khan et al[27]. Table-24 shows that the designs by Venkayya et al[12] and this thesis are quite different. Iteration history and comparisons with other results appear in Table-25.

Table-23 124-Bar Truss Case II

(a) Comparison of the Design Values

Member	This Thesis	Sheu(45) Method 2	Member	This Thesis	Sheu(45) Method 2
5-8	0.1363	0.1252	73-76	0.1	0.1021
13-16	0.1260	0.1229	77-80	0.4269	0.4056
21-24	0.1138	0.1267	81-84	0.1388	0.1566
25-28	0.7205	0.6865	85-88	0.2365	0.2370
29-32	0.6976	0.6977	89-92	0.1220	0.1255
33-36	0.5351	0.5645	103-104	0.1127	0.1358
37-40	0.2312	0.2738	109-110	0.2957	0.2815
41-44	0.1162	0.1168	111-112	0.1	0.1013
47-48	0.1	0.1459	113-114	0.1501	0.1401
49-50	0.2015	0.1657	115-116	0.2414	0.2208
51-52	0.1732	0.1317	117-118	0.2945	0.2815
57-58	0.1428	0.1718	119-120	0.1	0.1307
68	0.1	0.1307	121-122	0.1593	0.1822
69	0.1	0.1013	123-124	0.2323	0.1972
72	0.1340	0.1500			
Weight				126.77	127.29

Other member sizes are all at their minima (0.1) in both designs.

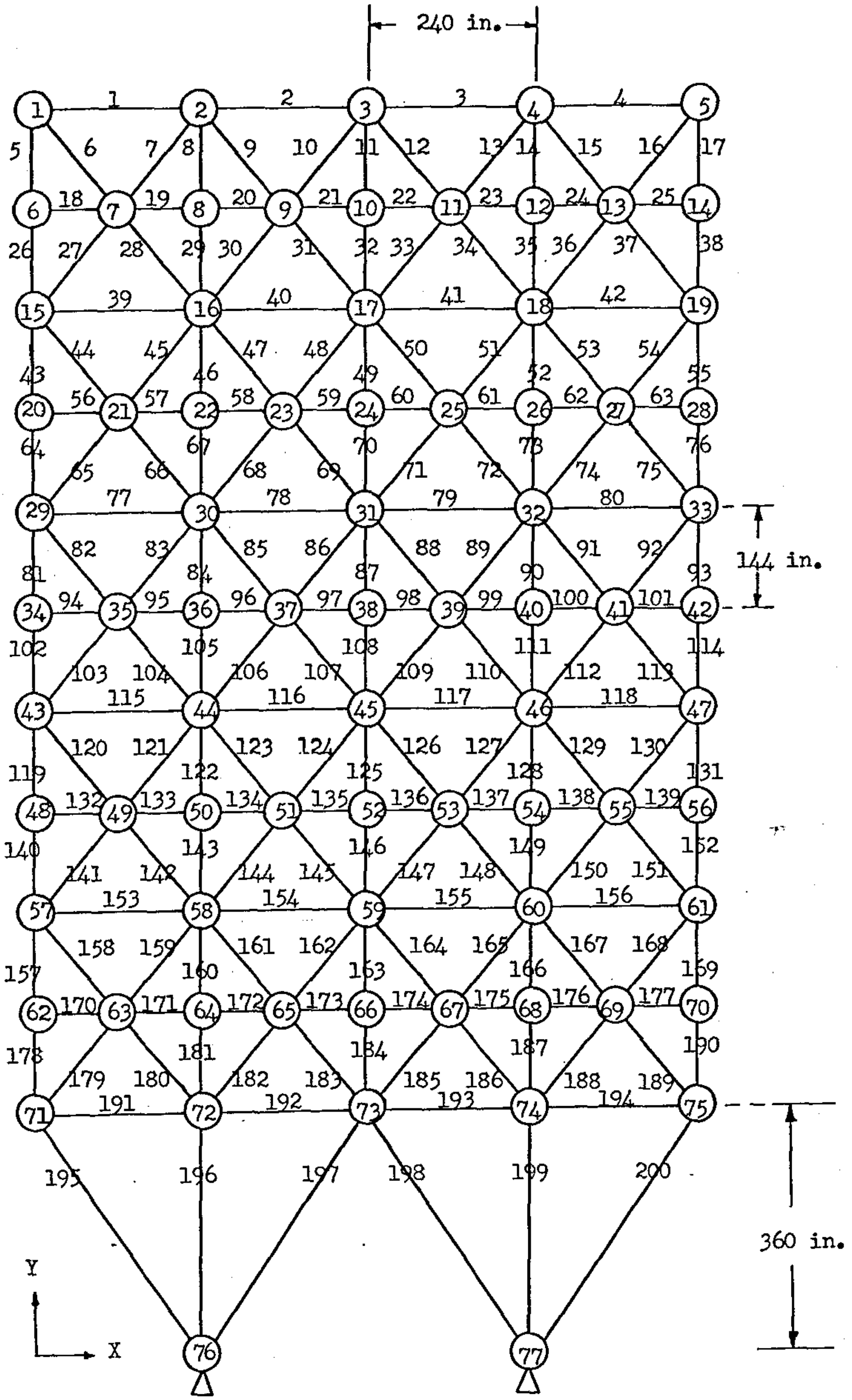
(b) Comparison of the Iteration Histories

Number of Analyses	Total Weight		
	This Thesis	Sheu(45) Method 1	Sheu(45) Method 2
1	204.39	204.09	204.09
2	134.84	186.78	186.78
3	129.85		154.87
4	128.33	143.92	141.80
5	127.63		136.20
6	127.33	136.81	
7	126.98		132.93
8	126.87		
9	126.84	130.68	130. 87
10	126.81		
11	126.79	129.04	128.85
12	126.77		
13			128.08
15			127.49
18			127.29

Table-24 Designs of 200-Bar Truss

El. No	Area	El. No	Area	El. No	Area	El. No	Area	El. No	Area
1	0.182 1.348	40	0.1 0.233	81	9.592 5.737	120	2.545 2.558	159	0.1 0.210
2	0.1 1.313	43	8.073 4.798	82	1.136 1.988	121	0.1 0.237	160	12.609 14.981
5	4.765 3.402	44	0.321 1.850	83	0.1 0.201	122	8.832 10.649	161	1.707 1.175
6	0.214 1.771	45	0.1 0.127	84	6.358 7.220	123	0.405 0.966	162	0.505 1.251
7	0.1 0.173	46	4.485 4.318	85	0.407 0.984	124	0.714 0.991	163	7.937 9.800
8	2.357 1.497	47	0.292 0.971	86	0.414 0.797	125	6.914 7.822	170	0.1 0.116
9	0.126 0.742	48	0.228 0.749	87	5.997 5.626	132	0.1 0.116	171	0.1 0.816
10	0.130 0.782	49	4.673 3.346	94	0.1 0.116	133	0.156 0.634	172	0.1 0.816
11	2.708 1.156	56	0.1 0.116	95	0.1 0.491	134	0.156 0.634	173	0.125 0.703
18	0.1 0.116	57	0.1 0.333	96	0.1 0.491	135	0.215 0.512	178	8.713 6.713
19	0.1 0.377	58	0.1 0.333	97	0.113 0.318	140	10.391 7.285	179	0.101 0.713
20	0.1 0.377	59	0.1 0.208	102	10.573 6.688	141	0.180 0.587	180	4.044 4.281
21	0.1 0.435	64	9.334 5.662	103	0.119 0.533	142	2.686 2.835	181	13.113 16.104
26	6.755 4.575	65	0.1 0.519	104	1.229 2.151	143	9.361 11.752	182	0.530 1.309
27	0.1 0.538	66	0.413 1.950	105	6.926 8.288	144	0.730 1.049	183	1.747 1.317
28	0.294 1.895	67	5.163 5.326	106	0.427 0.884	145	0.437 1.011	184	8.937 10.950
29	3.351 2.483	68	0.254 0.813	107	0.430 0.984	146	7.447 8.969	191	6.139 5.073
30	0.165 0.750	69	0.320 0.954	108	6.611 6.770	153	2.445 2.495	192	3.833 3.243
31	0.177 0.784	70	5.435 4.495	115	1.539 1.687	154	0.816 1.024	195	11.151 8.983
32	3.903 2.278	77	0.644 1.391	116	0.820 0.605	157	8.025 5.695	196	17.098 20.687
39	0.125 1.294	78	0.446 0.343	119	9.577 6.274	158	3.976 3.932	197	7.892 9.594

1 0.182 ; Design of this thesis
1.348 ; Design by Venkayya - Ref. 12



(a) Configuration and Node/Element Numbers

Fig. 24 200-Bar Truss

-continue

Material Data

Material ; Steel

$E = 30 \times 10^6$ psi

$\rho = 0.283$ pci

Stress Limit ;±10,000 psi

Min. Size ; 0.1 in^2

Applied Loads

Load Case 1 ; 1 Kips acting in positive X direction at nodes 1, 6, 15, 20, 29, 34, 43, 48, 57, 62, 71.

Load Case 2 ; 1 Kips acting in negative Y direction at nodes 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19, 20, 22, 24, ---- 71, 72, 73, 74, 75.

Load Case 3 ; Load Case 1 and 2 acting together.

Note

The original problem set by Venkayya 12) was subjected to 5 loading cases, but it can be redeuced by symmetry and design variable linking to 3.
0.5 in. of deflection limits were imposed on all nodes.

(b) Other Design Information

Fig. 24 200-Bar Truss

Table-25 200-Bar Truss - Iteration History and Comparisons with Other Results.

No. of Iter.	This Thesis Stress Limit (±10 Ksi)	Other Results		
		Total Weingt	Method	Remark
8	29,091	32,996	Khan (27)	Stress Limit; ±10 Ksi
9	29,073	-	-	
10	29,055	-	-	
11	29,067	-	-	
12	29,041	29,700	Venkayya(49)	Stress Limit; ±30 Ksi
13	29,020	29,037	Fleury (31)	max. stress ;10,623 psi
14	29,009	-	-	
15	29,001	28,963	Arora (42)	Stress Limit; ±30 Ksi
16	28,992	-	-	
-	-	31,020	Venkayya(12)	Stress Limit; ±10 Ksi No. of Iterations, unknown

8. CONTINUOUS BEAMS WITH TAPERED ELEMENTS

A range of beam problems and their solutions are presented in this chapter. They are 2 to 5 span continuous beams assembled with tapered beam elements. The design process decides the depths at the nodes and these depths decide the tapered configurations of elements so as to maintain continuity of structure at the element boundaries. Deflection and/or stress constraints are imposed on nodes and the loads are applied only to nodes. The shape of sections is rectangular or I-shape. The term "cost" throughout this chapter means either the total weight or the cost defined by Equ. (5.5) or Equ. (5.37).

The problems and their solutions are illustrated in the figures 25 to 34. The nodes at which deflection and/or stress constraints are active are indicated in the figures by *d and/or *s representing active deflection and stress constraints respectively. The design values also appear in tables 26 to 33 together with the values of the optimality criteria equations. These values are given to show to what extent the designs satisfy the optimality criteria. No other results are available in the literature for the results of this thesis to be compared with.

<u>Ex.B-1 2-Span Beam Case I</u>

A simple 2-span beam was taken first so as to show the ways of design variable linking and their effects. The problem and a number of solutions under various conditions appear in Fig. 25. The self-weight of the beam was neglected, its section was of I-shape having constant flange areas, 4000 mm^2, and the cost function was linear. The minimum size was 300 mm for all design variables. This problem, Ex.B-1, was

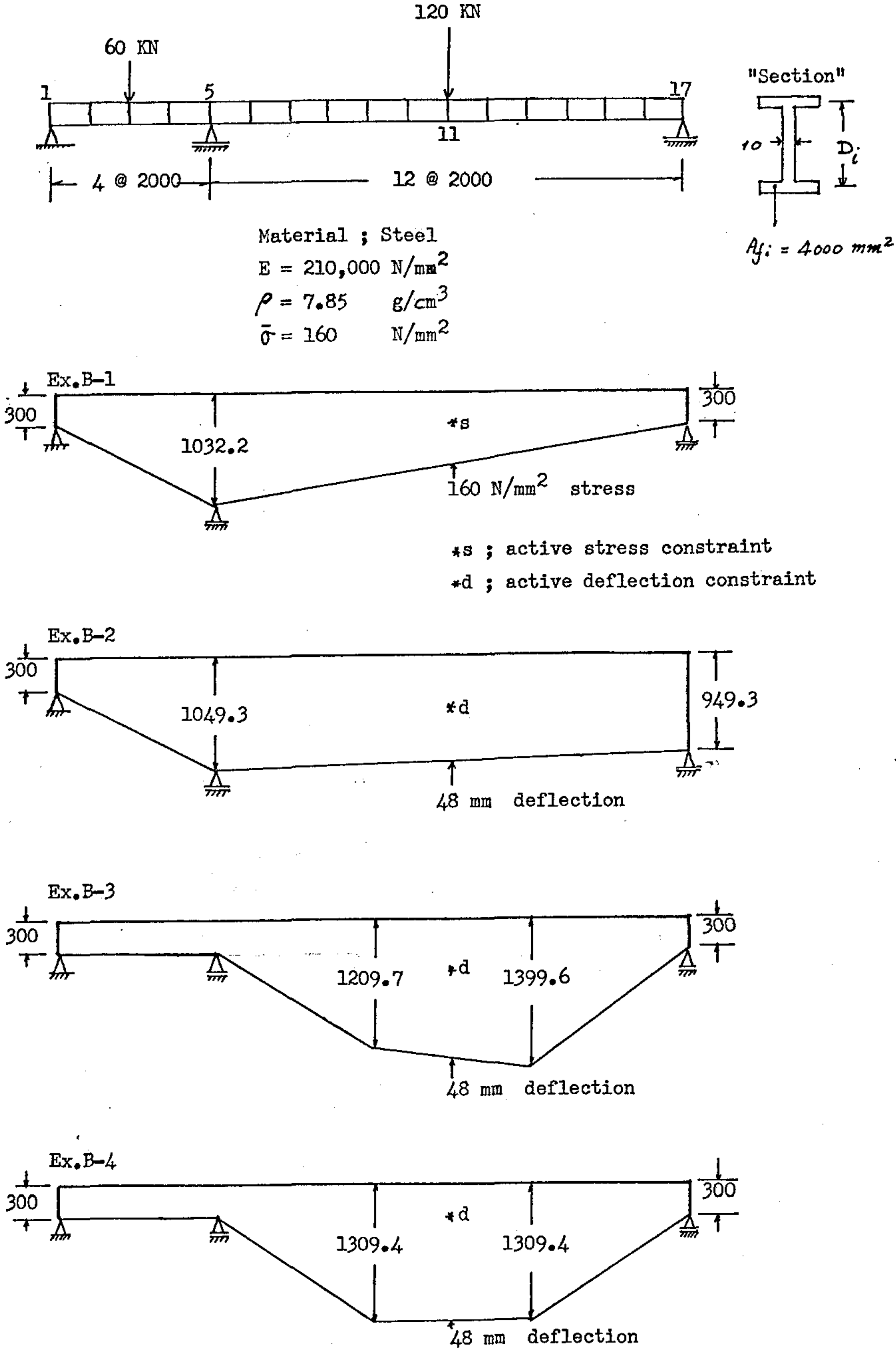


Fig 25 Designs of 2-Span Beam

subject only to stress constraints at nodes and as shown in Fig. 25 the stress at node 11, to which the point load 120 KN was applied, reached the permitted stress, 160 N/mm^2. The design values and other results are shown in Table-26 and Table-34.

Table-26 Designs of 2-Span Beam

Design Vari.	Ex.B-1		Ex.B-2		Ex.B-3		Ex.B-4	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	300	0.5654	300	0.4966	300	-0.0483	300	-0.0551
2	1032	0.09989	1049	1.0019	300	0.8909	300	0.8237
3	300	0.3021	949	0.9976	1210	1.0017	1309	1.0007
4	-	-	-	-	1400	0.9998	300	0.6489
5	-	-	-	-	300	0.5604	-	-
Cost of web	1673290		2306300		2015530		2021540	
No. of Iter.	16		20		22		14	

minimum size ; 300 mm

(1) Design values, i.e. depth of the beam

(2) Values of optimality criteria equation, Equ. (5.59).

Ex.B-2 2-Span Beam Case II

This problem is the same as Ex.B-1 but deflection constraints were imposed on the midspans, nodes 3 and 11, in addition. Each deflection limit was set to a five hundredth of the span length. The design values and other results are shown in Fig. 25 and Table-26, and Table-34. The deflection at node 11 reached the limit, 48 mm, and no other constraints, deflection or stress, were active.

Ex.B-3 2-Span Beam Case III

This is the same as Ex.B-2 but with a different way of design variable linking. The elements of the right span were divided into three groups, each having the same rate of tapering. Therefore, the number of design variables changed from 3 into 5. The deflection at node 11 was also active. The design values and other results of this problem are also shown in Fig. 25 and Table-26, and Table-34.

Ex.B-4 2-Span Beam Case IV

This is the same as Ex.B-3 but the middle part of the right span was made to have the same depth by linking further the two design variables governing the depths of the part. In consequence the number of design variables was reduced by one. As in Ex.B-2 and Ex.B-3, the deflection at node 11 was active, and the design values and other results are shown in Fig. 25, Table-26, and Table-34. It is interesting to note that the depth at node 5 has been set to the minimum in Ex.B-3 and Ex.B-4. The maximum bending stress at the node was 117 and 116 N/mm^2 respectively, both well below the permitted value 160 N/mm^2.

Ex.B-5 3-Span Beam Case I

A 3-span beam is shown in Fig 26. It is assembled with 30 elements and subject to 3 load cases. The self-weight of this beam was not taken into account. The minimum size restriction was 400 mm for all design variables.

The first problem concerning this beam, Ex.B-5, was assumed

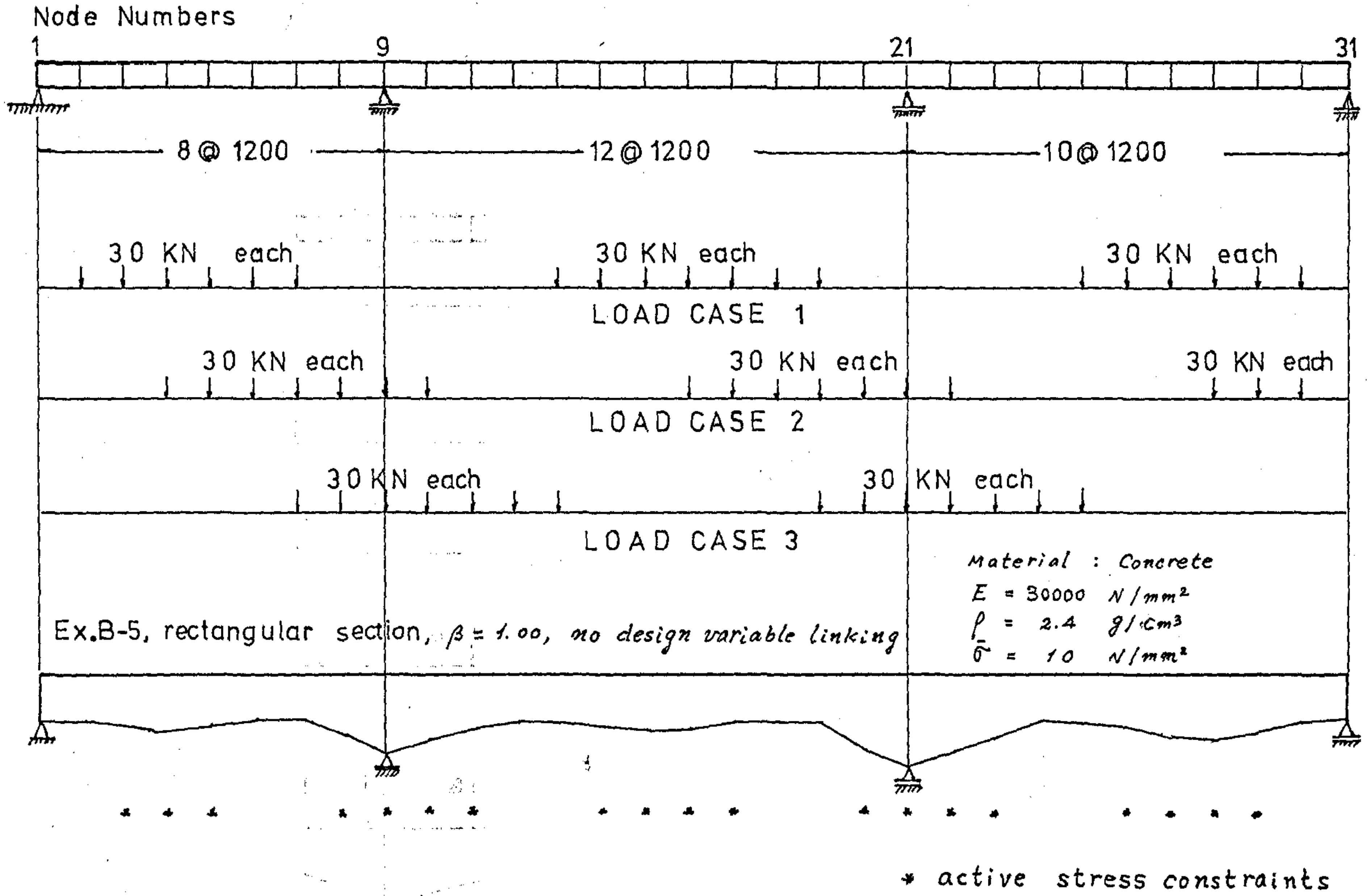


Fig. 26 3-Span Beam & Ex.B-5

to have rectangular sections with a constant breadth, 400 mm for all elements. Stress limit of 10 N/mm^2 was imposed on the sections of all nodes, but neither deflection constraints nor design variable linking were adopted. The cost function of this problem was assumed to be linear. The design values, the values of the optimality criteria equation, Equ. (5.57), and other results are given in Fig. 26, Table-27, and Table-34.

Table-27 Design of 3-Span Beam Case I, Ex.B-5.

(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
1	400	0.0111	11	432	1.0031	21	788	1.0005
2	400	0.0626	12	400	-0.0058	22	667	1.0007
3	437	0.9988	13	400	0.2421	23	519	1.0012
4	454	0.9985	14	443	0.9968	24	400	0.2113
5	423	0.9982	15	484	0.9972	25	400	-0.0595
6	400	0.0875	16	467	0.9969	26	451	0.9984
7	400	-0.2433	17	401	0.9974	27	520	0.9988
8	526	1.0018	18	400	-0.1306	28	533	0.9991
9	692	1.0013	19	400	0.1305	29	493	0.9992
10	577	1.0019	20	623	1.0011	30	400	-0.0442
						31	400	-0.0088

Cost of the beam ; 16507800

No. of iterations; 11

minimum size ; 400 mm

(1) Design variable numbers

(2) Depths of the beam, design values.

(3) Values of optimality criteria equation, Equ. (5.57)

Ex.B-6 3-Span Beam Case II

This problem is the same as Ex.B-5 but with deflection limits of 6 mm, 10 mm and 8 mm imposed on the midspans respectively and design variable linking. The design values and other results are shown in Fig. 27, Table-28 and Table-34. The deflections at the midspans were all active and the stress of one node (29) was the only active stress constraint. The stresses at the intermediate supports, nodes 9 and 21, were only 80% and 70% of the permitted value respectively. This fact shows that deflection constraints are rather predominant in this problem and the depths at the supports have been decided by the stiffness requirements.

Table-28 Designs of 3-Span Beam

Design Variable	Ex.B-6		Ex.B-7		Ex.B-8	
	(1)	(2)	(1)	(2)	(1)	(2)
1	400	0.1438	400	0.1951	400	-0.0005
2	521	0.9992	598	1.0008	432	0.9978
3	791	1.0031	1568	0.9978	1725	1.0013
4	443	0.9978	400	0.1564	400	-0.0015
5	1035	1.0009	2167	1.0008	2058	0.9977
6	499	0.9995	656	0.9985	689	1.0013
7	403	1.0004	450	0.9999	451	0.9981
Cost	19930600		230859		226411	
No. of Iter.	18		17		14	

minimum size ; 400 mm

(1) Design values.

(2) Values of optimality criteria equation, Equ. (5.57) or (5.59)

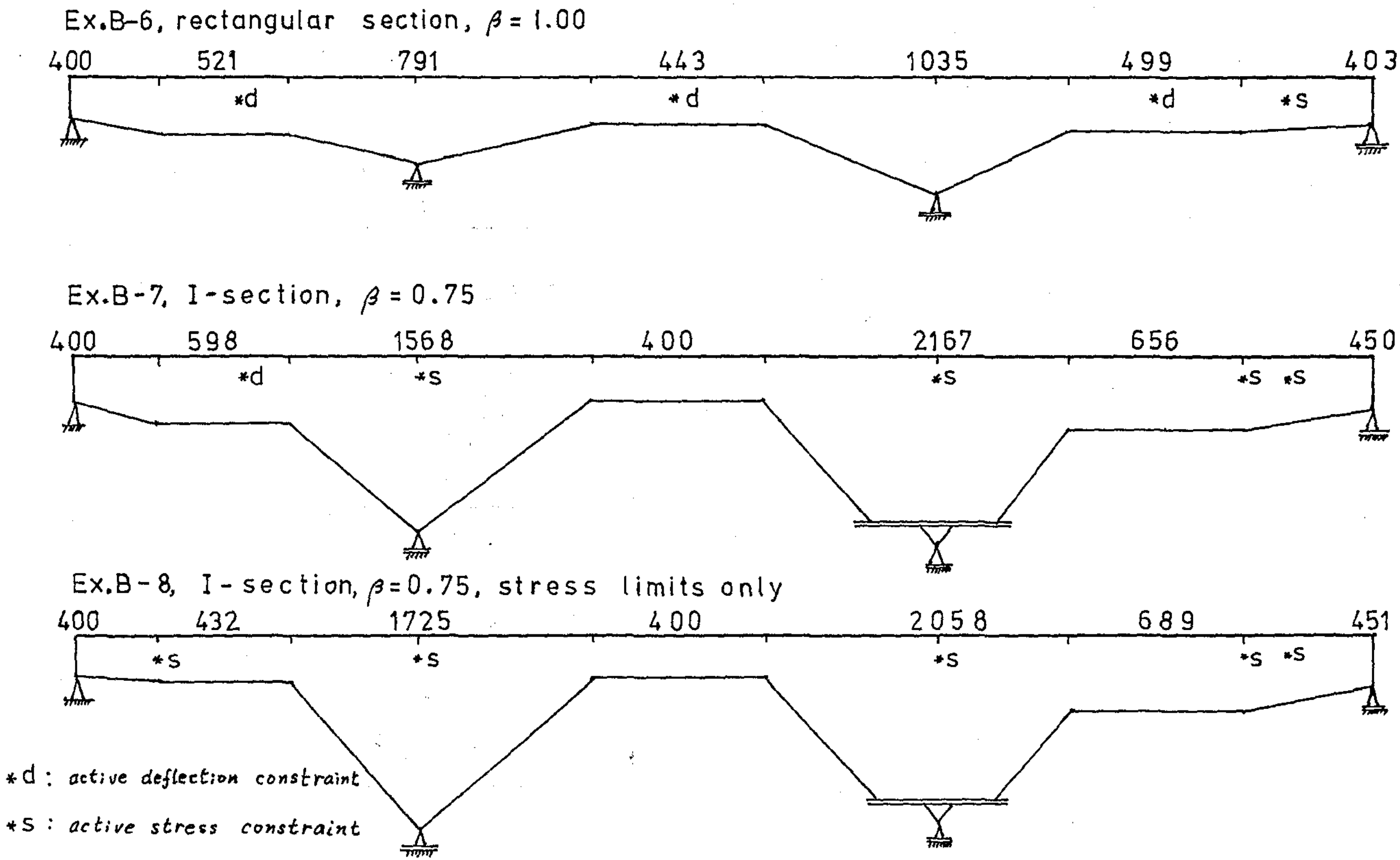


Fig. 27 Various Designs of 3-Span Beam

Ex.B-7 3-Span Beam Case III

This problem is the same as Ex.B-6 but with I-shape sections and the nonlinear cost function. The flanges of I-shape sections were assumed to have a constant cross sectional area, 24000 mm^2, and the exponent used in the nonlinear cost function, Equ. (5.37), was set to 0.75, i.e. $\beta = 0.75$. The design and other results are shown in Fig. 27, Table-28 and Table-34. In this design the stresses at the intermediate supports were all active and called for very deep sections whereas the section at the midst, node 15, was understressed (77% of the permitted) and set to the minimum depth. The design values were decided generally by strength requirements.

Ex.B-8 3-Span Beam Case IV

This problem is the same as Ex.B-7 except that no deflection constraints are imposed. The resulting design, shown in Fig. 27 and Table-28, is similar to that of Ex.B-7. Instead of the deflection constraint at node 5, the midst of the first span, the stress at a nearby node, node 3, was active.

Ex.B-9 4-Span Beam Case I

The configuration of this beam, the applied loads for each of the three load cases and other design conditions are illustrated in Fig. 28. The problems concerning this beam take into account the self-weight of the beam.

However, the first problem of this beam, Ex.B-9, was solved for two cases, neglecting and considering the self-weight. The beam was

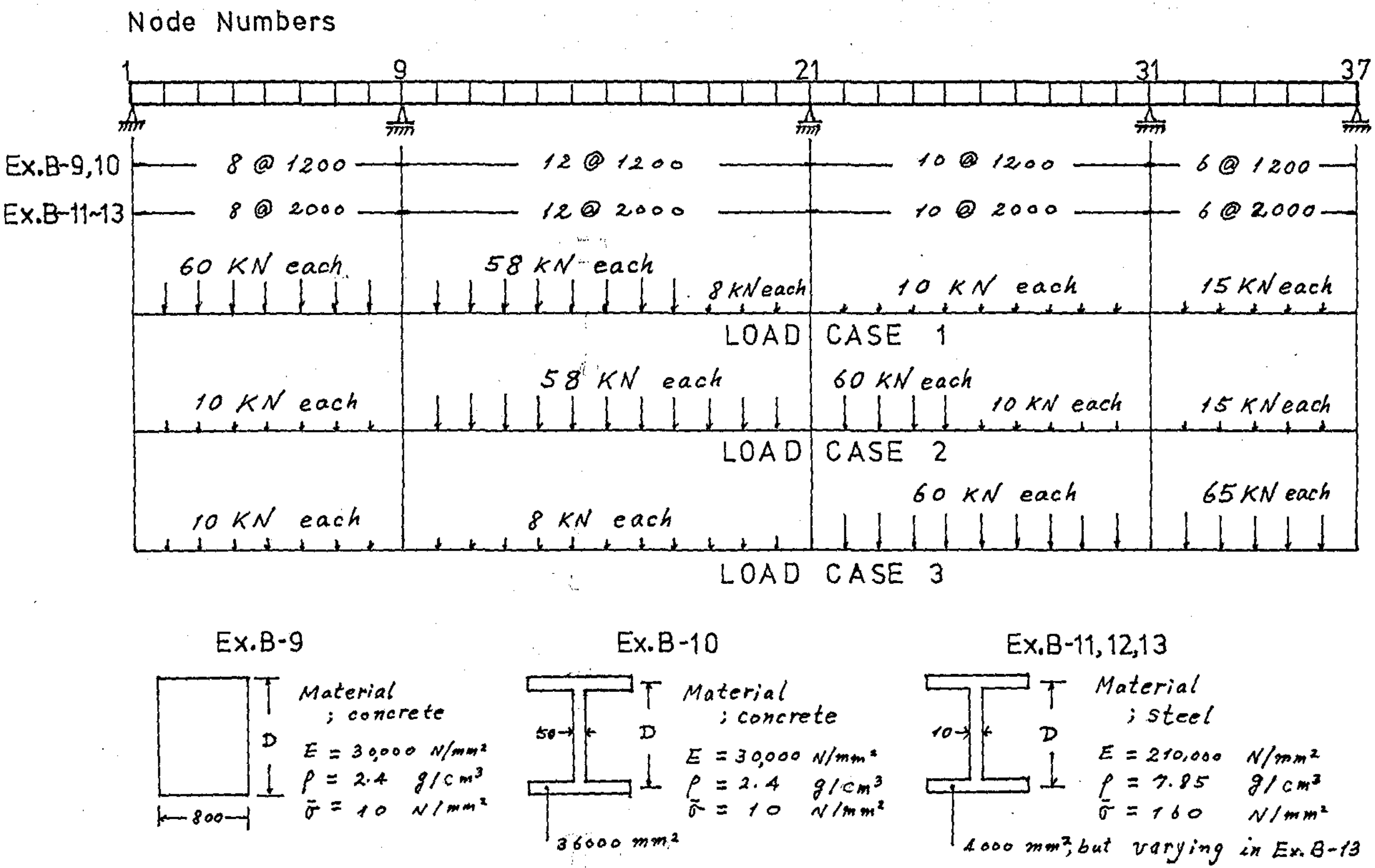


Fig. 28 Problems of 4-Span Beam

assumed to have rectangular sections with a constant breadth, 800 mm, and subject to deflection constraints of 8, 12, 10 and 6 mm at the midspans respectively as well as stress limits of 10 N/mm^2 at the nodes. The cost function was assumed linear. The resulting designs for the both cases are shown in Fig. 29 and Table-29. Considering self-weight naturally called for deeper sections but the middle part of the third span became shallower.

Table-29 Designs of 4-Span Beam Case I, Ex.B-9.

Design Variable	Ex.B-9-1		Ex.B-9-2	
	(1)	(2)	(1)	(2)
1	300	0.0509	300	0.0402
2	395	0.9979	409	0.9997
3	819	1.0006	916	1.0000
4	830	1.0005	892	0.9999
5	716	0.9983	848	1.0010
6	591	1.0008	568	0.9994
7	658	0.9993	704	0.9996
8	300	-0.0103	310	1.0019
9	300	-0.1012	300	-0.2872
Cost	501539		532967	
No. of Iter.	7		8	

Ex.B-9-1 ; neglecting self-weight.
Ex.B-9-2 ; considering self-weight.
minimum size ; 300 mm
(1) Design values.
(2) Values of optimality criteria equation, Equ. (5.57).

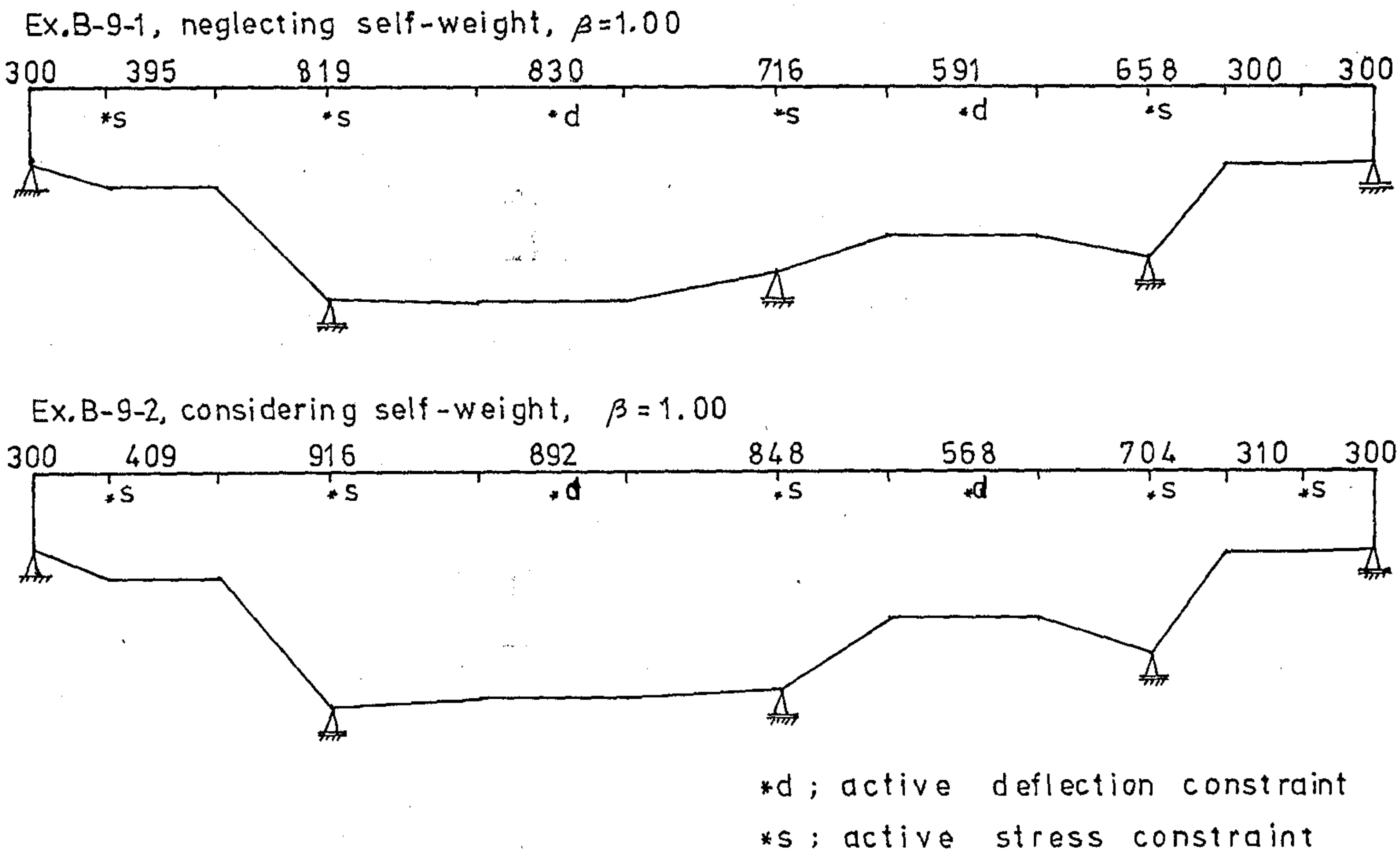


Fig. 29 Designs of 4-Span Beam
with rectangular section

Ex.B-10 4-Span Beam Case II

This problem is the same as Ex.B-9 except that the sections are of I-shape and both flanges of any section have a constant cross sectional area, 36000 mm^2, and the cost function is nonlinear with $\beta = 0.75$. Fig. 30 and Table-30 show apparently different two designs of this problem, each satisfying the optimality criteria. Design 1 was obtained by the optimality criteria method of this thesis including both deflection and stress constraints, whereas Design 2 was obtained by using the optimality criteria method for deflection constraints and the stress ratio method for stress constraints.

In order to explore the nature of the design space, a few trials were made. Firstly the design process of the first method was started using Design 2 as starting values. But the design did not move away from Design 2. Conversely the design process of the second method was started using Design 1 as starting values. Also in this case, the design did not move away from the initial design, Design 1. Secondly an initial design, other than a uniform design which had always been used, was selected by an engineer without looking at any of the two designs and used in both design methods. The design was on the side of Design 1 and it was hoped that both methods led to Design 1, but the use of a different initial design made no difference. Lastly linear cost function was used instead. This made no difference either and it became clear that the feasible region of the design space was non-convex and there were two local minima or even more.

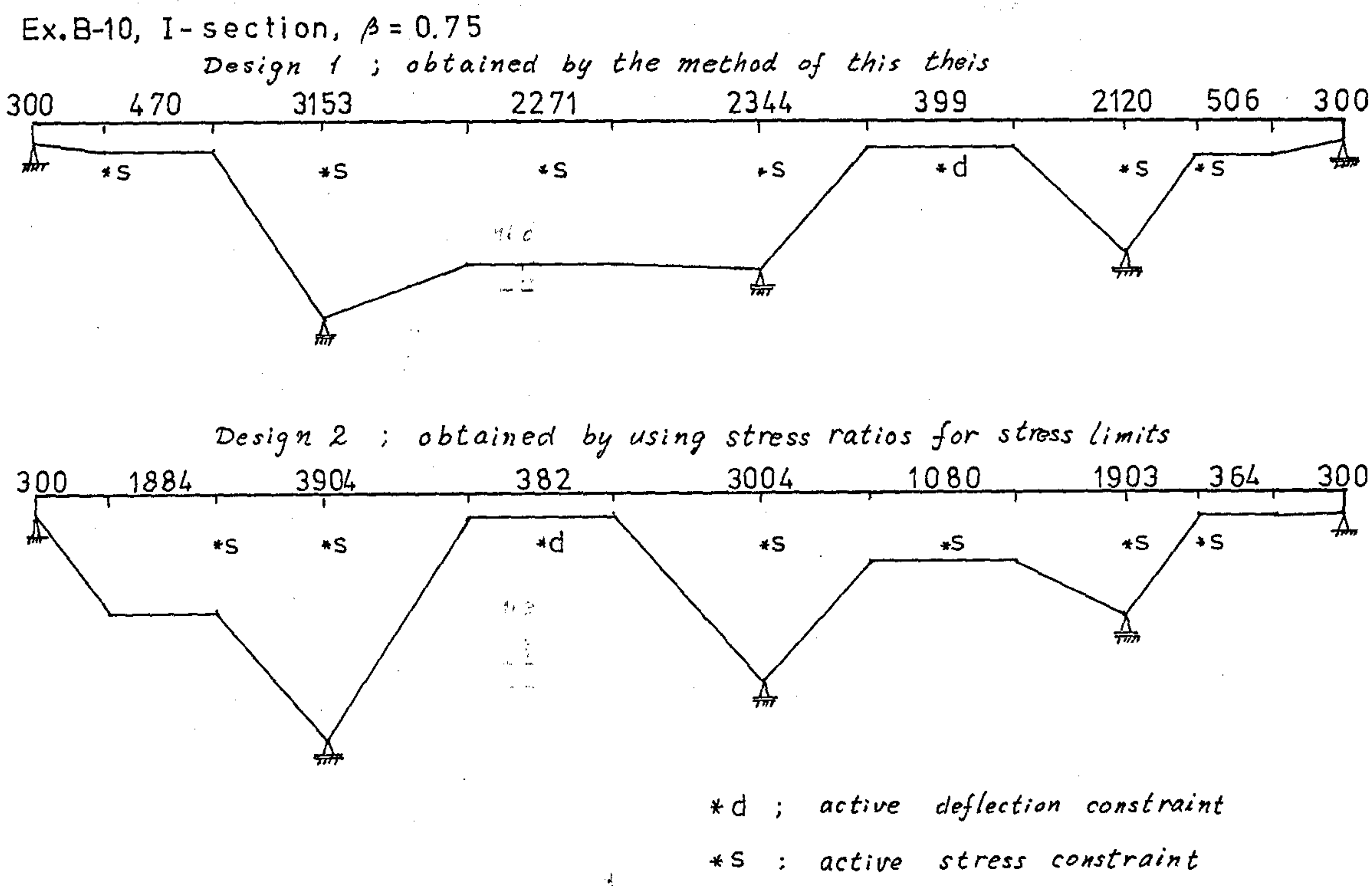


Fig. 30 Two Local Minimum Designs of 4-Span Beam with I-section

Table -30 Two Designs of Ex.B-10, 4-Span Beam Case II.

Design Variable	Design 1		Design 2
	(1)	(2)	(1)
1	300	0.3070	300
2	470	1.0004	1884
3	3153	1.0007	3904
4	2271	0.9999	382
5	2344	0.9990	3004
6	399	1.0016	1080
7	2120	1.0005	1903
8	506	1.0014	364
9	300	0.5105	300
Cost	2001470		2030390
No. of Iter.	27		22

minimum size ; 300 mm

(1) Design values

(2) Values of optimality criteria equation, Equ. (5.59).

Ex.B-11 4-Span Beam Case III

In this problem the beam has the same configuration as that of Ex.B-9 and Ex.B-10 but is made of steel instead of concrete and therefore subject to different design conditions. The length of an element is 2000 mm instead of 1200 mm. The flanges of any section have a constant cross sectional area, 4000 mm^2. Stress limits of 160 N/mm^2 are imposed on the sections of all nodes. Deflection limits imposed on the midspans are 16, 24, 20, 12 mm respectively. Each of them is equal to a thousandth of the length of the corresponding span. The elastic modulus of the material used is assumed 210,000 N/mm^2. The cost function, as in Ex.B-10, is nonlinear with $\beta = 0.75$.

The design and other results are shown in Fig. 31, Table-31, and Table-34. The deflection at each midspan reached the limit and the stress of the section at a support was active.

Table-31 Designs of 4-Span Beam, Ex.B-11, 12, 13.

Design Variable	Ex.B-11		Ex.B-12		Ex.B-13	
	(1)	(2)	(1)	(2)	(1)	(2)
1	300	0.5338	300	0.3174	300	0.4047
2	793	0.9999	439	0.9989	543	1.0000
3	2545	1.0000	2905	1.0004	1263	1.0001
4	2669	1.0003	2099	1.0022	1282	0.9999
5	1838	0.9982	1968	0.9981	1146	1.0020
6	1533	1.0007	863	1.0002	965	0.9990
7	1759	0.9998	1712	1.0000	1501	1.0002
8	423	1.0000	304	0.9998	305	1.0011
9	300	0.3215	300	0.4886	300	0.5325
Cost of Web	2496010		2133880		1678920	
No. of Iter.	21		15		19	

minimum size ; 300 mm

(1) Design values

(2) Values of optimality criteria equation, Equ. (5.59).

Ex.B-12 4-Span Beam Case IV

This is the same as ExB-11 except that the deflection constraints were relaxed. The deflection limits were doubled. In other words, each midspan was allowed to deflect up to a five hundredth of the length of its corresponding span. The design and other results are shown in the same figure and tables as those of Ex.B-11. In the design six stress constraints were active while only one deflection was active.

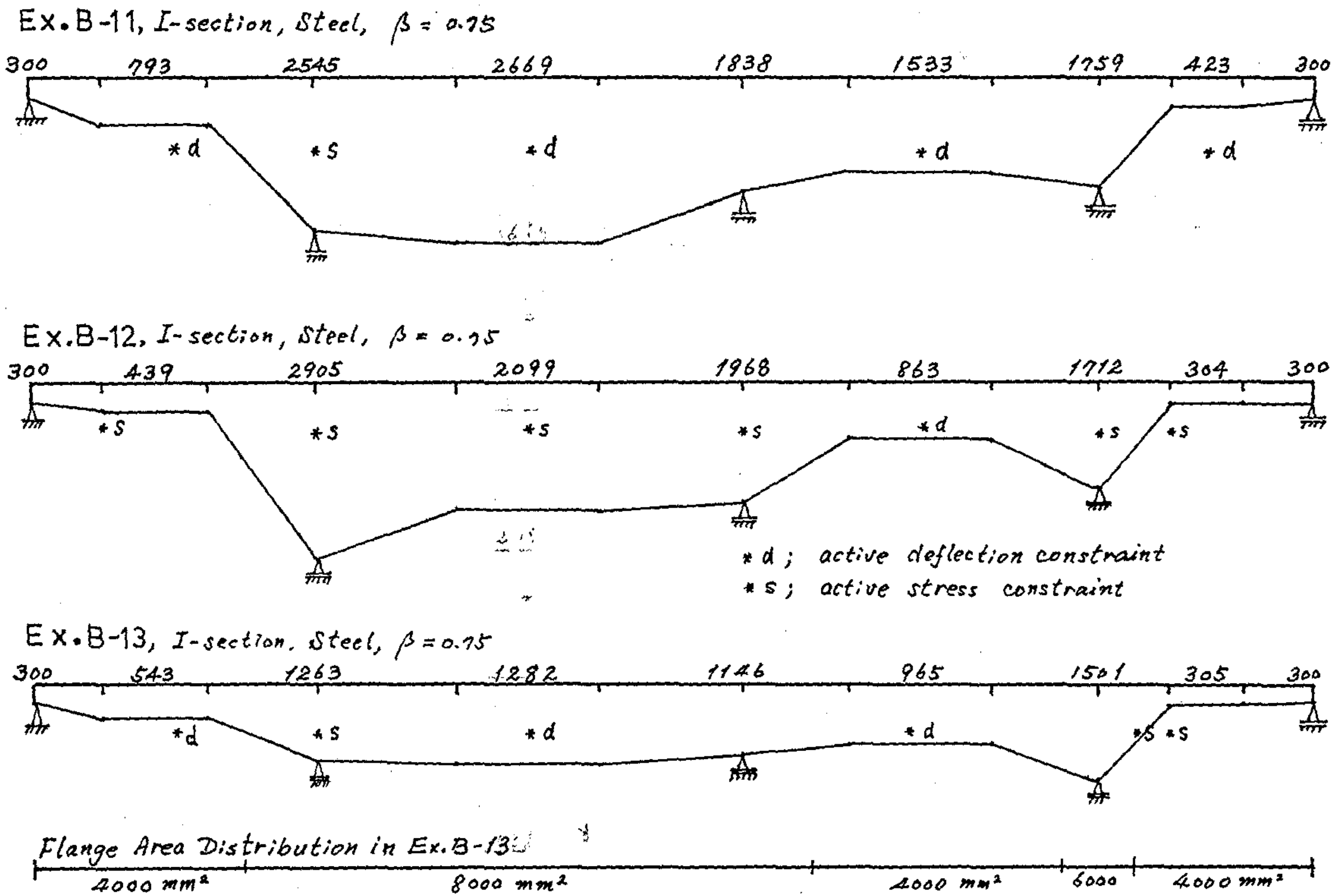


Fig. 31 Various Designs of 4-Span Beam made of Steel

Ex.B-13 4-Span Beam Case V

This problem is the same as Ex.B-12 except that the cross sectional areas of flanges of some elements were increased as shown in Fig. 31. Therefore the size of flanges in this problem varies from element to element. The design obtained and other results are shown in Fig. 31, Table-31 and Table-34.

Ex.B-14 4-Span Beam Case VI

This problem and the design resulted in after 12 redesign iterations are shown in Fig. 32. In the solution process, the stress at node 31 was included in and deleted from the set of active constraint alternately and thus its associated design variable was treated as of Group 3. However, the optimality test showed that the design was quite close to the optimum.

Ex.B-15 4-Span Symmetric Beam Case I

This beam is illustrated in Fig. 33. Since the structural configuration and the applied loads are both symmetric the resulting design must by symmetric, too. The beam is made of concrete, whose elastic modulus, mass density and permitted stress are 30,000 N/mm^2, 2.4 g/cm^3 and 10 N/mm^2 respectively. The deflection limits imposed on midspans are 8, 10, 10, 8 mm.

The first problem, Ex.B-15, is of rectangular section with a constant breadth, 400 mm. The cost function was assumed linear and the self-weight was neglected. In the resulting design, shown in Fig. 33 and Table-32, two midspans, nodes 15 and 27, were not only

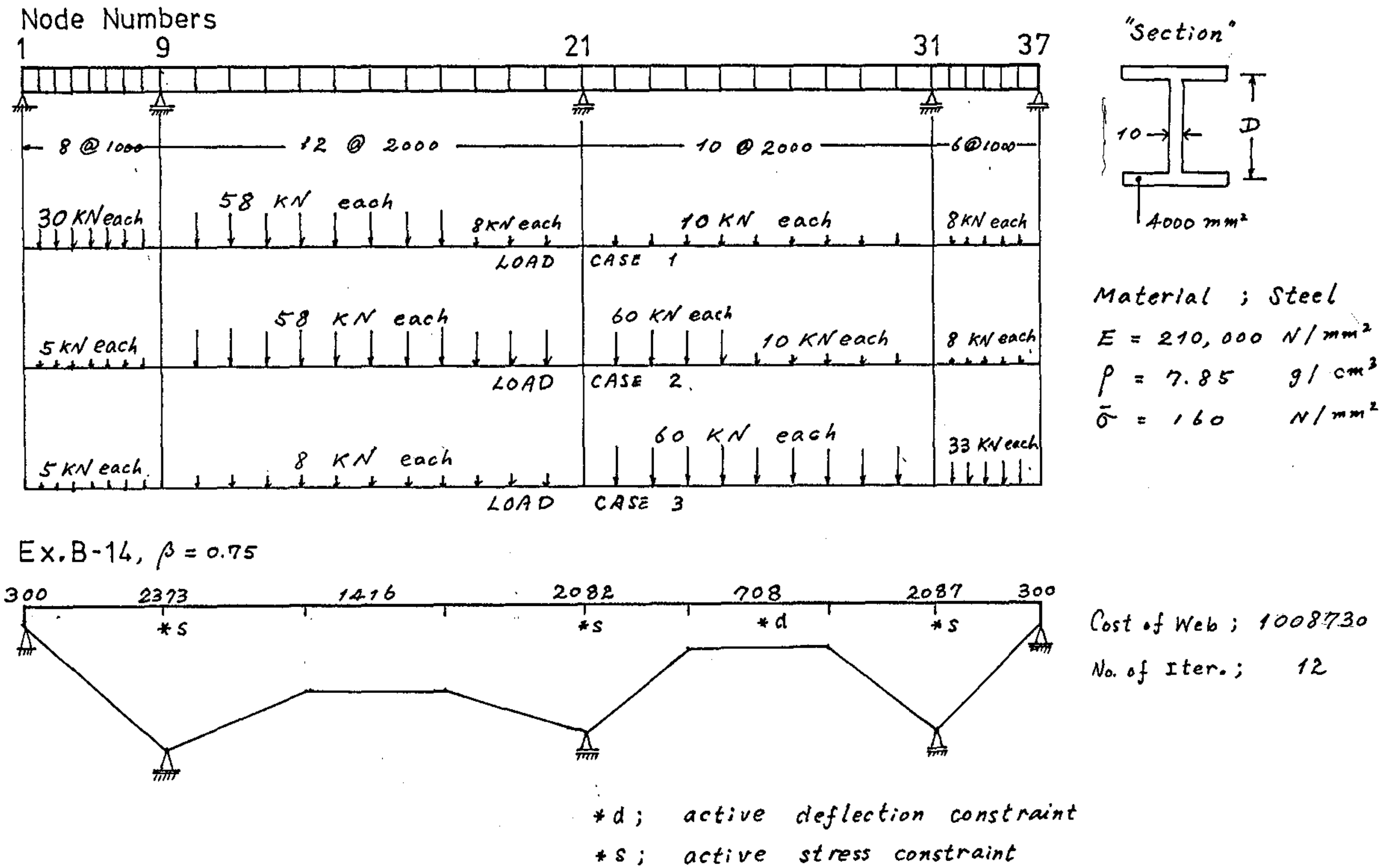


Fig. 32 Another Problem of 4-Span Beam & its solution

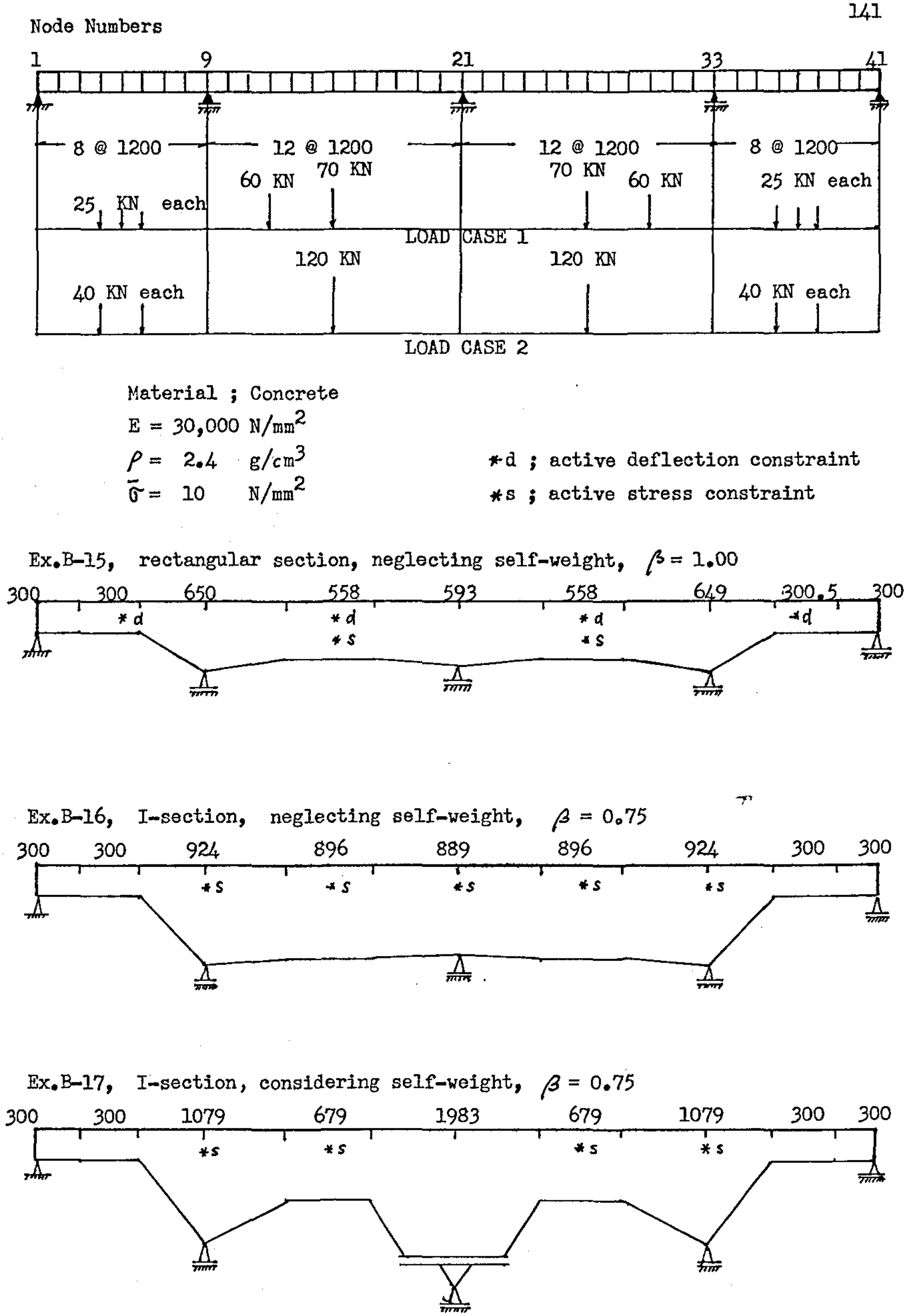


Fig. 33 4-Span Symmetric Beam and the Designs deflected up to the limit but fully stressed whereas none of the sections at the supports is fully stressed. It appears that the deflections were controlled more by the sections at the adjacent supports while the stresses were controlled more by the sections at the midspans.

The design is not strictly symmetric. The 8th design value is very close to the minimum size but it still stands in Group 1. It is easily foreseeable that the design variable should have gone to Group 2 making the design symmetric, but this implies a change of the nature of the design. Currently there are 9 active constraints, including 3 minimum size constraints, in the 9-dimensional design space. Therefore if design variable 8 had gone to Group 2, the active constraint gradient must have been linearly dependent and for this reason one of the active deflection or stress constraints should have been put aside as inactive, although it was in fact active, for the design process to be successful. It seems that the slightly unsymmetric design, which satisfied the optimality criteria very well, was resulted in due to rounding error, and the design process incidentally stopped at the design obtained.

Ex.B-16 4-Span Symmetric Beam Case II

In this problem the beam had I-shape sections with a constant flange area, 2400 mm^2. The self-weight and the deflection constraints were neglected. The cost function was assumed nonlinear with $\beta = 0.75$. In the resulting design none of the deflections at the midspans exceeded the limits. Therefore the same design could have resulted even if the deflection constraints had also been considered. However an adverse situation arose when the problem was solved considering the deflection constraints as well. The deflections at nodes 5 and 37 took part in the set of active constraints in iterations 2, 3 and 4 and caused deletion of some stress constraints, and particularly in iteration 4 the deletion became more violent and the design process failed to continue. This aspect needs further research on the interrelation between deflection and stress constraints.

Table-32 Designs of 4-Span Symmetric Beam

Design Variable	Ex.B-15		Ex.B-16		Ex.B-17	
	(1)	(2)	(1)	(2)	(1)	(2)
1	300	0.1184	300	-0.0000	300	0.0491
2	300	0.7328	300	-0.0003	300	0.1612
3	650	1.0001	924	1.0004	1079	1.0003
4	558	1.0000	896	1.0009	679	0.9906
5	593	1.0003	889	0.9975	1983	1.0059
6	558	0.9988	896	1.0009	679	0.9906
7	649	1.0036	924	1.0004	1079	1.0003
8	300.5	0.9971	300	-0.0003	300	0.1612
9	300	0.1590	300	-0.0000	300	0.0491
Cost	22756600		1378270		1426980	
No. of Iter.	21		18		25	

minimum size ; 300 mm

(1) Design values.

(2) Values of optimality criteria equation, Equ. (5.57) or (5.59).

Ex.B-17 4-Span Symmetric Beam Case III

This problem was the same as Ex.B-16, but the self-weight and deflection constraints were considered. However, the deflection constraints did not take part in the set of active constraints throughout the design process, therefore such a problem as explained in Ex.B-16 was not encountered. It is interesting to note that the design of this problem, shown in Fig. 33 and Table-32, is quite different from that of Ex.B-16 in spite of the fact that treating self-weight is the only difference between the two problems. Although the design process was made to stop after 25 iterations and yielded the design satisfying the optimality criteria fairly well as shown in Table-32, the cost was still decreasing and the section at node 21 was getting deeper. It is also notable that the section at node 21 is not fully stressed (82% of the permitted) and there is no active stress constraint associated with the 5th design variable, the depth at node 21.

Ex.B-18 5-Span Beam

The problem and its solution are shown in Fig. 34 and Table-33. It is interesting as well as foreseeable that active stress constraints occurred at supports and active deflection constraints occurred at midspans.

Table 34 lists the beam problems solved so far by the optimality criteria method of this thesis and other information concerning the resulting designs. The designs converged within 10 to 20 redesign iterations under a rather strict cutoff criterion, 0.001 times the current value for each design variable, and the last column of the table shows how fast the designs converged. It is also notable that the number of redesign iterations was scarcely sensitive either to the size of problems or to the number of active constraints. The tables 26 to 33 show the values of the optimality criteria equation

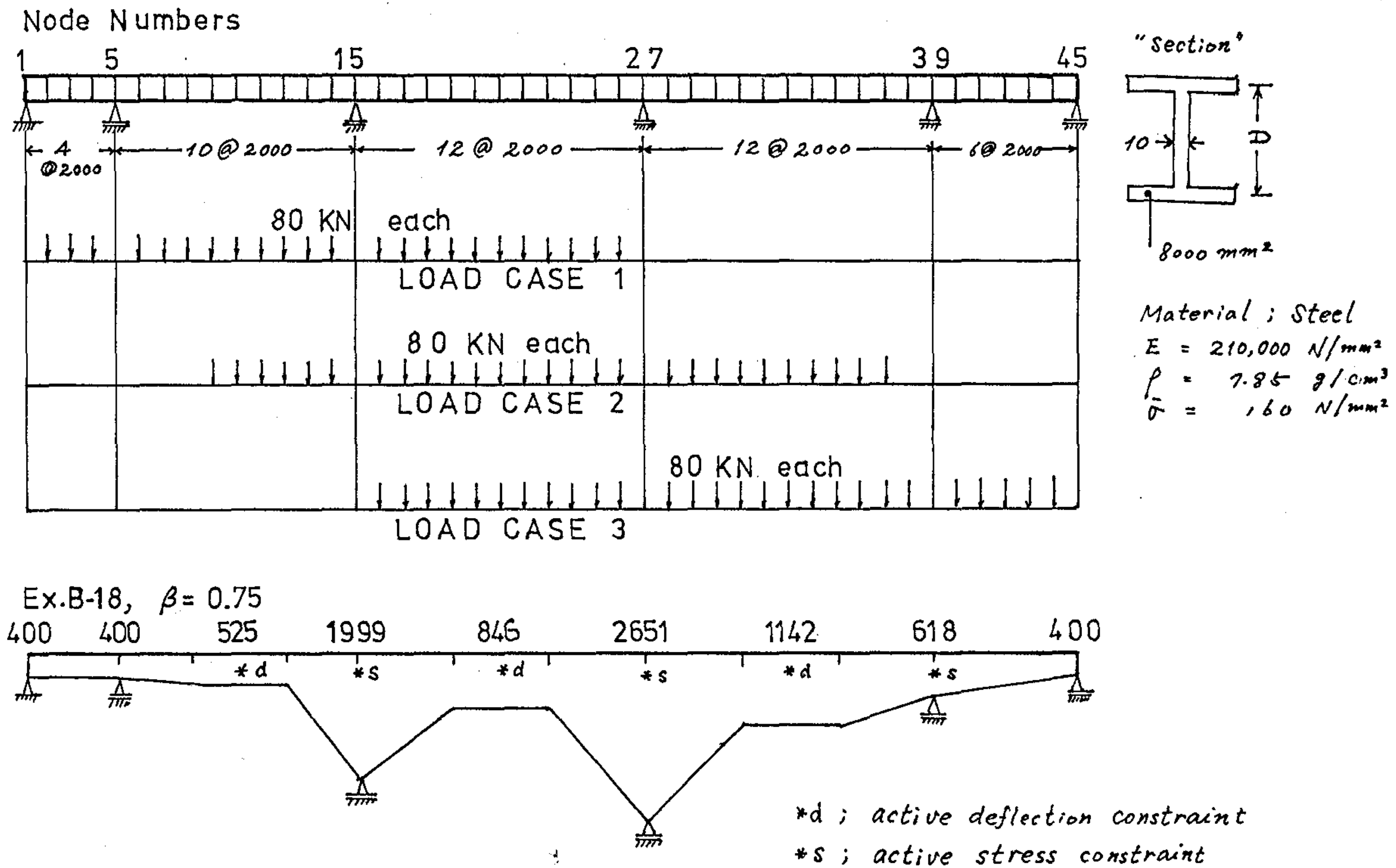


Fig. 34 5-Span Beam & the Design

of the resulting designs demonstrating how accurate the solutions are. The values were obtained by evaluating the optimality criteria equation using the Lagrange multipliers determined during the last redesign iteration and the constraint gradients evaluated from the final design.

Table-33 Design of 5-Span Beam, Ex.B-18

Design Variable	(1)	(2)
1	400	0.2280
2	400	0.2446
3	525	1.0003
4	1999	0.9999
5	846	0.9995
6	2651	1.0011
7	1142	1.0005
8	618	0.9972
9	400	0.3688
Cost of Web	2093400	
No. of Iter.	14	

minimum size ; 400 mm

(1) Design values

(2) Values of optimality criteria equation, Equ. (5.59).

Table-34 List of the Beam Problems solved and Some information concerning their Designs.

Example	No. of Spans	No. of Load Cases	No. of Design Variables		No. of Active Constraints		No. of Iterations	
			Gr. 1	Gr. 2	def.	str.	(1)	(2)
Ex.B-1	2	1	1	2	-	1	16	11
Ex.B-2	2	1	2	1	1	0	20	2
Ex.B-3	2	1	2	3	1	0	22	10
Ex.B-4	2	1	1	3	1	0	14	9
Ex.B-5	3	3	19	12	-	19	11	4
Ex.B-6	3	3	6	1	3	1	18	3
Ex.B-7	3	3	5	2	1	4	17	7
Ex.B-8	3	3	5	2	-	5	14	7
Ex.B-9-1	4	3	6	3	2	4	7	3
Ex.B-9-2	4	3	7	2	2	5	8	5
Ex.B-10	4	3	7	2	1	6	27	15
Ex.B-11	4	3	7	2	4	1	21	6
Ex.B-12	4	3	7	2	1	6	15	7
Ex.B-13	4	3	7	2	3	3	19	10
Ex.B-14	4	3	5	2	1	3	>12	
Ex.B-15	4	2	6	3	4	2	21	2
Ex.B-16	4	2	5	4	-	5	18	6
Ex.B-17	4	2	5	4	0	4	>25	
Ex.B-18	5	3	6	3	3	3	14	7

(1) required to make the design converge such that the change of any design value is less than 0.001 times its current value.

(2) required to make the cost of the design less than 1.01 times that of (1).

9. DISCUSSION AND SUGGESTIONS FOR FURTHER DEVELOPMENT

Major improvements achieved in this work are in two respects. Firstly the stability and efficiency have been improved substantially compared with Taig's method. The strategies of deleting inactive constraints during the Newton-Raphson process and removing passive design variables were entirely changed to eliminate possibilities of 'oscillating' and 'looping'. A different approach of finding active constraints has brought about not only stability but also efficiency. Secondly the scope of problems to be solved by the method has been extended. Stress constraints can take part in the Newton-Raphson process. This was possible due to the improved stability of the method, and in consequence exact solutions were almost always obtainable. Beam problems of a practical scale can also be solved by the method. This may be a notable improvement for the optimum design of civil engineering structures.

Nevertheless, further developments are necessary in the both respects. In the following sections, some difficulties encountered are explained and the areas of possible further developments are suggested.

9.1 Selecting Active Constraints

An important improvement was the way of selecting active constraints. The set of active constraints was made to expand gradually as the design process proceeded by taking more critical constraints if there were any. A possible criticism of this approach might be that the incorrect sets of active constraints at earlier stages of the design process could direct the design wrongly. However, the approach has been good enough to fix the correct set of active constraints within a reasonable number of redesign iterations and caused no adverse situations when coupled with stress ratio to cope with the absence of some stress constraints in the set. In the design space shown in Fig. 35, P_2 is the optimum design when Constraint 1 is the only constraint whereas P_3 is the optimum when both Constraint 1 and 2 are imposed. If the design process starts at P_1 considering Constraint 1 only, the moves will be towards P_2 rather than P_4 and eventually the design will pass by P_3. Then Constraint 2 will be included in the set of active constraints and make the design process find P_3.

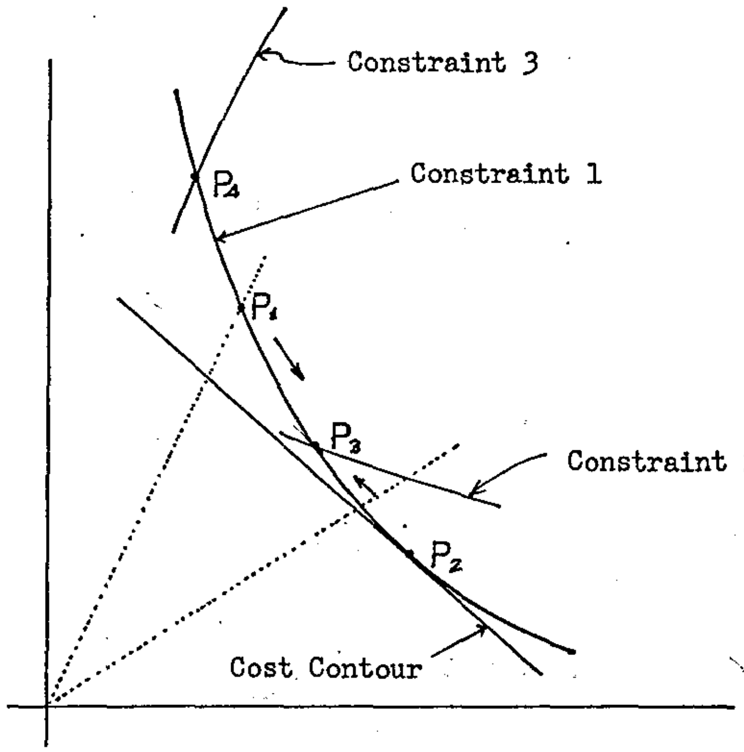


Fig. 35 Process of Finding Active Constraints

In the large-scale problems, 124-bar and 200-bar trusses, too many active constraints in earlier redesign iterations in fact caused disturbance during the Newton-Raphson process. It appeared that considering unnecessarily many constraints when the design was still remote from the optimum was not helpful. For these problems, therefore, it made the method more efficient to consider a constraint as active when it was violated by more than a certain amount and to reduce the amount gradually as the design process proceeded.

9.2 Initial Estimates of Lagrange multipliers

The main criticism of the method by Taig and Kerr[28] has been that the Newton-Raphson process requires good initial estimates of the Lagrange multipliers which are not always easy to obtain.[22][31] This also applies to the method of this thesis. However, the extensive numerical experiments made throughout this work showed that the estimates of the Lagrange multipliers obtained from Equ.(5.62) and (5.63) were good enough for the Newton-Raphson process to reach the solution within a reasonable number of iterations.

Nevertheless, a difficulty arose in connection with the design variable linking by ratio for the beam problems. Failure in estimating the Lagrange multipliers for beam problems was experienced, but only occasionaly. The method assumes equal contributions from each constraint and this sometimes lead to the values in the brackets of Equ.(5.70) - (5.73) becoming negative and making it impossible to obtain estimates of the Lagrange multipliers from the equations. It appears that this failure should be blamed on the assumption used for evaluating y_{ip} and z_{ip} included in Equ. (5.70) - (5.73).

The stress at node 31 in Ex.B-9 and the stress at node 15 in Ex.B-18 took part in the set of active constraints in the 5th and 6th redesign iteration respectively, and their associated Lagrange multipliers could not be estimated from Equ. (5.70) - (5.73). For these case an alternative way was used. Whenever a negative value is assigned to the brackets of Equ. (5.70) - (5.73) the design process automatically switches over to Equ. (5.88), the linear equations for optimality test, to obtain estimates of the Lagrange multipliers. Therefore the estimates of the Lagrange multipliers in iteration 5 for Ex.B-9 and in iteration 6 for Ex.B-18 were obtained from Equ. (5.88) and thereafter the problems were solved successfully as shown in the preceding chapter.

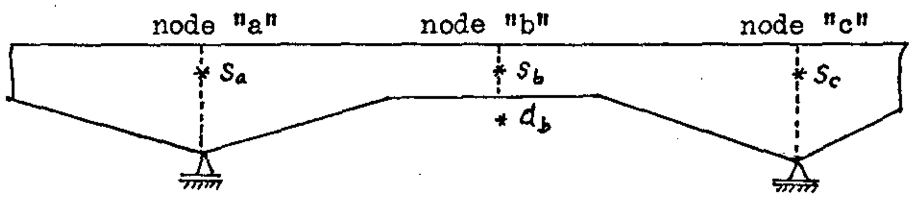
Since Equ. (5.88), used for optimality test of a design, is based on good mathematical grounds it is worth considering the sole use of this equation for estimating the Lagrange multipliers. However, the use of Equ. (5.88) in the earlier redesign iterations sometimes created another problem. It gave negative estimates to some of the Lagrange multipliers, and caused disturbance of the Newton-Raphson process. So far, the use of Equ. (5.70) - (5.73) coupled with the use Equ. (5.88) as an emergency measure has been satisfactory.

9.3 Functional Dependency of Constraints

Another difficulty is the possible singularity of the Jacobian matrix in Equ. (5.67). If functional dependency exists in the set of active constraints, the Jacobian matrix becomes singular and the Newton-Raphson process fails to proceed. In Ex.T-12, 25-Bar Truss Case III, and Ex.T-13, 72-Bar Truss Case II, the design variable linking keeps the designs doubly symmetric and in addition the applied loads are arranged symmetrically or doubly symmetrically. Therefore the X- and Y-directional deflections at node 1 of Ex.T-13 under load case 1 are kept the same at any design subject to the design variable linking. This fact led to severely ill-conditioned Jacobian matrices, and resulted in deletion of one of the two deflection components from the set of active constraints. The deletion did no harm to the Newton-Raphson process except requiring more computing time. However, it is sensible to consider one of the two deflection components as inactive throughout the design process. In Ex.T-12, the Y-directional deflections at node 1 and 2 were always the same under load case 1 and the same in magnitude under load case 2, and thus the same situation as in Ex.T-12 happened.

It is more than desirable to take only one member stress as an active constraint from the members controlled by a design variable. This approach prevents singularity of the Jacobian matrix, more fruitfully reduces the number of active constraints, and has raised no disturbances such as taking different member stresses as an active constraints from iteration to iteration. In Ex.T-8, 10-bar truss Case I-b, members 3 and 4 always carry forces with the same magnitude and thus it is very likely that both members are assigned the same area in an optimum design. If they have the same size, the derivatives of both member stresses with respect to each design variable except those associated with members 3 and 4 become the same. If the sizes of members 3 and 4 go to Group 2, the gradients of both member stresses in the subspace spanned by Group 1 design variables will become identical and thus one of the two member stresses will be deleted from the set of active constraints. In order to avoid this situation the sizes of members 3 and 4 were linked so as to be represented by one design variable and only one of the two member stresses was taken as an active constraint throughout the design process.

In the beam problems with design variable linking the stresses only at the boundary nodes and at only one node among the inside nodes of a group of elements having the same rate of tapering were allowed to take part in the set of active constraints for the same reason as in the truss problems. In addition, the deflections were taken into account only at one node per span. Nevertheless, there is a possibility of the number of active constraints exceeding the number of design variables and thus of functional dependency between the active constraints. It is possible that the stresses at nodes "a", "b" and "c" and the deflection at node "b" in Fig. 36 are all active. If this happens to all spans the number of active constraints exceeds the number of design variables by the number of spans. Even if this happens only to a particular span there still exist possibilities of nearly dependent constraints which may result in ill-conditioned Jacobian matrices during the Newton-Raphson process. Therefore it will be reasonable to prevent all the constraints in Fig. 36 from being active at the same time by dropping the least restrictive constraint, although no such a situation has yet been encountered.



d_b ; deflection constraint at node b,

s_a, s_b, s_c ; stress constraints at nodes a, b, c.

Fig. 36 Possible Constraints in a Span of Beam Problems.

However, a serious problem could arise from dropping the wrong constraint when all the constraints are really active and functionally dependent. Of the dependent constraints we have several different independent subsets, each calling for different sets of the Lagrange multipliers. Moreover, some of them could require some negative Lagrange multipliers and therefore may cause failure of the Newton-Raphson process or deletion of the constraints requiring negative Lagrange multipliers. Detecting functional dependency between the constraints considered as active and deciding which constraints should be dropped are the aspects which require further research for the method of this thesis to be completely successful.

9.4 Use of Stress Ratio

The stress raio algorithm is used in many methods to replace stress constraints, whereas the method of this thesis uses it only as a temporary measure as was explained in section 5.4.5. In section 7.1 it was demonstrated that the use of stress ratio could lead to a wrong design. In spite of this, the stress ratio algorithm has been popular due to its simplicity and, in many cases of truss problems, resulted in designs close to the optimum.

In the beam problems, however, failure of the design process was often experienced when stress ratio was used to replace stress constraints taking part in the Newton-Raphson process. When the deflection constraint at node "b" of Fig. 36 was active during the Newton-Raphson process, but later the depth at node "b" as well as the depths at nodes "a" and "c" were decided rather by stress ratio, the next round of the Newton-Raphson process was soon disturbed. The design variables governed predominantly by the constraint, deflection at node "b", were deemed as inactive variables and thus removed from the design space. In this consequence, the deflection gradient in the subspace spanned by the Group 1 design variables became almost null leading to nearly singular Jacobian matrices.

Moreover, the design variable linking by ratio makes it difficult to use the stress ratio method effectively. The required depths at individual nodes may be calculated by stress ratios with a certain accuracy, but accurate transformation of the depths into the design values is impossible since the transformation matrix in Equ. (5.34) is not invertable.

9.5 Damping of Newton-Raphson Step Sizes

One of the important improvements achieved in this work was eliminating possibilities of a "loop" forming during the Newton-Raphson process as mentioned in Chapter 5. When Taig's method was used for deflection constrained beam problems, it was sometimes experienced that the Newton-Raphson process neither converged nor diverged, but oscillated. The Jacobian matrix gave the same Newton direction in every 2 or even 30 iterations. This drawback was eliminated as was explained in section 5.4.6, but loops formed when the set of active constraints changed in Ex.T-10, 10-bar truss Case I-c, and in Ex.T-13, 72-bar truss Case II. Participation of a new constraint in the set of active constraints might have led to poor estimates of the Lagrange multipliers and thus a loop in the Newton-Raphson process. To overcome this problem we may consider damping the Newton-Raphson process, by scaling down the calculated step size but it is not desirable since damping only increases the number of Newton-Raphson steps required and moreover this situation happens very rarely. The method adopted in this thesis firstly allows the Newton-Raphson process to proceed without damping up to a certain number of steps, say 20 steps, and then, if the process does not converge, introduces a damping factor, say 0.5, and starts the Newton-Raphson process again. This procedure may be repeated, but only one round was enough to reach the solutions in the two problems which have had this difficulty, Ex.T-10 and Ex.T-13. In the later iterations this problem did not arise since the set of active constraints contained correct entries.

9.6 Reducing Computing Time

Although the method of this thesis can give exact solutions in a stable manner the most painful aspect of the method is the significant amount of computing effort involved in the Newton-Raphson process. Fleury [32)] suggested that a hybrid optimality criterion "characterized by a mix of zero and first order approximations of the constraints" was obtainable by applying the FSD concept for the less critical stress constraints. It may be helpful for truss problems but is doubtful if it can give correct solutions to the beam problems treated in this thesis.

An approach to reduce remarkably the amount of computing involved in the Newton-Raphson process is conceivable. Particularly in the beam problems with design variable linking, it may be a reasonable approximation to neglect X_{12} , X_{21} and the off-diagonal entries of X_{22} in Equ. (5.67). Using this approximation we can update the Lagrange multipliers associated with deflection and stress constraints separately and moreover finding the component of Newton-Raphson steps associated with each stress constraint is quite straightforward. Alternatively we can consider X_{21} and find the components associated with stress constraints after obtaining those associated with the deflection constraints by inverting X_{11}. This approximation does not affect the validity of the results provided no active constraints are incorrectly deleted. A few beam problems have been solved successfully using this approximation, but much more numerical experiment and improvements are necessary to use it with confidence.

9.7 Possible Other Structures

There is plenty of room, as is often the case, for further developments to improve the reliability and efficiency and to extend the scope of problems to be tackled by the method of this thesis. In particular the beam problems with I-shape sections can be formulated in a number of different ways and solved by the method with minor changes.

Firstly the flange areas can be taken as the design variables instead of the depths at nodes. The problem then becomes that of minimizing the cost of flanges assuming that the configuration of web is fixed. This problem can be solved in the same way as the truss problems. The bending stress can be expressed as a linear combination of the generalized rotational displacements with constant coefficients. The design variable linking by ratio is no longer bothersome.

Secondly the problem can be formulated as one of decentralized problems. The beam is first designed by considering the depths at nodes as the design variables and then, with the depths so determined, the flange areas are redistributed as explained in the preceding paragraph, and vice versa. In this problem, the cost of webs and the cost of flanges are minimized in seperate processes and this procedure may be repeated for several times. However, there is a little doubt whether the repeated processes will really reduce the total cost.

On the contrary, both the depths at nodes and the flange areas can be considered at the same time. Geometrical similarity existing between available I-section members or those reasonably proportioned makes it possible to establish relationships between various properties of a section. We assume that the area of a flange, A_f , can be expressed in terms of the depth, D , as

$$A_f = \alpha D^{\beta}$$

and determine the constants, α and β , by regression analysis from available or optimally proportioned I-shape sections. Then the flange areas vary during the design process, but only depending on the depths of nodes.

Application of the method to other types of structures such as rigid frames is also possible. The virtual loads to express bending stresses in terms of virtual work as illustrated in Fig. 5 are applicable, but in case of rigid frames it is desirable to obtain fixed end moments rather than slope deflections since the frames are analysed preferably by the displacement method.

10. CONCLUSION

An optimality criteria method has been presented. The method is to solve the design problems of structures built with either bar or beam elements subject to deflection, stress and minimum size constraints. The use of the Newton-Raphson method in problems of structural optimization first proposed by Taig and Kerr [28)] was improved in both respects of reliability and applicability.

Well known truss problems were solved by the new method and the results were compared favourably with other published results. A number of continuous beams with tapered elements were designed by the method. The resulting designs satisfied the optimality conditions very well, but there were no other solutions to these problems in the literature to be compared with. The stress constraints in both truss and beam problems were approximated within first order using the method of virtual work as was often the case with the deflection constraints, and it was demonstrated that a proper approximation should be used also to the stress constraints rather than the crude stress ratio approximation.

Since the design problems are usually subject to inequality constraints it is necessary to discriminate active constraints, which is generally known to be a difficult task. In the method of this thesis, however, the simple conviction that the design would move on towards the optimum even when not all the active constraint were taken into account proved successful in selecting the correct set of active constraints.

The Newton-Raphson method has been surprisingly good at solving the optimality criteria and constraint equations for the Lagrange multipliers. Large-scale systems of highly involved nonlinear equations were solved without raising severe difficulties such as looping or diverging. Failure was occasionally experienced, but the blame lay rather on the functional dependency existing between the constraint functions. It is generally known that the success of the Newton-Raphson method is too sensitive to the initial values for the method to be universally applicable to systems of nonlinear equations. Because of this drawback inherent in the Newton-Raphson method, the methods by Marquardt[50) or Jones [51) are sometime used for problems, such as nonlinear parameter estimation, where good initial values are not always possible to obtain. For the problems solved in this thesis, however, the Newton-Raphson method was so good that it was not necessary to resort to other methods.

The optimality criteria method presented in this thesis proves a promising method in many respects. It solves design problems rigourously and results in exact solutions in a stable manner. It is also possible to extend the scope of problems the method can tackle. Its ability to handle bending elements as well as bar elements and multiple deflection and stress constraints, as has been shown throughout this thesis, proves that it is feasible to develop the method to the extent of automated design procedures for practical civil engineering structures.

Although the Newton-Raphson process adopted by the method involves a large amount of computing effort, the stability of the design process makes the method efficient compared with some other crude methods. Moreover , ever increasing availability of computers will allow the engineer to be able to afford to use rigourous methods and obtain better solutions rather than to rely on crude methods.

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APPENDIX A

How to use the "TRUS" program

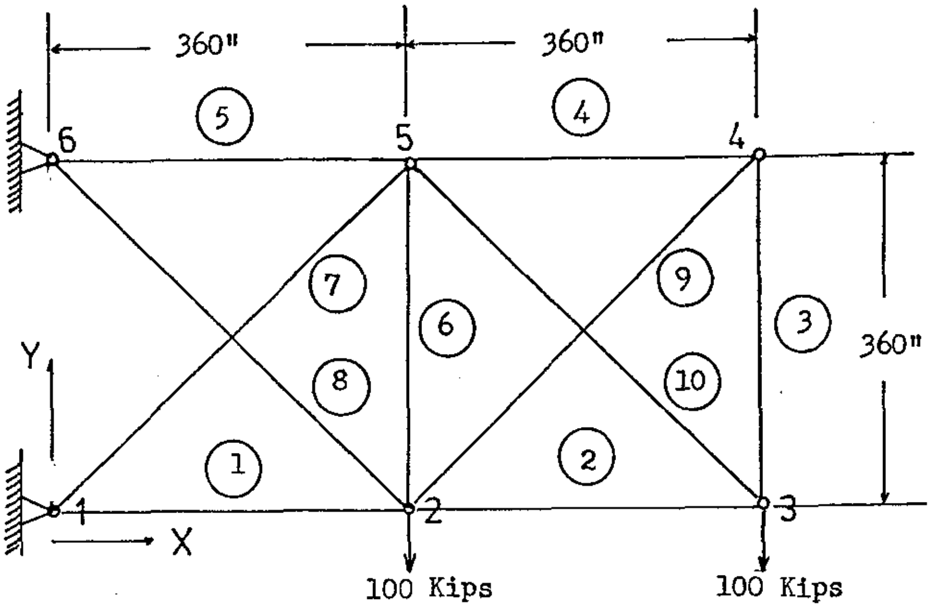
A.1. A Guide for the User

A.1.1 What the program does.

This program provides a preliminary design for plane and space trusses. A typical problem and its design obtained by this program are shown in Fig. 1. The design given is a minimum-weight design satisfying optimality conditions. Before using the program you should decide the following.

1) configuration of the truss,
2) material properties such as elastic modulus, mass density and permitted stress,
3) loading conditions,
4) deflection limit you wish to impose on the nodes,
5) minimum sizes of the members,
6) grouping of members such that the members belonging to a group are of the same material and cross sectional area,
7) your trial design.

If you prepare a set of data, the program will give you an optimum design minimizing the weight of material used. The printed results show the cross sectional area of each group of members, the member number mostly stressed among those belonging to the group, its stress and the load case number, and the deflection of each nodal point in each direction under each load case as given in Fig. 2.



Material Data

Elastic Modulus ; $E = 10^7$ psi

Mass Density ; $\rho = 0.1$ pci

Stress Limits ; $\bar{\sigma} = \pm 25{,}000$ psi

Design Conditions

Deflection Limits ; 2.5 in.

Minimum Size ; 0.1 in.^{2}

No. of Load Cases ; 1

(a) A 10-Bar Truss Problem

ITERATION 12 TOTAL WEIGHT IS 0.406800E 04

SCLNG FACTOR IS 1.0000503063

D.V. NO.	AREA	ELEMENT	LD CASE	STRESS
1	18.9982	1	1	-10.3958
2	11.9995	2	1	-8.3337
3	0.1000	3	1	0.0006
4	0.1000	4	1	0.0006
5	24.3840	5	1	8.3046
6	0.1000	6	1	24.9984
7	16.6253	7	1	-8.7191
8	6.8353	8	1	20.1727
9	0.1000	9	1	-0.0008
10	16.9697	10	1	8.3337

NODE	1 - LD CASE	
1	0.000	0.000
2	-0.374	-1.827
3	-0.674	-2.500
4	0.299	-2.500
5	0.299	-0.927
6	0.000	0.000

(b) Minimum-weight Design

Fig. 1 A Sample Truss Problem and its Solution.

A.1.2 Preparing a set of data

Most of the data input part and analysis part of this program have been taken from the program presented in "The Finite Element Method", 3rd edition, by O.C. Zienkiewicz, McGraw-Hill, London, 1977. Therefore, preparation of data for this program is much similar to that for the program in the book.

1. The title card - FORMAT(20A4)

The first four columns of this card must contain "TRUS" and the rest (columns 5-80) may be any alphanumeric information to be printed with output as page header

T	R	U	S	S			*	*		1	0	-	B	A	

Parts of the sample data taken from the problem of Fig. 1 will appear whenever appropriate.

2. Data for the problem size - FORMAT(16I5)

				6				1	0				1	0					2					1	

Columns	Description	Variable
1 to 5	Number of nodal points	NUMNP
6 to 10	Number of members	NUMEL
11 to 15	Number of groups of members	NUMMAT
16 to 20	Dimension of co-ordinate space	NDM
21 to 25	Max. number of members of any group	MX

3. Co-ordinate data - FORMAT(2I5,7F10.0)

COOR				
1	1	0.	0.	
3	0	720.	0.	
4	1	720.	360.	
6	0	0.	360.	

The first card must contain "COOR" in columns 1 to 4 and the following cards are for node numbers, generator increments and co-ordinates. Nodal co-ordinates can be generated along a straight line described by the values input on two successive cards. The value of the node number is computed using the N and NG on the first card to compute the sequence N, N+NG, N+2G, etc. NG may be input as a negative number and nodes need not be in order. The input of co-ordinate data terminates with blank card (s).

Columns	Description	Variable
1 to 5	Node number	N
6 to 10	Generator increment	NG
11 to 20	X-co-ordinate	X(1,N)
21 to 30	Y-co-ordinate	X(2,N)
31 to 40	Z-co-ordinate	X(3,N)

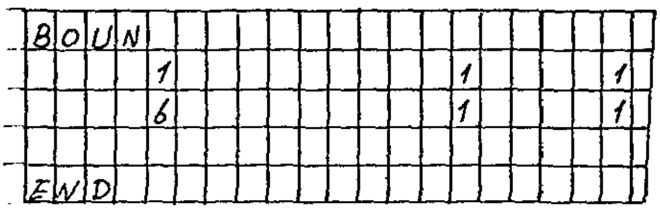
4. Member data - FORMAT(16I5)

ELEM				
1	1	2	1	
6	2	5		
7	1	5	1	
9	2	4	1	

The first card must contain "ELEM" in columns 1 to 4. The following cards contain the member number, two nodes connected to the member and generator increment. Members must be in order. If member cards are omitted the member data will be generated from the previous member with the nodes all incremented by the generation increment, LX, on the previous member. Generation to the maximum member number occurs when a blank card is encountered.

Columns	Description	Variable
1 to 5	Member number	L
6 to 10	not in use	-
11 to 15	Node 1 number	IX(1,L)
16 to 20	Node 2 number	IX(2,L)
21 to 25	Generation increment	LG

5. Boundary restraint data - FORMAT(16I5)



For each node which has been restrained on a support, a boundary condition card must be input, preceded by the first card containing "BOUN" in columns 1 to 4. The convention used for boundary restraints is = 0 for no restraint and ≠ 0 for restraint. The input of boundary restraint data terminates with blank card(s).

Columns	Description	Variable
1 to 5	Node number	N
6 to 10	not in use	-
11 to 15	Boundary code for X-direction	ID(1,N)
16 to 20	Boundary code for Y-direction	ID(2,N)
21 to 25	Boundary code for Z-direction	ID(3,N)

The input of the data for structural configuration terminates with "END" card.

6. Enter the design process - FORMAT(A4),FORMAT(2I5,7F10.0)

D	E	G	N																
				1				1	5						0	.	0	0	1

"DEGN" in columns 1 to 4 of the first card makes the program enter the design process. In columns 1 to 5 in the next card, you enter the number of load cases and in columns 6 to 10 a number up to which you wish the iterative design process to proceed. The figure 0.001 in columns 11 to 20 makes the design process terminate when the change of any design value is less than 0.001 times its current value. Recommended figures are 10 - 20 for the limit on the number of redesign iterations and 0.01 - 0.001 for the cutoff criteria.

7. Force data - FORMAT(2I5,7F10.0)

				2																					-	1	0	0	.
				3																					-	1	0	0	.

Force data can be generated in the same way as for the co-ordinate data. In columns 11 to 20, 21 to 30 and 31 to 40 you should input X-, Y- and Z-directional forces respectively being applied to the node numbered as in columns 1 to 5. Presence of a generator increment in columns 6 to 10 makes force data generated along a straight line described by the values input on two successive cards, but in the case of force data the magnitudes of the loads on the two cards may be the same and thus the same loads will be generated. The input of force data in each load case terminates with a blank card.

8. Grouping of member - FORMAT(16I5)

				1				1	0					1					1

The members numbered between that in columns 1 to 5 and that in columns 6 to 10 inclusively are made to belong to the groups numbered MA in columns 16 to 20 incremented each time by the number LK in columns 11 to 15. In this sample problem, each group contains only one member. If LK is zero, all the members, 1 to 10, will belong to one group.

Column	Description	Variable
1 to 5	Member number	N
6 to 10	Member number	NN
11 to 15	Increment for group number	LK
16 to 20	Group number	MA

9. Material properties and trial design - FORMAT(2I5,F10.0)

				1				1	0					1	0	0	0	0	.

							0	.	1								4	.	0

							0	.	1							2	5	.	0						-	2	5	.	0

The groups of members, numbered between that in columns 1 to 5 and that in columns 6 to 10 inclusively are assumed to have the same material properties and initial sizes described throughout the columns 21 to 70.

Columns	Description	Variable
1 to 5	Group number	MA
6 to 10	Group number	MAK
11 to 20	Elastic modulus	D(1,MA)
21 to 30	Mass density	D(3,MA)
31 to 40	The size of trial design	D(4,MA)
41 to 50	Mimimum size	D(5,MA)
51 to 60	Permitted tensile stress	D(6,MA)
61 to 70	Permitted compressive stress	D(7,MA)

10. Deflection limit - FORMAT(8F10.0), FORMAT(16I5)

							2	.	5
				2					2
				3					2
				4					2
				5					2

The deflection limit of the figure in the first card may be imposed on every node in every direction. However, it is desirable to specify the nodes and directions on which the deflection limit is to be imposed. You can easily select the nodes which are likely to deflect more than others and thus you may wish to check their deflections by imposing a deflection limit. In columns 1 to 5 you put node number and in columns 6 to 10 the number 1, 2 or 3 corresponding to X-, Y- or Z-direction respectively. Input of node numbers and directions terminates with a blank card. In the sample problem, the deflections at nodes 2, 3, 4 and 5 in Y-direction are limited to 2.5.

11. The last card - FORMAT(20A4)

S	T	O	P			

The last card contains the word "STOP" in columns 1 to 4, which stops the design process. However, you can start another design process by giving another title card instead.

A.2 Notes for the Programmer

A.2.1 Variables and arrays

Some interger variables defining the configuration of a truss being designed are listed below

Variable	Description
NUMNP	Number of nodes
NUMEL	Number of members
NUMMAT	Number of groups of members
NDM	Dimension of co-ordinate space
NEQ	Order of global stiffness matrix
LDCS	Number of load cases

These are defined from data input, except NEQ, and used throughout the design process without changing. Other integer variables of relative importance are listed below.

Variable	Description
ITER	Redesign iteration number
NC	Number of active constraints
NQC	Number of active deflection constraints
NSC	Number of active stress constraints

In this program, a big array "M" is declared in the blank COMMON area and partitioned to store a number of arrays mostly used for storing data and for work spaces at analysis stage. Among them some arrays of relative importance are listed below.

Pointer	Array	Description
N6	D(1,MA)	Elastic modulus
	D(2,MA)	not in use
	D(3,MA)	Mass density
	D(4,MA)	Current design value
	D(5,MA)	Minimum size
	D(6,MA)	Permitted tensile stress
	D(7,MA)	Permitted compressive stress
	D(8,MA)	Design values scaled until critical
	D(9,MA)	Design values in preceding iteration
	D(10,MA)	h_i of Equ.(5.1)
N7	ID	Boundary restraint code, equation number
N8	X	Co-ordinates of nodal points
N9	IX	Node numbers connected to a member
N12	JDIAG	Pointer array to locate the diagonal entries of global stiffness matrix
N15	MAT	Member numbers belonging to each group and the number of members
NA	A	Global stiffness matrix

Pointer	Array	Description
NE	DEA	c_{ik} and d_{ij} in Equ. (5.53)
N21	FK	Jacobian matrix in "REDEGN"

Other arrays declared in a number of labeled COMMON areas are listed below.

Label	Array	Description
DEGN	DISP	Deflections of nodes
DEGN	STRS	Stresses of members
DEGN	STRN(I,N)	= 0.0 when the stress of member N is inactive in load case I = 1.0 when the tensile stress of member N is active in load case I = - 1.0 when the compressive stress of member N is active in load case I
DEGN	W	Length of each member and its X-, Y-, Z-components
ACTV	ICOL	Equation number or member number concerning each active constraint
ACTV	IROW	Load case number concerning each active constraint arranged consistently with those for ICOL
ACTV	IGR	Group number of each design variable, 1, 2 or 3
BLUE	PLR	Lagrange multipliers
ACTD	SDISP	= 1 for active deflection constraint = 0 for inactive deflection constraint = - 1 for the deflection component on which no limit is imposed
ACTD	ICL	Member number concerning each active stress constraint
ACTD	IRW	Load case number concerning each active stress constraint arranged consistently with those for ICL

A.2.2 Flow diagram and subroutines

Fig. 2 shows the overall flow diagram of the design process and Fig. 3 shows the subroutines arranged according to their levels in the structure of the program. The iterative stage of the design process is described further in Fig. 4.

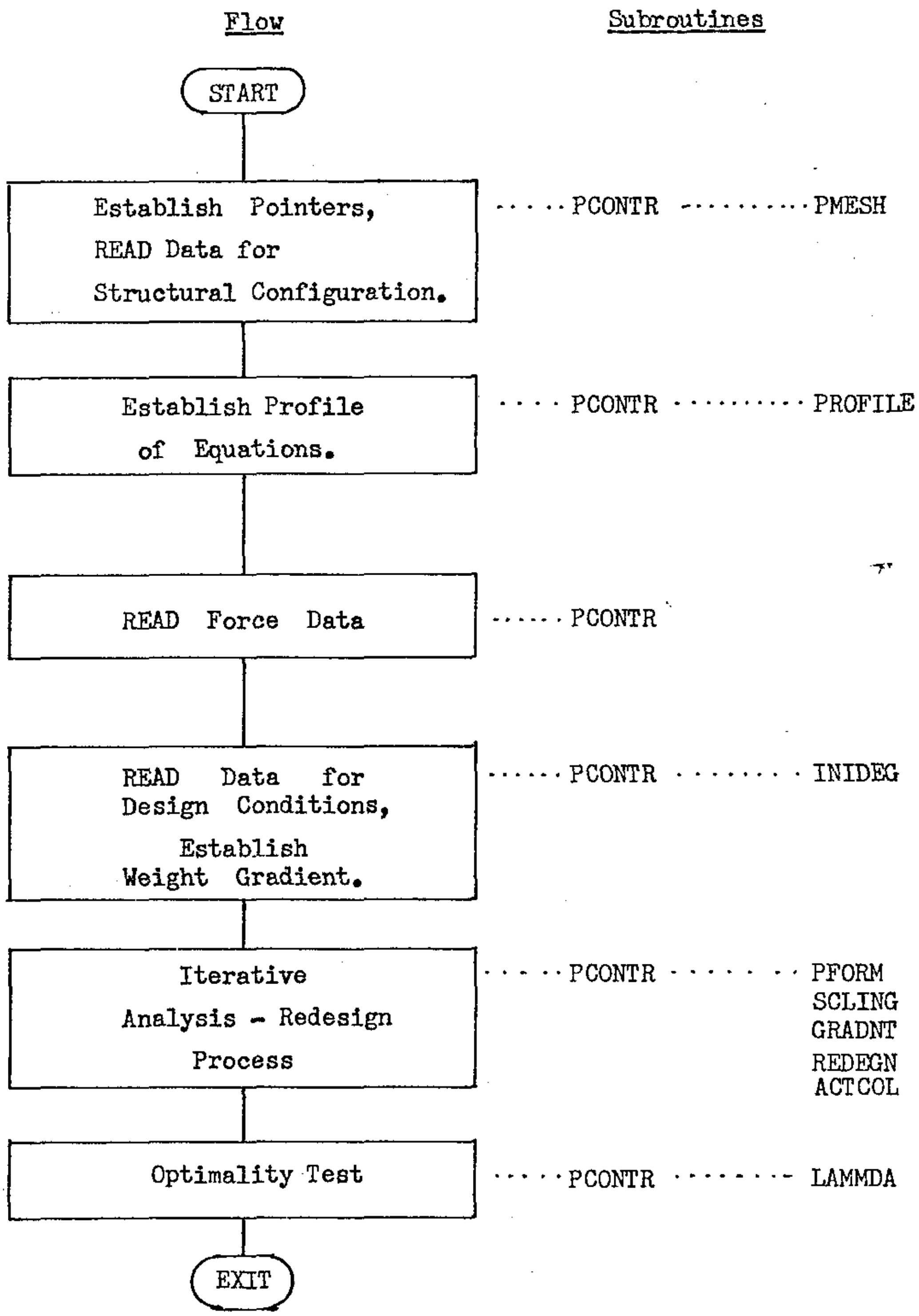


Fig. 2 Overall Flow Diagram of the "TRUS" Program

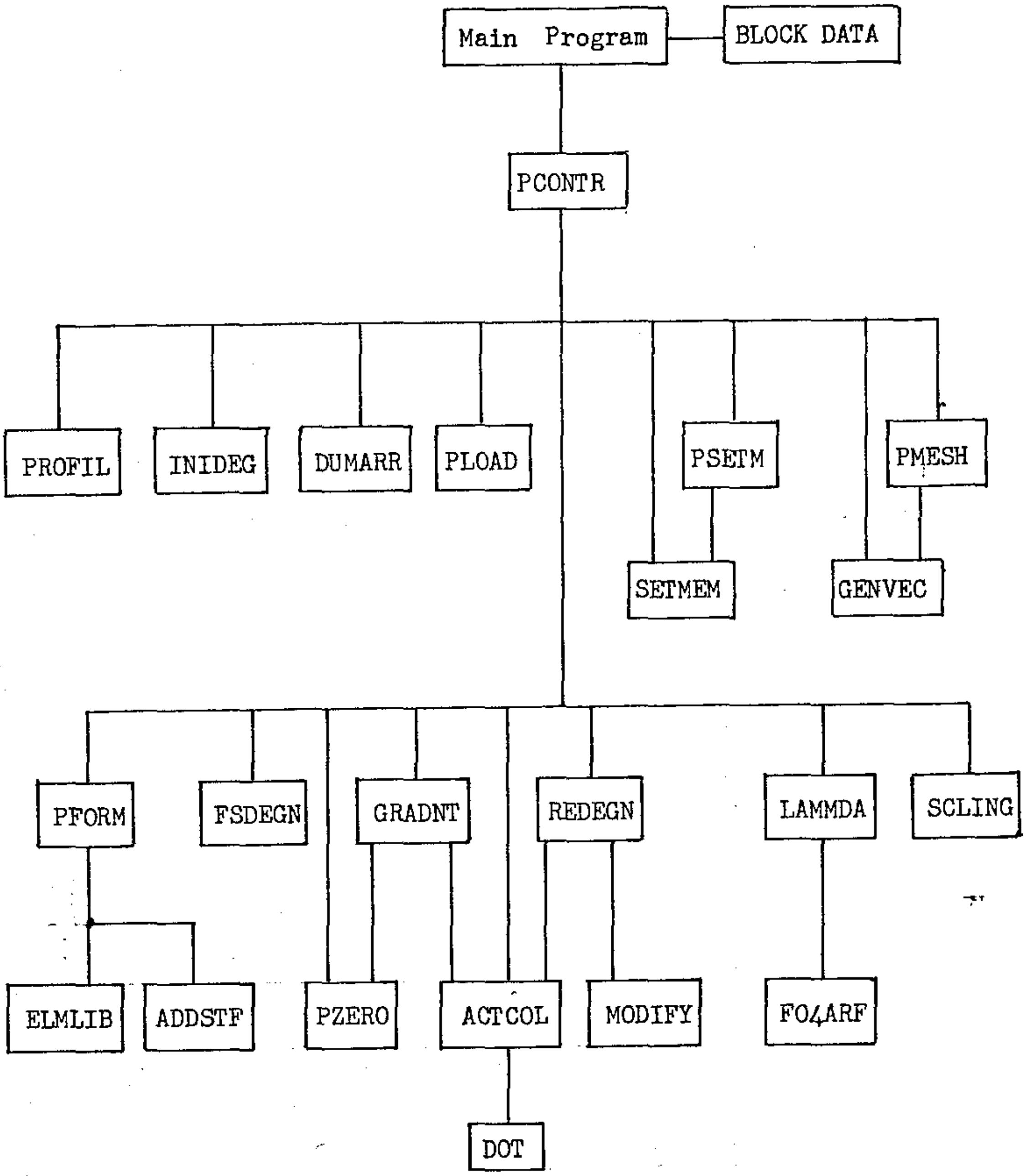


Fig. 3 Subroutines of the "TRUS" Program

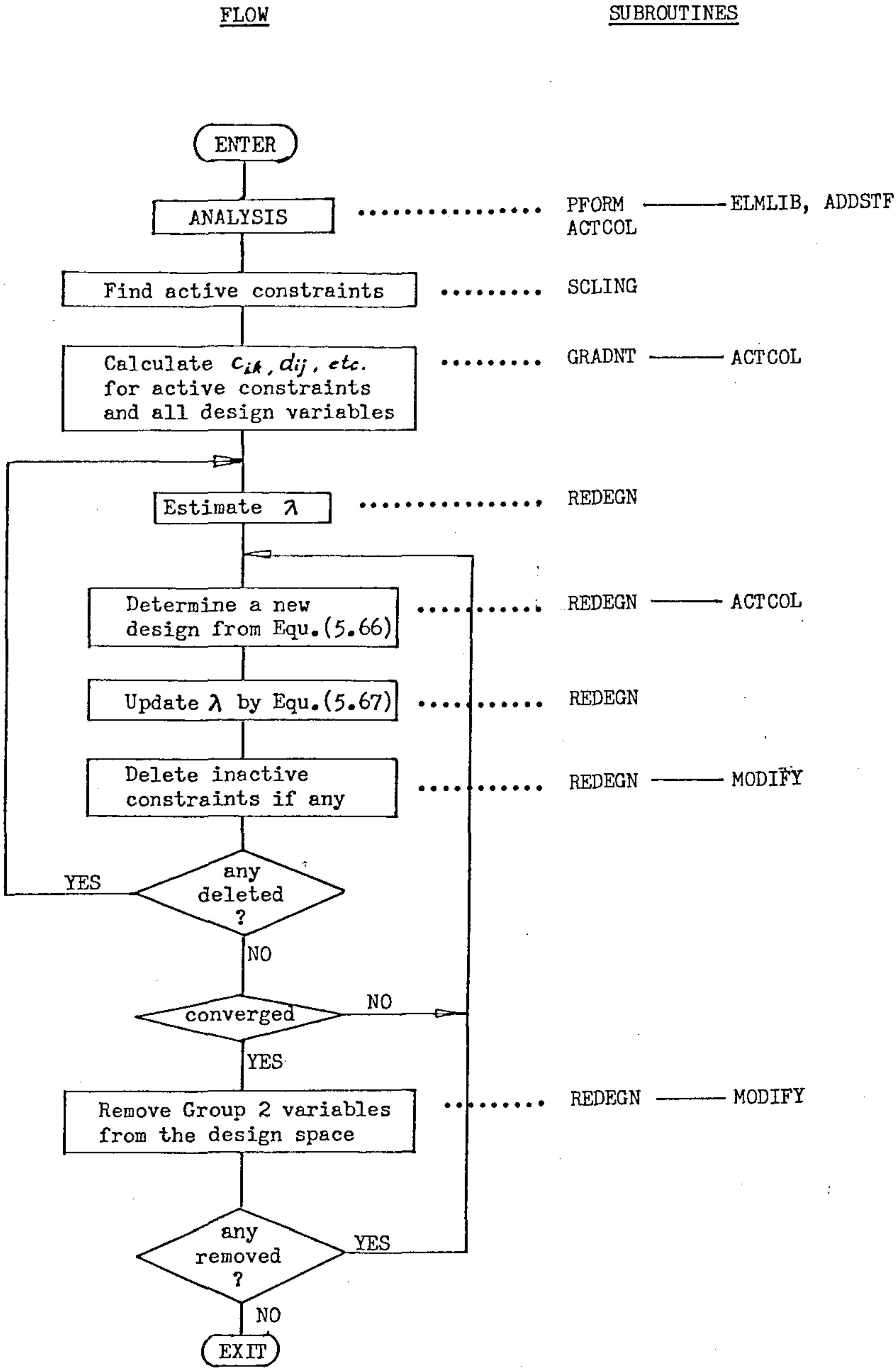


Fig. 4 Iterative Design Stage

APPENDIX B

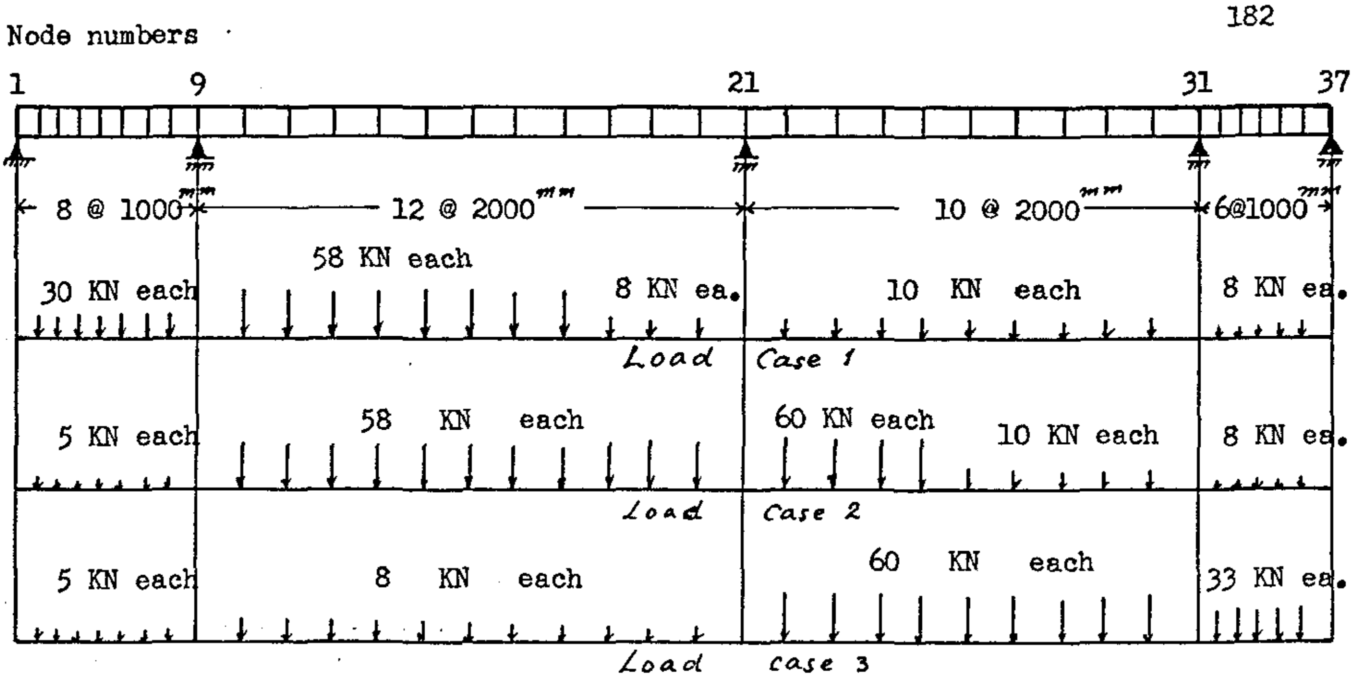
How to use the "BEAM" Program

B.1 A Guide for the User

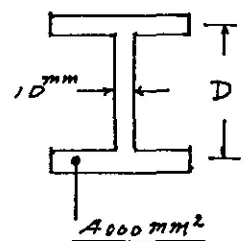
B.1.1 What the program does and what you should do before using it.

This program provides a preliminary design for continuous beams with rectangular or I-shape sections. A typical problem and its solution are shown in Fig. 1. Before using the program you should decide the following.

1) number of span, the length of each span,
2) material properties such as elastic modulus, mass density and permitted stress,
3) type of section, rectangular or I-shape,
4) breadth of rectangular section or cross sectional area of a flange of I-section (Both flanges, upper and lower, should have the same cross sectional area.),
5) thickness of web of I-section,
6) number of load cases,
7) for each load case, the magnitudes of concentrated loads (or couples) and their locations (Distributed loads must be replaced by concentrated loads. Downward loads and clockwise couples should have "+" sign.),
8) the points at which you would like to impose deflection and/or stress limits and check if the resulting deflections and/or stresses are within the limits,
9) the deflection limits at various points,
10) the minimum values you want to impose on the depths of the nodes,



" Section "



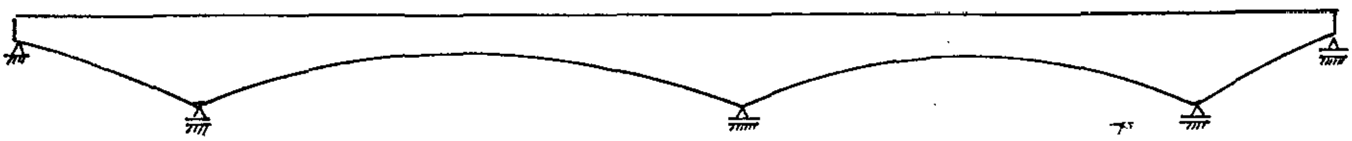
Material ; Steel

$E = 210{,}000\ N/mm^2$

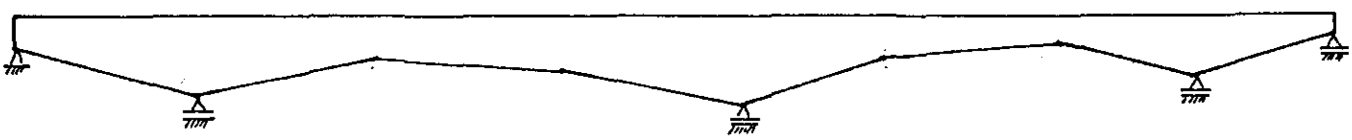
$\rho = 7.85\ g/cm^3$

$\bar{\sigma} = 160\ N/mm^2$

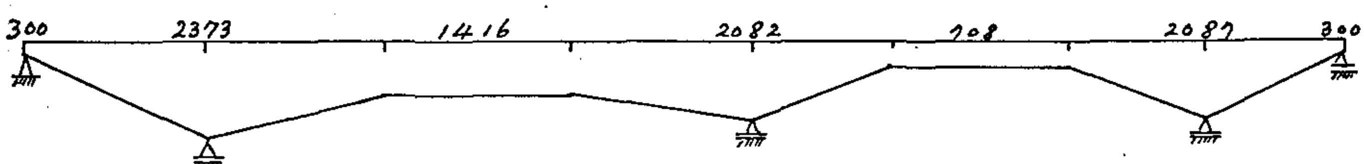
(a) The 4-Span Beam Problem



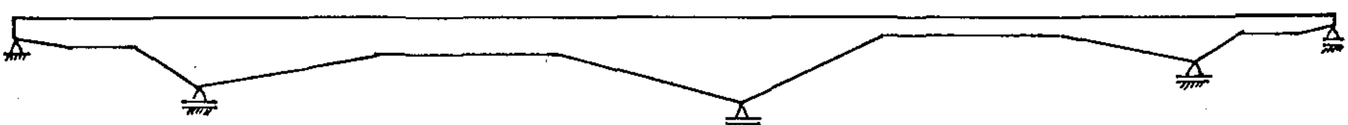
(b) A Possible Design, No linking of depths.



(c) A Possible Design, Linking of depths by ratio.



(d) A Design obtained from the Program.



(e) A Possible Design, 3 groups of segments per span,
the same depth at the middle part of each span.

Fig. 1 A Typical Beam Problem and Possible Designs

11) your trial design.

As shown in Fig. 1, the beam has many segments, each having a tapered or flat configuration. The boundaries of segments are called nodes, to which the program assigns their depths. You should now decide the nodes bearing in mind that;

1) the loads and couples can be applied only to the nodes,
2) deflection and stress limits are observed only at the nodes,
3) the segment between two adjacent nodes must be made of one material.

If you prepare a set of data, the program will give you an optimum design by printing the depth of each node. In addition the response quantities such as the deflections and the stresses of the extreme fibres at all nodes under the most critical load case are also given.

The depths of individual nodes can either vary independently as shown in Fig. 1-b or be linked as shown in Fig. 1-c, d and e. In order to obtain a design with linked depths you must divide the beam segments in groups, each containing a number of segments which all have either the same rate of tapering or a constant depth. In this case the number of design values, which make a design, reduces to the number of groups plus one. This number reduces further by one whenever a group of segments having a constant depth is assigned.

The resulting design will normally be the minimum weight design. However, you may obtain a minimum cost design if you can define a cost function as follows.

$$f = \alpha D^{\beta}$$

where

f ; cost per unit length of the beam,

D ; depth of the beam segment,

α, β ; constants.

Since the flanges of I-section beam are predetermined, the cost defined by f does not include the cost of flanges. The constants α and β should be decided by yourself.

B.1.2 Preparing a set of data

1. The title card - FORMAT(20A4).

The first four columns of this card must contain the word "BEAM", and the rest (columns 5-80) will be any alphanumeric information to be printed with output as page header. Since this card also serves as a start-of-problem-card, you must not miss this card.

BEAM ** EXAMPLE B-14 **

Parts of the sample data taken from the problem of Fig. 1 will appear whenever appropriate.

2. Data for the configuration of the beam and the design conditions - FORMAT(16I5)

4	37	7	3	8
1	4	1	1	
8	12	10	6	

Card 1, Columns	Description	Variable
1 to 5	Number of spans	NUMSP
6 to 10	Number of nodes	NUMNP
11 to 15	Number of design values	NUMDV
16 to 20	Number of load cases	NUMLC
21 to 25	Number of groups of segments	NGR

Card 2,	Columns	Description	Variable
	1 to 5	if =0, stress limits only	MD1
		if =1, deflection and stress limits	
	6 to 10	if =0, no linking of depths	MD2
		if =2, linking as in Fig. 1-e	
		if =3, linking as in Fig. 1-c	
		if =4, linking as in Fig. 1-d	
	11 to 15	if =0, rectanfular section	MD5
		if ≠0, I-section	
	16 to 20	if =0, not including self weight	MD8
		if ≠0, including self weght	

Card 3,	Columns	Description	Variable
	1 to 5	Number of segments in Span 1	SP(1)
	6 to 10	Number of segments in Span 2	SP(2)
	11 to 15	etc,	etc.

3. Co-ordinate data - FORMAT(I5,F10.0,I5,F10.0)

1	0.	9	8000.
9	8000.	31	52000.
31	52000.	37	58000.

Columns 1 to 5 and columns 16 to 20 contain two node numbers, each followed by its co-ordinate. The co-ordinates of the nodes lying between the two nodes are automatically generated along a straight line such that they are arranged at intervals of the same distance. Therefore it is required for you to provide a card whenever the interval changes. Input of co-ordinate data finishes when the last node number appears in columns 16 to 20.

4. Segment date - FORMAT(I5,F10.0,I5)

1	4000.	36
1	7.85	36
1	210000.	36
1	160.	36
1	10.	36

The first card is for the cross sectional area of flanges. Columns 1 to 5 and columns 16 to 20 contain two segment numbers, both assumed to have a flange area of that in columns 6 to 15. The segments whose numbers are lying between the two given in the card have the same flange area. The segments must be numbered from left to right in the same way as the nodes. Therefore the two nodes of the ith segment have the numbers i and i+1. If your beam is to be built with segments having different flange areas you will need more than one card for input of flange areas. Input of the data for flange areas terminates when the last element number appears in columns 16 to 20.

The following cards are for mass density, elastic modulus, permitted stress and thickness of web respectively. If your beam has elements of rectangular sections you must omit the last card, the card for thickness of web, and replace flange areas with breadths of segments.

5. Force data - FORMAT(2I5,2F10.0)

2	8	30.	
10	17	58.	
18	20	8.	
22	30	10.	
32	36	8.	
2	8	5.	
10	20	58.	
22	25	60.	
26	30	10.	

32	36	8.
2	8	5.
10	20	8.
22	30	66.
32	36	33.

Columns 1 to 5 and 6 to 10 contain two node numbers, and columns 11 to 20 and 21 to 30 contain a concentrated load and a couple respectively. The nodes lying between the two nodes are assumed to have the same concentrated load and couple. The force data for one load case terminates with a blank card. Since the example is subjected to three load cases, you need three blank cards as shown above. In the example there are no couples applied. In this case you simply leave the corresponding columns.

6. Data for deflection limits - FORMAT(I10,F10.0)

5	32.
15	48.
26	40.
34	24.

Columns 1 to 10 and columns 11 to 20 contain a node number and the deflection limit imposed on that node respectively. The data for deflection limits terminates with a blank card.

7. Data for your trial design - FORMAT(I5,F10.0,I5)

1	1000.	7

Usually, a uniform design, i.e. equal depth for any node, makes a good trial design as far as it is reasonable. However, you may decide your own design and feed it into the computer. The way of data generation is the same as that for segment data.

8. Data for minimum size - FORMAT(I5,F10.0,I5)

1	300.	7	

You may impose different minimum size restrictions on each design value. The way of data generation is the same as that for segment data.

9. Data for grouping segments - FORMAT(16I5)

8	4	4	4	3	4	3	6
3	6						

The number of segments belonging to each group should be provided if the depths of nodes are to be linked. The numbers should be arranged such that each of them corresponds to a group numbered from left to right. If the linking is to be as shown in Fig. 1-d, i.e. MD2 = 4, a further set of data is necessary as shown above. Each of the data represents a group number whose segments have the same depth.

10. Parameters for printing control - FORMAT(16I5)

12	12	0	

The design process is in an iterative way. In other words, the program derives a new design from the information obtainable by analysing the given trial design, and then takes the new design as the trial design at the next stage. This process is repeated until the optimality conditions are met. Therefore you can get the computer to print many designs, mostly the designs obtained during the iterative process, but you may not wish all of them. The number you put in columns 1 to 5 makes the computer not print the design values until the redesign iteration number reaches that number. The number in columns 6 to 10 is for the bending moments at nodes under each load case. When the number in columns 11 to 15 is other than zero, the computer prints various kinds of information obtained during the redesign process such as active constraints, Lagrange multipliers, and so on. These may not helpful for the ordinary user.

11. Enter the Design Process - FORMAT(A4),FORMAT(I10,3F10.0)

```
DEGN
        12     0.001    0.0785      0.75
```

"DEGN" in columns 1 to 4 of the first card makes the program enter the design process. In columns 1 to 10 of the next card, you enter a number up to which you wish the iterative design process to proceed. The figure 0.001 in columns 11 to 20 makes the design process terminate when the change of any design value is less than 0.001 times its current value. Recommended figures are 10 - 20 for the limit on the number of redesign iterations and 0.01 - 0.001 for the cutoff criteria. The following figures in columns 21 to 30 and 31 to 40 are α and β being used in the cost function.

If you want a minimum weight design, then you may leave these columns, columns 21 to 40

12. The last card - FORMAT(20A4)

```
STOP
```

The last card contains the word "STOP" in columns 1 to 4, which stops the design process you started by giving the word "BEAM". At this stage, however, you can start another design process by giving another title card instead.

B.1.3 The results printed

Fig. 2 shows the printed results of the problem of Fig. 1 obtained after 12 redesign iterations. Fig. 1-d shows the same design as that of Fig. 2. The depths of the nodes at which the rate of tapering changes, the depths of individual nodes, maximum bending stresses at nodes and deflections at nodes are given in Fig. 2. The stresses and deflections are those under the load case most critical to the node concerned and Fig. 2 also shows the load case numbers. It is also shown in the figure that the stresses at node 9 under load case 1, at node 21 under load case 2 and at node 31 at load case 3, and the deflection at node 26 under load case 3 reached the limits. "RGHT" under the heading "L-R" stands for the word "right" and means that the corresponding stress is that of the section located on the right hand of the corresponding node. As far as the problem of Fig. 1 is concerned "LEFT or RGHT" makes no difference. However, if your beam is subjected to couples or built with segments of different materials, "LEFT" will appear in the column when the stress at the left hand side section is more critical.

The units used are mm for depths and deflections, and N/mm^2 for stresses. The units for input data should follow those as shown in Fig. 1, i.e. mm for length, KN for point loads, KN-m for couples, N/mm^2 for stresses, and g/cm^3 for mass densities.

**ITERATION 12 TOTAL WEIGHT IS 0.100873E 07
SCLNG FACTOR IS 0.9999669790

DESIGN VARIABLES NO.	VALUES	NODE	MAXIMUM BENDING STRESS DEPTHS	STRESS	LOAD	L-R	DEFLECTIONS DISPL.	LOA
1	300.00							
		1	300.00	0.00	0		0.00	1
		2	559.09	-68.75	2	RGHT	-3.41	2
		3	818.18	-95.81	2	RGHT	-5.71	2
		4	1077.27	-111.30	2	RGHT	-6.89	2
		5	1336.36	-121.99	2	RGHT	-7.09	2
		6	1595.45	-130.25	2	RGHT	-6.42	2
		7	1854.54	-137.12	2	RGHT	-4.96	2
		8	2113.62	-143.70	1	RGHT	-2.81	2
2	2372.71							
		9	2372.71	-159.72*	1	RGHT	0.00	1
		10	2133.51	-95.81	1	RGHT	7.29	2
		11	1894.31	-32.27	1	RGHT	16.09	2
		12	1655.11	41.50	2	RGHT	25.34	2
3	1415.90							
		13	1415.90	103.89	2	RGHT	33.62	2
		14	1415.90	137.56	2	RGHT	39.31	2
		15	1415.90	149.54	2	RGHT	41.40	2
		16	1415.90	139.84	2	RGHT	39.56	2
3	1415.90							
		17	1415.90	108.46	2	RGHT	34.05	2
		18	1582.54	-72.20	3	RGHT	25.80	2
		19	1749.18	-90.48	3	RGHT	16.86	1
		20	1915.81	-108.69	3	RGHT	8.00	1
4	2082.45							
		21	2082.45	-159.72*	2	RGHT	0.00	1
		22	1624.37	-122.84	2	RGHT	8.47	3
		23	1166.30	-84.31	2	RGHT	18.81	3
5	708.22							
		24	708.22	-74.43	1	RGHT	29.79	3
		25	708.22	118.88	3	RGHT	37.89	3
		26	708.22	131.34	3	RGHT	40.00*	3
		27	708.22	99.79	3	RGHT	35.44	3
5	708.22							
		28	708.22	29.06	1	RGHT	25.90	3
		29	1167.86	-57.83	3	RGHT	15.18	3
		30	1627.50	-112.92	3	RGHT	6.27	3
6	2087.13							
		31	2087.13	-159.01*	3	RGHT	0.00	1
		32	1789.28	-142.45	3	RGHT	-2.05	3
		33	1491.42	-125.14	3	RGHT	-3.33	3
		34	1193.57	-106.48	3	RGHT	-3.82	3
		35	895.71	-85.04	3	RGHT	-3.44	3
		36	597.86	-56.58	3	RGHT	-2.16	3
7	300.00	37	300.00	0.00	0		0.00	1

Fig. 2 The Design Values of the Problem in figure 1.

B.2 Notes for the Programmer

B.2.1 Variables and arrays

Integer variables defining the configuration of a beam being designed and design conditions are listed below.

Variable	Description
NUMSP	Number of spans
NUMNP	Number of nodes
NUMEL	Number of beam segments
NUMDV	Number of design variables
NUMLC	Number of load cases
NGR	Number of groups of segments
NEQ	Order of flexibility matrix

These variables are defined from data input, except NUMEL and NEQ which are derivable, and used throughout the design process without changing. Other integer variables varying from iteration to iteration are listed below.

Variable	Description
ITER	Redesign iteration number
NAC	Number of active constraints
NAQC	Number of active deflection constraints
NASC	Number of active stress constraints

During a redesign iteration in the subroutine named REDEGN, the number of active constraints changes since some constraints formerly considered active may found inactive. Therefore NC, NQC and NSC replace NAC, NAQC and NASC respectively in "REDEGN" and they are allowed to change. Some other important scalar variables will be explained later when appropriate.

The majority of arrays share the blank COMMON area. A big array "M" is declared in the area and partitioned to store all the data arrays and most of the other arrays for pieces of information obtained or simply for working spaces. Each array in a subprogram is variably dimensioned to the exact size required for each problem by using a set of pointers established in the calling program. The partitioned arrays whose lengths are not varying from iteration to iteration are listed below.

Pointer	Array	Description
1	NES	Number of segments in each span.
N0	JD	Pointer array to locate the diagonal entries of flexibility matrix.
N1	X	Co-ordinate of each node.
N2	BR	Flange area of each I-shape segment or breadth of each rectangular segment.
N3	RHO	Mass density of each segment.
K9	WEB	Thickness of the web, later weight of the flanges of each I-shape segment.
N4	E	Elastic modulus of each segment.
N5	SA	Permitted stress of each segment.
N6	F	Point load applied to each node in each load case.
N7	C	Couple applied to each node in each load case.
N8	SP	Length of each span.
N9	SL	Length of each segment.
N10	HE	RHO(I)*WEB(I)*SL(I) for I-section, RHO(I)*BR(I)*SL(I) for rectangular section.
N11	CON	Deflection limit imposed on any node.
N12	NED	Number of segments linked as a group.
N13	D(I,1)	Design values obtained from stress ratio.
	D(I,2)	Design values in the preceding iteration.
	D(I,3)	Current design values.
	D(I,4)	Minimum size restrictions.
	D(I,5)	Values of Equ. (5.57) or (5.59), work space.
	D(I,6)	Scaled design values.
N14	H	h_i in Equ. (5.37).
N15	FLX	SL(I)/(3.0*E(I)*BR(I)) for I-section, 2.0*SL(I)/(E(I)*BR(I)) for rectangular section.
N16	RIG	Flexural rigidities, EI, of three sections of each segment, multiplied by 2 for I-section or 12 for rectangular section.

Pointer	Array	Description
N17	ACOE	Entries of flexibility matrix and later those of its decomposed matrix.
N18	WK	Work space, stress at each node in most critical load case.
N19	DR	Work space.
N20	PK	Work space, response ratio of stress at each node.
N21	BMT	Work space.
N22	BM	Bending moment at each node in each load case.
N23	DISP	Deflection at each node in each load case.
N24	A	Work space, design variable in "REDEGN".
N25	AN	Work space, minimum or stress ratio size in"REDEGN"
N26	SGN(I,J)	= 0.0 when the stress at node J is inactive in load case I, = 1.0 when the stress at node J is active and positive in load case I, = -1.0 when the stress at node J is active and negative in load case I.
N27	IGR	Group number of each design variable, 1, 2 or 3.
N28	IMD	Work space
N29	RD(1,J)	Depth at each node
	RD(2,J)	RD(1,J+1)/RD(1,J)
N30	BETA	Work space.
N31	RNED	Sum of the lengths of the segments linked as a group.
N32	TRAN	Tranformation matrix in Equ. (5.34).
N33	LINKF	Group numbers of segments so linked as to have the same depth.

These arrays, except "RHO", and an interger variable "NEND" make the set of the parameters of the subroutine "DESIGN". "NEND" has been set to "N33" plus the length of "LINKF" in the calling program "PCONTR" and transferred to "DESIGN" to be used as the pointer when a new array is defined and made to share the blank COMMON area. In "DESIGN" some more partitioned arrays with fixed lengths are defined as follows,

Pointer	Array	Description
N40	QK	Depth of each node when scaled by stress ratio.
N41	FK	Work space.
N42	F1	Work space.
N43	IQK	Node number whose stress decides D(I,1) for each design variable.

and used in the subroutine "SCLNG". "QK", "FK" and "F1" are used only when D(I,1) are decided from the depths of individual nodes, QK(J), by minimizing the sum of squares of the differences between QK(J) and those obtainable from D(I,1). This approach of deciding D(I,1) has been generally unsuccessful and therefore made not to take part in the ordinary design process. However, the set of the FORTRAN statements for this approach still resides in the program and can be used under a certain condition, being explained later, for possilde further developments.

Now, the pointer "NEND" is set to "N43" plus the length of "IQK" and the design process enters the iterative stage. In each iteration a number of arrays are defined by partitioning the array "M" starting at the location "NEND". As the number of active constraints varies from iteration to iteration, so do the lengths of these arrays. The following arrays are used in the subroutine "GRADST".

Pointer	Array	Description
N50	PLR	Lagrange multipliers
N51	ICL	Node number concerning each active constraint, deflection constraint and lower node number first.
N52	IRW	Load case number concerning each active constraint, arranged consistently with those for ICL.
N53	ILR	= 1 when the stress concerned is at the left hand side element of the node. = 2 when at the right hand side element. when both elements are equally stressed, 2 is assigned.
N54	CEA	Values of a_{tk} and b_{tj} appearing in Equ. (5.15), (5.16), (5.19), (5.20), (5.26), (5.27), (5.30), (5.31).

Pointer	Array	Description
N55	DR1	Values of $(1+r_i)^3$ or $(1+r_i)^2$ used for evaluating a_{tk} and b_{tj}.
N56	DR2	Values of $(1+1/r_i)^3$ or $(1+1/r_i)^2$ used for evaluating a_{tk} and b_{tj}.
N57	GMM(J,I)	RD(1,I)/D(J,3), i.e. the ratio of depth of node to design value defined in Equ. (5.39) and (5.41), later $\dfrac{T_i M_j}{\rho h_i A_{tj} T_i^2}$ or $\dfrac{6 T_i M_j}{\rho h_i B_j T_i^3}$ in Equ. (5.74), (5.75), (5.80), (5.81).
N58	TMP	Work space.

In the subroutine "REDEGN" the following arrays are used.

Pointer	Array	Description
N70	B	Prescribed limit of each active deflection/stress constraint.
N71	RP	To store initial estimates of the Lagrange multipliers
N72	AB	Values of ν_i in Equ. (5.74) or (5.75) for Group 1 design variables.
N73	TPK	Work space
N74	OP	Values of the Lagrange multipliers in the preceding Newton iteration.
N75	F	Residuals of the active constraint equations when evaluated using the Lagrange multipliers.
N76	FK	Entries of Jacobian matrix.
N77	INE	Work space for deleting inactive constraints.
N78	ISY	Work space for deleting inactive constraints.
N79	JDG	Work space for removing inactive design variables.

If another redesign iteration is to be carried out, the partitioning of the array "M" is repeated starting at "NEND". Whereas the array "M" in the calling programs is an integer array, many of the arrays in the subprograms are real arrays. For this reason an integer variable "IPR" is used to adjust the length of each partitioned real array. For instance, the required length of "F" is "NAC" and thus the pointers are set such that the length of "F" is equal to "NAC * IPR" and "IPR" is give an appropriate value. In this program "IPR" is set to 2 in "BLOCK DATA" since the program has been run on PRIME machines and these machines use 2-byte interger variables and 4-byte single precision real variables. If you wish to use double precision real variables, "IPR" has to be set to 4. On the contrary, if you are to use long integers by using the option -INTL, you should set "IPR" to 1 to save the core size.

There are many other arrays not taking part in the array "M" but declared in a number of labeled COMMON areas. Some of them cannot take part in "M" since they should keep the information obtained in the preceding iteration. The others may go into the array "M", but they have remained in the labeled COMMON areas merely for the reason of simplicity during the development of this program. Some arrays of importance are listed below.

Label	Array	Description
INDI	DEA	Values of C_{ik} and d_{ij} appearing in in Equ. (3.38) - (5.45).
INDI	DR3	Values of $(1+r_t)^4/2.0$ or $(1+r_t)^3/2.0$ used for evaluating c_{tk}' and d_{tj}' in Equ. (5.18), (5.22), (5.29), (5.33).
INDI	DR4	Values of $(1+1/r_t)^4/2.0$ or $(1+1/r_t)^3/2.0$ used for evaluating c_{tk}' and d_{tj}' .
NONLIN	HH	ρ_i in Equ. (5.57) - (5.60), (5.74), (5.75), (5.80), (5.81).
DEFLCT	LCL(I)	= 1 when deflection at node I is active. = 0 when deflection at node I is inactive.
DEFLCT	LRW(I)	Load case number making node I most deflected.
DEFLCT	DK	Max. response ratio of deflection at each node.
ACTV	JCL,JRW, JLR,KCL, KRW,KLR	To store the information stored in ICL, IRW, ILR in the preceding iteration.
ACTV	PI	The values of the Lagrange multipliers obtained at the end of the first round of the Newton-Raphson process in the preceding redesign iteration.

B. 2.2 Flow diagrams and subroutines

Fig. 3 shows the overall flow diagram of the design process and Fig. 4 shows the subroutines arranged according to their levels in the structure of the program. The design process can be divided largely into four stages as indicated in Fig. 3. Among them the iterative stage is described further in Fig. 5.

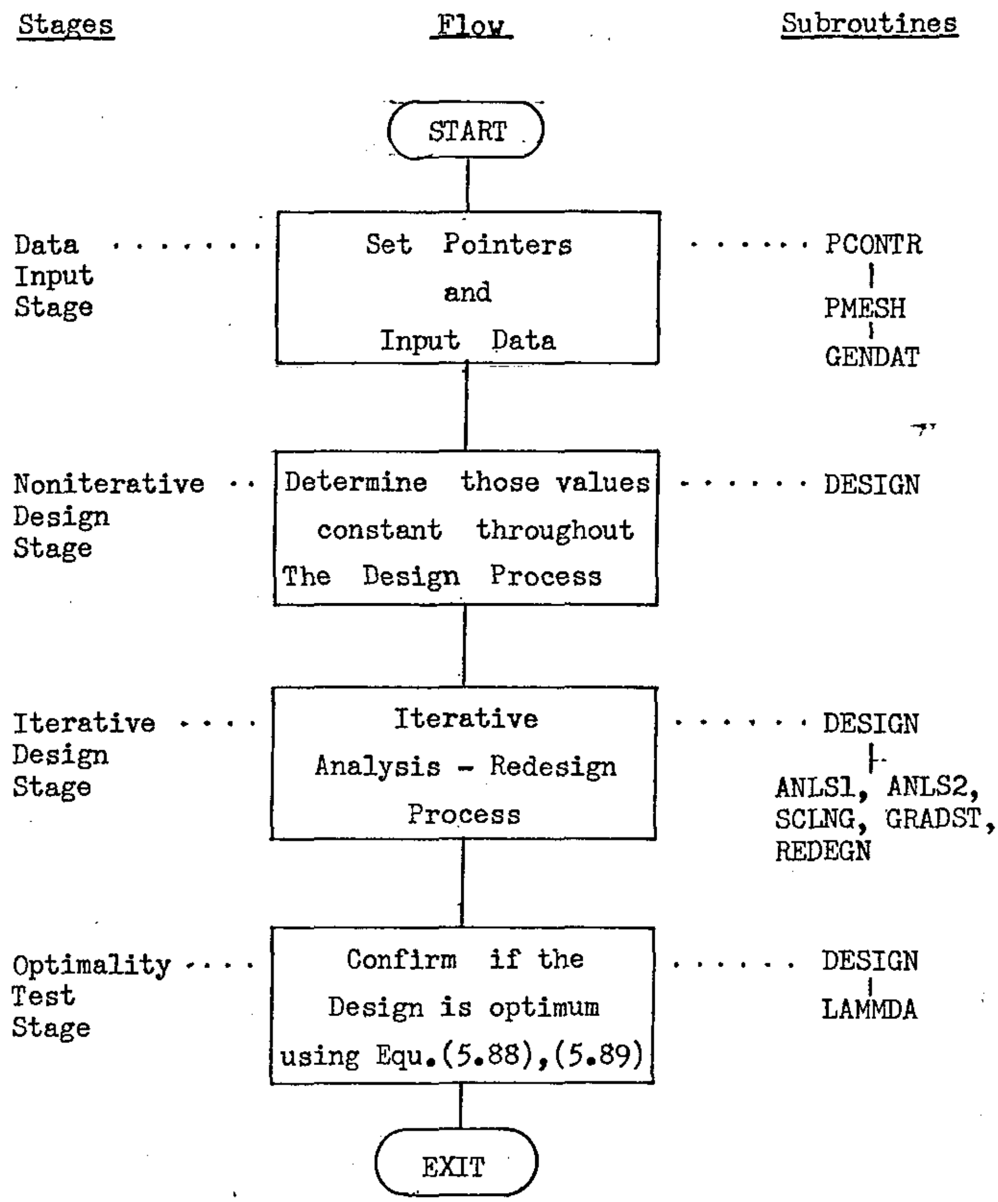


Fig. 3 Overall Flow Diagram

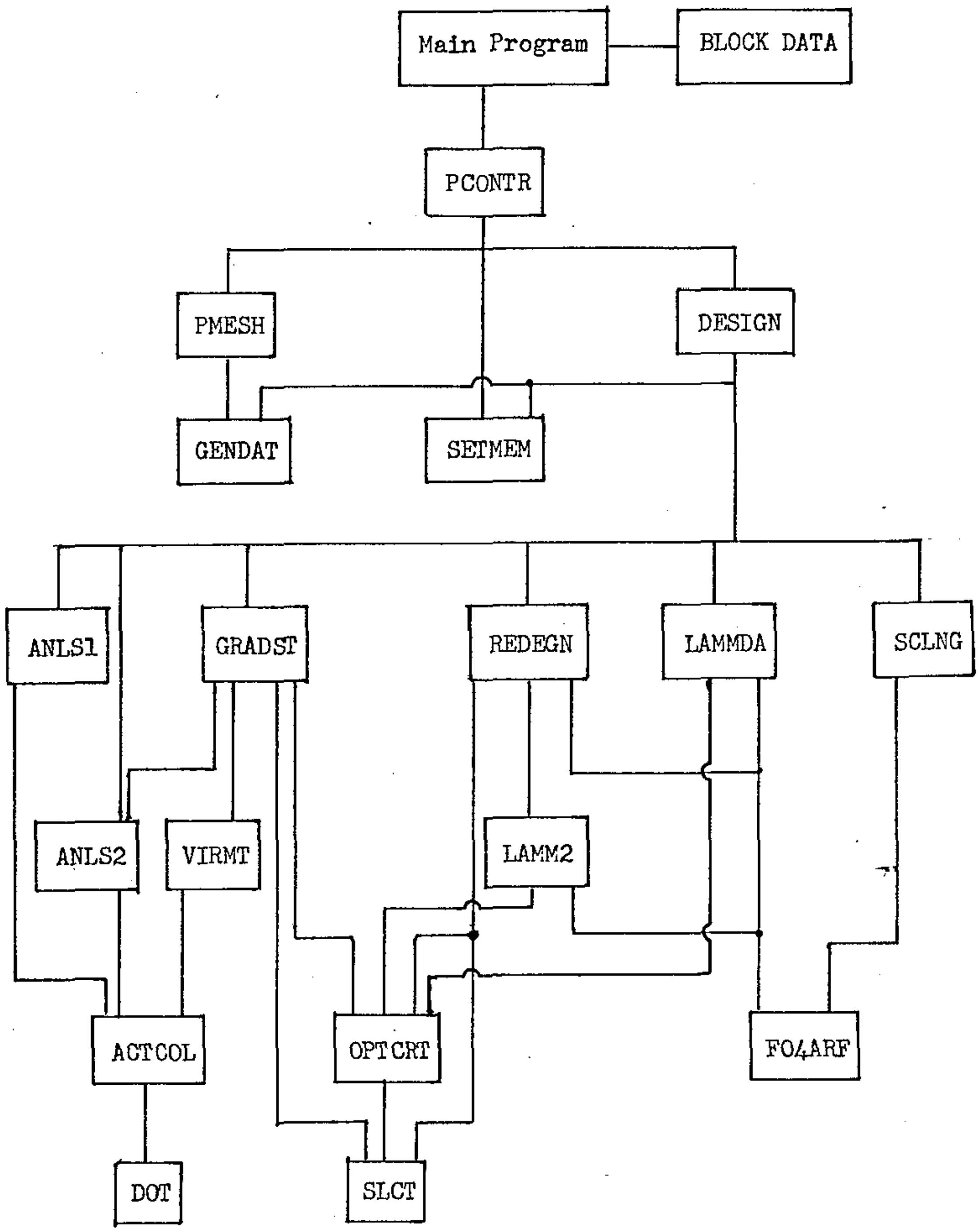


Fig. 4 Subroutines of the "BEAM" Program

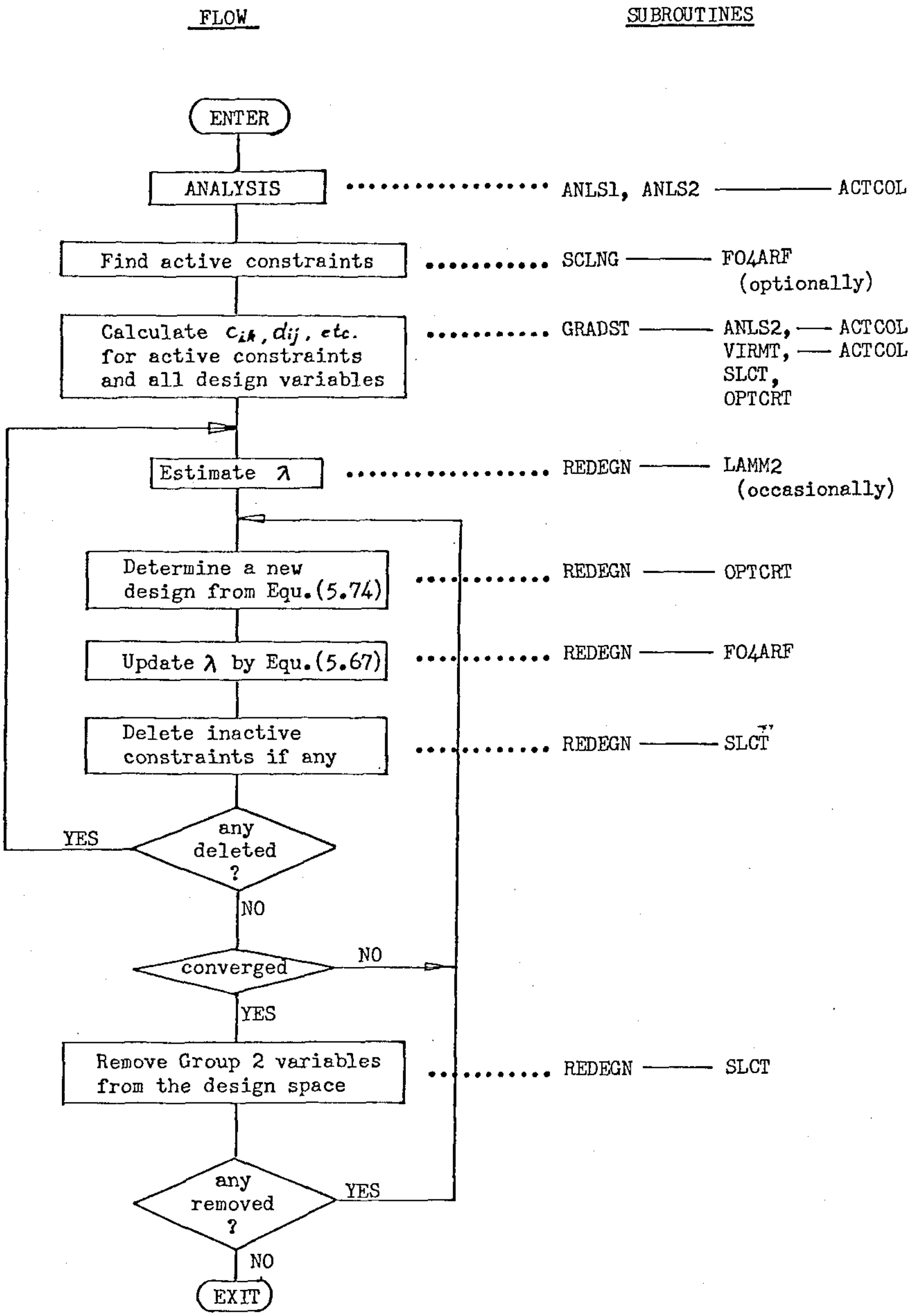


Fig. 5 Iterative Design Stage

The roles of important subroutines are described below.

PCONTR first establishes the pointers to partition the array "M" in the blank COMMON area and reads data by calling PMESH. Then PCONTR calls DESIGN to enter the design process.

DESIGN determines firstly those values constant throughout the design process before entering the iterative process. The constant coefficients and exponents used in the solution process are determined according to whether the beam is of rectangular section or of I-section. The transformation matrix, $\{T_{ti}\}$ in Equ. (5.34) and the cost gradient vector, $\{h_i\}$ are also established at this stage. Then it carries out the iterative process by establishing pointers and calling ANLS1, ANLS2, SCLNG, GRADST and REDEGN repeatedly. Upon completion of the iterative process it calls LAMMDA to confirm the optimality of the resulting design. The last process is in fact unnecessary since the optimality of the design is tested in each iteration in GRADST when there is no change in the set of active constraints. This optimality test is useful only when the design process terminates before the set of active constraints is fixed.

ANLS1 generates the depths of nodes from the new design by using the tranformation matrix, $\{T_{ti}\}$, establishes the flexibility matrix of the new design, and decomposes it by calling ACTCOL.

ANLS2 establishes the right hand side from the given load case, determines the redundant moments by calling ACTCOL, and determines the bending moment distributions under the given load case and stores them in array BMT.

SCLNG determines the responseratios for deflections and stresses, finds scaling factor, finds more active constraints if any, determines stress ratio sizes for Group 3 variables, and prints design values and response quantities, deflection and stress, scaled until critical.

GRADST prints node numbers and load case numbers at which the deflection and/or stress are active, establishes DR1, DR2, DR3, DR4 and GMM, calculates such coefficients as CEA and DEA, and evaluates the optimality equation (Equ. (5.57) or (5.59)), if the set of active constraints has not changed, and prints the evaluated values.

REDEGN estimates the Lagrange multipliers, determines the design values from the values of the Lagrange multipliers using Equ. (5.74) or (5.75), evaluates the constraint equations, Equ. (5.76) - (5.79), and updates the Lagrange multipliers if necessary by using Equ. (5.67) and calling FO4ARF. It deletes inactive constraints and removes inactive design variables if any by calling SLCT.

ACTCOL carries out triangular decomposition and/or forward reduction of linear equations.

FO4ARF gives solutions to systems of linear equations.

VIRMT establishes virtual loads to express stress constraints in terms of virtual work, calculates the bending moment distributions under the virtual loads, and returs them to GRADST.

OPTCRT calculates the values of ν_i in Equ. (5.74), (5.75), (5.82) or (5.81) and returns them to REDEGN, GRADST, LAMMDA or LAMM2.

SLCT is used to rearrange various arrays when inactive constraints or variables are to be deleted and to sort arrays.

LAMMDA tests the optimality of a design using Equ. (5.88) and (5.89).

LAMM2 provides estimates of Lagrange multipliers, as an alternative but used only occasionally, using Equ. (5.88).

B.2.3 Use of alternative approaches.

For the purpose of test or further development, we can use alternative approaches by giving some values to the control parameters other than those declared in BLOCK DATA. If we feed "DATA" card before "DEGN" card the program flow enters a module to change the values of the parameters. The next card "MTHD" makes the program expect data for MD10 and MD11 with FORMAT(16I5), and "JCOB" is for MD3 and MD16. Different approaches effected by different values of these parameters are listed below.

1. If MD10 = 10, the Lagrange multipliers are estimated by LAMM2.
2. If MD11 $\geqslant$ ITER, the values of D(I,1) are determined differently as explained in section B.2.1.
3. If MD3 $<$ 0, or MD3 $>$ 0 the entries of the Jacobian matrix, Equ. (5.67), (5.80), (5.81), are evaluated differently. The effect of this is not yet understood.
4. If MD16 $\neq$ 0, the entries of X_{12} and X_{21}, and the off-diagonal entries of X_{22} in Equ. (5.67), (5.80) and (5.81) are neglected. This approach reduces computing time significantly and was successful in a few problems. This aspect together with that of simplifying Jacobian matrices requires further research.

When finding new active constraints such factors as "SFAC" and "SFAC3" are used. A constraint, formerly inactive, becomes an active constraint in the subroutine SCLNG when its response ratio time SFAC is greater than that of the most critical constraint among those considered active so far. In the program SFAC is initially set to 0.99 and approaches to unity as follows iteration by iteration.

$$SFAC = (1.0 + SFAC)/2.0$$

SFAC3 is for those stress constraints associated with Group 3 design variables and updated by

SFAC3 = (SFAC3+SFAC4)/2.0

In the program both SFAC3 and SFAC4 are set to 0.98 and therefore SFAC3 is kept 0.98 all the time. We can change the values of these parameters by feeding first "FCTR" card and then the values of SFAC, SFAC3, SFAC4 with FORMAT(8F10.4).

The Newton-Raphson process is made to stop when the change of any Lagrange multiplier is less than "DRIM" times the current value and allowed to go up to "IRIM" iterations before damping of the step sizes is introduced. In the program IRIM and DRIM are set to 40 and 0.0015 respectively. We can also change these values by feeding "NEWT" card and desired values with FORMAT(I5 ,F10.4). An "END" card makes the program flow get out of the module, and we can start the design process by feeding "DEGN" card.