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The 1887 committee meets again. Subject: freedom and constraint.

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Abstract

The work of the original Committee is described in Appendix 2 of Ball's book. This time the Committee is supplemented by distinguished guests attending as observers and new committee members intent on gaining a thorough understanding of freedom and constraint.

Led by Mr Duality, the newcomers find that the route to the answers to their problems uncovers a beautiful symmetry in kinestatics. The Law twins, Circuit and Cutset, explain the principles involved; the Coordinates twins, Motion and Action, quantify the problems; and Miss Topology and Miss Matrix perform mathematical magic to devise simple equations for both kinematics and statics of identical form. Mr Virtual Power then repeats their work using his methods to obtain two more dual equations. Mr Querulous remains as sceptical as ever, until the very end.

The new committee members are asked to provide an example. They choose one that is topologically complex but geometrically trivial as befits a gathering that has no need for lessons in screw theory.

The Chairman welcomed the original members of the committee: Messrs Anharmonic, Cartesian, Commonsense, Helix, One-to-One and Querulous. He commented that they did not look a day older than when they attended the previous meeting. He also spoke warmly of the achievements of four scholars who had been invited as observers at this meeting: Mr Synthetic Geometry, Mr Freedom, Mr Exceptional Mobility and Mr Seven R. Spatial. Knowing them to be talkative, indeed argumentative, gentlemen with an enthusiasm for screw theory, he urged them to show restraint on this occasion while others had their say.

The Chairman reminded the gathering that, at the end of the previous meeting in Manchester, one member showed how the theory of screws could be applied to any mechanical system whatsoever and had introduced the concept of a screw-chain. "Today we shall return to that theme."

"We shall consider any collection of rigid bodies some or all pairs of which are directly coupled together. We shall refer to them collectively as coupling networks. Although there is a variety of direct couplings we shall restrict attention to those that involve contact at a point, along a line, or over surface, between areas of the surfaces of the two bodies sometimes called kinematic pairs. In the absence of all other couplings each direct coupling would allow some relative motion of the two bodies they couple but this freedom might be wholly or partially destroyed by the existence of the other couplings. If some excess freedom remains we call these coupling networks kinematic chains. For kinematic chains we shall aim to find what relative motions are possible for every pair of bodies. There are also some coupling networks within which actions can be locked in as a consequence of the closure of direct couplings. For these we shall seek the actions that can be transmitted by every direct coupling in conditions of equilibrium."

Mr Commonsense opened the discussion by stating that the first objective could be to find the *nett* degree of freedom F_N for the kinematic chain as a whole. This is the number of variables required to describe all motions when all couplings are in place, he explained. "The nett degree of freedom f_{ij} of two bodies i and j, allowed by the kinematic chain, has been referred to by Mr Freedom as their connectivity. The degree of freedom f_{ij} can be less than or equal to F_N, but never greater, so finding F_N would make a useful start."

Committee members were aware of formulae that existed to find F_N by means of a count of the number of bodies and of the constraints imposed by couplings. They examined several kinematic chains. Although in many instances these formulae worked satisfactorily, for some kinematic chains they failed. Mr Anharmonic said that the reasons for failure could be blamed on special geometric circumstances. He said, "Formulae involving terms that are simply counts of bodies, freedoms and constraints are never universally applicable". To help them make progress it was resolved that new members should supplement the committee. The Chairman proposed that the meeting should be adjourned to enable him to find others to join them. This received unanimous approval. It was resolved that they would reconvene the following morning.

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On resumption the next day the Chairman announced that Mr Duality had kindly agreed to join them on condition that he could bring some of his friends with him. The Chairman then invited Mr Duality to reopen the discussion and introduce those who had accompanied him.

Mr Duality began by acknowledging the help they had received from the observers present through the contributions they had made to screw theory. We have benefited especially from the classification of special screw systems provided by Mr Synthetic Geometry and the insights provided by Mr Freedom, in particular, those he demonstrates with his superb illustrations. Mr Duality expressed only one reservation. It was that whereas these authors have a remarkable facility for spatial awareness the majority does not share this skill. With other pressures on mathematics syllabi, for example, set theory and Boolean Algebra, geometry has suffered. Mr Anharmonic observed that even for the older generation it was the wrong geometry anyway. They needed line geometry, if not screw geometry, and did not get it. Mr Duality expressed the wish to make things easier for the younger generation by using concepts that they were more familiar with.

Mr Duality also acknowledged the advice they had received from Mr Seven R. Spatial. They had to admit that they had not given the learned works of that gentleman the detailed attention that they no doubt deserved. However, he felt sure that the author would understand that their relevance to today's discussion was tenuous. He also spoke of the helpful discussions that had taken place with Mr Exceptional Mobility. In particular an example provided by that gentlemen had proved to be of great value. Mr Duality added that no doubt throughout the day opportunities would arise to speak of the contributions these visitors had made.

Turning then to the problem under discussion Mr Duality thanked the Chairman for the briefing he had received on progress so far. He was reminded of a remark made by Mr Freedom to a short course for mechanism engineers in 1968. “We kinematicians may feel like second-class citizens because, whilst a subject like thermodynamics has laws to give it respectability, kinematics apparently does not.” Mr Duality said that there were laws that govern kinematics but they were rarely called laws. “To explain one of them, I would like you to welcome Mr Circuit Law.” Mr Duality ended by explaining that there were two sets of twins who had accompanied him and that Mr Circuit Law was one of twins.

Mr Circuit Law told the gathering that he and his twin brother Mr Cutset Law had been around for even longer than some of the original committee members. “However, we have been so closely associated with electrical circuits since Dr Kirchhoff brought us into the world that our roles in other areas had often been overlooked. The usual form of my law is that the potential differences between a close sequence of points of different electrical potential sum to zero. The same is true of the relative motions of a closed sequence of bodies. One interesting point, he added, is that the law is true for potential differences regardless of whether the points are electrically connected or not. Likewise, the bodies in motion need not be coupled.”

A committee member interrupted to say that the concept that Mr Circuit Law spoke of was well known and formed the basis of so-called loop equations. Mr Circuit Law agreed. He went on, “Loop equations come in many forms. Often they are written in terms of motions relative to a frame. We shall use relative motions of directly coupled bodies, both of which can be in motion relative to earth. Furthermore we aim to formulate and solve a sufficient number of circuit equations simultaneously.”

Mr Anharmonic asked what exactly was being meant by motion. Mr Circuit Law replied that although this could mean infinitesimal displacement, the concept Sir Robert Ball had used, he would divide all such quantities by an infinitesimal interval of time to provide the first time derivative of displacement. He explained that the distinction is unimportant but ‘motion’ was briefer than the expression ‘infinitesimal displacement’.

Mr Helix wanted to know why twist rate about a screw was not the appropriate term. Mr Circuit Law assured him that indeed it could be but, because pitch was most often in practice either zero or infinite, the term twist rate would be confined to cases where the pitch was *neither* zero *nor* infinite. Motion would be the brief generic term he would use for the first time derivative of any displacement.

Mr Cutset Law added that what his twin brother had said about potential difference and motion was also true, using his law, for electrical current and action. Mr Anharmonic, becoming more irascible, responded, “And what do you mean by action then?” “Action is the generic term I shall use for force, torque and their combination, the wrench”, replied Mr Cutset Law. “It is the dual of motion. There are precedents”, he continued. “Newton began his third law using the Latin word *actioni*, the dative singular of the third declension noun *actio*. He could have used *vi*, the dative of *vis* meaning force, indeed he did so in his second law, but chose action in his third law. Perhaps he anticipated the later contributions of Euler and Poinsot. Incidentally, Mr Freedom also uses the term action.” Mr Anharmonic sat down, flushed and muttering to himself, wishing that he had held his tongue.

Mr Querulous observed that whilst electrical current and potential difference were simple concepts requiring only one quantity, motion and action were far from simple. He added that to use the laws, motion and action would need to be quantified, a task that he believed was beyond the capabilities of the Law twins. Mr Duality responded by stating that coordinates can be used based on Ball's screw coordinates, themselves adaptations of Plücker's line coordinates. He explained that, using lower case for unit screw coordinates, these were:

$$l, m, n; p^*, q^*, r^*,$$

when written in axis formation.

Mr Duality then invited Mr Motion Coordinates, one of the Coordinates twins, to elaborate. Mr Motion Coordinates began by explaining that he had replaced these dimensionless screw coordinates by identical unit motion coordinates:

$$\hat{r}, \hat{s}, \hat{t}; \hat{u}, \hat{v}, \hat{w} \ .$$

He added that for motion that was not translational velocity these six unit coordinates could be multiplied by angular velocity magnitude ψ to give

$$\psi(\hat{r}, \hat{s}, \hat{t}; \hat{u}, \hat{v}, \hat{w}\ ) = (r, s, t; u, v, w\ ).$$

“These motion coordinates are the three components of angular velocity followed by those of the velocity of the point at the global origin. For translational velocity the first three coordinates are absent and the multiplier is the velocity magnitude. We shall use the symbol ψ for generalised velocity magnitude, regardless of whether it is angular velocity or translational velocity.”

"And where can this global origin be?" inquired Mr Commonsense. "Anywhere", replied Mr Motion Coordinates. "However, our task can be made far easier by choosing the origin location and frame axes carefully. For example consideration should be given to any symmetry that exists. Also, there are obvious advantages in locating the origin on an axis characteristic of at least one coupling and choosing frame axes parallel to the directions characteristic of as many couplings as possible." Mr Cartesian, who had anticipated having no part to play in the proceedings after the treatment he received at the last meeting, suddenly started to pay attention.

Mr Action Coordinates added that by transposing the first three and the last three unit screw coordinates they become expressed in ray formation as:

$$p^*, q^*, r^*; l, m, n.$$

He then explained that, like his twin brother, he had replaced these unit screw coordinates by identical unit action co-ordinates,

$$\hat{R}, \hat{S}, \hat{T}; \hat{U}, \hat{V}, \hat{W}\ .$$

Mr Action Coordinates continued to speak. "For action that is not torque the unit action co-ordinates can be multiplied by force magnitude Ψ to provide the action coordinates:

$$R, S, T; U, V, W.$$

These are the three components of moment at the origin followed by components of force. For pure torque, or if you prefer, a couple, the last three coordinates are absent and the multiplier is torque magnitude."

The twins both expressed their gratitude to Mr Seven R. Spatial for drawing their attention to the distinction between ray and axis formation.

Mr Querulous wanted to know what purpose was served by introducing yet more sets of symbols. Mr Duality responded by saying that the reciprocity condition could thereby be expressed as:

$$rR + sS + tT + uU + vV + wW = 0,$$

with or without caps, and in this form it was easily remembered. Several of the old guard had been nodding off but mention of the word reciprocity caused them to renew their attention. "Furthermore", Mr Duality continued, "Now we can dispense with those wretched asterisks."

"And the symbol *p* becomes available to be used for the pitch of a screw rather than *h*", interjected Mr Helix. Mr Freedom smiled but Mr Synthetic Geometry did not. Forgetting that he had been asked to remain silent, Mr Synthetic Geometry attempted to catch the chairman's attention in order to raise an objection. The Chairman forestalled him by addressing the meeting.

“Gentlemen, no, sorry ladies, ladies and gentlemen. We can be pleased with the progress so far but let us engage in one task at a time. I should like to hear again from Mr Cutset Law but can we please focus first on the kinematics problem. We have yet to employ the ideas of Mr Circuit Law.” The Chairman then asked Mr Circuit Law to explain his law in more detail.

Mr Circuit Law responded. “For some kinematic chains we need be concerned with only *d* motion coordinates, a number less than six. The number *d* is the minimal order of the motion system to which all relevant motions belong. There is an obvious simplification if the geometry permits a lower value than six to be used. For example, *d* can be three for a so-called planar kinematic chain: only the unit motion coordinates $\hat{t};\hat{u}$ and $\hat{v}$ are required. When no other coupling is present the motions allowed by a direct coupling are linear combinations of *f* independent motions spanning an *f*-system of motions, a subsystem of the *d*-system. This number, *f*, is the gross degree of freedom of the direct coupling and the sum $\Sigma f = F$, for all of the *e* direct couplings, could be called the gross degree of freedom of the kinematic chain. Each of the *f* motions of a direct coupling can be quantified by one unknown generalised magnitude and *d* known unit motion coordinates. These *d* unit motion coordinates are known from the type, orientation and location of the coupling. For each of the *d* motion components, the components quantifying the motions allowed by couplings belonging to a closed circuit sum to zero. For *l* independent circuits there are therefore *dl* equations that can be written that impose conditions on the *F* unknown magnitudes.”

The committee then spent some time trying to create a single matrix equation, but without success. It was agreed *nem con* that a working party should be set up comprising Mr Circuit Law, Ms Topology and Miss Matrix to find out how this could be done. Mr One-to-One abstained on the grounds that “It was not women’s work”. Ms Topology said something about his remark not being PC. Mr One-to-One heard her but did not understand so chose to say no more.

After an adjournment the working party returned. Ms Topology rose to reopen the discussion. She smiled at the august gathering, and told them that the working party had found a way forward. She explained that they had found it helpful to start by producing the coupling graph G_C of the kinematic chain. "In G_C each of the n nodes represents a body and the e edges connecting pairs of nodes represent direct couplings between those bodies. G_C needs to be a directed graph with every edge assigned a positive sense indicated by an arrow. We have decided that all bodies and their corresponding nodes should be labelled by number and that, because the directions of edges were arbitrary decisions, it might prove convenient to ascribe the positive sense of an edge from the terminal node of lower number towards the node of higher number." Ms Topology continued by explaining that it was necessary to select a tree of G_C. "A tree of a graph is a connected subgraph of that graph that contains all nodes of the graph but no circuits." She went on to say that a graph could have several possible trees, the choice being an arbitrary one. "The edges of G_C retained in the tree are called branches and the edges that are omitted are called chords. For every chord of G_C there is a corresponding circuit of G_C and a positive sense needs to be ascribed to each circuit. It will be found convenient to choose the positive sense for the circuit that corresponds to the positive sense of its corresponding chord."

Miss Matrix then made a contribution. She explained that the circuit matrix $[\mathbf{B}]_{l,e}$ of G_C can represent the circuits. She asked the committee to take note of the subscripts of all matrices that are used because they were so very revealing. "The subscripts indicate the number of rows and columns respectively. In $[\mathbf{B}]_{l,e}$, l is the number of loops or circuits and e is the number of edges. Each element b_{ij} of $[\mathbf{B}]_{l,e}$ is 0, +1, or -1: b_{ij} is zero if circuit i does not include edge j; +1 if the positive sense of circuit i is in the same direction as the positive sense of the edge j that it includes; and -1 if those positive senses are opposed."

Ms Topology took over again. "Next, G_C is modified to create the motion graph G_M. An edge of G_C that represents a direct coupling of freedom f is replaced in G_M by f edges. If f is equal or greater than two these edges are arranged in series and are interspersed by (f - 1) new dummy nodes. These dummy nodes have no significance other than to terminate the (f - 1) internal pairs of the f edges in series. These nodes are additional to the n nodes of G_M, reproduced from G_C, that represent bodies."

"Each of the f edges of G_M that replace one edge of G_C represents a single motion. Together these f motions must span the f-system of motions allowed by the direct coupling represented by the edge of G_C that the f edges of G_M replace. The total number of edges of G_M is then F. The chosen tree of G_C can be used as a basis for the tree of G_M. Thereby all edges of G_M replacing branches of G_C become branches of G_M. Also, any one of a set of edges of G_M replacing a chord of G_C can be a chord of G_M and the remaining members of the set of edges become branches of G_M. Thus l, the number of chords and circuits in G_C, remains the same in G_M. The positive senses of the edges and circuits of G_M can be the same as those assigned to the edges and circuits of G_C that they replace."

Mr Circuit Law spoke next. "We must be quite clear about what Ms Topology means by the f-system of motions of a direct coupling. This is the system of motions that the bodies coupled by the direct coupling can enjoy *if they are not also indirectly coupled.* To use an electrical analogy this is an open circuit condition with only the two bodies and their direct coupling present. Except for some trivial kinematic chains for which our investigations are not required, two directly coupled bodies are also coupled by an indirect coupling provided by other bodies and direct couplings. This indirect coupling imposes further restrictions on the motions of which the directly coupled bodies are capable. Thus, the degree of freedom f of directly coupled bodies i and j may be reduced to f_{ij} by the other couplings in a kinematic chain. We shall aim to find f_{ij} for every direct coupling and the f_{ij}-system of motions, a subsystem of the f-system."

Miss Matrix added that G_M also has a circuit matrix $[\mathbf{B_M}]_{l,F}$. "We use the internal subscript **M,** signifying motion, to distinguish this matrix from the circuit matrix $[\mathbf{B}]_{l,e}$ of the coupling graph G_C. Notice that F, the number of edges in G_M, may be greater than e, the number of edges in G_C; but it could not be less than e."

Miss Matrix asked Mr Motion Coordinates whether he would mind if she described another matrix that was needed as this one contained unit motion coordinates. Mr Motion Coordinates agreed. She continued, "This is the unit motion matrix $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ for direct couplings. To distinguish it from other matrices the internal subscript **D** provides a reminder that it is for motion allowed by direct couplings only. As Mr Circuit Law has explained it is *not* the motions of directly coupled bodies when the constraints imposed by *all* couplings of the kinematic chain are considered. Each column of $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ contains the d unit motion coordinates corresponding to the single motion represented by an edge of G_M, where d is the minimal order of the screw system to which all F motions belong. Assembling $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ can be the difficult part for anyone not conversant with screw theory. It is the essential geometric heart of the problem that cannot be avoided."

"The next step is to combine these two matrices $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ and $[\mathbf{B_M}]_{l,F}$", continued Miss Matrix. "There are l circuits and d components and the circuit law requires that for each circuit each of the components sum to zero thereby providing *dl* equations. From the circuit matrix $[\mathbf{B_M}]_{l,F}$ extract *l* diagonal matrices $[\mathbf{B_i}]_{F,F}$, $i = 1,2,\ldots,l$, where the diagonal elements in $[\mathbf{B_i}]_{F,F}$, are the elements of row *i* of $[\mathbf{B_M}]_{l,F}$." Miss Matrix then asked for a piece of paper, which was quickly supplied. She wrote on it what she said the working party had decided to call the network unit motion matrix.

$$\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{dl,F} = \begin{bmatrix} \left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}\left[\mathbf{B_1}\right]_{F,F} \\ \left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}\left[\mathbf{B_2}\right]_{F,F} \\ \cdots \\ \left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}\left[\mathbf{B_l}\right]_{F,F} \end{bmatrix}_{dl,F}$$

Mr Circuit Law thanked the ladies for their valuable contributions and explained that it had been agreed that he should have the honour of presenting his law in its final form. "It is that

$$\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{dl,F}\left[\psi\right]_{F,1} = \left[0\right]_{dl,1},$$

where $\left[\psi\right]_{F,1}$ is a vector of unknown motion magnitudes, angular or translational velocity. The *dl* conditions on the *F* unknowns means that those unknowns can be expressed in terms of F_N of them, where F_N is the nett degree of freedom they were seeking. F_N has also been called the degree of mobility in the past but we no longer use the term because mobility also means the complex velocity response to a unit force."

The committee took a long while to assimilate what they had heard. Several of them spent some of that time studying examples to test the effectiveness of the equation. Mr One-to-One took no part in this. Having in his youth divested himself of the axioms of both Euclidean and affine geometry in seeking the relatively rarefied abstractions of projective geometry, Mr One-to-One was preoccupied in marvelling how Ms Topology had discarded even those last vestiges of shape, including his beloved anharmonic ratio, to leave only connectedness.

Mr Commonsense was the first to speak. He observed that for several of the examples that they had studied one or more of the *dl* equations were redundant. Miss Matrix agreed but, before she could continue, Mr Duality intervened to save his colleague possible embarrassment. He said, "Where there are redundant equations it is because the kinematic chain is overconstrained. The cause might be that we have used a larger value for *d* than was necessary. This will happen for example if we analyse a planar kinematic chain using a *d* value of six. If we find the rank *m* of $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{dl,F}$ then the nett degree of constraint C_N, in other words the degree of overconstraint, is (dl - m). This number of rows can be removed from $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{dl,F}$ to leave $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F}$ and Mr Circuit Law's equation then becomes:

$$\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F}[\psi]_{F,1}=[\mathbf{0}]_{m,1}.$$

Now at last we do have a formula for F_N, specifically $F_N = F - m$. Perhaps Miss Matrix would like to explain how she would set about solving this equation."

Miss Matrix responded. "To obtain a solution it is necessary to identify a suitable set of F_N primary variables from among the F unknowns in $[\psi]_{F,1}$. In practice, this will usually be simple; for example, whenever the actuators of a manipulator are known. If suitable primary variables are not self-evident, trial and error may be required. Once a suitable set of F_N primary variables have been identified the vector $[\psi]_{F,1}$ can be partitioned into a vector $[\psi]_{m,1}$ of m secondary variables and a vector $[\psi]_{F-m,1}$ of $F_N = (F - m)$ primary variables. The columns of $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F}$ must also be rearranged and partitioned into corresponding submatrices $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,m}$ and $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F-m}$ in the same manner."

Miss Matrix then took another sheet of paper and began to write the rearranged form of the equation.

$$\left[\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,m}\ \vdots\ \left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F-m}\right]\begin{bmatrix}[\psi]_{m,1}\\ \cdots\\ [\psi]_{F-m,1}\end{bmatrix}=[\mathbf{0}]_{m,1},$$

$$\therefore \left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,m}[\psi]_{m,1}=-\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F-m}[\psi]_{F-m,1},$$

$$\therefore [\psi]_{m,1}=-\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,m}^{-1}\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{m,F-m}[\psi]_{F-m,1}.$$

Mr Circuit Law took over again. "This equation results in the secondary variables of $[\psi]_{m,1}$ being expressed in terms of the F_N primary variables of $[\psi]_{F-m,1}$. The vector of motion magnitudes $[\psi]_{F,1}$ can now be reconstructed from the vector $[\psi]_{m,1}$ of secondary variables and the vector $[\psi]_{F-m,1}$ of F_N primary variables. Before the solution of the equation was found $[\psi]_{F,1}$ had F different elements. Now those F elements are either zero or are expressed in terms of the F_N primary variables."

"A new motion matrix $[\mathbf{M}]_{d,F}$ can now be created by multiplying each column of $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ by the corresponding element of $[\psi]_{F,1}$. Whereas $[\mathbf{M_D}]_{d,F}$ is the motion matrix for direct couplings expressed in terms of F variables, the gross degree of freedom of the kinematic chain, the new matrix $[\mathbf{M}]_{d,F}$ is expressed in terms of F_N variables, the nett degree of freedom of the kinematic chain. Each of the F columns of $[\mathbf{M}]_{d,F}$ have either zero elements or elements that are functions of one of the F_N primary variables."

"Both $[\mathbf{M_D}]_{d,F}$ and $[\mathbf{M}]_{d,F}$ can be contracted to $[\mathbf{M_D}]_{d,e}$ and $[\mathbf{M}]_{d,e}$. This is done by adding the sets of f columns of $[\mathbf{M_D}]_{d,F}$ and $[\mathbf{M}]_{d,F}$ representing the motions associated with each direct coupling of gross freedom f. Consider one of the e edges of G_C representing the direct coupling of freedom f between bodies i and j. The column of $[\mathbf{M_D}]_{d,e}$ corresponding to this direct coupling contains the coordinates of the f-system of motions allowed by that direct coupling disregarding the remainder of the kinematic chain."

"Now contrast matrix $[\mathbf{M_D}]_{d,e}$ with the new matrix $[\mathbf{M}]_{d,e}$ in which the internal subscript **D** is absent. The column of $[\mathbf{M}]_{d,e}$ corresponding to the same direct coupling that we have just considered contains the d components that could all be zero. In this event all motion of j relative to i is inhibited. This is an unlikely event in a kinematic chain. Alternatively the column will have d components that represent the f_{ij}-system of motions of bodies i and j allowed by the compound coupling provided by the whole kinematic chain. This number f_{ij} is the number of primary variables in which these d components are expressed. Because additional constraints are imposed $f_{ij} \leq f$ and, because there are F_N primary variables, it follows that $f_{ij} \leq F_N$. The number f_{ij} can be said to be the nett degree of freedom of the compound coupling between i and j provided by the kinematic chain or, to use Mr Freedom's expression, the connectivity of i and j."

"There will usually be pairs of bodies of a kinematic chain that are not directly coupled. To find the motion system of order f_{ij} for two indirectly coupled bodies i and j, select *any* path of G_M leading from node i to node j. Taking note of the positive sense given to the edges of that path, we need only sum each of the d components for those edges. The d components of this sum are the components of the f_{ij}-system of motions the indirect coupling allows. These components are also expressed in terms of f_{ij} of the F_N primary variables, where f_{ij} is the nett degree of freedom of the indirect coupling."

Mr Cutset Law then turned to the Chairman and said; "Now that this problem has been solved may I speak?" "You said earlier that you hoped to hear from me again. I just need a little time with the ladies to prepare something similar on statics." Mr Cartesian, anxious to escape for an early lunch, sensed some reluctance in the Chairman. "Do we really need to hear about statics? It has all been done before. After all statics is easy." Mr Cutset Law snapped back, "If I knew as much about statics as Mr Cartesian knows about statics I would think it was easy too".

"Gentlemen, gentlemen, please", the Chairman said. "Yes Mr Cutset Law, you quote me correctly, but please make your presentation as brief as you can."

After a short interlude for discussions Mr Cutset Law began. "We have been examining closed overconstrained coupling networks. We will call them overconstrained chains because we hope to demonstrate that they are dual to kinematic chains. We have also considered machines and simply stiff structures that can be converted into overconstrained chains by internalising external actions. A mechanism is a kinematic chain with two or more ports. A port is a pair of the bodies of the kinematic chain that are designed to be coupled by active couplings external to the mechanism when it is incorporated in a machine. Actuators and loads provide these active couplings, internal to the machine. Equivalent passive couplings can replace these active couplings to create an overconstrained network. Likewise, a weight supported by a simply stiff structure is usually regarded as an external action on the structure. By including the earth as a member of the extended closed system the weight becomes an internal gravitational coupling between the structural member carrying the weight and earth. A gravitational coupling is an active coupling that can also be replaced by an equivalent passive coupling to create an overconstrained chain."

"For some overconstrained chains we need be concerned with only d action coordinates, a number less than six. This number d is the minimal order of the action system to which all actions belong that are of interest to us. The actions that could be transmitted by a direct coupling when that direct coupling is the only coupling bridging a gap in what otherwise would be a rigid ring are linear combinations of c independent actions spanning a c-system of actions. This number, c, is the gross degree of constraint of the coupling and the sum $\Sigma c = C$, for all of the e direct couplings, could be called the gross degree of constraint of the overconstrained chain. Each of the c actions of a direct coupling can be quantified by one unknown magnitude and d known unit action coordinates. These d unit action coordinates are known from the type, orientation and location of the coupling. In an overconstrained chain there will be other direct couplings that, together with each direct coupling, belong to a set, a cutset, the removal of which would create two disconnected networks. For each of the d action components, the components quantifying the actions allowed by couplings belonging to a cutset, sum to zero. For k independent cutsets there are therefore dk equations that can be written that impose conditions on the C unknown magnitudes. I would like Ms Topology to explain to you how these dk equations can be assembled."

Before Ms Topology could speak Mr Commonsense asked for, and was given, permission to ask a question.

"Your name has been puzzling me. I thought you would be Mr Node Law. Why Cutset and not Node?"

Mr Cutset Law replied, "many do call me Mr Node Law. I said that the removal of a cutset of couplings leads to two subnetworks. If one of the subnetworks comprises a single body then node law would be appropriate. But that would be a special case and an unnecessary restriction on what we can do. In elementary statics a common technique is to separate a network into free body diagrams. However the expression free body diagram is a misnomer in the sense that the best strategy sometimes can be to isolate a group of two or more bodies."

Mr Commonsense had not quite finished. "The cutset reminds me of a strategy in elementary structural analysis called *The Method of Sections* whereby a cut is made through several members. Why does that require cutting members while you cut couplings?"

Mr Cutset answered again. "For a truss the members are all binary links with two couplings. All actions are forces along those slender members so the forces are the same as those transmitted by the couplings. Generally, in machinery for example, actions are not always forces and the bodies are not all binary ones. Cuts must be made through couplings."

Ms Topology then spoke of the coupling graph of the overconstrained chain. "Like the kinematic chain we produce the coupling graph G_C of the overconstrained chain, assign positive senses to its e edges, and select a tree. Instead of the chords and circuits of this tree, we identify its k branches and corresponding k cutsets. Each cutset is then given a positive sense that corresponds to the positive sense given to the only branch the cutset traverses."

Miss Matrix added her contribution. “These cutsets can be represented in the cutset matrix $\mathbf{[Q]}_{k,e}$. Each element q_{ij} of $\mathbf{[Q]}_{k,e}$ is 0, +1, or -1: q_{ij} is zero if cutset i does not include edge j; +1 if the positive sense of cutset i is in the same direction as the positive sense of the edge j that it includes; and -1 if those positive senses are opposed.”

Ms Topology continued. “Next, G_C is modified to create the action graph G_A. An edge of G_C that represents a direct coupling of constraint c is replaced in G_A by c edges. If c is equal or greater than two these edges are arranged in parallel, that is to say they share the same terminal nodes. Each of the c edges of G_A that replace one edge of G_C represents a single action. Together, these c actions must span the c-system of actions that can be transmitted by the direct coupling represented by the edge of G_C that the c edges of G_A replace. The total number of edges of G_A is then C, where C can be said to be the gross degree of constraint of the overconstrained chain. The chosen tree of G_C can be used as a basis for a tree of G_A. Thereby all edges of G_A replacing a chord of G_C become chords of G_A. Also, any one of a set of edges of G_A replacing a branch of G_C can be a branch of G_A and the remaining members of the set of edges become chords of G_A. Thus k, the number of branches and cutsets in G_C, remains the same in G_A. The positive senses of the edges and cutsets of G_A can be the same as those assigned to the edges and cutsets of G_C that they replace.”

Mr Cutset Law asked to intervene. “My brother has made clear what is meant by the f-system of motions of a direct coupling of a kinematic chain. I must do the same for the c-system of actions of an overconstrained chain. This c-system of actions contains all the actions that a direct coupling can transmit *if the coupled bodies are also integral with one another*. By making the bodies integral a closed circuit is created broken only by the direct coupling itself. Using an electrical analogy again, this is the short circuit condition. Overconstrained chains must contain circuits. Two directly coupled bodies are thereby also coupled by an indirect coupling provided by other bodies and direct couplings. However, this indirect coupling may not be rigid. It therefore may be incapable of transmitting all the actions of the c-system of actions of the direct coupling. Thus the actions that can be transmitted by a direct coupling within an overconstrained chain belong to an action system that is the subsystem of the characteristic c-system of that coupling. We aim to show how this action subsystem can be found for every direct coupling of an overconstrained chain.”

Miss Matrix returned to the subject of matrices. “The cutset matrix $\mathbf{[Q_A]}_{k,C}$ of G_A is needed. Note the internal subscript **A** of $\mathbf{[Q_A]}_{k,C}$. It indicates that it is derived from the action graph to distinguish it from the cutset matrix $\mathbf{[Q]}_{k,e}$ of the coupling graph G_C. We need also the unit action matrix $\left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}$ for direct couplings, in which each column contains the d unit action coordinates of the single action represented by an edge of G_A. From $[\mathbf{Q_A}]_{k,C}$ we find the diagonal matrices $[\mathbf{Q_i}]_{C,C}$, $i = 1,2,\ldots,k,$ as we did for the circuit matrix of the motion graph of a kinematic chain." Taking another sheet of paper Miss Matrix then wrote the network unit action matrix

$$\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{dk,C} = \begin{bmatrix} \left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}[\mathbf{Q_1}]_{C,C} \\ \left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}[\mathbf{Q_2}]_{C,C} \\ \cdots \\ \left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}[\mathbf{Q_k}]_{C,C} \end{bmatrix}_{dk,C} .$$

Mr Cutset Law now provided his equation. "The cutset law can be expressed as:

$$\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{dk,C} \; [\Psi]_{C,1} = [\mathbf{0}]_{dk,1} ,$$

where $[\Psi]_{C,1}$ is the vector of unknown action magnitudes being either force or torque. If the rank a of $\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{dk,C}$ is less than dk it is because the overconstrained chain is also a kinematic chain with a nett degree of freedom $F_N = dk - a$. In this event we can remove F_N equations to create

$$\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{a,C} \; [\Psi]_{C,1} = [\mathbf{0}]_{a,1} .$$

Thus the nett degree of constraint, $C_N = C - a$. This has sometimes been called the degree of overconstraint or redundancy, but we prefer to use the latter term solely for unwanted equations. This equation can be solved in the same way that Miss Matrix solved the motion equation. We can then assemble the matrix $[\mathbf{A}]_{d,e}$ in the same way as $[\mathbf{M}]_{d,e}$, and, from the columns of $[\mathbf{A}]_{d,e}$, find the components of the action subsystem for each of the e direct couplings. These action subsystems contain all actions that the direct couplings can transmit when all the freedoms of other couplings are present. As mentioned earlier, the order of each subsystem for a direct coupling is less than or equal to the degree of constraint c of that coupling. The order of each action subsystem is also equal to or less than C_N."

"Chairman, please, I must speak", pleaded Mr Commonsense. "I cannot go on listening to all this. Is Mr Cutset Law seriously suggesting that statical analysis can be performed on statically indeterminate structures? That would be the *non-sequitur* of all time." Mr Duality had feared that this point might be raised. Indeed, he had been having misgivings himself about the use of the word statics. He quickly hit on a plan. It was to stall and placate and then to throw Mr Cutset Law in at the deep end, temporarily, just to give himself time to think. He began, “Mr Chairman, we have all benefited from the contributions Mr Commonsense has made. The strong feelings he has expressed deserve, indeed require, an adequate response. Perhaps Mr Cutset Law would like the opportunity to provide the explanation.” Mr Cutset Law then spoke. “The brief answer is that the term statical indeterminacy is itself flawed because, as I have shown, it is possible to find relationships between the unknowns. When my brother spoke of kinematics Mr Commonsense raised no objections despite the fact that we spoke of kinematic chains such as manipulators having a nett degree of freedom F_N or, if we may use a new expression dual with statical indeterminacy, having kinematic indeterminacy.”

Mr Duality was impressed by the way Mr Cutset Law was handling himself, but then, suddenly he exploded. “Of course, how silly of me. It's quite obvious. Why didn't I think of that before? Yes...” He was interrupted by the Chairman. “Ladies and gentlemen the meeting is in danger of getting out of hand. Would members please address the Chair in future when they want to speak.” “Mr Chairman”, replied Mr Duality, “my apologies.” “The reason for my outburst is that I have just had an insight that had previously eluded me. I have always presumed that statics is the dual of first order kinematics, the topic that Mr Circuit Law introduced. But this is not true; at least it is not true of statics in the sense that we usually understand that term. Statics requires a statically determinate system on which external actions act. What Mr Cutset Law has been talking about requires a self-contained overconstrained system in which all actions that exist are internal. Kinematic chains are self-contained systems and their dual must be also. Known external actions we speak of in statics must be converted to known internal actions in the self-contained system. Mr Cutset Law said that the expression statically indeterminate is flawed. I understand what he means but I disagree that it is flawed in the context of statics. The truth that has just dawned on me is that Mr Cutset Law has not been talking about statics at all. His analysis provides relationships between actions that exists in statically indeterminate networks.”

Mr Duality continued, “You will not be surprised to learn that I like to find dual pairs of terms and, if one does not exist, I am prepared to create a neologism. For example, that overfreedom is the dual of overconstraint. I like Mr Cutset Law's use of the expression kinematic indeterminacy because it is dual to statical indeterminacy. The French have a pair of terms *hypostatique* (underconstraint) and *hyperstatique* (overconstraint) but I would hesitate to call them dual. The danger with these is the implication that a system must be one or the other. I think they are used in the theory of structures, wherein underconstraint is of course quite unacceptable, rather than in the theory of machines. But kinematic chains can have both excess freedom and excess constraint, overfreedom and overconstraint if you like. I prefer these terms to underconstraint and overconstraint which, when they both exist in a kinematic chain, seem inappropriate. Finally, and then I promise to shut up, I prefer the expression kinematic network to kinematic chain, but the latter now has a long history. Again to emphasise duality I suggest overconstrained chains rather than overconstrained networks, the expression Mr Cutset Law has been using."

Mr Commonsense, now calmer, asked the Chairman if he could put a supplementary question. This was agreed. "I want to ask about actions that exist in overconstrained structures and machines that are not attributable to active couplings. I mean actions that can arise from manufacturing error, from differential expansion as a consequence of temperature gradients, or from materials that have different coefficients of expansion. Can the analysis provided by Mr Cutset Law be helpful in order to measure these?"

Mr Cutset Law addressed the Chair before replying. "Mr Duality spoke earlier of manipulators. My twin brother's analysis enables us to recognise special pairs of bodies of a manipulator namely those pairs that can be coupled by active couplings. Dually, there are special bodies in an overconstrained chain that can be identified from my analysis. Suppose we have an overconstrained chain with a nett degree of constraint C_N. It is possible to select C_N of the bodies of the system on which to affix judiciously located strain gauges. From the strains measured before and after assembly, or before and after temperature change, it is possible to find the actions existing in those bodies attributable to those causes. With this information my method enables us to find the corresponding actions transmitted by all direct couplings."

The Chairman decided that it was an appropriate time to adjourn for lunch, much to the relief of Mr Cartesian. Before doing so he spotted another newcomer who had remained silent.
"And who are you?"
"Mr Virtual Power", was the reply.
"Don't you mean Virtual Work?"
"My friend Mr Virtual Work would have come if we had been talking about infinitesimal displacements but, because first time derivatives are being used, I am here instead."
"And I suppose you have something to say as well", said the Chairman.
"Indeed I do", responded Mr Virtual Power.
"And do you also have a twin?"
"No, I can do the job of two myself."
"No modesty then. Well, you may speak after lunch."

When they returned from lunch Mr Virtual Power began his presentation. “I want to tell you about another dual pair of methods that achieve the same results as those provided by the Law twins. I call these internal virtual power methods. I shall start where this morning's session finished by finding relationships between actions and then later turn to relationships between motions. I have chosen to present my findings in this order because the concepts regarding actions that I shall use will be more familiar to you. Like Mr Cutset Law I shall be concerned with what Mr Duality has called overconstrained chains. Also, as Mr Cutset Law as explained, we can find actions resulting from external active couplings by internalised them and replacing them with equivalent passive couplings.”

“First, I want you to consider a strip of steel. Please ignore its weight; alternatively think of the strip having zero mass. We do not want to be concerned here with irrelevant gravitational couplings. Imagine now that the strip is heated up until it becomes ductile. Two people each grasp an end of the strip with tongs and then bend, stretch, twist and generally distort the strip. They then bring the edges at the ends of the strip together still exerting actions on those ends. A third person now welds the two edges together to create a distorted ring. The shape could be that of a Möbius strip. The ring is then left to cool. Locked into the ring is a circuit action. I call it a circuit action because, if it were possible to measure stresses in *any* cross-section of the ring that has been created, it would be found that the resultant action is identical, and is a wrench on a screw. The ISA of the wrench will have a fixed location with respect to the ring. Incidentally, I can confidently use the term wrench here because it would be most extraordinary if the pitch of the screw happened to be either exactly zero or exactly infinite. However, I shall continue to use the term action as the others have done. We know nothing about this circuit action locked into the ring; it could be any one among the actions that belong to the 6-system of all actions.”

“The expression circuit action may be new to you but Maxwell introduced the concept of circuit currents in electrical networks long ago. The circuit action is analogous. Because there are always fewer circuits than edges in a graph information about electrical currents flowing in conductors is more concisely expressed by circuit currents than edge currents. Likewise, circuit actions provide the same information as coupling actions, but do so more concisely.”

"The two flanges of a hinge are then bolted to the strip now formed into a ring. The countersunk holes in the flanges of the hinge are used as guides for a drill so that, when the bolts are inserted and nuts tightened, no change takes place in the circuit action. Now think of a global frame of reference; most conveniently for our purposes with the origin on the axis of the hinge and with the z-axis coinciding with the axis of that hinge. The following action coordinates represent the 6-system to which the circuit action belongs:

$$(R, S, T; U, V, W),$$

all of which are unknown. Next, the original ring is carefully cut between the flanges of the hinge, but without disturbing the hinge or its flanges. The ring remains a single integral body but is now hinged to itself. We must expect a small rotation of the hinge to take place when this cut is made. We must expect also the circuit action to be affected by this cut because the hinge, which must now transmit the circuit action, is incapable of transmitting torque T parallel to the z-axis. The circuit action now is one that belongs to a 5-system of actions characterised by the action coordinates

$$(R, S; U, V, W).$$

This is now the ring with a break bridged by a direct coupling that Mr Cutset Law spoke about."

"Another way of arriving at this conclusion is to examine the unit motions that can normally be permitted by the hinge. There is only one such motion; the unit motion coordinates that can represent it are

$$(0, 0, \hat{t}\,; 0, 0, 0),$$

the 1-system representing a unit angular velocity about the z-axis. Once any torque T about the z-axis has been relieved as a consequence of the cut no further motion can take place about the z-axis. However, we can describe the capability of the hinge to allow rotation as a virtual motion and $\hat{t}$ as the virtual unit motion coordinate. The circuit action cannot expend power on unit virtual motions and so the reciprocity condition can be evoked. For this coupling

$$(0, 0, \hat{t}\,; 0, 0, 0) \,.\, (R, S, T; U, V, W)^{\mathrm{T}} = 0.$$

This condition places no restrictions on R, S, U, V, or W but does require that $T = 0$."

"Another hinge, hinge number two, is bolted elsewhere to the ring. Because of the distortion of the ring the axes of the hinges are skew to one another. Again the material of the ring is cut without disturbing the hinge and again a small rotation of this second hinge is likely. A second condition now applies to the circuit action. This time the single virtual unit motion cannot be so easily expressed because it must be written with respect to the global frame we have chosen. The new unit motion coordinates could be written first with respect to a frame local to hinge number two, and then transformed. This is the difficult geometric part of the problem that cannot be avoided. A consequence of this second condition is that the circuit action is confined to one that belongs to a 4-system of actions. Subsequently, each new hinge and cut that is introduced reduces the order of the action system to which the circuit action belongs by one. When five hinges transmit the circuit action there is only one circuit action it could be, that is the one reciprocal to the motion screws of zero pitch of those five hinges. A sixth hinge and cut results in a statically determinate structure and no circuit action; a seventh hinge and cut introduces excess freedom."

"I want to turn now to the general case where the overconstrained chain has l independent circuits. Mr Circuit Law as stated that there is a total of F motions associated with direct couplings of a kinematic chain. For an overconstrained chain some or all of these F motions might be virtual motions. Each of the f known unit motions of a direct coupling of an overconstrained chain has a zero scalar product with the unknown circuit actions of the circuits to which the direct coupling belongs. There are therefore a total of F equations that impose conditions on the dl unknown circuit action components."

"To create a single matrix equation I need to produce the coupling graph G_C and label all edges. Select a tree thereby identifying the l chords and the corresponding l circuits. Label the circuits with the same letter as the corresponding chords. Also, give all edges a positive sense and assign a positive sense to the circuits that corresponds with the positive sense given to their associated chords. For each circuit d circuit action coordinates can represent the unknown circuit action system where, in general, d is six as we have seen. For l circuits there are therefore a total of dl unknown circuit action co-ordinates. All chords of G_C and possibly some branches belong to only one circuit. Other branches belong to two circuits. The action transmitted by couplings represented in G_C by branches belonging to two circuits is the resultant of the two circuit actions. Furthermore, all actions that these couplings are *capable* of transmitting belong to the action system that is the union of the two circuit action systems."

"I do not need to keep you much longer on this problem because we can now assemble the necessary reciprocity equation. It is necessary to assemble the dl unknown action coordinates in an action vector $[\mathbf{A}_l]_{dl,1}$. We must now multiply each of these circuit action coordinates by the unit virtual motion coordinates of the couplings that are represented by edges that belong to the same circuit as the circuit actions. To do this we need the motion graph G_M of the overconstrained chain."

Mr Commonsense interrupted. “Chairman, did I hear Mr Virtual Power correctly? How can an overconstrained chain have a motion graph?” “Yes, the motion graph”, replied Mr Virtual Power. “If the overconstraint chain is a structure rather than a kinematic chain then the motions are virtual motions not real ones. The graph could then be called a virtual motion graph. For an overconstrained kinematic chain only some of the motions are virtual as I shall explain in a moment.” Mr Commonsense did not seem convinced but let Mr Virtual Power continue.

“All we need to do now is to assemble the network unit motion matrix $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}\right\rfloor_{dl,F}$ in the way explained by Miss Matrix, and then transpose it to give $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{F,dl}$. To assemble the equation the circuit action coordinates in the vector $[\mathbf{A_l}]_{dl,1}$ are premultiplied by $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{F,dl}$ to give

$$\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{F,dl}\ [\mathbf{A_l}]_{dl,1} = [\mathbf{0}]_{F,1}\ .$$

Remember that the overconstrained chain could be a kinematic chain of nett freedom F_N if our purpose is to study the actions arising from the overconstraint present. In that event the rank m of $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{F,dl}$ will be less than F because $F = F_N + m$, as we saw earlier. If so, F_N redundant rows can be removed from $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{F,dl}$ to leave $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{m,dl}$. Now the only unit motions recorded in the unit network motion matrix $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{m,dl}$ are virtual unit motions. Furthermore, for an overconstrained structure $F = m$ so that there are no real motions and no redundant rows, the motion graph can be called a virtual motion graph in its entirety. The number dl of columns of $\left\lfloor\hat{\mathbf{M}}_{\mathbf{N}}^{\mathbf{T}}\right\rfloor_{m,dl}$ exceeds m, the number of rows. This excess, $C_N = dl - m$, is the nett degree of constraint. So, the solution expresses all dl unknowns in terms of C_N primary ones.” A gesture from Mr Commonsense indicated that he was satisfied now with the explanation.

Mr Virtual Power continued. “Please note that from this equation we find the components of circuit action systems, not those of couplings, except for the couplings represented by chords of G_C. To obtain the components for all couplings the dl components of $[\mathbf{A_l}]_{dl,1}$, now expressed in terms of C_N primary variables, must be reassembled first in matrix form as $[\mathbf{A_l}]_{d,l}$. This matrix can then be postmultiplied by $[\mathbf{B}]_{l,e}$, the circuit matrix of G_C, to give

$$[\mathbf{A}]_{d,e} = [\mathbf{A_l}]_{d,l}\ [\mathbf{B}]_{l,e},$$

where $[\mathbf{A}]_{d,e}$ provides a column of d components for each of the e couplings. This is the same matrix as the one produced by Mr Cutset Law."

"Thank you, Mr Virtual Power. That has been very interesting", said the Chairman.
"I don't suppose he can use internal virtual power for kinematics though", muttered Mr One-to-One.
"Oh yes I can", replied Mr Virtual Power.
"Can you do it quickly? It will soon be time for tea and biscuits", the Chairman responded.

Mr Virtual Power began. "Like Mr Circuit Law's contribution this work concerns kinematics chains that also can be overconstrained. For a kinematic chain there is a total of C actions. Each of the c known unit actions of a direct coupling of a kinematic chain has a zero scalar product with the unknown cutset motions of the cutsets to which the direct coupling belongs. For all C unit actions there are therefore C equations that impose conditions on the dk unknown circuit motion components. To obtain a single matrix equation I need to draw the coupling graph G_C again, identifying a tree and..."

The Chairman interrupted. "Yes, I think we know all that, label, assigned positive senses, and so on. Please move on quickly."

"For each branch there is a unique cutset", Mr Virtual Power continued. "Produce a vector $[\mathbf{M_k}]_{dk,1}$ of the motion coordinates for each of the k cutsets. When $d = 6$, these motion coordinates are $\{r, s, t; u, v, w\}$. A cutset motion is identical to the motion allowed by the coupling represented by the branch corresponding to the cutset. Now produce the action graph G_A and from it produce the network unit action matrix $[\hat{\mathbf{A}}_{\mathbf{N}}]_{dk,C}$ as Miss Matrix did, and then its transpose $[\hat{\mathbf{A}}_{\mathbf{N}}^{\mathbf{T}}]_{C,dk}$."

Mr Commonsense could not restrain himself. "The action graph of a kinematic chain? I can see it coming. You are going to tell us these are virtual actions aren't you?"
Mr Virtual Power responded. "Yes, at least some of them are, and why not? There can be some real actions if the kinematic chain is also an overconstrained chain. But the other actions, like the virtual motions I spoke of in my analysis of overconstrained chains, are virtual actions that do virtual work on the cutset motions."

"Please, no more questions or interruptions", said the Chairman. "Please carry on Mr Virtual Power."

Mr Virtual Power continued, "Now the reciprocity equation becomes

$$\left\lfloor \hat{\mathbf{A}}_{\mathbf{N}}^{\mathbf{T}} \right\rfloor_{C,dk} \left[\mathbf{M}_{\mathbf{k}}\right]_{dk,1} = \left[\mathbf{0}\right]_{C,1} .$$

If there is overconstraint the rank a of $\left\lfloor \hat{\mathbf{A}}_{\mathbf{N}}^{\mathbf{T}} \right\rfloor_{C,dk}$ will be less than C, the number of rows, by $C_N = C - a$, the nett degree of constraint. The equation then becomes:

$$\left\lfloor \hat{\mathbf{A}}_{\mathbf{N}}^{\mathbf{T}} \right\rfloor_{a,dk} \left[\mathbf{M}_{\mathbf{k}}\right]_{dk,1} = \left[\mathbf{0}\right]_{a,1} .$$

We have now eliminated any *real* actions that could exist and what remain are virtual actions. The number of columns of $\left\lfloor \hat{\mathbf{A}}_{\mathbf{N}}^{\mathbf{T}} \right\rfloor_{a,dk}$ exceeds the rank a by $F_N = dk - a$. Thus the dk unknown cutset motion components can be expressed in terms of F_N primary ones. To obtain the motions that all directly coupled bodies are capable of, we now reassemble vector $\left[\mathbf{M}_{\mathbf{k}}\right]_{dk,1}$ as matrix $[\mathbf{M}_{\mathbf{k}}]_{d,k}$ and postmultiply it by $[\mathbf{Q}]_{k,e}$, the cutset matrix of G_C, to give:

$$[\mathbf{M}]_{d,e} = [\mathbf{M}_{\mathbf{k}}]_{d,k}\,[\mathbf{Q}]_{k,e},$$

where $[\mathbf{M}]_{d,e}$ contains a column of d components for each of the e couplings. This provides the same results as Mr Circuit Law's method."

"Do you take sugar Miss Matrix?" The Chairman had spent the previous few minutes shuffling teacups to committee members. Without acknowledging the shake of her head the Chairman's glanced around at the others who had been speaking. "While we have a break could you prepare an example illustrating all four methods we have heard described?" Without waiting for an answer he passed a pad of paper to Mr Duality. "Here is some more paper." Mr Duality and his friends took their teacups to a vacant table and were soon engaged in animated discussion.

When the cups and saucers had been removed Mr Duality addressed the meeting. "We are aware that those gathered here today are conversant with the geometry of our subject, but some may be less familiar with the topological aspects of networks. For this reason we have chosen as an example one that is topologically complex but geometrically simple. The example is one that will be readily understood and no doubt you will have seen examples of a similar kind before. It has the merit that our results can be easily checked. The example is that of a 2-stage epicyclic gear train shown in sectional view like this which we have labelled Figure 1." Mr Duality passed round the first drawing.

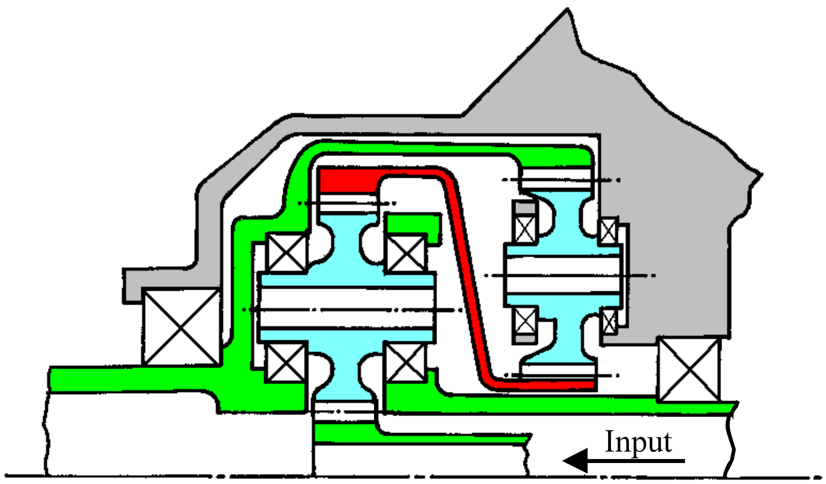

Fig. 1: A section through a two-stage epicyclic gear train

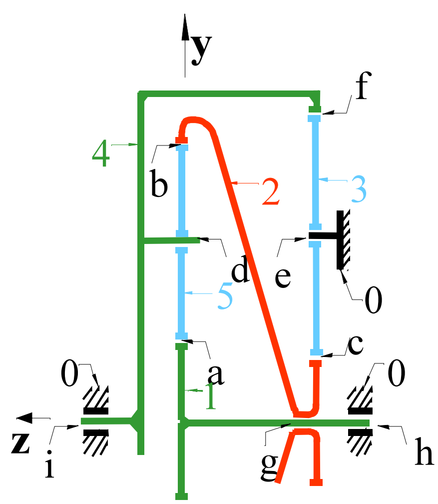

Fig. 2: A skeleton diagram representing the epicyclic gear of Fig. 1, with bodies and couplings labelled

“For our purposes much of the information in this drawing is unnecessary. We can replace it by the skeletal representation shown in this second figure that we have called Figure 2. In Figure 2 we have labelled all bodies by numbers and all couplings by lower case letters. We also show two of the axes of the global frame we have chosen.”

“Our objective is to find relationships between motions in this mechanism and to find relationships between actions that are attributable to active couplings. Notice that I use the term mechanism rather than kinematic chain. This is because ports have been identified that are to be coupled with active couplings. One port comprises members numbered zero and one; the other comprises members zero and four.”

Mr Cutset Law then intervened. “What Mr Duality has said about active couplings must be emphasised. As we have explained, our methods allow us to analyse actions attributable to overconstraint. Epicyclic gear trains are greatly overconstrained and our methods could be used to find relationships between actions attributable to those overconstraints. However, for the part of the problem that concerns actions this is not the task we have set ourselves. Instead, it is a task that could be achieved by conventional methods of statics. We will remove or ignore the existing overconstraints and then introduce new overconstraint arising from active couplings that are not shown in Figures 1 and 2. If we do not remove the existing overconstraints we shall be analysing actions caused by both the active couplings *and* the existing overconstraints. This would be an unnecessary complication, and can be avoided. Some of the existing overconstraint is removed by dispensing with all but one of the planets in each planet set. There still remains some overconstraint that is easily recognised; we only need to think of the implications if one of the planets is oversized. This overconstraint is ignored. Think of the planets being perfectly made to size.”

“Mr Cutset Law has made an important point”, continued Mr Duality. “To make our task easier we make three more assumptions.”

“Firstly, we shall assume that the pressure angles of all gear wheels are zero. This is never true; it is an assumption made temporarily to avoid a proliferation of notation. Later we shall show how the results are easily adjusted to take account of the true pressure angle.”

“Secondly, we shall assume that the gear train is 100% efficient. This is invalid because of friction but the small errors can be corrected using experimental data.”

"Thirdly and lastly, we shall assume that the lines of action of all forces lie in the plane $z = 0$. This is untrue in practice because of the offset in the z direction of planet numbered three. Torque parallel with the y-axis that these offsets cause is ignored but could be measured later should we want to do so."

Mr Motion Coordinates then explained that the only motions were angular velocities with axes of rotation parallel with the z-axis in the plane $x = 0$. "Collectively these motions belong to a second special 2-system of motions. Thus the appropriate value of d is two, and the only motion coordinates required are $\{t; u\}$."

Mr Action Coordinates added that a consequence of the assumptions explained by Mr Duality was that the actions attributable to active couplings were torques parallel to the z-axis and forces with lines of action parallel to the x-axis in the plane $z = 0$. "Collectively these actions also belong to a second special 2-system of actions but not the same system, geometrically, as the motion system. Therefore we can use $d = 2$ also for action analysis and all actions can be quantified using two action coordinates; $\{T; U\}$."

Mr Duality spoke again. "In what follows we intend to use two columns on the paper you provided to present figures, matrices and equations side by side. The intention is to emphasise the dual nature of the problems." He then invited Ms Topology to provide the coupling graphs.

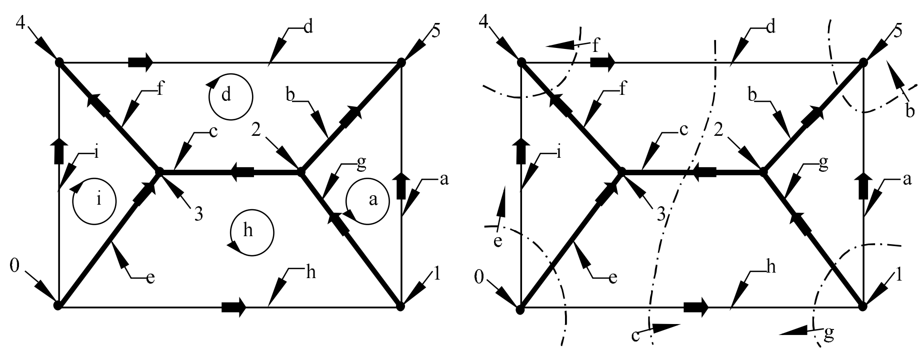

Fig. 3: The coupling graph G_C of the mechanism with circuits and the coupling graph G_C of the overconstrained chain with cutsets.

Ms Topology began by saying that Figure 3 shows the two coupling graphs G_C. "On the left G_C is for the mechanism and, on the right, G_C for the overconstrained chain. The two graphs appear to be identical but I have drawn separate graphs to avoid any confusion that could result by having both circuits and cutsets shown on the same graph. For both graphs the six nodes are numbered with the labels given to the bodies they represent and the nine edges are labelled with letters assigned to the couplings they represent. Heavier lines show the selected tree and an arrow on each edge indicates its positive sense. The difference between the two graphs concerns edges *h* and *i*. For the mechanism, the edges *h* and *i* represent bearings coaxial with the *z*-axis. For the overconstrained chain there are also two active couplings. One couples bodies 0 and 1, the other couples bodies 0 and 4. So, for the overconstrained chain, edges *h* and *i* each represent a bearing and an active coupling in parallel with one another."

Mr Cutset Law then contributed again. "I have said that active couplings can be replaced by equivalent passive couplings. Imagine a hole being drilled through the frame 0 and shaft 1, and then a pin being inserted to prevent the relative rotation of those bodies. By doing that alone a simply stiff structure is created. Now imagine someone grasping the output shaft 4 and twisting the shaft slightly relative to the frame. This is possible owing to the elasticity of the parts. Thereby a torque is transmitted between the frame and shaft 4. While still exerting this torque a colleague drills another hole, this time through members 0 and 4 and inserts another pin. Shaft 4 is then released leaving actions locked into the system. The network is now an overconstrained chain and the only internal actions that can be sustained our present. The actions transmitted by couplings are no different, except in scale, from those they would experience in normal operation that are attributable to active couplings. Torque ratios and force ratios are unaffected by the substitution of these passive couplings."

Ms Topology continued. "For the mechanism each of the four circuits in left hand graph of Fig. 1 is identified by a circle that is labelled by the letter corresponding to the only chord belonging to the circuit. The arrows on the arcs indicate the positive sense of the circuits. For the overconstrained chain, each of the five cutsets is identified by a chain-dotted line that is labelled by the letter corresponding to the only branch belonging to the cutset. Arrows piercing these chain-dotted lines indicate the positive sense of the cutsets."

Miss Matrix then provided the circuit and cutset matrices of the coupling graphs.

$[\mathbf{B}]_{l,e} = [\mathbf{B}]_{4,9} =$	$[\mathbf{Q}]_{k,e} = [\mathbf{Q}]_{5,9} =$
$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}.$	$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$

She explained that, in both matrices, the nine columns from left to right correspond to edges *a-j* of G_C in alphabetic order. She went on to say that, from top to bottom, the four rows of $[\mathbf{B}]_{4,9}$ corresponded to circuits ordered *a, d, h, i*; and the five rows of $[\mathbf{Q}]_{5,9}$ corresponded to cutsets ordered *b, c, d, f, g*.

Ms Topology then drew the motion and action graphs. She labelled this Figure 4.

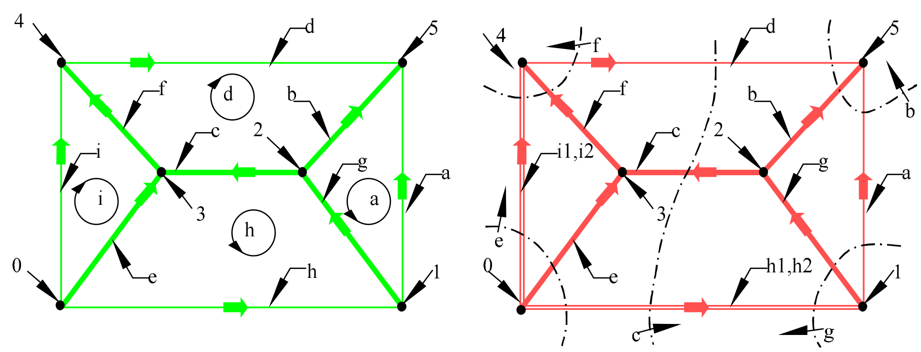

Fig. 4: The motion graph G_M of the mechanism and the action graph G_A of the overconstrained chain.

"The first thing you will notice is my use of colour. I have used green edges in G_M as a reminder that each edge represents a motion. The red edges in G_A represent actions. The colours also provide a distinction with the corresponding coupling graphs G_C. Apart from colour, G_C and G_M for the mechanism are identical. This exceptional circumstance arises because we are concerned with a motion system of order two. In this 2-space the freedom *f* of every direct coupling is one. Thus a single motion represents the motion system of every coupling. Also, with two exceptions, the action system for each coupling is represented by a single action. The exceptions are couplings *h* and *i* where each are represented in G_A by two edges in parallel. If we have need to, we can refer to these edges as $h1$, $h2$, $i1$ and $i2$. Let $h1$ represent the active coupling or the torque transmitted by the pin and $h2$ the bearing force: likewise for $i1$ and $i2$."

Miss Matrix then provided the circuit and cutset matrices of these graphs.

The circuit matrix $[\mathbf{B_M}]_{l,F}$ of G_M is $[\mathbf{B_M}]_{4,9}$ =	The cutset matrix $[\mathbf{Q_A}]_{k,C}$ of G_A is $[\mathbf{Q_A}]_{5,11}$ =
$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix}.$	$\left[\begin{array}{ccccccc\|cc\|cc} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right].$

"You will note that whereas $[\mathbf{B_M}]_{4,9}$ is the same as $[\mathbf{B}]_{4,9}$; $[\mathbf{Q_A}]_{5,11}$ differs from $[\mathbf{Q}]_{5,9}$. I have used column separators in $[\mathbf{Q_A}]_{5,11}$ to draw attention to the two identical columns needed for the actions of couplings h and i. For an example wherein the motion and action system was of order greater than two I would expect more identical columns to be present."

The Coordinates brothers then collaborated to provide the direct coupling motion and action matrices and the vectors of magnitude. Mr Motion Coordinates explained that they would use the symbols a, b, c, f for the distances from the z-axis to the pitch points of the gear contacts at couplings labelled with those letters. Also he said that the symbols d and e would be used for the distances from the z-axis to the axes of bearings d and e. "In general, an angular velocity t about an axis parallel to the z-axis and distant y from that axis causes a velocity u at the origin O directed along the x-axis where,

$$u = yt.$$

The two motion coordinates $\{t; u\}$ are therefore $\{t; yt\}$. Thus the magnitude of the motion is the angular velocity t and the unit motion coordinates are $\{1; y\}$."

Mr Action Coordinates then added that in general, a force U with a line of action parallel to the x-axis and distant y from that axis caused a moment T at the origin O directed along the z-axis where,

$$T = -yU.$$

"The two action coordinates $\{T; U\}$ are therefore $\{-yU; U\}$. Thus the magnitude of the action is the force U and the unit action coordinates are $\{-y; 1\}$."

The Coordinates brothers then wrote the direct coupling unit motion and action matrices and associated vectors.

$\lfloor \hat{\mathbf{M}}_{\mathbf{D}} \rfloor_{d,F} = \lfloor \hat{\mathbf{M}}_{\mathbf{D}} \rfloor_{2,9} =$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e & f & 0 & 0 & 0 \end{bmatrix},$ and the vector of generalised motion magnitudes, $[\psi]_{F,1} = [\psi]_{9,1} =$ $\{t_a\ t_b\ t_c\ t_d\ t_e\ t_f\ t_g\ t_h\ t_i\}^{\mathrm{T}}$.	$\lfloor \hat{\mathbf{A}}_{\mathbf{D}} \rfloor_{d,C} = \lfloor \hat{\mathbf{A}}_{\mathbf{D}} \rfloor_{2,11} =$ $\begin{bmatrix} -a & -b & -c & -d & -e & -f & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$ and the vector of generalised action magnitudes, $[\Psi]_{C,1} = [\Psi]_{11,1} =$ $\{U_a\ U_b\ U_c\ U_d\ U_e\ U_f\ U_g\ T_h\ U_h\ T_i\ U_i\}^{\mathrm{T}}$.

Mr Action Coordinates drew attention to the mix of torques T and forces U in the vector $[\Psi]_{11,1}$. "It is for this reason we refer to them as generalised action variables."

Mr Commonsense, as alert as ever, asked whether velocity u could be used equally well, instead of angular velocity t, for the first six generalised action magnitudes of $[\psi]_{9,1}$. Also, he asked whether torque T could be used instead of force U as elements in $[\Psi]_{11,1}$ for the actions represented by each of the edges a to f. Mr Action Coordinates responded by saying that these choices could indeed be made. "However, the consequence is fractional terms in the unit motion and action matrices. This is something we chose to avoid."

Miss Matrix then wrote down the network unit motion and unit action matrices.

$\lfloor \hat{\mathbf{M}}_{\mathbf{N}} \rfloor_{dl,} = \lfloor \hat{\mathbf{M}}_{\mathbf{N}} \rfloor_{8,9} =$ $\left[\begin{array}{ccccccccc} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ a & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -b & c & d & 0 & f & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & c & 0 & -e & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -e & -f & 0 & 0 & 0 \end{array}\right]$	$\lfloor \hat{\mathbf{A}}_{\mathbf{N}} \rfloor_{dk,C} = \lfloor \hat{\mathbf{A}}_{\mathbf{N}} \rfloor_{10,11} =$ $\left[\begin{array}{ccccccccccc} -a & -b & 0 & -d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -c & d & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -e & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hdashline 0 & 0 & 0 & d & 0 & -f & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hdashline -a & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right]$

"I have used row separators to make it easy to identify the four circuits and five cutsets. Please note that the ranks m and a of the left-hand and right-hand matrices respectively are the same as the numbers of rows in those matrices. This is because there is no overconstraint remaining in the mechanism and no excess freedom in the overconstrained chain. If you will be so kind as to give us a minute we shall provide the solutions." Miss Matrix then became engrossed in a flurry of activity."

After a few seconds the Chairman's curiosity got the better of him. "What is that on your lap Miss Matrix?"
"A laptop."
"Can I sit on it?" interjected Mr One-to-One with a leer in her direction.
"Behave yourself, One-to-One. Really, at your age", snapped back the Chairman.
Ms Topology whispered to Miss Matrix, "I had been wondering how he acquired the name One-to-One. Now I think I know."

The Chairman turned again to Miss Matrix. "No, not your lap top. I mean what is it you are resting on your lap?" Miss Matrix, recognising the misunderstanding, laughed. "Oh I see. That is my laptop computer. It's a portable PC."

"That is called PC but I am not apparently. How odd." Mr One-to-One had not forgotten Ms Topology's earlier remark.

"And now it is producing the results", announced Miss Matrix. "I shall write them down. I won't go into the computational details, but just write down these matrices and then explain them."

$(1/\,t_h)\ [\mathbf{M^T}]_{e,d} = (1/\,t_h)\ [\mathbf{M^T}]_{9,2} =$ $$\begin{bmatrix} 4bde/(a-b)A & 4abde/(a-b)A \\ 4ade/(a-b)A & 4abde/(a-b)A \\ 2aef/(f-c)A & 2acef/(f-c)A \\ 4abe/(a-b)A & 4abde/(a-b)A \\ 2acf/(f-c)A & 2acef/(f-c)A \\ 2ace/(c-f)A & 2acef/(c-f)A \\ -4de/A & 0 \\ 1 & 0 \\ ac/A & 0 \end{bmatrix},$$ where $A = ac + bc + bf$.	$(1/\,T_h)\ [\mathbf{A^T}]_{e,d} = (1/\,T_h)\ [\mathbf{A^T}]_{9,2} =$ $$\begin{bmatrix} 1 & -1/a \\ b/a & -1/a \\ -b/a & b/ac \\ -2d/a & 2/a \\ 2be/ac & -2b/ac \\ bf/ac & -b/ac \\ 0 & (b-c)/ac \\ 1 & (b-2c)/ac \\ -A/ac & (b+2c)/ac \end{bmatrix},$$ where $A = ac + bc + bf$.

“I have transposed the matrices only in order to find enough room to present them side by side. For the mechanism all quantities are divided by t_h the input angular velocity; so that column one contains angular velocity ratios and column two contains velocity / angular velocity ratios that have the dimensions of length. For the overconstrained chain…”

Mr Querulous stopped her. “Just a moment, Miss Matrix. What you have just done must have involved the inversion of 8x8 and 10x10 matrices. How long did that take you?” “Since the Chairman asked me about my laptop”, she replied.

Eyebrows rose, jaws dropped, mouths fell open.

When Mr Querulous had recovered sufficiently he continued. “But that is barely a minute. And that’s not all. These are not matrices with only numerical elements. They also contain symbols.” “I know”, Miss Matrix continued. “It still amazes me. It only requires some mathematics software on a CD that is provided with a book costing £30.”

“Software, CD, I don't understand”, said the Chairman.

Mr Circuit Law, anxious to press on, ignored this conversation. He said, “Please note that we can use the geometric identities, $2d = a + b$ and $2e = c + f$, to eliminate d and e from the terms in these matrices. Then, for the non- dimensional angular velocity ratios in the left hand column of the motion matrix we can replace distances a, b, c and f by the numbers of teeth on gears. These are the teeth on sun gear 1, the annulus of 2, the sun gear of 2, and the annulus of 4, respectively. We cannot make this substitution for elements of the second column because they have dimensions of length.”

The Chairman, temporarily bewildered by the pace of this interchange, remembered his responsibilities. “Miss Matrix, my apologies, you were interrupted when about to talk about the actions in the right hand matrix.”

“I was only going to say”, said Miss Matrix, “that, in the matrix on the right, the rows contain the two components of the actions transmitted by direct couplings, both terms being divided by T_h. Rows again correspond to edges of G_C in the order a - i. Column one provides moments about the z-axis or, in the absence of force, torques parallel to the z-axis. Column two provides force parallel to the x-axis.

Mr Cutset Law added that for the torque ratios in the left-hand column of the matrix of actions lengths can be replaced by numbers of teeth in the way his brother had explained. “If we are interested in the force/torque ratios of the right-hand column, having dimensions that are the inverse of length, we must remember that a zero pressure angle has been assumed. To obtain the correct results these expressions must be divided by cos ϕ, where ϕ is the pressure angle.”

“I have just two more observations to make, Mr Chairman, and I shall be brief”, said Mr Duality. “The first is that the output/input angular velocity ratio t_i/t_h found in the first column and ninth row of left hand matrix is ac/A, and the output/input torque ratio T_i/T_h found in the first column and tenth row of right hand matrix is $-A/ac$. The nett power entering the system, $t_hT_h + t_iT_i$ must be zero if there is 100% efficiency, which is something we assumed. The entries in the matrices confirm this.”

“My second point is that I do not think there is any interest here in the motions of indirectly coupled bodies, despite that being one of the original objectives of our investigation. Clearly, for all such pairs i and j, $f_{ij} = 1$. If we should want the velocity of the output shaft 4 relative to the input shaft 1 for example, one of only two pairs of indirectly coupled bodies, then we can take any path of G_C, from node 1 to node 4, and add the corresponding rows of $[\mathbf{M}^\mathbf{T}]_{9,2}$, changing signs wherever the path traverses an edge in the direction opposite to its positive sense. The answer is that the two components, when divided by t_h, are $\{-b(c+f)/A;\ 0\}$, describing an angular velocity about the z-axis of course.”

The Chairman thanked all the speakers for their contributions. He then turned to Mr Virtual Power. “My guess is that you will want to say something.”

“Yes please, Chairman.”

“Very well then, but again I must ask you to be as brief as possible. Dinnertime is approaching. Will you want help from the others?”

Ms Topology silently mouthed the reply from Mr Virtual Power in perfect synchronism. “No thank you. I can do it all myself.” Mr Virtual Power then began his presentation. “I shall not repeat the coupling graphs but I must provide the virtual action graph G_A of the mechanism with cutsets shown and the virtual motion graph G_M of the overconstrained chain with the circuits shown. These I shall call Figure 5.”

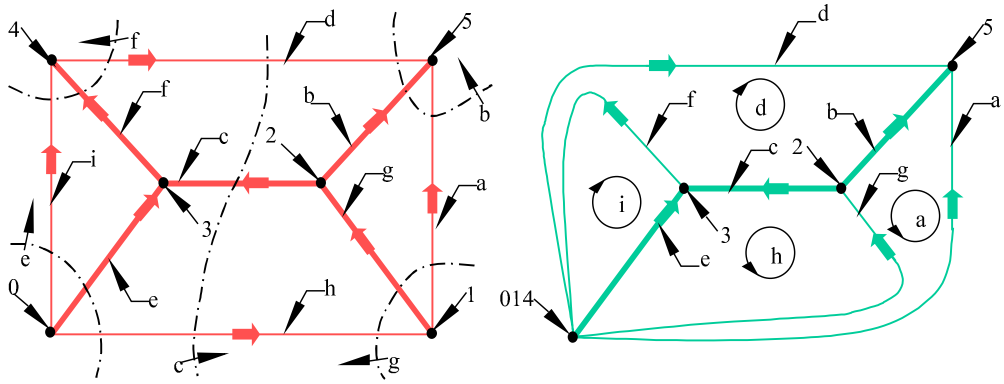

Fig. 5: The virtual action graph G_A of the mechanism and the virtual motion graph G_M of the overconstrained chain.

"The virtual action graph G_A for the mechanism, drawn on the left of Figure 5, has five cutsets and nine edges. Thus $k = 5$ and the gross degree of constraint $C = 9$. The nine edges are two less than the number in the action graph of the overconstrained chain that Mr Cutset Law needed. Remember that those two extra edges represented the torque provided by active couplings or the pin equivalent. They are absent in G_A for the mechanism. The consequences are that, for the mechanism, the cutset matrix $[\mathbf{Q_A}]_{k,C}$ is $[\mathbf{Q_A}]_{5,9}$; the unit action matrix $\left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}$ is $\left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{2,9}$; and the network unit action matrix $\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{dk,C}$ is $\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{10,9}$. These matrices can be obtained by deleting the eighth and tenth columns of $[\mathbf{Q_A}]_{5,11}$, $\left[\hat{\mathbf{A}}\;\;\right]_{2,11}$ and $\left[\hat{\mathbf{A}}_{\mathbf{N}}\right]_{10,11}$, the cutset, unit action and unit network action matrices respectively of the overconstrained chain used by Mr Cutset Law."

"The virtual motion graph G_M for the overconstrained chain, drawn on the right of Figure 5, requires some explanation. As we have seen, couplings h and i for the overconstrained chain are $c = 2$ couplings. They are also $f = 0$ couplings of the kind Mr Duality has called rigid. In other words, kinematically, they don't exist. Members numbered zero, one and four can be regarded as integral with one another where virtual motion is concerned. Thus, in G_M for the overconstrained chain, edges h and i of G_C are removed and the nodes 0, 1 and 4 coalesce to the single node I have labelled 014. The tree comprises only three branches but there are four chords and four circuits as before. The chords are not the same as the chords of G_C but I have used the same labels for the four circuits as those of used previously by Mr Circuit Law. Because the motion graph G_M for the overconstrained chain has four circuits and seven edges it follows that $l = 4$, and gross degree of virtual freedom $F = 7$. The consequences are that the circuit matrix $[\mathbf{B_M}]_{l,F}$ is $[\mathbf{B_M}]_{4,7}$ for the overconstrained chain; the unit motion matrix $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ is $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{2,7}$; and the unit network motion matrix $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{dl,F}$ is $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{8,7}$. These matrices can be obtained by deleting the eighth and ninth columns of $[\mathbf{B_M}]_{4,9}$, $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{2,9}$ and $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{8,9}$,the circuit, unit motion and unit network motion matrices respectively, used by Mr Circuit Law for the mechanism."

"We need the transpose of the network unit action and motion matrices." He then wrote down these matrices.

$\left[\hat{\mathbf{A}}_{\mathrm{N}}^{\mathbf{T}}\right]_{C,dk} = \left[\hat{\mathbf{A}}_{\mathrm{N}}^{\mathbf{T}}\right]_{9,10} = \begin{bmatrix} -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 1 \\ -b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -d & 1 & d & -1 & 0 & 0 & d & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$	$\left[\hat{\mathbf{M}}_{\mathrm{N}}^{\mathbf{T}}\right]_{F,dl} = \left[\hat{\mathbf{M}}_{\mathrm{N}}^{\mathbf{T}}\right]_{7,8} = \begin{bmatrix} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -b & -1 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & 1 & c & 0 & 0 \\ 0 & 0 & 1 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -e & -1 & -e \\ 0 & 0 & 1 & f & 0 & 0 & -1 & -f \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$

He followed this by writing down the vectors of cutset motions and circuit actions.

$[\mathbf{M_k}]_{dk,1} = [\mathbf{M_k}]_{10,1} = \{t_b, u_b, t_c, u_c, t_e, u_e, t_f, u_f, t_g, u_g\}^{\mathrm{T}}.$	$[\mathbf{A_l}]_{dl,1} = [\mathbf{A_l}]_{8,1} = \{T_a, U_a, T_d, U_d, T_h, U_h, T_i, U_i\}^{\mathrm{T}}.$

"Let me remind you of the equations in their general form and then the form they take for this example."

$\left[\hat{\mathbf{A}}_{\mathrm{N}}^{\mathbf{T}}\right]_{C,dk} [\mathbf{M_k}]_{dk,1} = [\mathbf{0}]_{C,1}$ becomes $\left[\hat{\mathbf{A}}_{\mathrm{N}}^{\mathbf{T}}\right]_{9,10} [\mathbf{M_k}]_{10,1} = [\mathbf{0}]_{9,1}$.	$\left[\hat{\mathbf{M}}_{\mathrm{N}}^{\mathbf{T}}\right]_{F,dl} [\mathbf{A_l}]_{dl,1} = [\mathbf{0}]_{F,1}$ becomes $\left[\hat{\mathbf{M}}_{\mathrm{N}}^{\mathbf{T}}\right]_{7,8} [\mathbf{A_l}]_{8,1} = [\mathbf{0}]_{7,1}$.

"Of course, if I had the laptop Miss Matrix has been using I could solve these equations."

"So you cannot do it all on your own after all", Miss Matrix responded sarcastically. "You would like my help?"

"Yes please."

Miss Matrix then went to work. She soon found a snag. "For the kinematics problem I find I cannot use the input angular velocity t_h as the primary variable as I did for Mr Circuit Law; I am going to use t_e instead." She then wrote down the results.

$(1/\,t_e)\,\left[\mathbf{M}_\mathbf{k}^\mathbf{T}\right]_{k,d} = (1/\,t_e)\,\left[\mathbf{M}_\mathbf{k}^\mathbf{T}\right]_{5,2} = \begin{bmatrix} 2de(f-c)/cf(a-b) & 2bde(f-c)/cf(a-b) \\ e/c & e \\ 1 & e \\ -e/f & -e \\ 2de(c-f)/acf & 0 \end{bmatrix}.$	$(1/\,T_h)\,\left[\mathbf{A}_\mathbf{l}^\mathbf{T}\right]_{l,d} = (1/\,T_h)\,\left[\mathbf{A}_\mathbf{l}^\mathbf{T}\right]_{4,2} = \begin{bmatrix} 1 & -1/a \\ -2d/a & 2/a \\ 1 & (b-2c)/ac \\ -A/ac & (b+2c)/ac \end{bmatrix},$ where A=ac+bc+bf as in previous results.

"I have arranged the results in matrix form because I know this is the form in which Mr Virtual Power needs them to have for the last stage of his analysis."

"Thank you Miss Matrix." Mr Virtual Power did not want Miss Matrix to continue to be the centre of attention now that the result he needed was available. "The rows of the matrix on the left provide the components of cutset motions divided by the angular velocity t_e. The first column contains angular velocities divided by t_e and the second column contains velocities at the origin divided by t_e. Rows in descending order correspond to cutsets in the order: *b, c, e, f,* and *g*."

"In the matrix on the right, the rows are the components of circuit actions divided by input torque T_h. The first column contains moments or torques divided by T_h and the second column contains forces divided by T_h. Rows in descending order correspond to circuits in the order: *a, d, h,* and *i.*"

Mr Commonsense then spoke. "I have a question for Mr Virtual Power. It is a pity that Miss Matrix was unable to use input angular velocity as the primary variable in motion analysis. Why was this not possible? After all she was able to use input torque as the primary variable for the action analysis."

"It is a consequence of our choice of tree", replied Mr Virtual Power. "To keep things simple we chose the same tree for both problems. Our choice means that edge h is a chord of G_C which corresponds to circuit h. Although in G_M for the overconstrained chain edge h is absent we retain the circuit labelled h and the circuit action for this circuit is the action transmitted by coupling h. However, with this choice of tree, edge h in G_A for the mechanism is a chord whereas it needs to be a branch of the tree if t_h is to be used as the primary variable. There are several other trees of G_A we could have used wherein edge h is a branch. Ideally we should have chosen different trees for the two different tasks. However, I shall soon be able to show that our choice of tree was not the disaster it may appear to be."

"To convert the results to coupling motions and actions we now use my last equations." Mr Virtual Power then wrote these equations.

$[\mathbf{M}]_{d,e} = [\mathbf{M_k}]_{d,k}\,[\mathbf{Q}]_{k,e}$, which, for this problem becomes $[\mathbf{M}]_{2,9} = [\mathbf{M_k}]_{2,5}\,[\mathbf{Q}]_{5,9}$. The solution $[\mathbf{M}]_{2,9}$ when transposed and divided by t_e is $(1/t_e)\,[\mathbf{M^T}]_{9,2} =$ $\begin{bmatrix} 2bde(f-c)/acf(a-b) & 2bde(f-c)/cf(a-b) \\ 2de(f-c)/cf(a-b) & 2bde(f-c)/cf(a-b) \\ e/c & e \\ 2be(f-c)/cf(a-b) & 2bde(f-c)/cf(a-b) \\ 1 & e \\ -e/f & -e \\ 2de(c-f)/acf & 0 \\ (f-c)A/2acf & 0 \\ (f-c)/2f & 0 \end{bmatrix}$, where $A = ac + bc + bf$.	$[\mathbf{A}]_{d,e} = [\mathbf{A_l}]_{d,l}\,[\mathbf{B}]_{l,e}$, which, for this problem becomes $[\mathbf{A}]_{2,9} = [\mathbf{A_l}]_{2,4}\,[\mathbf{B}]_{4,9}$. The solution $[\mathbf{A}]_{2,9}$, when transposed and divided by T_h, is $(1/T_h)\,[\mathbf{A^T}]_{9,2} =$ $\begin{bmatrix} 1 & -1/a \\ b/a & -1/a \\ -b/a & b/ac \\ -2d/a & 2/a \\ 2be/ac & -2b/ac \\ bf/ac & -b/ac \\ 0 & (b-c)/ac \\ 1 & (b-2c)/ac \\ -A/ac & (b+2c)/ac \end{bmatrix}$, where $A = ac + bc + bf$.

"First consider the matrix on the left hand side. The rows of the matrix contain the two components, divided by t_e, of the motions of pairs of directly coupled bodies. These pairs of bodies are those that are represented in G_C by the pairs of nodes that are both incident with the same edge. Rows, in descending order, correspond to edges of G_C in alphabetic order a - i. When this matrix is divided by

$$t_h / t_e = (f - c)A / 2acf,$$

the element in the first column and eighth row, the result is identical to the final matrix in the left-hand column of Mr Circuit Law's result where all motions are a ratio of t_h, the input angular velocity. So, not having t_h available as the primary variable is not a disaster; the matter is easily remedied The matrix on the right is identical with the matrix provided by Mr Cutset Law."

A telephone rang. The Chairman looked startled. "What's that?"
"It's my mobile", said Mr Virtual Power putting it to his ear.
"Mobile? What is a mobile?" The Chairman asked.
"My mobile phone: it's a One-2-One."
"A call for me?" Mr One-to-One looked astonished.
Ms Topology whispered to Mr Virtual Power, "Tell him it's his wife."
Ignoring her, Mr Virtual Power told the Chairman that it was a message to say that dinner was ready.
"Very well then, no time for questions now but we shall have a summing-up and final questions after dinner."

When the gathering had reassembled after dinner the Chairman called on Mr Duality to provide a summary of what he thought had been achieved during the day.

Mr Duality began. "A long term objective of those of us invited here today by the Chairman has been to gain a thorough understanding of freedom and constraint. We now feel we have achieved this. The route we found to attain this objective has required us also to meet your objectives."

"We can represent the coupling networks we have studied by a Venn diagram." Mr Duality then sketched Figure 6.

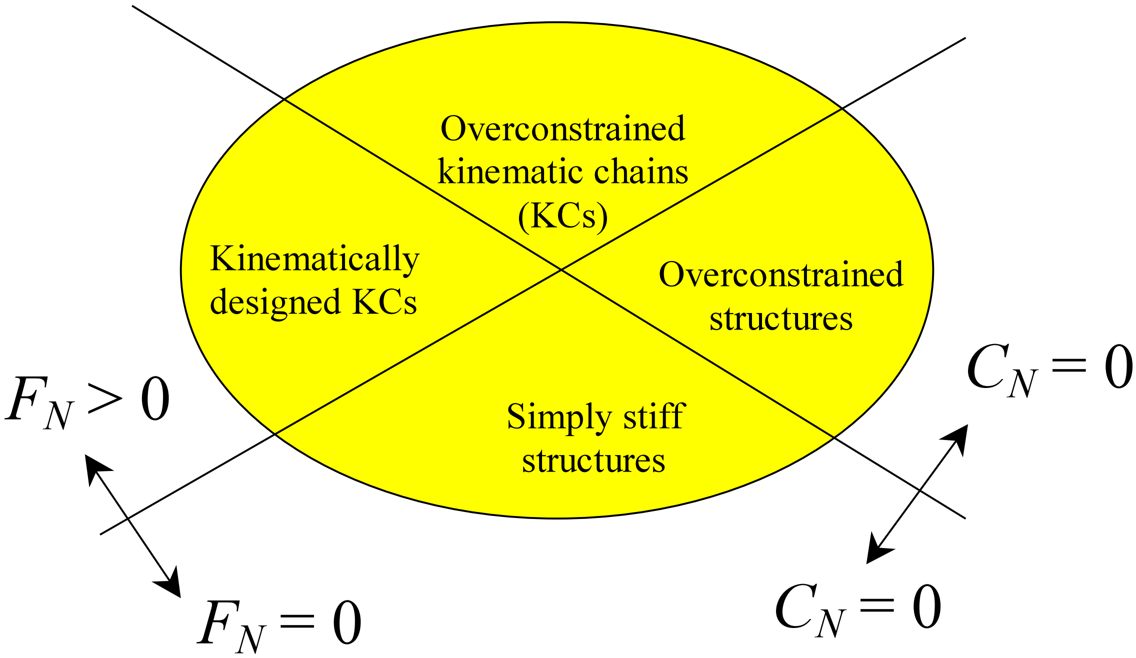

Fig. 6 Closed passive coupling networks

"By passive couplings we mean couplings such as kinematic pairs in contrast to active couplings that export or import power. The set is divided into two by the line from bottom left to top right separating kinematic chains with a positive nett degree of freedom F_N from structures with a zero value for F_N. The set is also divided into two by a line from top left to bottom right separating overconstrained chains that have a positive nett degree of constraint C_N from coupling networks with a zero value for C_N. These two lines divide the set into four subsets that are labelled in Figure 6. Please note that there is no such thing as negative freedom or negative constraint."

"We have derived two matrix equations for kinematic chains enabling the F unknown motion components to be expressed in terms of F_N of them. One equation is an adaptation of Kirchhoff's circulation law, the second is an internal virtual power equation. The number of rows and columns of the matrices provide two formulae for F_N of general applicability."

"Also, we have derived two matrix equations for overconstrained chains enabling the C unknown action components to be expressed in terms of C_N of them. One equation is an adaptation of Kirchhoff's node law, the second uses an internal virtual power equation. The number of rows and columns of the matrices provide two formulae for C_N of general applicability."

"For overconstrained kinematic chains, the subset represented at the top of Figure 6, all four methods can be used but some redundant equations will exist unless the chain is modified beforehand to one that belongs to an adjacent subset. For structures and machines that incorporate active couplings external to a simply stiff structure or mechanism, the active couplings can be replaced by equivalent passive couplings to create an overconstrained chain. We have provided an example of a machine wherein such replacements are made."

After thanking Mr Duality the Chairman asked the original committee members to think of one question each that they would like to ask. "To give you time to think I shall ask the first question myself." He turned towards the others. "Do any of you have any research proposals that might interest a Ph. D. candidate?"

Mr Circuit Law was anxious to say something before his twin brother who he had always felt was an attention seeker. "I have a proposal, Mr Chairman. I think those present today may have a greater interest in overfreedom than in overconstraint so this concerns kinematics. A short title for the research might be: A Unified Approach to Inverse Kinematics."

"Please imagine a manipulator with F_N degrees of freedom. It can be a series, parallel or hybrid manipulator; it does not matter. Now imagine an additional virtual coupling between the frame and the end-effector. This virtual coupling is an indirect coupling between these bodies. It comprises a serial chain of F_N couplings each allowing one degree of freedom and (F_N - 1) intermediate bodies. The important thing is that this additional coupling does not impose any further restraint on the end-effector."

"For a six degree of freedom manipulator these direct couplings can be three prismatic (*P*) kinematic pairs and three revolute (*R*) pairs. The direction of each of the *P* pairs can be parallel with a different axis of the global frame. Also, the rotation axes of each of the *R* pairs will be parallel with a different axis." Mr Cartesian, having dozed off, was now wide awake. "The introduction of this extra virtual coupling introduces an additional circuit", continued Mr Circuit Law. "However, that is a small price to pay because it enables us to conduct inverse kinematic analysis in the same manner as direct kinematic analysis. This additional circuit is of least significance for a parallel manipulator because it will have several circuits without the additional one."

"You may remember my saying that, when selecting primary variables, we have some freedom of choice. The difference between direct and inverse kinematics for the modified manipulator is solely in the choice of primary variables. For direct kinematics the primary variables are the magnitudes of motions of the F_N directly coupled pairs of bodies that are the ports for actuators. For inverse kinematics the primary variables are the magnitudes of the motions of the F_N pairs of directly coupled bodies in the virtual indirect coupling. The researcher's task could be to compare the procedures we have explained today with existing methods of inverse kinematics to see if there are advantages."

Mr Helix volunteered the second question. "It seems to me that in achieving your aim of finding the freedom of pairs of bodies you have stumbled across new ways of looking at coupling networks. I am wondering if you have a name in mind for these dual studies. Is dual mechanics appropriate?"

Mr Duality responded. "I am not going to quibble with Mr Helix's use of the word 'stumbled'. You cannot stumble by standing still. It has been more like research in the arts rather than the sciences; finding analogies, patterns and connections with other disciplines. A valuable guide has been that when the mathematics becomes elegant it is usually correct. We think this is true of the symmetry of duality. As for a name I think 'dual mechanics' is inappropriate because mechanics is too broad a term. We are concerned here with a limited part of mechanics. We have had discussions on the choice of name with Mr Seven R Spatial. We have rejected kinetostatics on the grounds that it is understood to mean the study of actions arising from acceleration. We prefer kinestatics."

Mr Anharmonic spoke next. "You stated clearly that the kinematics was confined to first derivative kinematics. Do you agree that it is important to stress that everything you have said is only instantaneously valid? For a mechanism or machine in motion all these calculations must be repeated for every configuration it moves to for which answers are required. You do not provide an analysis that produces the geometric constants that are needed for the direct coupling unit motion and action matrices."

Miss Matrix chose to reply. "That is quite correct. The example of the gear train we provided is not typical in this respect. The geometry of a gear train does not change following a finite displacement. For other coupling networks wherein the geometry does change with displacement, what would be nice would be to set up the network unit matrices for one configuration and then premultiply them with a matrix that is an operator for a finite displacement."

Mr Querulous then spoke. "I am interested in how the thought processes that went into these studies all started." Mr Circuit Law looked around and recognised everyone was expecting him to speak. "As I have been involved longer than anyone else here I shall attempt an answer to your question. The investigation started in 1964 when a Lecturer in Machine Theory together with his colleague Robert Macmillan, introduced a discussion on 'The Teaching of Kinematics' at the headquarters of the Institution of Mechanical Engineers in London. Robert Macmillan had previously written about the freedom of linkages. During the subsequent discussion one member of the audience stated that, whatever was taught, it was vital that freedom and constraint featured in it. How to teach freedom and constraint became an important theme. It was necessary first to thoroughly understand freedom and constraint and this has taken a surprisingly long time."

"Why?" asked Mr Querulous.

"There are several reasons. There were no guaranteed landmarks for research students to aim for. It was important to seek research funding but obtaining an understanding of something that was thought to be already understood did not appeal to funding bodies. So this activity we have been talking about has had to be confined to spare time available between teaching and other funded research obligations. A labour of love if you like. There was much to study even after I had been recruited: the immense output of Gabriel Kron for one. How he would have enjoyed it here today. Furthermore, we entered, or were led into, several blind alleys. There are structural engineers who use the nodes of graphs to represent couplings instead of bodies. Also, there are electrical engineers who insist that it is motion not action that is analogous to electrical current. Mr Bond Graph is a persistent offender in this regard. He wanted to be here today but we just had to say no."

Mr Duality thought that Mr Cutset Law was being too gloomy. "But it was not all bad news. Remember the two circuit spatial kinematic chain that Mr Exceptional Mobility proposed as an example. By good fortune it happened to be an overconstrained chain as well. Indirectly that led to my involvement, and your brother's."

“Why is this Lecturer in Machine Theory not here with you today?” continued Mr Querulous.

“He couldn't be could he?” replied Mr Circuit Law. “Unlike all of us here, he is a mere mortal like Sir Robert Ball was. We are figments of his imagination just as you were of Sir Robert Ball. We cannot have mortals at a meeting like this. What we can achieve in a day takes them years.”

Mr Commonsense asked the next question. “You provide two methods for motion analysis and two methods for action analysis. Can you tell us which of the two you believe to be the superior method for both tasks?”

Mr Virtual Power volunteered to reply. “This is the sort of good question one has come to expect from Mr Commonsense. It is difficult to generalise. Usually, the most difficult part is assembling the matrices $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ and $\left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}$. This difficulty is not apparent from the example we chose because the geometry of that example is exceptionally simple with d being only two. A measure of difficulty could be the number of elements in the matrices, dF and dC respectively. One of the most common couplings is the bearing, the revolute kinematic pair, or its indirect counterpart the rolling contract bearing. In general, that is to say when $d = 6$, these couplings have gross degrees of freedom and constraint $f = 1$ and $c = 5$. My personal view is that, if these couplings predominate, assembling $\left[\hat{\mathbf{M}}_{\mathbf{D}}\right]_{d,F}$ will generally be easier than assembling $\left[\hat{\mathbf{A}}_{\mathbf{D}}\right]_{d,C}$. Consequently, my guess is that Mr Circuit Law's method will generally be the easier one for motion analysis and my internal virtual power method the easier one for action analysis. The fact that we often speak of virtual motions but rarely, if ever, of virtual actions might be evidence of this.”

Mr One-to-One wanted to know what everyone would do next. “If you were just starting your research careers equipped with what you now know what new areas would you investigate?”

Ms Topology said that she might tackle dynamics. “I ought to have mentioned that Graph Theory is used in dynamics, notably by Jens Wittenburg in Germany, Gordon Andrews in Canada and Josef Wójnarowski in Poland. Of course the aims of those gentlemen differ from ours. For example, Wittenburg needs a base to body matrix in which to assemble inertia terms. I would seek to integrate our different approaches.”

Mr Cutset Law interrupted. “Can I just say that if we include couplings that we might call inertial couplings or D'Alembert couplings there is another law: **you cannot have an action without a circuit**. This then takes into account the free fall problem: Newton's apple and the earth had two couplings; one gravitational, the second inertial. Together they formed a circuit.” His twin brother was furious. “You used the word circuit. That should make it my law” “It could have been if you had said it first”, was the response. Mr Circuit Law glowered.

Mr Cutset Law continued by providing his own wish list. “I should like to invert the structural problem. Instead of asking what action arises from a given overconstrained chain I would ask what overconstrained chain is required to create a required action. I would like to study conformational and pharmaceutical chemistry. What shapes are required by viruses and drugs to attach themselves to cells?”

The Coordinates twins, unlike the Law twins, thought alike and displayed no envy or malice towards one another. Mr Motion Coordinates was quite happy when his brother offered to give their joint opinions. “We would have preferred you to have been given an example of a machine with structural overconstraint rather than the one chosen that can be solved by statics. Unfortunately we were outvoted. We believe that the speaker in 1964 would be aware that constraint, or rather overconstraint, is a more important topic than freedom. Finding the nett degree of freedom is a fascinating theoretical problem. However, the nett degree of freedom of a kinematic chain is never a great mystery in practice; models can be made, or simulated, and the true nature of freedom readily reveals itself. Essentially this is because whilst freedom can be seen, overconstraint is hidden. Ignorance of the existence of overconstraint can lead to premature fatigue failure in machines caused by alternating stresses of unknown amplitude. Furthermore these unknown stresses are additional to stresses caused by known actions that are themselves often alternating. We hear a lot about fatigue and stress raisers. But if there is no stress to raise, or it is known, then the problem is lessened. Kinematic design, in other words design with an absence of overconstraint, works well in lightly loaded instruments. In machines that transmit substantial power overconstraint is commonplace in order to maintain sufficient stiffness for vibration frequencies to be well above cycle frequencies. For such machines it is important that engineers are aware of the benefits of judiciously located compliance to limit stresses arising from overconstraint. Michael French gets the balance right between kinematic and elastic design. His chapters on that subject are most valuable. We would wish to draw attention to overconstraint, its dangers, and the remedies available.”

Mr Duality said he wanted to continue to spread the message about duality. “Can you imagine a University Department of Electrical Engineering advertising for two posts; one for a teacher of Electrical Circuit Theory (electrical currents) and another for a teacher of Electrical Circuit Theory (potential differences)? Yet often statics and kinematics are taught using separate textbooks and very often by different teachers. You will not be surprised to learn that I find that odd. Finally, may I just add that what I like about the work we have been talking about, and that of our hero Sir Robert Ball, is its timelessness. Whatever we and he have said that is true has always been true and will always remain so.”

The Chairman spoke. “I think that must end our discussion. It is time to go home now before Mrs One-to-One telephones to ask where her husband is.”

Mr Helix had one more thing to say. “There have been many different symbols, vectors and matrices mentioned. Could we be provided with a notation?” “And a bibliography?” added Mr Cartesian.

The Chairman explained that there was insufficient time but suggested that these could be sent to him. This was agreed. The Chairman then closed the meeting. "Thank you everyone. I hope we all meet again in the year 2100, no doubt with more new faces. I wonder what they will bring with them? A laptop indeed!”

NOTATION

a the rank of the network unit action matrix $\lfloor \hat{\mathbf{A}}_{\mathbf{N}} \rfloor_{dk,C}$

c the degree of constraint (d.o.c.) of a direct coupling and normally subscripted by the label of that coupling; the order of the minimal action system to which all actions under consideration belong that can be transmitted by a direct coupling when those directly coupled bodies are fully constrained by one or more other couplings

C the gross d.o.c of a coupling network equal to the sum $\sum_{1}^{e} c$ of the d.o.c of all direct couplings of a coupling network and the number of edges of its action graph G_A

C_N the nett d.o.c of the coupling network, equal to (C - a) and (dl - m); the number of primary variables needed to provide the magnitudes of all actions when those actions are limited by the freedoms allowed by all the couplings

d ($1 \leq d \leq 6$), the order of the minimal action or motion system that spans the action or motion systems of order c or f that are characteristic of all direct couplings of a coupling network

e the number of couplings in a coupling network and the number of edges of its coupling graph G_C

f the gross degree of freedom (d.o.f.) of a direct coupling and normally subscripted by the label of that coupling; the order of the minimal motion system to which all motions under consideration belong that are allowed by a direct coupling when those directly coupled bodies are otherwise unconstrained

f_{ij} the nett d.o.f. of bodies i and j; the order of the minimal motion system to which all motions under consideration of bodies i and j belong when these motions are limited by constraints imposed by couplings of the coupling network to which the bodies belong; for directly coupled bodies, $f_{ij} \leq f$, where f is the gross d.o.f. of that direct coupling

F the gross d.o.f. of a coupling network and the number of edges of its motion graph G_M equal to the sum $\sum_{1}^{e} f$ of the gross d.o.f of all direct couplings of the coupling network

F_N the nett d.o.f of the coupling network, equal to (F - m) and (dk - a); the number of primary variables needed to provide the magnitudes of all motions when those motions are limited by the constraints imposed by all the couplings

G_C, G_A, G_M the coupling, action and motion graphs of a coupling network

i, j indices

k the number of cutsets of graphs G_C and G_A

l the number of circuits (loops) of graphs G_C and G_M

m the rank of the network unit motion matrix $\left[\hat{\mathbf{M}}_{\mathbf{N}}\right]_{dl,F}$

n the number of bodies in a coupling network and the number of nodes of graphs G_C and G_A

ψ the magnitude of a motion; a generalised coefficient that can be angular velocity or translational velocity

Ψ magnitude of an action; a generalised coefficient that can be force or torque

Coordinates

$\{L, M, N; P, Q, R\}$ line coordinates, and…
$\{l, m, n; p, q, r\}$ … unit line coordinates, in axis formation
$\{P, Q, R; L, M, N\}$ line coordinates, and…
$\{p, q, r; l, m, n\}$ … unit line coordinates, in ray formation

$\{L, M, N; P^*, Q^*, R^*\}$ screw coordinates, and…
$\{l, m, n; p^*, q^*, r^*\}$ … unit screw coordinates†, in axis formation
$\{P^*, Q^*, R^*; L, M, N\}$ screw coordinates, and…
$\{p^*, q^*, r^*; l, m, n\}$ … unit screw coordinates‡, in ray formation

$\{r, s, t; u, v, w\}$ motion coordinates, and…
$\{\hat{r}, \hat{s}, \hat{t}; \hat{u}, \hat{v}, \hat{w}\}$ … unit motion coordinates†
$\{R, S, T; U, V, W\}$ action coordinates, and…
$\{\hat{R}, \hat{S}, \hat{T}; \hat{U}, \hat{V}, \hat{W}\}$ … unit action coordinates‡

† Unit screw coordinates in axis formation and unit motion coordinates are identical

‡ Unit screw coordinates in ray formation and unit action coordinates are identical

Vectors

$[\mathbf{A_l}]_{dl,1}$	*dl* components of all circuit actions[†] that can be partitioned into vectors of …
$[\mathbf{A_l}]_{m,1}$	… *m* secondary circuit action components, and …
$[\mathbf{A_l}]_{dl\text{-}m,1}$	… $C_N = (dk - m)$ primary circuit action components
$[\mathbf{M_k}]_{dk,1}$	*dk* components of all cutset motions[‡] that can be partitioned into vectors of …
$[\mathbf{M_k}]_{a,1}$	… *a* secondary cutset motion components, and …
$[\mathbf{M_k}]_{dk\text{-}a,1}$	… $F_N = (dk - a)$ primary cutset motion components
$[\Psi]_{C,1}$	*C* generalised action magnitudes that can be partitioned into vectors of …
$[\Psi]_{a,1}$	… *a* secondary action magnitudes, and …
$[\Psi]_{C\text{-}a,1}$	… $(C - a)$ primary action magnitudes
$[\psi]_{F,1}$	*F* generalised motion magnitudes that can be partitioned into vectors of …
$[\psi]_{m,1}$	… *m* secondary motion magnitudes, and …
$[\psi]_{F\text{-}m,1}$	… $(F - m)$ primary motion magnitudes

[†] Vector $[\mathbf{A_l}]_{dl,1}$ and matrix $[\mathbf{A_l}]_{d,l}$ contain identical elements, the vector in a single column in circuit order

[‡] Vector $[\mathbf{M_k}]_{dk,1}$ and matrix $[\mathbf{M_k}]_{d,k}$ contain identical elements, the vector in a single column in cutset order

Matrices

$[\mathbf{A}]_{d,C}$, $[\hat{\mathbf{A}}]_{d,C}$ the action, and unit action, matrices of a coupling network; each column of $[\mathbf{A}]_{d,C}$ provides the same d components of action as those of the corresponding column of $[\mathbf{A_D}]_{d,C}$ (see below) with the important difference that within $[\mathbf{A}]_{d,C}$ those components are subject to further restrictions by the freedom permitted by the indirect coupling created by *other* couplings of the coupling network

$[\mathbf{A}]_{d,e}$ a condensed version of $[\mathbf{A}]_{d,C}$ in which columns of $[\mathbf{A}]_{d,C}$ representing the actions that can be transmitted by a direct coupling are added thereby providing in a single column the componentsof the *system* of actions for each of the direct couplings

$[\mathbf{A_D}]_{d,C}$, $[\hat{\mathbf{A}}_\mathbf{D}]_{d,C}$ the action, and unit action, matrices of the direct couplings of a coupling network; each column of $[\mathbf{A_D}]_{d,C}$ contains the d components of an action chosen as one of c actions that span the c-system of actions that can be transmitted by a direct coupling when the coupled bodies are made integral by another rigid coupling

$[\hat{\mathbf{A}}_\mathbf{N}]_{dk,C}$ network unit action matrix of a coupling network,…

$[\hat{\mathbf{A}}_\mathbf{N}]_{a,C}$ … with redundant rows removed, and partitioned into submatrices of…

$[\hat{\mathbf{A}}_\mathbf{N}]_{a,a}$ … secondary and …

$[\hat{\mathbf{A}}_\mathbf{N}]_{a,C-a}$ … primary coefficients

$[\mathbf{A_l}]_{d,l}$ matrix of the d components of l circuit actions†

$[\mathbf{B}]_{l,e}$ circuit matrix of G_C

$[\mathbf{B_i}]_{F,F}$ $i = 1, 2, \ldots, l$, diagonal matrices with diagonal elements corresponding to row i of $[\mathbf{B_M}]_{l,F}$

$[\mathbf{B_M}]_{l,F}$ circuit matrix of G_M

$[\mathbf{M}]_{d,F}$, $[\hat{\mathbf{M}}]_{d,F}$ the motion, and unit motion, matrices of a coupling network; each column of $[\mathbf{M}]_{d,F}$ provides the same d components of motion as those of the corresponding column of $[\mathbf{M_D}]_{d,F}$ (see below) with the important difference that within $[\mathbf{M}]_{d,F}$ the components are subject to further restrictions by the constraints imposed by the indirect coupling created by *other* couplings of the coupling network

$[\mathbf{M}]_{d,e}$ a condensed version of $[\mathbf{M}]_{d,}$ in which columns of $[\mathbf{M}]_{d,}$ representing the motions that can be permitted by a direct coupling are added thereby providing in a single column the coordinates of the *system* of motions for each of the direct couplings

$[\mathbf{M_D}]_{d,F}$, $[\hat{\mathbf{M}}_\mathbf{D}]_{d,F}$ the motion, and unit motion, matrices of the direct couplings of a coupling network; each column of $[\mathbf{M_D}]_{d,F}$, contains the d components of a motion chosen as one of f motions that span the f-system of motions that can be permitted by a direct coupling when indirect couplings are ignored.

$[\hat{\mathbf{M}}_\mathbf{N}]_{dl,F}$ network unit motion matrix of a coupling network…

$[\hat{\mathbf{M}}_\mathbf{N}]_{m,F}$ … with redundant rows removed, and partitioned into submatrices of…

$[\hat{\mathbf{M}}_\mathbf{N}]_{m,m}$ … secondary and …

$[\hat{\mathbf{M}}_\mathbf{N}]_{m,F-m}$ … primary coefficients

$[\mathbf{M_k}]_{d,k}$ matrix of the d components of k cutset motions‡

$[\mathbf{Q}]_{k,e}$ cutset matrix of G_C

$[\mathbf{Q_i}]_{C,C}$ $i = 1, 2, \ldots, k$, diagonal matrices with diagonal elements corresponding to row i of $[\mathbf{Q_A}]_{k,C}$

$[\mathbf{Q_A}]_{k,C}$ cutset matrix of G_A

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