Periodic measures, transitions and exit times of stochastic differential equations

PUBLISHER

Loughborough University

Department of Mathematical Sciences

Periodic Measures, Transitions and Exit Times of Stochastic Differential Equations

by

Johnny Zhong

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University

Acknowledgements

Firstly, I would like to express my sincere gratitude to my supervisors Prof. Huaizhong Zhao and Dr. Chunrong Feng for their continued guidance and support over the years. Their academic brilliance and encouragement has been invaluable in my academic and personal development and has indeed lead me to push the envelope throughout my studies.

I would like to thank my fellow colleagues Yu Liu, Yujia Liu and Yan Luo for the helpful discussions and friendship during our studies.

Special thanks to my companion in life, Akhi Qureshi, thank you for your encouragement to embark on this PhD in the beginning, your support throughout the studies and for the bright future that lays ahead.

Finally I would like to give my warm heartfelt gratitude to my brothers, Jack and Lex, and my mother and father and my grandmother for their support, understanding and unconditional love. This thesis would not be possible without you. Dad, thank you also for the wonderful and precious time we shared on this Earth and being the best dad I could ask for.

Abstract

Periodic measures are the time-periodic counterpart to invariant measures for dynamical systems that can characterise the long-term periodic behaviour of stochastic dynamical systems. In this thesis, sufficient conditions are given for the existence, uniqueness and geometric convergence of periodic measures of time-periodic Markovian systems on locally compact metric spaces. The results will be applied specifically to time-periodic weakly dissipative stochastic differential equations (SDEs), gradient SDEs and Langevin equations. We show that the periodic measure density sufficiently and necessarily satisfies a time-periodic Fokker-Planck equation. We will also rigorously derive that the expected exit duration of time-periodic SDEs is the time-periodic solution of a second-order linear parabolic partial differential equation (PDE). Collectively, this rigorously establishes two novel Feynman-Kac dualities for time-periodic SDEs. Casting the time-periodic solution of the PDE as a fixed point problem and a convex optimisation problem, we give sufficient conditions in which the PDE is well-posed in a weak and classical sense. With no known closed formulae, we show that these approaches can be readily implemented to compute the expected exit time numerically. Periodic measures and expected exit times are novel tools to understand physical phenomena exhibiting periodicity. Particular application towards stochastic resonance will be discussed.

Keywords: periodic measures, geometric ergodicity, expected exit time, stochastic differential equations, time-periodic parabolic partial differential equations, stochastic resonance, Feynman-Kac duality

Contents

1 Introduction and Motivation

The theory of dynamical systems provides insight and understanding of the evolution of physical systems in time. Indeed many physical systems from all disciplines of science are modelled and studied as dynamical systems, particularly difference and differential equations. It is well-known that, in general, there are no analytical closed-form solutions to many dynamical systems. Resultantly, to understand dynamical systems is to identify key features such as fixed points, stationary solutions and periodic solutions. Intuitively, fixed points and stationary solutions capture the idea of equilibria of the system and periodic solutions for periodic behaviour. For many physical systems, its asymptotic or long time behaviour are fully characterised by these key features. This has the interpretation any trajectory of the system "settle" to these equilibria or periodic equilibria in the long term. Asymptotic behaviour of dynamical systems is still an active area of research.

Since the dawn of the 20th century, there has been continued interest modelling physical systems as stochastic dynamical systems. Particular attention was given to Markov chains and Markov processes with much focus on stochastic differential equations (SDEs). There are two main motivations for the interest. The first is that deterministic systems do not account for external randomness or "noise" in the realities of physical systems. The second is that complex deterministic systems can be simplified and better described when treated as stochastic system rather than a deterministic one. Statistical mechanics, a pillar of modern physics initiated by Boltzmann, takes this particular viewpoint. In particular, Einstein and Smoluchowski pioneered the theory of thermodynamics in the context of (now known as) SDEs. Independently, Bachelier pioneered this viewpoint for the financial markets. In these fields, the stochasticity typically models the complex interactions between the large number of particles and market participants respectively.

For stochastic dynamical systems, stationary processes play the stochastic counterpart of fixed points and stationary solutions in deterministic dynamical systems. Their existence are well-studied for SDEs as well as stochastic partial differential equations (SPDEs) e.g. [CKS04, EKMS00, Sch98, Sin96, ZZ07]. The behaviour of stochastic systems can be better understood via probability measures to capture the distribution of the process. Then the equilibria of stochastic systems can be characterised by invariant (probability) measures. In particular, if an invariant measure is limiting i.e. the law of any trajectory converges to the invariant measure, then the asymptotic behaviour of stochastic systems is essentially known. Indeed, it is well-known that there is an "equivalence" (possibly on an enlarged probability space) between stationary processes and invariant measures [Arn98, OO99]. In existing literature, there are many results on the existence, uniqueness of limiting invariant measure of time-homogeneous Markovian systems in many cases including Markov chains and processes of finite and infinite dimensional state spaces see e.g. [DZ96, MT92, Nor98, MSH02, Has12, Hai06, MT09, Mat03, MT93].

Until recently, due to the delicate interplay between periodicity and stochasticity, there was not a rigorous formalism for periodic solution for stochastic systems. Thus, while asymptotic periodic behaviour of deterministic systems are well understood, there was no formal definitions nor framework to discuss asymptotic periodic behaviour of stochastic systems. We briefly discuss the main difficulty. While it is clear that deterministic periodic solutions returns to the initial state after multiples of the period, we cannot expect likewise for stochastic periodic solutions with absolute certainty. Instead, we intuitively expect the stochastic periodic solutions to return to a neighbourhood of the initial state after period. However, it may also happen that after a period the process is significantly different from the initial state. To rigorously grapple with the various possibilities has been difficult. Having the mathematical framework to study periodic solution for stochastic systems is crucial because many processes in the physical reality possesses both periodicity and stochasticity. We promptly look at some examples.

Consider for instance daily average temperature of a fixed geographic location and season. It is clear that the temperature is a periodic process as the Earth rotates around the sun. Of course the temperature is a stochastic process as it is subject to a range of uncertainties including prevailing winds, cloudiness, reaction within the sun and humidity. Resultantly, the temperature generally does not return to the exact temperature at the same time on a daily basis, furthermore it may be substantially different. Another broad example are commodity prices in the financial markets. Many commodities have a seasonal trend e.g. coffee beans, cocoa, wheat, soya bean, corn and rice due to its natural growth cycle. Many uncertainty factors can affects its supply and demand including draught, sunlight, soil quality, pollution and geopolitical factors. It is worth mentioning that while these examples portray periodicity, in general, the observed processes of periodic systems may not appear periodic to the eye. This is particularly true for nonlinear periodic systems and will be apparent in the forthcoming discussion of stochastic resonance.

In a series of paper [FZZ11, FZ12, LZ12, ZZ09, FZ16], Zhao et al developed the formal framework to understand periodicity and stochasticity via the rigorous definition of random periodic solution. In [ZZ09], Zhao and Zheng gave a rigorous definition of random periodic solutions for C^1-cocycles in the context of random dynamical systems (RDS). In [FZZ11] and [FWZ16], the existence of random periodic solutions were shown for periodic semilinear SDEs. Numerical approximations of random periodic paths of SDEs were studied in [FLZ17]. In [FZ16], Feng and Zhao defined rigorously periodic measures as the time-periodic counterpart of invariant measures. Furthermore, just as there is an equivalence between stationary processes and invariant measures, in [FZ16], the equivalence between random periodic solution and periodic measures was proved. It was also shown in [FZ16] that an invariant measure can be obtained by lifting the periodic measure on a cylinder and considering its average over one period. The concept of periodic measures and ergodicity provides a rigorous framework and new insight to understanding time-periodic physical phenomena such as the ones mentioned above.

In this thesis, we look at the key ingredients for the existence and uniqueness of periodic measures for (time-inhomogeneous) time-periodic Markovian processes on locally compact state spaces. Furthermore, we show when periodic measure is limiting and have geometric convergence. We apply these abstract results specifically to time-periodic SDEs where verifiable coefficient conditions are given. Requiring only locally Lipschitz coefficients and non-degenerate noise, we show that there exists a unique geometric periodic measure for a broad range of time-periodic weakly dissipative SDEs, gradient SDEs and Langevin equations. Since the Markov process is time-inhomogeneous, in this thesis, we discuss convergence in the conventional sense, along multiples of the period and in the "pullback" sense. Pullback convergence is where one takes initial time further and further back in time rather than the forward time and is typically studied in the theory of non-autonomous dynamical systems [KR11] and RDS [CH16]. There has been some works in a similar direction for periodic measure. Invariant measures of the grid process on multiple integrals of the period were studied in [HL11] for time-periodic SDEs with (globally) Lipschitz coefficients. Since periodic measures were defined some years after

[HL11], the authors were not aware of periodic measures nor their ergodicity as introduced in [FZ16]. Furthermore, it seems that the Lipschitz coefficients are too restrictive for some applications. In [CLRS17], the authors obtained the existence and uniqueness of the periodic measure for stochastic overdamped Duffing Oscillator in one dimension via the theory of non-autonomous RDS. To the author's knowledge, this thesis contains the first proof to deduce further that this periodic measure is geometric. It is worth noting that the approach taken in this thesis applies in the multidimensional case and is completely different to that of [CLRS17].

We briefly discuss the approach taken in this thesis. The thesis extend classical time-homogeneous Markov chains results [MT09, MT92, MT93] to attain invariant measures to time-periodic Markovian systems to attain periodic measures. The crucial yet simple observation is that time-periodic Markov chain or process possesses a time-homogeneous subchain. We show that if the subchain has an invariant measure, one can construct the periodic measure. The underlying approach is utilising the existence of a Foster-Lyapunov function and the coupling method [Lin92, Tho00, MT92, MT09, MT93]. Foster-Lyapunov functions are utilised to ensure stability for the Markov process.

For successful coupling of Markov chains, it is typical to show the Markov transition probability satisfies the local Doeblin condition. We show that the local Doeblin condition can be essentially decomposed into irreducibility and the strong Feller property of the Markov transition probability. Indeed these two properties are key ingredients for the existence and uniqueness of invariant measures, even in the infinite dimensional case see e.g. [DZ96]. In the context of SDEs, it is well-known that the strong Feller property holds provided the coefficients are globally Hölder and bounded with uniformly elliptic diffusion [Fri64, SV06], however these conditions are too restrictive for applications from a SDE perspective. The celebrated Hörmander's condition is a weak condition to deduce the strong Feller property for autonomous SDE [Hör85, Mal78, Hai11, RW00]. In the recent paper [HLT17], the authors extended Hörmander's condition to sufficiently imply the strong Feller property holds for non-autonomous SDEs. The smooth SDE conditions of this thesis is to invoke the result of [HLT17] while flexible enough for applications.

Another contribution of this thesis is the rigorous derivation that the density of the periodic measure sufficiently and necessarily is the time-periodic solution of the Fokker-Planck equation. Attaining the periodic measure density evidently gives another approach to obtain the periodic measure. It is expected the density would be useful in applications much like stationary distributions has been in physical applications. As an example, an explicit formula will be given for the periodic measure and its density of Ornstein-Uhlenbeck processes with a periodic forcing.

Owing to the physical intuition of the Fokker-Planck equation, time-periodic solution of the Fokker-Planck has been studied previously [Jun89, CHLY17, JQSY19], however its relationship to periodic measure is formally established here. On a more fundamental level, in this thesis, the periodic measure density PDE establishes our first “time-periodic Feynman-Kac” duality. It appears this viewpoint was not taken in existing literature and was limited to the Fokker-Planck equation mentioned. Indeed the author expects using methods in this thesis for instance, further time-periodic Feynman-Kac dualities can be attained for other quantities associated to time-periodic SDEs. Conversely, the author expects this duality provides stochastic insight into existing time-periodic solutions of parabolic PDEs.

We promptly discuss our second novel time-periodic Feynman-Kac duality; we provide a rigorous derivation that the expected exit time of time-periodic SDEs is the time-periodic solution of a second-order linear parabolic PDE. To the author's knowledge, the derived PDE and its interpretation is novel in literature. Expected exit time is another key tool to understanding and approaching physical phenomena. Indeed in many disciplines of sciences, (expected) exit time of stochastic processes from domains is an important quantity to model the (expected) time for certain events to occur. For example, time for chemical reactions to occur [Kra40, Gar09, Zwa01], biological neurons to fire [RS79, Sat78], companies to default [BC76, BR04], ions crossing cell membranes in molecular biology [Bre04] are all broad applications of exit times. With the example of daily temperature and commodity price mentioned earlier, the average time it takes for the temperature or commodity price to reach particular threshold can be phrased as an expected exit time problem. For autonomous stochastic differential equations (SDEs), the expected exit time from a domain has been well-studied in existing literature. In particular, it is well-known that the expected exit time satisfies a second-order linear elliptic PDE [Has12, Gar09, Zwa01, Pav14, Ris96]. However, in existing literature, it appears that the expected exit time PDE is absent for non-autonomous SDEs and in particular time-periodic SDEs. The current thesis fills in this gap.

Via the Feynman-Kac duality, we discuss briefly the ill-posedness of the PDE for the general non-autonomous SDE case and thereby explaining its absence in literature.

We note that the rigorous derivation of the expected exit time PDE can be done under conditions even weaker than that needed for the existence and uniqueness of a limiting or geometric periodic measure. Therefore it applies to an even wider range of physical systems. We note that the conditions required to solve the PDE are weaker than that to derive the PDE from the SDE. This is expected because from a PDE perspective, weak solutions of PDE on bounded domains can often be attained requiring coefficients to only be L^p or Hölder, and classical solutions may be obtained via Sobolev embedding. On the other hand, as a priori, it is not known if the process would exit the bounded domain in finite time or indeed have finite expectation. By considering the SDE and its Markov transition probability on the entire unbounded domain, we show that if the exit time has finite second moment then the PDE derivation can be rigorously justified. We show that again irreducibility and the strong Feller property are the key ingredients to conclude the exit time has finite second moment. Foster-Lyapunov functions are not necessary in this problem.

We provide two complementary approaches to prove that the parabolic PDE has a unique solution in a weak and classical sense. In the proofs, we keep as much generality as convenient to show the main ingredients for the PDE's well-posedness and for straightforward application in the future for similar PDEs. In one approach, we show that the time-periodic solution can be casted as a fixed point of the parabolic PDE evolution operator after a period. We prove that if the associated bilinear form is coercive, then the time-periodic solution exists and is unique by Banach Fixed Point Theorem. As coercivity can be difficult to verify in practice, we also take a calculus of variations approach. Specifically, we cast the problem as a convex optimisation problem by defining a natural cost functional and show that a unique minimiser exists and satisfies the PDE.

We emphasise that while our core results are theoretical in nature, the Banach fixed point and convex optimisation approach can be readily implemented by standard numerical schemes. Acquiring the tools to numerically compute the expected exit time is vital because explicit or even approximate closed form formulas for the expected time are rarely known, even in the autonomous case. The known cases include (autonomous) one-dimensional gradient SDEs with additive noise, where the expected exit time can be expressed as a double integral [Gar09] and has an approximate closed form solution given by Kramers' time, when the noise is small [Kra40]. Kramers' time has since been extended to higher dimensional gradient SDEs [Ber11]. However, to our knowledge, there are currently no known exact formulae for the time-periodic case. Therefore, particularly for applications, there is an imperative to solving the PDE numerically.

It is worth mentioning the clear advantages to numerically computing the expected exit time by solving the PDE than direct Monte Carlo simulations. In simulations, one can be concerned with the quantity of simulations needed for a confident result. This contrasts with solving a PDE where its solution is deterministic and the result is as accurate as the numerical schemes and its parameters allow. As the Monte Carlo computational time is undoubtedly proportional to the exit time, in problems where the exit time are large e.g. stochastic resonance (see below), computational time can be large. Coupled with computing large number of simulations, solving the PDE can be significantly faster to compute.

Periodic measures and expected exit times together can provide a deeper insight and understanding of time-periodic systems such as "periodically forced" stochastic systems. These are systems in which the drift is perturbed by a periodic term. Periodically forced systems have a wide range of applications in the sciences, we refer readers to the monograph [Jun93] for examples and analysis on the such systems. Specifically, we apply the theory developed of periodic measures and expected exit times toward stochastic resonance, a physical phenomena typically modelled as a periodically forced system. We briefly describe this phenomena.

In a series of papers [BPSV81, BPSV82, BPSV83, Nic82], the paradigm of stochastic resonance was introduced to explain Earth's cyclical ice ages. In particular, the authors proposed a double-well potential SDE with periodic forcing to model scientific observation that Earth's ice age transitions from "cold" and "warm" climate occurs abruptly and almost regularly every 10^5 years. The wells model the two metastable states, states in which the process generally stays at for relatively large periods of time. We note that both periodic forcing and the noise are essential to this model in the following way. It has been observed (see e.g. [BPSV82]) that in the power spectrum of paleoclimatic variations in the last 700,000 years, there is a strong peak at a periodicity of 10^5 years and smaller peaks at periods of 2×10^4 and 4×10^2 years. The addition of the periodic forcing reproduced these peaks. From a physical viewpoint, as suggested by Milankovich [Mil30], the periodic forcing corresponds to the annual mean variation in insolation due to changes in ellipticity of the Earth's orbit. On the other hand, in the absence of noise (with or without the periodic forcing), the model does not produce any transitions between the stable states. The noise stimulates the global effect of relatively short-term fluctuations in the atmospheric and oceanic circulations on the long-term temperature behaviour [Has76]. Thus indeed both periodicity and noise are essential ingredients for modelling Earth's ice ages. Furthermore, it is the delicate balance between periodicity and noise level that collectively yield transitions between the metastable states to occur regularly in a periodic manner. Since the seminal papers, stochastic resonance has found applications in many other physical systems including optics, electronics, neuronal systems, quantum systems [GHJM98, JH07, ZMJ90, Jun93, HIP05]. The unique geometric periodic measures attained in this thesis give a rigorous description of the asymptotic periodic equilibria as observed by physicists. Furthermore, its uniqueness is significant in explaining the transition between the two wells; otherwise there should be two periodic measures instead of one.

It is noted that stochastic resonance occurs for the right set of parameters in the double well model, as suggested by numerical simulations [GHJM98, MW89, ANMS99, CLRS17]. Given the model and periodic forcing, noise intensity is the only free parameter. While there is no standard definition [JH07, HI05], stochastic resonance is said to occur if the expected transition time between the metastable states is (roughly) half the period [CLRS17]. Typical approaches to fine tune the noise intensity has been to maximise indicators such as spectral power amplification (SPA) coefficient and the signal-to-noise ratio (SNR) [GHJM98, HIP05] for an overview. The approach in this thesis is to solve the expected exit time PDE numerically. By applying the theory developed in this thesis, we first show that computationally solving the PDE and stochastic simulation for the expected transition time agrees. We then fine tune the noise intensity until the double-well stochastic model exhibits the stochastic resonance phenomena.

In existing stochastic resonance literature, Kramers' time is often used for analytic expressions. Note that Kramers' time applies only to autonomous gradient SDE case and in the small noise limit. For example in [MW89, CGM05] reduced the dynamics to "effective dynamics" two-state time-homogeneous Markov process and invoked a time-perturbed Kramers' time. More generally, utilising large deviation and specifically Wentzell–Freidlin theory [FW98], stochastic resonance and related estimates can be attained in the small noise limit. For example, [MS01] attained estimates for escape rates, a closely related quantity to expected transition time. Similarly, in [IP01] and [HI05, HIP05, HIPP14], the authors obtained estimates for the noise intensity for stochastic resonance by reducing to two-state Markov process and time-independent bounds respectively. In this thesis, we retain the explicit time-dependence of the coefficients and furthermore, small and large noise are permissible. In fact, the noise may even be state-dependent and exact exit duration is obtained.

2 Periodic Measures of Markovian Systems

2.1 Definitions and Preliminaries

In this section, we recall some basic definitions, notation and standard results of Markovian processes on locally compact separable metric space $(E, \mathcal{B})$ where $\mathcal{B}$ is the natural Borel σ-algebra and time indices $\mathbb{T} = \mathbb{N} := \{0, 1, ..., \}$ or $\mathbb{R}^+$. By convention, when $\mathbb{T} = \mathbb{N}$, the Markov process is referred to as a Markov chain. The objective of this section is to state important results for time-homogeneous Markov chains that would be crucial in proving vital results for T-periodic Markovian systems.

Let $P : \mathbb{T} \times \mathbb{T} \times E \times \mathcal{B} \to [0, 1]$ be a two-parameter Markov transition kernel. It is well-known P satisfies

(i) $P(s, t, x, \cdot)$ is a probability measure on $(E, \mathcal{B})$ for all $s \leq t$ and all $x \in E$.

(ii) $P(s, t, \cdot, B)$ is a $\mathcal{B}$-measurable function for all $s \leq t$ and $\Gamma \in \mathcal{B}$.

(iii) (Chapman-Kolmogorov) For all $s \leq r \leq t$, one has

$$P(s, t, x, \Gamma) = \int_E P(s, r, x, dy) P(r, t, y, \Gamma), \quad x \in E, \Gamma \in \mathcal{B}.$$

(iv) $P(s, s, x, B) = 1_\Gamma(x)$ for all $s \in \mathbb{T}$, $x \in E$ and $\Gamma \in \mathcal{B}$,

where 1_Γ denotes the indicator function.

For $s \leq t$, define linear operators $P(s, t)$ acting on $\mathcal{B}_b(E)$, the space of bounded measurable functions by

$$P(s, t) f(x) = \int_E f(y) P(s, t, x, dy), \quad f \in \mathcal{B}_b(E), x \in E.$$

We say that $P(\cdot, \cdot)$ is Feller if for all $s \leq t$, $P(s, t) f \in C_b(E)$ when $f \in C_b(E)$ and strong Feller if $P(s, t) f \in C_b(E)$ when $f \in \mathcal{B}_b(E)$. For $s \leq t$, we define adjoint operator $P^*(s, t)$ acting on $\mathcal{P}(E)$, the space of probability measures on $(E, \mathcal{B})$ by

$$\left(P^*(s, t)\mu\right)(\Gamma) = \int_E P(s, t, x, \Gamma) \mu(dx), \quad \mu \in \mathcal{P}(E), \Gamma \in \mathcal{B}.$$

Where not ambiguous, we write the left hand side as $P^*(s, t)\mu(\Gamma)$.

It is well-known that $P(s,t)$ and $P^*(s,t)$ forms a two-parameter semigroup on $\mathcal{B}_b(E)$ and $\mathcal{P}(E)$ respectively and satisfies $P(s,t) = P(s,r)P(r,t)$ and $P^*(s,t) = P^*(r,t)P^*(s,r)$. We endow on $\mathcal{P}(E)$ the total variation norm defined by

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{\Gamma\in\mathcal{B}} |\mu_1(\Gamma) - \mu_2(\Gamma)|, \quad \mu_1, \mu_2 \in \mathcal{P}(E).$$

It is easy to compute that the operator $P^*(s,t) : (\mathcal{P}(E), \|\cdot\|_{TV}) \to (\mathcal{P}(E), \|\cdot\|_{TV})$ has unit operator norm. This is clear because for any $\mu \in \mathcal{P}(E)$,

$$\|P^*(s,t)\mu\|_{TV} = \sup_{A\in\mathcal{B}} \int_E P(s,t,x,A)\mu(dx) = \int_E P(s,t,x,E)\mu(dx) = 1,$$

and since $\|\mu\|_{TV} = 1$, it follows that

$$\|P^*(s,t)\| = \sup_{\mu\in\mathcal{P}(E)} \frac{\|P^*(s,t)\mu\|_{TV}}{\|\mu\|_{TV}} = 1. \tag{2.1}$$

While many of the convergence results presented here holds in more general norms than the total variation norm (such as f-norms [MT09]). For clarity and simplicity, we shall only consider convergence in the total variation norm. Some results require only weak convergence of measures. Hence we occasionally consider $\mu \in \mathcal{P}(E)$ as a linear functional on $C_b(E)$ by

$$\mu(f) = \int_E f(x)\mu(dx), \quad f \in C_b(E).$$

And we say $\mu, \nu \in \mathcal{P}(E)$ are equal if $\mu(f) = \nu(f)$ for all $f \in C_b(E)$. Then for any $\mu \in \mathcal{P}(E)$, $f \in C_b(E)$ and $s \le t$, we have the following identity by Fubini's theorem,

$$\begin{aligned} P^*(s,t)\mu(f) &:= \int_E f(x)(P^*(s,t)\mu)(dx) \\ &= \int_E \int_E f(x)P(s,t,y,dx)\mu(dy) \\ &= \int_E P(s,t)f(y)\mu(dy) \\ &= \mu(P(s,t)f). \end{aligned} \tag{2.2}$$

We give the definition of a time-periodic Markov transition kernel. We also introduce the stronger definition of minimal time-periodic. Note that in the following definition, time-periodic Markov kernels depends on initial and terminal time.

Definition 2.1. The two-parameter Markov transition kernel $P(\cdot,\cdot,\cdot,\cdot)$ is said to be T-periodic for some $T > 0$ if

$$P(s,t,x,\cdot) = P(s+T,t+T,x,\cdot), \quad \text{for all } x \in E, s \leq t. \tag{2.3}$$

Moreover, we say P is minimal T-periodic if for every $\delta \in (0,T) \cap \mathbb{T}$

$$P(s,t,x,\cdot) \neq P(s+\delta,t+\delta,x,\cdot), \quad \text{for all } x \in E, s \leq t. \tag{2.4}$$

Lastly, we say P is time-homogeneous if

$$P(s,t,x,\cdot) = P(0,t-s,x,\cdot), \quad \text{for all } x \in E, s \leq t.$$

The definition of T-periodic should be clear and intuitive. Observe that the minimal T-periodic assumption is stronger than T-periodic. It rules out the possibility of being time-homogeneous by enforcing non-trivial period for every state. Equation (2.3) on the other hand allows the possibility for states to have trivial period. This implies that the results of this thesis assuming T-periodic P recovers results for the usual time-homogeneous case. We shall always assume $T > 0$.

As a convention, we denote by $P(t)$ for the time-homogeneous Markov semigroup and $P^*(t)$ for its adjoint depending only on the elapsed time $0 \leq t \in \mathbb{T}$. Specifically for $\mathbb{T} = \mathbb{N}$, we denote $P := P(1)$ and $P^* := P^*(1)$ for the "one-step" semigroup and adjoint semigroup respectively. We now define our central objects of study characterising stationary and periodic behaviour.

Definition 2.2. A probability measure $\pi \in \mathcal{P}(E)$ is called an invariant (probability) measure with respect to $P(s,t)$ if

$$P^*(s,t)\pi = \pi \quad \text{for all } 0 \leq s \leq t.$$

When P is time-homogeneous, π satisfies $P^*(t)\pi = \pi$ for all $t \geq 0$. In particular, when $\mathbb{T} = \mathbb{N}$, π need only to satisfy the one-step relation $P^*\pi = \pi$.

Invariant measures has been well-studied for many decades in many general settings. For example time-homogeneous Markov chains on finite dimensional state space [MT92, Nor98, MT09], and Markov processes on finite state space [Str05, Nor98], on infinite dimensional state spaces [DZ96]. On the other hand, the formulation of periodic measure below is new and was first formally defined in [FZ16].

Definition 2.3. A measure-valued function $\rho : \mathbb{T} \to \mathcal{P}(E)$ is called a T-periodic (probability) measure with respect to $P(\cdot,\cdot)$ if for all $0 \leq s \leq t$

$$\begin{cases} \rho_{s+T} = \rho_s, \\ \rho_t = P^*(s,t)\rho_s. \end{cases}$$

Similarly if P is time-homogeneous, then ρ satisfies

$$\begin{cases} \rho_{s+T} = \rho_s, \\ \rho_{s+t} = P^*(t)\rho_s. \end{cases}$$

Note that periodic measures are invariant measures when the period is trivial. We shall give sufficient conditions to ensure the periodic measure has a minimal positive period.

In classic literature, see [DZ96, Has12, MT92, MT09, Str05, Nor98] for instance, appropriate assumptions yields asymptotic convergence of the Markov kernel towards a unique invariant measure. However, these classical asymptotic results seems to have neglected the possibility of asymptotically periodic behaviour. While conceptually simple, it seems that asymptotic stochastic periodic behaviour was first rigorously formalised and defined by periodic measures by Feng and Zhao in [FZ16]. Nonetheless, these limiting invariant measures results can still be utilised for time-periodic Markovian system. We end this section by quoting, without proof, two now-classical results for time-homogeneous Markov chain result taken as special cases from [MT92, MT09]. To state the results, we require the following definitions.

Definition 2.4. Let P be a one-step time-homogeneous Markov transition kernel. We say that P satisfies the "minorisation" or "local Doeblin" condition if there exists a non-empty measurable set $K \in \mathcal{B}$, constant $\eta \in (0, 1]$ and probability measure φ such that

$$P(x,\cdot) \geq \eta\varphi(\cdot), \quad x \in K. \tag{2.5}$$

Definition 2.5. A function $U : E \to \mathbb{R}^+$ is said to be norm-like (or coercive) $U(x) \to \infty$ as $\|x\| \to \infty$ i.e. the level-sets $\{x \in E | V(s,x) \leq r\}$ are pre-compact for each $r > 0$. A time-dependent function $V : \mathbb{T} \times E \to \mathbb{R}^+$ is norm-like (or coercive) if $V(s,\cdot)$ is norm-like for every fixed $s \in \mathbb{T}$.

Lemma 2.6. *(Theorem 4.6 [MT92]) Let P be a one-step time-homogeneous Markov transition kernel and assume there exists a norm-like function $U : E \to \mathbb{R}^+$, a compact set $K \in \mathcal{B}$ and $\epsilon > 0$ such that*

$$PU - U \leq -\epsilon \quad \text{on } K^c, \tag{2.6}$$

$$PU < \infty \quad \text{on } K. \tag{2.7}$$

Then there exists a unique invariant measure π with respect to P. Moreover if P satisfies the local Doeblin condition (2.5) then the invariant measure is limiting i.e. for any $x \in E$,

$$\|P^n(x,\cdot) - \pi\|_{TV} \to 0, \quad \text{as } n \to \infty.$$

In literature, conditions (2.6) and (2.7) are typically referred to as the Foster-Lyapunov drift criteria and has the interpretation that the process moves inwards on average when outside the compact set. And U is referred as (Foster-)Lyapunov function. Lemma 2.6 is a qualitative result and does not give any rate of convergence. The following result from [MT92, MT09] gives sufficient condition for a time-homogeneous Markov chain to possess a unique invariant measure that converges geometrically.

Lemma 2.7. *(Theorem 6.3 [MT92], Theorem 15.0.1 [MT09]) Let P be a one-step time-homogeneous Markovian transition kernel satisfying (2.5). Assume there exists a norm-like function $U : E \to \mathbb{R}^+$, $\alpha \in (0,1)$ and $\beta > 0$ such that*

$$PU \leq \alpha U + \beta \quad \text{on } E. \tag{2.8}$$

Then there exists a unique geometric invariant measure $\pi \in \mathcal{P}(E)$ i.e. there *exist constants* $0 < R < \infty$ *and* $r \in (0,1)$ *such that*

$$\|P^n(x,\cdot) - \pi\|_{TV} \leq R(U(x)+1)r^n, \quad x \in E, n \in \mathbb{N}.$$

2.2 Time-Homogeneous Markovian Systems

In classic literature, results such as Lemma 2.6 and Lemma 2.7 are concerned with asymptotic convergence of the Markov kernel towards a unique invariant measure. However, these classical asymptotic results are “all-or-nothing” in that either there exists an limiting invariant measure or not and often neglected the possibility of asymptotically periodic behaviour. While conceptually simple, it seems that this was only formally pointed by Feng and Zhao in [FZ16] and formalised under the definition of periodic measures.

In this brief section, we study periodic measures in the context of time-homogeneous Markovian systems on a locally compact metric space E with sigma-algebra $\mathcal{B}$. For clarity, we mostly look at the Markov chain case. While many of the statements in this section are known in the context of RDS, the purpose is to show the differences with the time-periodic Markovian case. We shall see that periodic measures are only unique up to cyclic sets or Poincaré sections [FZ16]. As these sections are disjoint, the implication is that the support of the periodic measures at different points in time are disjoint. Also in the case of time-homogeneous Markovian systems, periodic measures and invariant measures co-exist. In fact, the time-average of the periodic measure always yields an invariant measure. We will see the situation differs significantly in the time-periodic case where periodic measures and invariant measures may be mutually exclusive and furthermore, the supports of periodic measures at different moments in time are generally not disjoint.

For clarity in portraying the main points of this section, we focus on time-homogeneous Markov chains i.e. $\mathbb{T} = \mathbb{N}$ and assume that the period $1 < T \in \mathbb{N}$. Analogous statements and proofs for Markov processes can be readily attained by replacing sums with integrals in many proofs.

We begin with a easy lemma showing that in this time-homogeneous case the time-average of the periodic measure is an invariant measure. Thus periodic measures and invariant measures always coexist. This is already known even in the context of Markovian cocycle in the theory of RDS, see Theroem 1.2.6 [FZ16].

Lemma 2.8. *Suppose there exists a T-periodic measure ρ with respect to time-homogeneous Markov transition kernel P. Then the time-average measure $\pi = \frac{1}{T}\sum_{n=0}^{T-1}\rho_n$ is an invariant measure with respect to P.*

Proof. By the linearity of P^* and definition of periodic measure,

$$P^*\pi = P^*(\frac{1}{T}\sum_{n=0}^{T-1}\rho_n) = \frac{1}{T}\sum_{n=0}^{T-1}P^*\rho_n = \frac{1}{T}\sum_{n=1}^{T}\rho_n = \pi.$$

□

For any $\alpha = (\alpha_0, ..., \alpha_{T-1}) \in \mathbb{R}^T$ and T-periodic measure ρ, define the linear combination $\alpha \cdot \rho := \sum_{n=0}^{T-1}\alpha_n\rho_n$. With this notation, we have the follow non-uniquness lemma of periodic measures.

Lemma 2.9. *Let P be a time-homogeneous Markov transition kernel. If there exists a T-periodic measure with respect to P, then there are infinitely many T-periodic measures with respect to P.*

Proof. Let $\rho = (\rho_n)_{n\geq 0}$ be a T-periodic measure then for any $\alpha \in \mathbb{R}^T$ with $\alpha_n \geq 0$ and $\sum_{i=1}^{T}\alpha_n = 1$, define the measure

$$\tilde{\rho}_0 = \alpha \cdot \rho.$$

It is easy to see that $\tilde{\rho}_0 \in \mathcal{P}(E)$. Define recursively

$$\tilde{\rho}_{n+1} = P^*\tilde{\rho}_n, \quad n \geq 1.$$

Then by the semigroup property,

$$\tilde{\rho}_n = P^*(n)\rho_0 = P^*(n)\sum_{i=1}^{T}\alpha_i\rho_i = \sum_{i=1}^{T}\alpha_i P^*(n)\rho_i = \sum_{i=1}^{T}\alpha_i\rho_{i+n}. \tag{2.9}$$

We verify that $\tilde{\rho} = (\tilde{\rho}_n)_{n\in\mathbb{N}}$ is a periodic measure. From (2.9) and the T-periodicity of ρ, we see that

$$\tilde{\rho}_{n+T} = \sum_{i=1}^{T}\alpha_i\rho_{i+n+T} = \sum_{i=1}^{T}\alpha_i\rho_{i+n} = \tilde{\rho}_n,$$

is also T-periodic. Finally, as ρ is a T-periodic measure, it is trivial to see for any $n, m \in \mathbb{N}$

$$\tilde{\rho}_{n+m} = \sum_{i=1}^{T} \alpha_i \rho_{i+n+m} = \sum_{i=1}^{T} \alpha_i P^*(m)\rho_{i+n} = P^*(m) \sum_{i=1}^{T} \alpha_i \rho_{i+n} = P^*(m)\tilde{\rho}_n.$$

i.e. $\tilde{\rho}$ is a periodic measure. As there are infinitely many such α, the result follows. □

It is easy to show that the time-average of $\alpha \cdot \rho$ is independent of α i.e. the constructed periodic measures by linear combinations does not yield a new unique invariant measure. We note that for Markov process, the above lemma can be done by replacing the sums with integrals and let α be any non-negative piecewise continuous function such that $\int_0^T \alpha(r)dr = 1$.

To discuss uniqueness of periodic measure, we define T-cyclic sets or Poincaré sections [FZ16] as follows.

Definition 2.10. The measurable sets $\{L_n\}_{n=0}^{T-1}$ are called T-cyclic sets or Poincaré sections of T-periodic Markov kernel P if they have the following properties:

(i) $E = \cup_{n=0}^{T-1} L_n$,

(ii) $L_n = L_{n+T}$ for all $n \in \mathbb{N}$,

(iii) $L_n \cap L_m = \emptyset$ if $n \neq m$.

(iv) For any $n, m \in \mathbb{N}$, $P(m, x, L_{n+m}) = 1$ for any $x \in L_n$.

The nomenclature Poincaré sections is derived from the fact $P(T) : L_n \to L_n$ for any $0 \leq n \leq T-1$ reflecting its classical deterministic counterpart in dynamical systems. We say that a periodic measure ρ is supported by Poincaré sections if

$$\mathrm{supp}(\rho_n) \subseteq L_n, \tag{2.10}$$

i.e. at any time the periodic measure is supported by exactly one Poincaré section. Following [FZ16], P is said to satisfy the kT-irreducibility condition Poincaré section if for any fixed $0 \leq n \leq T-1$, there exists $k_i \in \mathbb{N}$ such for any non-empty relatively open $\Gamma \in L_i$, we have that $P(k_i, x, \Gamma) > 0$, for $x \in L_i$. Lemma 1.3.10 of [FZ16] showed that periodic measures (if exist) are unique up to Poincaré sections and time-shift i.e. allowing for the possibility that there exists a $j \in \mathbb{N}$, $\mathrm{supp}(\rho_n) \subseteq L_{n+j}$.

For concreteness, we take E to be a finite state space and enumerate its states. Then we can represent the Markov transition kernel P by a $|E| \times |E|$ matrix. In this finite case, $\lambda \in \mathcal{P}(E) \cong \mathbb{R}^{|E|}$ is represented by a row vector of length $|E|$. Adopting conventional classical Markov chain notation, we have $P^*(n)\lambda = \lambda P^n$. We consider Example 1.1.6 from [FZ16], where there was a detailed analysis of random periodic path and from which a periodic measure was constructed.

Example 2.11. Let $E = \{1, 2, 3\}$ and consider Markov transition probability matrix

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that P is irreducible and is 2-periodic since $T(i) = \gcd(\{n \in \mathbb{N} | P_{ii}^n > 0\}) = 2$ for all $i \in E$. It is easy to compute that this example has the unique cyclic sets $L_0 = \{1\}$ and $L_1 = \{2, 3\}$. It is to verify that $\rho = (\rho_n)_{n \geq 0}$ defined by

$$\rho_n = \begin{cases} (1, 0, 0) & n \quad \text{even}, \\ (0, 0.5, 0.5) & n \quad \text{odd}, \end{cases}$$

is a periodic measure for the system. Indeed, identifying by index $\{0, 1\}$, it is easy to see $\rho_0 P = \rho_1$ and $\rho_1 P = \rho_0$ holds. Hence ρ is the unique periodic measure with the support in the Poincaré sections. Uniqueness follows from considering the subsystem via classical Markov chain theory taking $T = 2$ steps. Relaxing the support condition, periodic measures are not unique - indeed for any $\alpha \in (0, 1)$, by Lemma 2.9 it is easy to see

$$\begin{cases} \tilde{\rho}_n = \alpha \rho_0 + (1 - \alpha)\rho_1 & n \quad \text{even}, \\ \tilde{\rho}_n = \alpha \rho_1 + (1 - \alpha)\rho_0 & n \quad \text{odd}, \end{cases}$$

is another periodic measure. We see that the time-average of the periodic measure (for any α), $\pi = (0.5, 0.25, 0.25) \in \mathcal{P}(E)$ is the unique invariant measure with respect to P. Note that π is not limiting for any arbitrary initial distribution.

For time-homogeneous Markov chains, it is well-known that non-uniqueness of invariant measures may hold if we relax irreducibility. This remains true for periodic measures even given Poincaré sections. We demonstrate this with the following example.

Example 2.12. Let $E = \{1, 2, 3, 4, 5\}$ and consider the Markov transition probability matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Observe that $\{1, 2\}$ and $\{3, 4, 5\}$ forms a closed subsystem that are disjoint from each other hence P fails to be irreducible (and kT-irreducible). It is easy to see that

$$\begin{aligned} \pi^1 &= (0.5, 0.5, 0, 0, 0), \\ \pi^2 &= (0, 0, 0.5, 0.25, 0.25), \end{aligned}$$

are two invariant measures. Hence π^i is also a periodic measure with a trivial period. Since this example has Example 2.11 as a sub-system, it is clear that

$$\rho_n = \begin{cases} (0, 0, 1, 0, 0) & n \quad \text{even}, \\ (0, 0, 0, 0.5, 0.5) & n \quad \text{odd}, \end{cases}$$

is a 2-periodic measure for the system. Therefore, indeed we have non-uniqueness (up to Poincaré sections) of periodic measures.

2.3 Time-Periodic Markovian Systems

2.3.1 Existence and Uniqueness of Periodic Measures

The aim of this section is to give novel results to establish the existence and uniqueness of a periodic measure in the general setting of time-periodic Markovian systems on locally compact metric spaces E. We will give sufficient conditions in which a periodic measure to have a minimal positive period (hence not an invariant measure) for T-periodic Markov processes on Euclidean space. We start with the following basic existence and uniqueness lemma.

Lemma 2.13. *Let P be a two-parameter T-periodic Markov transition kernel. Assume for some fixed $s_* \in \mathbb{T}$, there exists an invariant measure ρ_{s_*} with respect to the one-step Markov transition kernel $P(s_*, s_* + T)$. Then there exists a T-periodic measure ρ with respect to $P(\cdot, \cdot)$. If ρ_{s_*} is unique then ρ is also unique.*

Proof. Given ρ_{s_*}, define the following measures

$$\rho_s := P^*(s_*, s)\rho_{s_*}, \quad s \geq s_*. \tag{2.11}$$

Extend ρ_s by periodicity for $0 \leq s \leq s_*$. Then it is clear that $\rho : \mathbb{T} \to \mathcal{P}(E)$. Furthermore, we show $\rho = (\rho_s)_{s \in \mathbb{T}}$ is a periodic measure with respect to $P(\cdot, \cdot)$. Since $P(\cdot, \cdot)$ is T-periodic then so is $P^*(\cdot, \cdot)$. Due to the periodicity, without any loss of generality, we prove only for $s \geq s_*$. By the semigroup property and the invariance of ρ_{s_*} with respect to $P(s_*, s_* + T)$, we have

$$\begin{aligned}
\rho_{s+T} &= P^*(s_*, s+T)\rho_{s_*} \\
&= P^*(s_* + T, s+T)P^*(s_*, s_* + T)\rho_{s_*} \\
&= P^*(s_*, s)\rho_{s_*} \\
&=: \rho_s.
\end{aligned}$$

By construction (2.11), for any $s \geq s_*$ and for any $t \in \mathbb{T}$

$$\begin{aligned}
\rho_{s+t} &:= P^*(s_*, s+t)\rho_{s_*} \\
&= P^*(s, s+t)P^*(s_*, s)\rho_{s_*} \\
&= P^*(s, s+t)\rho_s.
\end{aligned}$$

This concludes ρ is a periodic measure with respect to $P(\cdot, \cdot)$. If ρ_{s_*} is the unique invariant measure with respect to $P(s_*, s_* + T)$, we prove ρ is also unique. Suppose there are two T-periodic measures $\rho^i = (\rho^i_s)_{s \in \mathbb{T}}$ with respect to $P(\cdot, \cdot)$ for $i = 1, 2$. By definition of periodic measures, ρ^i_s satisfies $\rho^i_s = P^*(s_*, s)\rho^i_{s_*}$ for any $s \geq s_*$, hence by the linearity of P^* and (2.1)

$$\begin{aligned}
\|\rho^1_s - \rho^2_s\|_{TV} &= \|P^*(s_*, s)(\rho^1_{s_*} - \rho^2_{s_*})\|_{TV} \\
&\leq \|P^*(s_*, s)\| \|\rho^1_{s_*} - \rho^2_{s_*}\|_{TV}.
\end{aligned}$$

The result follows by the assumption of uniqueness i.e. $\rho^1_{s_*} = \rho^2_{s_*}$. □

Recall that, by definition, an invariant measure is always a periodic measure with a trivial period. For applications, it is important to distinguish periodic measures of minimal positive period and those of a trivial period. However, it is not immediate whether the periodic measure constructed in Lemma 2.13 has a trivial period. The distinction can be subtle since T-periodic Markovian transition kernel (that is not time-homogeneous) does not necessarily yield a periodic measure with a non-trivial period.

We demonstrate this with a SDE example. In the example, we consider T-periodic coefficients hence it is straightforward to see that the Markovian transition kernel is T-periodic. Let W_t be a one-dimensional Brownian motion and S is a continuously differentiable T-periodic function and consider the following SDE

$$dX_t = (-\alpha X_t^3 + S(t)X_t)dt + \sigma dW_t, \quad \alpha > 0, \sigma \neq 0.$$

We will see that the results from Section 3.3 will yield the existence and uniqueness of a periodic measure with a minimal positive period when S has a minimal period $T > 0$. On the other hand, it is clear that the same SDE with multiplicative linear noise,

$$dX_t = (-\alpha X_t^3 + S(t)X_t)dt + X_t dW_t, \quad \alpha > 0, \tag{2.12}$$

has δ_0 (Dirac mass at the origin) as its invariant measure and hence is a periodic measure with a trivial period.

For the time-homogeneous Markovian case, Lemma 2.8 yields that time-average of a periodic measure always yields an invariant measure. In this time-inhomogeneous case, it is not necessarily the case that the time-average yields an invariant measure. W do note however, that one can obtain an invariant measure by lifting the periodic measure on a cylinder and considering its average over one period [FZ16]. We demonstrate that time-average is not an invariant measure by the following simple example.

Example 2.14. Let $E = \{1, 2\}$ and P be a Markov transition matrix given by

$$P(n,n+1) = \begin{cases} \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} & \text{odd } n, \\ \\ \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix} & \text{even } n. \end{cases}$$

Then the measure $(\rho_n)_{n\in\mathbb{N}}$ defined by

$$\rho_n = \begin{cases} (0.25, 0.75) & \text{odd } n, \\ (0.5, 0.5) & \text{even } n, \end{cases}$$

is the unique 2-periodic measure with respect to P. For $n \in \mathbb{N}$, it is indeed easy to verify

$$P^*(n,n+1)\rho_n = \rho_{n+1}, \quad n \in \mathbb{N}.$$

Hence showing ρ is a periodic measure with respect to $P(\cdot,\cdot)$. However, there does not exist an invariant measure with respect to $P(\cdot,\cdot)$. Indeed, if there was an invariant measure π then it must simultaneously satisfy

$$\pi = P^*(n,n+1)\pi \quad \text{and} \quad \pi = P^*(n+1,n+2)\pi.$$

Equating implies $\left(P^*(n,n+1) - P^*(n+1,n+2)\right)\pi = \pm\pi \begin{pmatrix} 0.3 & -0.3 \\ 0.2 & -0.2 \end{pmatrix} = 0$. The only non-negative solution is $\pi = 0 \notin \mathcal{P}(E)$. Hence there does not exist an invariant probability measure with respect to $P(\cdot,\cdot)$. In particular, the time-average of ρ does not yield an invariant measure.

Summarising briefly, we have demonstrated that in this time-inhomogeneous T-periodic case, periodic measures may or may not have a trivial period. Unlike the time-homogeneous case where periodic measure implies the existence of an invariant measure, periodic measures and invariant measures may be mutual exclusive. Therefore, since we are interested in asymptotic periodic behaviour, we wish to ensure this mutually exclusivity. If we exclude the possibility of invariant measures and deduce that periodic measures (if any) have a minimal period and is limiting, then we can conclude that the long term behaviour can be characterised by strictly periodic behaviour (rather than trivial period). The following proposition gives sufficient condition in which an invariant measure cannot exist, thus if there exists any periodic measure, it has a minimal positive period. We only state and prove it for Euclidean state space, it will be apparent that it can hold in more general topological spaces.

Proposition 2.15. *Let $T > 0$ and $E = \mathbb{R}^d$ and assume P is minimal T-periodic and strong Feller. Then periodic measures with respect to P (if exists) has a minimal positive period.*

Proof. We prove by contradiction and assume there exists periodic measure with a trivial period i.e. there exists an invariant measure $\pi \in \mathcal{P}(\mathbb{R}^d)$. Then it must be that for all fixed $\delta \in (0, T) \cap \mathbb{T}$

$$P^*(s,t)\pi = \pi = P^*(s+\delta, t+\delta)\pi.$$

By duality, linearity and (2.2), this is equivalent to

$$\pi(P(s,t)f - P(s+\delta, t+\delta)f) = 0, \quad \text{for all } f \in C_b(\mathbb{R}^d). \tag{2.13}$$

To prove the result, we construct an $f \in C_b(\mathbb{R}^d)$ such that

$$\pi\left(P(s,t)f - P(s+\delta, t+\delta)f\right) > 0.$$

By minimal T-periodic assumption (2.4), for every fixed $x \in \mathbb{R}^d$, there exists $f_x \in C_b(\mathbb{R}^d)$ such that $P(s,t,x,\cdot)(f_x) \neq P(s+\delta, t+\delta, x, \cdot)(f_x)$. Hence, without loss of generality, there exists $\epsilon > 0$ such that

$$P(s,t)f_x(x) - P(s+\delta, t+\delta)f_x(x) > 2\epsilon.$$

For $x = (x_1, .., x_d) \in \mathbb{R}^d$, define the half-open cube of length $r > 0$ centred at x by

$$C(x,r) := \prod_{i=1}^{d} [x_i - r, x_i + r) \in \mathcal{B}(\mathbb{R}^d),$$

where $\prod$ denotes the standard Cartesian product of Euclidean space. The (not necessarily strong) Feller assumption yields that $P(s,t)f_x \in C_b(\mathbb{R}^d)$ and $P(s+\delta, t+\delta)f_x \in C_b(\mathbb{R}^d)$ hence there exists a $r_x > 0$ such that

$$P(s,t)f_x(y) - P(s+\delta, t+\delta)f_x(y) > \epsilon, \quad \text{for all } y \in C(x, r_x).$$

Let $r := \min_x r_x > 0$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}^d$ and define the half-open cube

$$C_\alpha = \prod_{i=1}^{d} [\alpha_i r, (\alpha_i + 1)r) \in \mathcal{B}(\mathbb{R}^d).$$

Clearly $\{C_\alpha\}_{\alpha\in\mathbb{Z}^d}$ is a countable disjoint covering of $\mathbb{R}^d$. For every $\alpha \in \mathbb{Z}^d$, define $x_\alpha \in \mathbb{R}^d$ with components $x_i = (\alpha_i + 0.5)\, r$ for each $1 \leq i \leq d$ i.e. the midpoint of the cube and define the functions

$$f_\alpha := f_{x_\alpha}, \quad \alpha \in \mathbb{Z}^d.$$

And define the piecewise continuous functions over all possible tuples

$$f(x) := \begin{cases} f_{(1,0,...,0)}(x) & \text{if } x \in C_{(1,0,...,0)}, \\ f_{(0,1,...,0)}(x) & \text{if } x \in C_{(0,1,...,0)}, \\ \vdots & \vdots \\ f_\alpha(x) & \text{if } x \in C_\alpha. \end{cases}$$

By construction, $f \in \mathcal{B}_b(\mathbb{R}^d)$. Let $g := P(s,t)f - P(s+\delta, t+\delta)f$. Then $g > \epsilon$. The strong Feller assumption implies that $g \in C_b(\mathbb{R}^d)$. Hence for any $\pi \in \mathcal{P}(\mathbb{R}^d)$ (hence non-empty support), it follows that $\pi(g) > 0$ contradicting (2.13). □

2.3.2 Limiting and Geometric Ergodicity of Periodic Measures

We now discuss ergodicity of time-periodic Markovian systems. Classically, ergodic (time-homogeneous) Markov processes have the property that the Markov transition kernel converges to an invariant measure as time tends to infinity. In this sense, the invariant measure characterises the long-time behaviour of the system. On the other hand, for periodic measure (with a minimal positive period) cannot be limiting in the same way because the periodic measures evolves over time. However, it is possible that the Markov transition kernel can converge along integral multiples for T-periodic Markovian processes. This captures the idea that the periodic measure describes long-time periodic behaviour of the system. This shall be apparent and rigorously written in the forthcoming theorem. We remark that the forthcoming theorem can be regarded as the time-periodic generalisation of Lemma 2.6.

Before we state and prove the theorem, we make the following trivial but important observation. If $(X_t)_{t\in\mathbb{T}}$ is a T-periodic Markov chain or process, then $(Z_n^s)_{n\in\mathbb{N}} := (X_{s+nT})_{n\in\mathbb{N}}$ is a time-homogeneous Markov chain. This enables the usage of classical time-homogeneous Markov chain theory that is already well-established.

Theorem 2.16. *Let P be a T-periodic Markov transition kernel. Assume there exists $s_* \in \mathbb{T}$, norm-like function $U_{s_*} : E \to \mathbb{R}^+$, a non-empty compact set $K \in \mathcal{B}$, $\epsilon > 0$, $\eta_{s_*} \in (0,1]$ and $\varphi_{s_*} \in \mathcal{P}(E)$ such that*

$$P(s_*, s_*+T)U_{s_*} - U_{s_*} \leq -\epsilon \quad \text{on } K^c, \tag{2.14}$$

$$P(s_*, s_*+T)U_{s_*} < \infty \quad \text{on } K, \tag{2.15}$$

$$P(s_*, s_*+T, x, \cdot) \geq \eta_{s_*}\varphi_{s_*}(\cdot), \quad x \in K. \tag{2.16}$$

i.e. (2.5), (2.6) and (2.7) are satisfied for $P(s_*, s_*+T)$. Then there exists a unique T-periodic measure ρ that satisfies all the convergences below:

(i) For any fixed $x \in E$ and $s \in \mathbb{T}$

$$\|P(s, s+nT, x, \cdot) - \rho_s\|_{TV} \to 0, \quad \text{as } n \to \infty. \tag{2.17}$$

(ii) For any fixed $x \in E$ and $s \in \mathbb{T}$, the following "moving" convergence holds,

$$\|P(s, t, x, \cdot) - \rho_t\|_{TV} = 0, \quad \text{as } t \to \infty. \tag{2.18}$$

(iii) Allowing for negative initial time, for any fixed $x \in E$, $s, t \in \mathbb{T}$, the following pullback convergence holds

$$\|P(s-nT, t, x, \cdot) - \rho_t\|_{TV} = 0 \quad \text{as } n \to \infty. \tag{2.19}$$

Proof. Since $P(s, s+T, \cdot, \cdot)$ is a one-step time-homogeneous Markov kernel for all $s \in \mathbb{T}$, by Lemma 2.6, there exists a unique $\rho_{s_*} \in \mathcal{P}(E)$ with respect to $P(s_*, s_*+T)$. Moreover, by Lemma 2.13, there exists a unique periodic measure ρ. To show the convergences, we show that $P(s, s+T)$ satisfies (2.5), (2.6) and (2.7) for all $s \in \mathbb{T}$. By T-periodicity of P and the semigroup properties of

P, observe that for $s \le s_*$

$$\begin{aligned} P(s,s_*)P(s_*,s_*+T) &= P(s,s_*)P(s_*,s+T)P(s+T,s_*+T) \\ &= P(s,s+T)P(s,s_*). \end{aligned}$$

Hence applying $P(s,s_*)$ to both sides of (2.14) yields

$$P(s,s+T)P(s,s_*)U_{s_*} - P(s,s_*)U_{s_*} < -\epsilon, \quad \text{on } K^c.$$

i.e. $U_s := P(s,s_*)U_{s_*}$ satisfies (2.6) with respect to $P(s,s+T)$. Analogously, U_s satisfies (2.7). It is easy to verify that $U_s \ge 0$. We extend U_s for all $s \in \mathbb{T}$ by periodicity. We claim that for any $s \ge s_*$, $\eta_s := \eta_{s_*} \in (0,1]$ and $\varphi_s := P^*(s_*,s)\varphi_{s_*} \in \mathcal{P}(E)$ satisfies

$$P(s,s+T,x,\cdot) \ge \eta_s\varphi_s(\cdot), \quad x \in K. \tag{2.20}$$

i.e. $P(s,s+T)$ satisfies (2.5). Should this not be the case, then there exists some $x \in K$ and $\Gamma \in \mathcal{B}$ such that $P(s,s+T,x,\Gamma) < \eta_s\varphi_s(\Gamma)$. Then

$$P(s_*,s)P(s,s+T,x,\Gamma) = P(s_*,s+T,x,\Gamma) < \eta_s\varphi_s(\Gamma),$$

by applying $P(s_*,s)$ to both sides and Chapman-Kolmogorov equation. However by assumption (2.16),

$$\begin{aligned} \eta_s\varphi_s(\Gamma) &> P(s_*,s+T,x,\Gamma) \\ &= P^*(s_*+T,s+T)P(s_*,s_*+T,x,\Gamma) \\ &= \eta_{s_*}P^*(s_*,s)\varphi_{s_*}(\Gamma), \end{aligned}$$

which is a contradiction. We again extend by periodicity for all $s \in \mathbb{T}$. Thus, the assumptions of Lemma 2.6 are satisfied to deduce (2.17) for all $s \in \mathbb{T}$. By (2.1), observe that for $t \ge s+nT$,

$$\begin{aligned} \|P(s,t,x,\cdot) - \rho_t\|_{TV} &= \|P^*(s+nT,t)P(s,s+nT,x,\cdot) - P^*(s+nT,t)\rho_{s+nT}\|_{TV} \\ &= \|P^*(s+nT,t)P(s,s+nT,x,\cdot) - P^*(s+nT,t)\rho_s\|_{TV} \\ &\le \|P(s,s+nT,x,\cdot) - \rho_s\|_{TV}. \end{aligned}$$

Hence (2.18) follows by (2.17), by taking $t \to \infty$ followed by $n \to \infty$. Using

(2.17), convergence (2.19) holds due to

$$\begin{aligned}P(s-nT,t,x,\cdot) &= P(s,t+nT,x,\cdot)\\ &= P^*(s+nT,t+nT)P(s,s+nT,x,\cdot)\\ &= P^*(s,t)P(s,s+nT,x,\cdot).\end{aligned}$$

□

We elaborate on the convergences given in Theorem 2.16. The first convergence (2.17) is clear where the convergence is along integral multiples of the period towards a fixed measure. That is, ergodicity of the grid chain. Convergence (2.18) extends (2.17) by allowing the convergence to be taken continuously in time. Observe that (2.18) captures the idea that long-term behaviour is characterised by the periodic measure. Note that (2.18) convergence is towards a "moving target" as the periodic measure evolves over time. It is typical in the theory of non-autonomous dynamical systems [KR11] and RDS (random dynamical systems) [CH16] to study "pullback" convergence. This is convergence where one takes initial time further and further back in time rather than the forward time. The advantage is that the convergence will be to a fixed target rather than a moving one. This is the content of convergence (2.19). In general, (forward) convergence and pullback convergence do not coincide (see [KR11, CH16] for examples). In this T-periodic case, we see that the convergences coincide.

Remark 2.17. In the time-homogeneous case, Lemma 2.9 show existence of one periodic measure is sufficient to deduce the existence of infinitely many periodic measures by linear combination; the uniqueness is up to Poincaré sections and time-shift. This contrasts with Theorem 2.16, where in the time-inhomogeneous T-periodic case, periodic measures are unique in the entirely in $\mathcal{P}(E)$ and in particular not necessarily supported by Poincaré sections i.e. (2.10) generally does not hold.

Further to Theorem 2.16, we now provide a theorem for the existence and uniqueness of a geometric periodic measure. Observe in the theorem that the geometric convergence intrinsically depends on the initial time and state. This is akin to the autonomous case where the convergence depends on initial state.

Theorem 2.18. *Let P be a T-periodic Markov transition kernel. Assume there exists $s_* \in \mathbb{T}$, a norm-like function $U_{s_*} : E \to \mathbb{R}^+$, a non-empty compact* *set $K \in \mathcal{B}$ constants $\alpha \in (0,1)$ and $\beta > 0$ such that $P(s_*, s_* + T)$ satisfies the local Doeblin condition (2.16) and*

$$P(s_*, s_* + T)U_{s_*} \le \alpha U_{s_*} + \beta \quad \text{on } E.$$

Then there exists a unique geometric periodic measure ρ (with respect to P). Specifically, there exists a norm-like function $V : \mathbb{T} \times E \to \mathbb{R}^+$, constants $R_s < \infty$ and $r_s \in (0,1)$ such that the following all holds

(i) For any $s \in \mathbb{T}$ and $x \in E$, we have

$$\|P(s, s, +nT, x, \cdot) - \rho_s\|_{TV} \le R(V(s,x) + 1)r_s^n, \quad n \in \mathbb{N}. \tag{2.21}$$

(ii) For any $s \le t$, $x \in E$, we have

$$\|P(s, t, x, \cdot) - \rho_t\|_{TV} \le R_s(V(s,x) + 1)r_s^n, \quad \mathbb{N} \ni n \le \lfloor \frac{t-s}{T} \rfloor.$$

(iii) Allowing for negative initial time, for any $s \le t$, $x \in E$, we have

$$\|P(s - nT, t, x, \cdot) - \rho_t\|_{TV} \le R_s(V(s,x) + 1)r_s^n, \quad \mathbb{N} \ni n \le \lfloor \frac{t-s}{T} \rfloor.$$

(iv) The periodic measure is uniformly geometric convergence over initial time i.e. there exist constants $R > 0, r \in (0,1)$ and a norm-like function $V : E \to \mathbb{R}^+$ such that

$$\|P(s, s + nT, x, \cdot) - \rho_s\|_{TV} \le R(V(x)+1)r^n, \quad \text{for all } x \in \mathbb{R}^d, s \in \mathbb{T}, n \in \mathbb{N}. \tag{2.22}$$

Proof. Define $V(s,x) := P(s, s_*)U_{s_*}(x)$ for all $s \le s_*$ and extend by periodicity for all $s \in \mathbb{T}$. Then analogous to Theorem 2.16, the function $V(s, \cdot)$ satisfies (2.8) with respect to $P(s, s+T)$. Likewise from Theorem 2.16, the local Doeblin condition holds. Then (2.21) holds immediately by Lemma 2.7. Convergence (2.22) is obvious from (2.21) by defining $R = \sup_{s \in [0,T)} R_s$, $V(x) = \sup_{s \in [0,T)} V(s,x)$ and $r = \sup_{s \in [0,T]} r_s < 1$. The remaining converges are proven in the way as Theorem 2.16. $\square$

2.3.3 Key Ingredients for Periodic Measures

Assuming we have a Foster-Lyapunov function for a T-periodic Markovian kernel, Theorem 2.16 and Theorem 2.18 yields a limiting and geometric periodic measure respectively, provided the local Doeblin condition (2.16). In this section, we show that the local Doeblin condition can be essentially decomposed into a (local) irreducibility condition and strong Feller condition. A sufficient condition for the strong Feller condition to hold is the existence of a continuous density. These are the main indgredients for the periodic measure.

The following two results gives sufficient conditions in which (2.16) holds. We denote for convenience $\mathcal{M}(E)$ to be the space of measures on $(E, \mathcal{B})$.

Proposition 2.19. *Let P be a T-periodic Markov transition kernel and assume there exists some $s_* \in \mathbb{T}$, a non-empty set $K \in \mathcal{B}$, $\epsilon > 0$ and $\Lambda \in \mathcal{M}(E)$ such that $\Lambda(K) > 0$, $P(s,t,x,\cdot)$ possesses a density $p(s,t,x,y)$ with respect to Λ and*

$$\inf_{x,y\in K} p(s_*, s_* + T, x, y) > 0. \tag{2.23}$$

Then the local Doeblin condition (2.16) of Theorem 2.16 holds.

Proof. By Theorem 2.16, it suffices to show $P(s_*, s_* + T)$ satisfies (2.5) for some $s_* \in \mathbb{T}$. By assumption that $\Lambda(K) > 0$,

$$\begin{aligned}
\eta_{s_*} :&= \int_E \inf_{x\in K} p(s_*, s_* + T, x, y) dy \\
&\geq \int_K \inf_{x\in K} p(s_*, s_* + T, x, y) dy \\
&\geq \int_K \inf_{x,y\in K} p(s_*, s_* + T, x, y) dy \\
&= \inf_{x,y\in K} p(s_*, s_* + T, x, y) \Lambda(K) \\
&> 0.
\end{aligned}$$

Clearly, $\eta_{s_*} \in (0, 1]$. Define the measure

$$\varphi_{s_*}(\Gamma) := \frac{1}{\eta_{s_*}} \int_\Gamma \inf_{x\in K} p(s_*, s_* + T, x, y) dy, \quad \Gamma \in \mathcal{B}.$$

It is easy to verify that $\varphi_{s_*} \in \mathcal{P}(E)$ and for any $x \in K$ and any $\Gamma \in \mathcal{B}$

$$\begin{aligned} P(s_*, s_* + T, x, \Gamma) &= \int_\Gamma p(s_*, s_* + T, x, y)dy \\ &\geq \int_\Gamma \inf_{x \in K} p(s_*, s_* + T, x, y)dy \\ &= \eta_{s_*}\varphi_{s_*}(\Gamma). \end{aligned}$$

Thereby (2.5) holds with constant η_{s_*} and probability measure φ_{s_*}. □

In practice, assumption (2.23) in Proposition 2.19 can be difficult to verify as well as being stronger than required. By assuming the Markov transition kernel possesses a continuous density, we can relax (2.23).

For the forthcoming theorem, we define

$$\mathcal{M}^+(E) = \{\mu \in \mathcal{M}(E) | \mu(\Gamma) > 0, \quad \text{non-empty open } \Gamma \in \mathcal{B}\}.$$

We will make explicit use of the metric d on $(E, \mathcal{B})$ and define $B_r(x) := \{y \in E | d(x, y) < r\}$ to be the open ball of radius $r > 0$ centred at $x \in E$.

Theorem 2.20. *Let P be a T-periodic Markov transition kernel and assume there exists some $s_* \in \mathbb{T}$, a compact set $K \in \mathcal{B}$ with a non-empty interior, $0 \leq r \leq T$ and $\Lambda \in \mathcal{M}^+(E)$ such that $P(s, t, x, \cdot)$ possesses a (local) density $p(s, t, x, y)$ with respect to Λ and is jointly continuous on $K \times K$. Assume further that for any non-empty open set $\Gamma_1, \Gamma_2 \subset K$ and $x \in K$*

$$P(s_*, s_* + r, x, \Gamma_1) > 0, \quad P(s_* + r, s_* + T, x, \Gamma_2) > 0. \tag{2.24}$$

Then the local Doeblin condition (2.16) of Theorem 2.16 holds.

Proof. Fix any $y' \in K$, by (2.24), then for any non-empty open set $\Gamma \subset K$,

$$P(s_* + r, s_* + T, y', \Gamma) > 0.$$

By the existence of a density, there exists $z' \in \Gamma$ such that

$$p(s_* + r, s_* + T, y', z') \geq 2\epsilon,$$

for some $\epsilon > 0$. Joint continuity assumption implies there exists $r_1, r_2 > 0$

such that

$$p(s_*+r,s_*+T,y,z)\geq\epsilon,\quad \text{for all } y\in B_{r_1}(y')\subset K, z\in B_{r_2}(z')\subset K.$$

Hence for any $\Gamma\in\mathcal{B}$ and $y\in B_{r_1}(y')$,

$$\begin{aligned}P(s_*+r,s_*+T,y,\Gamma)&=\int_\Gamma p(s_*+r,s_*+T,y,z)\Lambda(dz)\\&\geq\int_{\Gamma\cap B_{r_2}(z')}p(s_*+r,s_*+T,y,z)\Lambda(dz)\\&\geq\epsilon\Lambda(\Gamma\cap B_{r_2}(z')).\end{aligned}$$

By (2.24), we have

$$P(s_*,s_*+r,x,B_{r_1}(y'))>0,\quad \text{for all } x\in K.$$

As $p(s_*,s_*+r,x,y)$ is a continuous function of x, by dominated convergence theorem, $P(s_*,s_*+r,x,\Gamma)$ is also continuous function of x. By the compactness of K,

$$\inf_{x\in K}P(s_*,s_*+r,x,B_{r_1}(y'))\geq\gamma',$$

for some $\gamma'>0$. In particular,

$$\inf_{x\in K}P(s_*,s_*+r,x,B_{r_1}(y'))\geq\gamma:=\min\left\{\gamma',\frac{1}{\epsilon\Lambda(B_{r_2}(z'))}\right\}.$$

Putting them together via Chapman-Kolmogorov equation, we have for any $x\in K$ and $\Gamma\in\mathcal{B}$,

$$\begin{aligned}P(s_*,s_*+T,x,\Gamma)&=\int_E P(s_*+r,s_*+T,y,\Gamma)p(s_*,s_*+r,x,y)\Lambda(dy)\\&\geq\int_{B_{r_1}(y')}P(s_*+r,s_*+T,y,\Gamma)p(s_*,s_*+r,x,y)\Lambda(dy)\\&\geq\epsilon\Lambda(\Gamma\cap B_{r_2}(z'))\int_{B_{r_1}(y')}p(s_*,s_*+r,x,y)\Lambda(dy)\\&=\epsilon\Lambda(\Gamma\cap B_{r_2}(z'))P(s_*,s_*+r,x,B_{r_1}(y'))\\&\geq\epsilon\gamma\Lambda(\Gamma\cap B_{r_2}(z')).\end{aligned}$$

Thus, the probability measure

$$\varphi(\cdot) = \frac{\Lambda(\cdot \cap B_{r_2}(z'))}{\Lambda(B_{r_2}(z'))},$$

and the constant $\eta = \epsilon\gamma\Lambda(B_{r_2}(z')) \in (0, 1]$ collectively satisfy the local Doeblin condition (2.16). $\square$

Remark 2.21. Note that in Theorem 2.20, if E is a locally compact metrisable topological group, then any Haar measure Λ (for which a local density exist and jointly continuous) will suffice. Moreover, note that Theorem 2.20 holds provided the Markov transition kernel is (local) irreducible and possesses the strong Feller property.

3 Periodic Measures of Stochastic Differential Equations

Using the developed theory from Section 2, we apply the results specifically in the context of T-periodic SDEs evolving on Euclidean state spaces. In this section, with asymptotic behaviours in mind, we are particularly interested in results that can be verified to possess a limiting and geometric periodic measure. It should be clear from Section 2 that non-limiting periodic measures results hold when the local Doeblin condition fails to hold. We note that Theorem 2.16 accommodates other types of noise e.g. Lévy noise, in this thesis, we focus specifically in studying T-periodic SDEs with white noises as its source of randomness.

We first fix some nomenclature and notation that will be used. We refer to SDE as non-autonomous when its coefficients depend explicitly on time and SDE as autonomous when the coefficients are time-independent. We always denote by $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ the Euclidean space where $\mathcal{B}(\mathbb{R}^d)$ denote the standard Borel σ-algebra on $\mathbb{R}^d$ and let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote the standard inner-product and norm on $\mathbb{R}^d$. Then we can define $B_r(y) := \{x \in \mathbb{R}^d | \, \|x - y\| < r\}$ for the open ball of radius $r > 0$ centred at y. And denote for convenience $B_r := B_r(0)$. On $\mathbb{R}^d$, we re-use Λ as the Lebesgue measure. We let $GL(\mathbb{R}^d)$ denote the space of invertible $d \times d$ matrices and let $L_2(\mathbb{R}^d) := \{\sigma \in \mathbb{R}^{d\times d} | \|\sigma\|_2 < \infty\}$, where $\|\sigma\|_2 = \sqrt{\mathrm{Tr}(\sigma\sigma^T)} = \sqrt{\sum_{i,j=1}^d \sigma_{ij}^2}$ is the standard Frobenius norm.

We let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ denote the space of functions which are continuously differentiable in the first variable and twice differentiable in the spatial variables. We let $C_b^\infty(B_n)$ denote the space of infinitely differentiable real-valued bounded functions on B_n. Functions $b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are said to be locally Lipschitz if for any compact set $K \subset \mathcal{B}(\mathbb{R}^d)$ there exists a constants $L = L(K)$ and $M = M(K)$ such that $\|b(t,x) - b(t,y)\| \le L \|x - y\|$ and $\|\sigma(t,x) - \sigma(t,y)\|_2 \le M \|x - y\|_2$ for $x, y \in K$. They are (globally) Lipschitz if $K = E$. Define for ease, $\|\sigma\|_\infty := \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} \|\sigma(t,x)\|_2$. For tuple $\beta = (\beta_0, \beta_1, ..., \beta_d) \in \mathbb{N}^{d+1}$, define the partial derivatives $\partial^\beta := \frac{\partial^{|\beta|}}{\partial_t^{\beta_0} \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}}$ where $|\beta| := \sum_{i=0}^d \beta_i$.

3.1 Limiting Periodic Measures

We study time-inhomogeneous Markov processes $X_t = X_t^{s,x}$ satisfying T-periodic SDEs of the form

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & t \geq s, \\ X_s = x, & x \in \mathbb{R}^d. \end{cases} \tag{3.1}$$

Here $b \in C(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C(\mathbb{R}^+ \times \mathbb{R}^d, GL(\mathbb{R}^d))$ are T-periodic i.e.

$$b(t, \cdot) = b(t + T, \cdot), \quad \text{and} \quad \sigma(t, \cdot) = \sigma(t + T, \cdot),$$

and W_t is a d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. To avoid triviality, we assume the coefficients collectively have a minimal period i.e. at least one of the coefficients have a minimal period. The infinitesimal generator of (3.1), $\mathcal{L}(t)$ is given by

$$\mathcal{L}(t)f(t,x) = \partial_t f(t,x) + \sum_{i=1}^{d} b_i(t,x)\partial_i f(t,x) + \frac{1}{2}\sum_{i,j=1}^{d} \left(\sigma\sigma^T\right)_{ij} \partial_{ij}^2 f(t,x), \quad f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d). \tag{3.2}$$

We use the short hand notation $\mathbb{P}^{s,x}$ and $\mathbb{E}^{s,x}$ for the associated probability measure and expectation respectively for the process starting at $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. When a unique solution exists, one can define the Markov transition kernel

$$P(s,t,x,\Gamma) := \mathbb{P}^{s,x}(X_t \in \Gamma), \quad s < t, \Gamma \in \mathcal{B}. \tag{3.3}$$

A unique solution exists when the Markov process X_t is regular i.e. for any $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$,

$$\mathbb{P}^{s,x}\{\eta_\infty = \infty\} = 1, \tag{3.4}$$

where

$$\eta_\infty := \lim_{n\to\infty} \eta_n, \quad \eta_n := \inf_{t \geq s}\{\|X_t\| \geq n\}, \quad n \in \mathbb{N}.$$

It is well-known that when b and σ are Lipschitz, in a construction a la Picard–Lindelöf theorem, a unique solution exists. Relaxing the coefficients to be locally Lipschitz, it is well-known also that if there exists a norm-like function V and constant $c > 0$ such that

$$\mathcal{L}(t)V \leq cV, \tag{3.5}$$

then a unique solution exists. For completeness, see Appendix A for proof.

Finally, it is easy to prove and intuitively true that the Markov transition kernel (if it exists) is T-periodic i.e. (2.3) holds when the SDE (3.1) is T-periodic.

In our study we require the following set of assumptions also. We say that σ has linear growth if there exists a constant $C > 0$ such that

$$\|\sigma(t, x)\|_2^2 \leq C(1 + \|x\|^2), \quad t \in \mathbb{R}^+, x \in \mathbb{R}^d. \tag{3.6}$$

We say σ is bounded with bounded inverse if

$$\max\{\|\sigma\|_\infty, \|\sigma^{-1}\|_\infty\} < \infty. \tag{3.7}$$

We say that the functions $b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are locally smooth and bounded if for all $n \in \mathbb{N}$

$$\sigma_{ij} \in C_b^\infty(B_n), \quad 1 \leq i, j \leq d, \tag{3.8}$$

and

$$b(t, x) + \partial^\beta b(t, x) \quad \text{bounded on } \mathbb{R}^+ \times B_n, \beta \in \mathbb{N}^{d+1}, |\beta| = d. \tag{3.9}$$

Note that (3.8) and (3.9) imply the respective functions are locally Lipschitz. Whenever we assume (3.8), we always demand that σ is a function of spatial variables only.

From Proposition 2.19 and Theorem 2.20, we saw that the main ingredients for the local Doeblin condition (2.20) is (local) irreducibility and the existence of a jointly continuous density. For diffusion processes on $\mathbb{R}^d$, we shall use the Lebesgue measure Λ (a Haar measure, see Remark 2.21). Sufficient conditions for these two properties to hold will be given.

We note in passing that it is indeed possible to show both properties simultaneously. For instance, for non-autonomous SDEs with Hölder and bounded coefficients with non-degenerate bounded diffusion, Aronson's heat kernel estimates [Aro67] yields

$$p(s, t, x, y) \geq C_1 \frac{1}{(4C_2\pi(t - s))^{d/2}} e^{-\frac{\|x-y\|^2}{4C_2(t-s)}} \quad x, y \in \mathbb{R}^d, 0 \leq s \leq t < \infty,$$

for some positive constants C_1, C_2. So it follows that that $p(s,t,x,y)$ is strictly positive when $x, y \in K$ and $t - s < \infty$. Hence sufficiently implies the Proposition 2.19 hold. With typical SDE applications in mind, in this thesis, we study the two irreducibility and density separately while allowing for the possibility of unbounded coefficients.

It is well-known that autonomous SDEs satisfying (3.4) and Hörmander's condition possesses a smooth density (globally with respect to Λ) for the Markov transition kernel [Mal78, Hör85, RW00] (see Appendix B for details). However it is generally insufficient to yield irreducibility i.e. Hörmander's condition does not imply the process can reach any given non-empty open set with positive probability. We refer readers to Remark 2.2 of [Hai11] for a counter-example. This suggests some degree of non-degeneracy is required to imply irreducibility. We emphasise that in existing literature, Hörmander's condition is often applied for autonomous SDEs with relatively few existing results for the non-autonomous case. Observe also that Theorem 2.20 requires density of the transition kernel to exist locally rather than globally. Recent advances by Höpfner, Löcherbach and Thieullen gave the existence of a smooth local density of non-autonomous SDEs under a time-dependent Hörmander's condition in [HLT17].

Since the intention of this thesis is to introduce main ideas and approach to deduce the existence and uniqueness of periodic measures, we shall show (global) irreducibility under the assumption that the diffusion matrix and its inverse are bounded and utilise the results of [HLT17] for a local density. It will be the subject of future works to generalise the results in the direction of local time-dependent Hörmander's condition and relaxing the non-degeneracy assumption to attain a local irreducibility.

Consider the following associated control system to (3.1)

$$\begin{cases} dZ_t = \varphi(t)dt + \sigma(t, Z_t)dW_t, \quad t \geq s, \\ Z_s = x, \end{cases} \tag{3.10}$$

for some bounded adapted process $\varphi : \mathbb{R}^+ \to \mathbb{R}^d$. Inspired by the irreducibility argument of [DZ96], we have the following lemma:

Lemma 3.1. *Assume b and σ are locally Lipschitz and moreover σ satisfies (3.6) and (3.7). Let $X_t = X_t^{s,x}$ and $Z_t = Z_t^{s,x}$ satisfy (3.1) and (3.10) respectively. Then the laws of X_t and Z_t are equivalent.*

Proof. By Lemma A.1, the locally Lipschitz coefficients and (3.5) yields that X_t exists and is unique. Since φ is a bounded adapted process and σ is locally Lipschitz with linear growth, by Theorem 3.1 of [Mao07], Z_t also exists and is unique. Set

$$\eta_n = \inf_{t \geq s}\{\|Z_t\| \geq n\}, \quad Z_t^n = Z_{t \wedge \eta_n},$$

and

$$\mathbb{P}^n(d\omega) = \mathbb{P}(d\omega)M_t^n,$$

where

$$M_t^n = \exp\left(-\frac{1}{2}\int_s^{t\wedge\eta_n} \alpha^2(r)dr - \int_s^{t\wedge\eta_n} \alpha(r)dW_r\right),$$

and $\alpha(r) = \sigma^{-1}(r, Z_r)[\varphi(r) - b(r, Z_r)]$. It is clear that $\alpha(r)$ is bounded for $s \leq r \leq \eta_n$, hence Novikov condition is satisfied. Then Girsanov theorem implies

$$\widetilde{W}_t^n = W_t + \int_s^t \alpha(r)dr$$

is a Brownian motion on $\mathbb{R}^d$ under the probability measure $\mathbb{P}^n$. It is clear that

$$d\widetilde{W}_r^n = dW_r + \alpha(r)dr$$

and by rearranging

$$\varphi(t) = \sigma(t, Z_t)\alpha(t) + b(t, Z_t)$$

hence

$$\begin{aligned}
Z_t^n &= x + \int_s^{t\wedge\eta_n} \varphi(r)dr + \int_s^{t\wedge\eta_n} \sigma(r, Z_r^n)dW_r \\
&= x + \int_s^{t\wedge\eta_n} \left[\sigma(r, Z_r^n)\alpha(r) + b(r, Z_r^n)\right]dr + \int_s^{t\wedge\eta_n} \sigma(r, Z_r^n)\left[d\widetilde{W}_r^n - \alpha(r)dr\right] \\
&= x + \int_s^{t\wedge\eta_n} b(r, Z_r^n)dr + \int_s^{t\wedge\eta_n} \sigma(r, Z_r^n)d\widetilde{W}_r^n.
\end{aligned}$$

i.e. Z_t^n is a solution of (3.1) on $(\Omega, \mathcal{F}, \mathbb{P}^n)$. As the law of the solution does not depend on the choice of probability space, we have that

$$\mathbb{P}(X_t^n \in \Gamma) = \mathbb{P}^n(Z_t^n \in \Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

As $\mathbb{P}$ and $\mathbb{P}^n$ are equivalent, the laws of X_t^n and Z_t^n are equivalent. This implies that

$$\begin{aligned}\mathbb{P}^n(\eta_n > t) &= \mathbb{P}^n(\sup_{s\le r\le t}\|Z_r\| \le n)\\ &= \mathbb{P}^n(\sup_{s\le r\le t}\|X_r\| \le n)\\ &\to 1, \quad n\to\infty.\end{aligned}$$

Define

$$M_t = \exp\left(-\frac{1}{2}\int_s^t \alpha^2(r)dr - \int_s^t \alpha(r)dW_r\right).$$

Then

$$\mathbb{E}[M_t] \ge \mathbb{E}[M_t^n I_{\{\eta_n > t\}}] = \mathbb{P}^n(\eta_n > t) \to 1 \quad \text{as } n\to\infty.$$

Moreover, we can prove that $\mathbb{P}(\eta_n > t) \to 1$ as $n \to \infty$. This suggests from Borel-Cantelli Lemma that there is a subsequence n_k such that $\eta_{n_k} \to \infty$ almost surely where $n_k \to \infty$ as $k \to \infty$. Thus

$$M_t^{n_k} \to M_t, \quad \text{as } k\to\infty$$

almost surely. Now, by Fatou's lemma

$$\lim_{k\to\infty} \mathbb{E}\left[M_t^{n_k}\right] \ge \mathbb{E}\left[\lim_{k\to\infty} M_t^{n_k}\right] = \mathbb{E}[M_t],$$

and $\mathbb{E}[M_t^{n_k}] = 1$ for each k since $M_t^{n_k}$ is a martingale. So $\mathbb{E}[M_t] \le 1$. Thus, we have that $\mathbb{E}[M_t] = 1$. Now we apply Girsanov theorem [DZ92] to yield that

$$\widetilde{W}_t = W_t + \int_s^t \alpha(r)dr$$

is a Brownian motion on $\mathbb{R}^d$ under the probability measure $\widetilde{\mathbb{P}}$,where $\widetilde{\mathbb{P}}(d\omega) = \mathbb{P}(d\omega)M_t$. As before,

$$\begin{aligned}Z_t &= x + \int_s^t \varphi(r)dr + \int_s^t \sigma(r, Z_r)dW_r\\ &= x + \int_s^t b(r, Z_r)dr + \int_s^t \sigma(r, Z_r)d\widetilde{W}_r\end{aligned}$$

is a solution to (3.1) on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. As the law of the solution does not depend on the choice of probability space, we have that

$$\mathbb{P}(X_t \in \Gamma) = \widetilde{\mathbb{P}}(Z_t \in \Gamma), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

As $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, the laws of X_t and Z_t are equivalent. □

Theorem 3.2. *Consider SDE (3.1) (not necessarily periodic) and assume the same conditions as Lemma 3.1. Then the Markov transition kernel $P(s,t,\cdot,\cdot)$ for $s<t<\infty$ is irreducible i.e. $P(s,t,x,\Gamma)>0$ for all $x \in \mathbb{R}^d$ and non-empty open $\Gamma \in \mathcal{B}(\mathbb{R}^d)$.*

Proof. By Lemma 3.1, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. Hence it is sufficient to show that for any $\delta > 0$ and $x, a \in \mathbb{R}^d$ that

$$\mathbb{P}(\|Z_t^{s,x} - a\| < \delta) > 0.$$

We consider the auxiliary system

$$\begin{cases} dY_t = \sigma(t, Y_t)dW_t, \\ Y_s = x. \end{cases} \tag{3.11}$$

Since σ is Lipschitz, then (3.11) has a unique solution Y_t satisfying

$$Y_t = x + \int_s^t \sigma(r, Y_r)dW_r. \tag{3.12}$$

For $u \in [s,t)$, $R>0$ and $\widetilde{a} \in \mathbb{R}^d$ all to be chosen later, pick a bounded function $f : [u,t] \times \mathbb{R}^d \to \mathbb{R}^d$ such that f is Lipschitz and

$$f(r,y) = \begin{cases} 0 & \text{if } \|y\| > 2R, \\ \frac{\widetilde{a}-y}{t-u} & \text{if } \|y\| \le R. \end{cases}$$

Define the integral

$$I_1(y) = y + \int_u^t f(r,y)dr, \quad y \in \mathbb{R}^d.$$

Observe that if $\|y\| \le R$,

$$I_1(y) = y + \frac{1}{t-u}\int_u^t (\widetilde{a} - y)dr = \widetilde{a}. \tag{3.13}$$

Set

$$\varphi(r) = \begin{cases} 0 & \text{if } r \in [s,u), \\ f(r, Y_u) & \text{if } r \in [u,t]. \end{cases}$$

Then it is clear that $Z_r^{s,x} = Y_r$ for $r \in [s,u)$. Hence, by sample-path continuity of Y_t, Z_t can be represented as an initial-valued SDE in terms of Y_u namely

$$Z_t^{s,x} = Y_u + \int_u^t f(r, Y_u)dr + \int_u^t \sigma(r, Z_r)dW_r.$$

Let $I_1 = I_1(Y_u)$ and $I_2 = \int_u^t \sigma(r, Z_r)dW_r$. Then $Z_t^{s,x} = I_1 + I_2$. Choose any fixed $\widetilde{a} \in \mathbb{R}^d$ such that

$$\|a - \widetilde{a}\| \le \frac{\delta}{3}.$$

Suppose the events $\{I_1 = \widetilde{a}\}$ and $\{\|I_2\| \le \frac{\delta}{3}\}$ holds then

$$\begin{aligned} \|Z_t^{s,x} - a\| &= \|(I_1 - \widetilde{a}) + (I_2 + \widetilde{a} - a)\| \\ &\le \|I_2\| + \|\widetilde{a} - a\| \\ &\le \frac{2}{3}\delta. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{P}(\|Z_t^{s,x} - a\| \le \delta) &\ge \mathbb{P}(I_1 = \widetilde{a} \text{ and } \|I_2\| \le \frac{\delta}{3}) \\ &\ge \mathbb{P}\left(I_1 = \widetilde{a}\right) - \mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right) \end{aligned} \tag{3.14}$$

where we used the elementary inequality $\mathbb{P}(A \cap B) \ge \mathbb{P}(A) - \mathbb{P}(B^c)$ for any event $A, B \in \mathcal{F}$. Thus the proof is complete provided the right hand side of inequality (3.14) is positive. By Chebyshev's inequality and Itô's isometry,

$$\begin{aligned} \mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right) &\le \frac{9}{\delta^2}\mathbb{E}[\|I_2\|^2] \\ &\le \frac{9}{\delta^2}\int_u^t \|\sigma(r, Z_r)\|_2^2 dr \\ &\le \frac{9}{\delta^2}(t-u)\|\sigma\|_\infty^2. \end{aligned}$$

Hence, one can fix a $u \in [s,t)$ such that

$$\mathbb{P}\left(\|I_2\| > \frac{\delta}{3}\right) \leq \frac{1}{4}.$$

Similarly, for the fixed u and any $R > 0$, from (3.12) we have

$$\begin{aligned}\mathbb{P}(\|Y_u\| > R) &= \frac{1}{R^2}\mathbb{E}[\|Y_u\|^2] \\ &= \frac{1}{R^2}\left(\|x\|^2 + \|\int_u^s \sigma(r, Y_r)dr\|^2\right) \\ &\leq \frac{1}{R^2}\left[\|x\|^2 + \|\sigma\|_\infty^2 (u-s)^2\right].\end{aligned}$$

Hence one can fix a sufficiently large $R > 0$ such that

$$\mathbb{P}(\|Y_u\| \leq R) \geq \frac{3}{4}. \tag{3.15}$$

By (3.13), we have the inclusion $\{\|Y_u\| \leq R\} \subset \{I_1 = \widetilde{a}\}$. Hence by (3.15)

$$\mathbb{P}(I_1 = \widetilde{a}) \geq \mathbb{P}(\|Y_u\| \leq R) \geq \frac{3}{4}.$$

The proof is complete by the following inequality for irreducibility

$$\begin{aligned}\mathbb{P}(\|Z_t^{s,x} - a\| \leq \delta) &= \mathbb{P}(I_1 = \widetilde{a}) - \mathbb{P}(\|I_2\| > \frac{\delta}{3}) \\ &\geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.\end{aligned}$$

In the next theorem, we apply Theorem 1 of [HLT17] to attain a smooth density of transition probabilities in extension of classical results by Aronson [Aro67] for parabolic equations with bounded time-dependent coefficients. We assume that σ is time-independent as in [HLT17]. □

Theorem 3.3. *Consider SDE (3.1) and assume the same conditions as Lemma 3.1. Assume further that (3.8) and (3.9) holds that there exists a compact set $K \in \mathcal{B}(\mathbb{R}^d)$ such that (2.14) and (2.15) hold. Then the results of Theorem 2.16 hold.*

Proof. The invertibility of σ implies linear independent columns hence our collective assumptions satisfy Theorem 1 of [HLT17]. Hence there exists a smooth density $p(s,t,x,y)$ with respect to Λ. Then using Theorem 3.2, we have that Theorem 2.20 holds. Hence the assumptions of Theorem 2.16 are satisfied. □

Remark 3.4. With global irreducibility of P implied by Theorem 3.2, further to Remark 2.17, it is clear by the convergences given in Theorem 2.16 that that the unique periodic measure attained in Theorem 3.3 has global support i.e. $\mathrm{supp}(\rho_s) = \mathbb{R}^d$ for all $s \in \mathbb{R}^+$. This contrasts significantly to the time-homogeneous case where (2.10) holds.

3.2 Geometric Ergodic Periodic Measures

In the previous section, we studied limiting periodic measures of SDEs in a qualitative manner. We extend this for geometrically ergodic periodic measures. That is, the Markov transition kernel convergence towards the periodic measure is exponentially fast. We recall the geometric drift condition for SDEs.

Definition 3.5. The SDE (3.1) is said to satisfy the geometric drift condition if there exists a function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^+)$ and constants $C \geq 0$ and $\lambda > 0$ such that

$$\mathcal{L}(t)V \leq C - \lambda V \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \tag{3.16}$$

where $\mathcal{L}(t)$ is given by (3.2).

Note that if (3.16) is satisfied then the SDE is regular. Specifically, since $V \geq 0$ and $\mathcal{L}(t)[\text{const}] = 0$, it is easy to see that

$$\mathcal{L}(t)(V+1) \leq C - \lambda V \leq C \leq C(V+1),$$

hence the regularity condition (3.5) is satisfied.

Using the geometric drift condition, we give one of the main results on the existence, uniqueness and geometric ergodicity of a periodic measure. It is worth noting that if the SDE coefficients have a trivial period, then the theorem recovers known results of invariant measures. Hence, the results here presented can be regarded as time-periodic generalisations of such theorems of invariant measures for autonomous SDEs.

Theorem 3.6. *Assume T-periodic SDE (3.1) coefficients satisfies (3.6), (3.7), (3.8) and (3.9). Assume further that there exists a T-periodic norm-like $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^+)$ satisfying the geometric drift condition (3.16). Then Theorem 2.18 follows.*

Proof. By Itô's formula and the regularity of V, one has

$$d\left(e^{\lambda t}V(t,X_t)\right) = e^{\lambda t}(\lambda V + \mathcal{L}(t)V)dt + e^{\lambda t}\langle \nabla V, \sigma dW_t\rangle,$$

hence by the geometric drift condition (3.16),

$$\begin{aligned} V(t,X_t) &= e^{-\lambda(t-s)}V(s,X_s) + \int_s^t e^{-\lambda(t-r)}(\lambda V + \mathcal{L}(r)V)dr + \int_s^t e^{-\lambda(t-r)}\langle \nabla V, \sigma dW_r\rangle \\ &\leq e^{-\lambda(t-s)}V(s,X_s) + \frac{C}{\lambda}\left(1 - e^{-\lambda(t-s)}\right) + \int_s^t e^{-\lambda(t-r)}\langle \nabla V, \sigma dW_r\rangle. \end{aligned} \tag{3.17}$$

By (3.4) and the regularity of V, $\int_s^t e^{\lambda(t-r)}\langle \sigma^T(r,X_r)\nabla V(r,X_r), dW_r\rangle_{\mathbb{R}^d}$ is a martingale. Hence

$$\mathbb{E}^{s,x}[V(t,X_t)] \leq e^{-\lambda(t-s)}V(s,x) + \frac{C}{\lambda}(1 - e^{-\lambda(t-s)}) \quad s \leq t. \tag{3.18}$$

Specifically,

$$\mathbb{E}^{s,x}[V(s+T, X_{s+T})] \leq e^{-\lambda T}V(s,x) + \frac{C}{\lambda}(1 - e^{-\lambda T}). \tag{3.19}$$

Define the functions $U_s(\cdot) := V(s,\cdot) \geq 0$ for $s \geq 0$. Since V is T-periodic, we have that (3.19) is equivalent to

$$P(s,s+T)U_s(x) \leq e^{-\lambda T}U_s(x) + \frac{C}{\lambda}(1 - e^{-\lambda T}) \tag{3.20}$$

That is to say (2.8) is satisfied for each $s \geq 0$. Subtracting $U_s(x)$ from (3.20) yields

$$P(s,s+T)U_s(x) - U_s(x) \leq (1 - e^{-\lambda T})\left(\frac{C}{\lambda} - U_s(x)\right).$$

Since U_s is norm-like assumption, define for $\epsilon > 0$

$$K = \bigcap_{s\in[0,T]} K_s, \quad \text{where } K_s := \left\{x \in \mathbb{R}^d | U_s(x) \leq \frac{C}{\lambda} + \frac{\epsilon}{1 - e^{-\lambda T}}\right\}.$$

For sufficiently large ϵ, K is non-empty compact set. Since the SDE is regular, the same proof from Theorem 3.3 implies that Theorem 2.20 holds i.e. $P(s,s+$

$T, x, \cdot)$ satisfies the local Doeblin condition (2.16) for each $s \geq 0$ and the compact set K. Thus the conditions of Theorem 2.18 are met. □

3.3 Weakly Dissipative SDEs

Theorem 3.6 depends crucially on finding a suitable Foster-Lyapunov function V. Dissipative SDEs are special cases where the Euclidean norm is a such Foster-Lyapunov function. This has the advantage that it is easier to verify directly than the geometric drift condition, where there may not be a clear Foster-Lyapunov function to use. Examples of weakly dissipative systems in physical systems will be given and discussed. The definition of dissipativity in this thesis coincides with that of Hale [Hal10] when the SDE is deterministic ($\sigma = 0$).

Definition 3.7. SDE (3.1) is weakly dissipative if there exists constants $c, \lambda > 0$ such that

$$2\langle b(t,x), x\rangle \leq c - \lambda \|x\|^2 \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \tag{3.21}$$

and dissipative if $c = 0$.

Corollary 3.8. *Assume T-periodic SDE (3.1) coefficients satisfies (3.6), (3.7), (3.8) and (3.9) and is weakly dissipative. Then Theorem 2.18 holds.*

Proof. By Theorem 3.6, it suffices to show $V(t,x) = \|x\|^2$ satisfies the geometric drift condition. We compute that

$$\mathcal{L}(t)\|x\|^2 = 2\langle b(t,x), x\rangle + \mathrm{Tr}(\left(\sigma\sigma^T\right)(t,x)) \leq c - \lambda\|x\|^2 + \|\sigma\|_\infty^2.$$

i.e. $\|x\|^2$ satisfies the geometric drift condition with $C = c + \|\sigma\|_\infty^2$ and same λ from (3.21). □

Theorem 3.9. *Consider T-periodic SDE (3.1) with σ satisfying (3.6), (3.7) and (3.8) and drift*

$$b(t,x) = \begin{pmatrix} \sum_{k=0}^{2p_1-1} S_k^1(t) x_1^k \\ \vdots \\ \sum_{k=0}^{2p_d-1} S_k^d(t) x_d^k \end{pmatrix},$$

where $\{p_i\}_{i=1}^d \in \mathbb{N}\backslash\{0\}$, $\{S_k^i\}_{i=1\dots d}^{k=1\dots 2p_i-2}$ are continuously differentiable T-periodic functions and constants $S_{2p_i-1}^i < 0$. Then Theorem 3.6 holds.

Proof. Clearly b satisfies (3.9). Hence by Corollary 3.8, it suffices to show that the SDE is weakly dissipative. We compute that

$$\langle b(t,x), x\rangle = \sum_{i=1}^{d}\sum_{k=1}^{2p_i} S_{k-1}^i x_i^k.$$

For each fixed $1 \le i \le d$, $\sum_{k=1}^{2p_i} S_{k-1}^i x_i^k$ is an even degree polynomial with leading negative coefficient. By assumption, $\{S_k^i\}$ are all bounded hence, fixing a $\lambda \in (0, -2\min_i S_{2p_i-1}^i)$, define the constants

$$\widetilde{c}_i := \sup_{x_i\in\mathbb{R}, t\in[0,T]} \left(2\sum_{k=1}^{2p_i} S_{k-1}^i x_i^k + \lambda x_i^{2p_i}\right) < \infty, \quad c_i := \widetilde{c}_i + \sup_{x_i\in\mathbb{R}} \lambda\left(x_i^2 - x_i^{2p_i}\right) < \infty,$$

then we deduce the SDE is weakly dissipative by

$$2\langle b(t,x), x\rangle \le \sum_{i=1}^{d}\left(\widetilde{c}_i - \lambda x_i^{2p_i}\right) \le \sum_{i=1}^{d}\left(c_i - \lambda x_i^2\right) = \sum_{i=1}^{d} c_i - \lambda\|x\|^2.$$

□

As it would be more apparent in the next section of gradient SDEs, Theorem 3.9 has many physical applications.

We give two specific examples of Theorem 3.9. First, we consider periodically forced mean-reverting Ornstein-Uhlenbeck processes. In this example, we compute the density of the process, periodic measure and its geometric convergence rate explicitly. The periodic measure is of minimal period and will be clear that the process does not have a limiting invariant measure. While the computations are straightforward, it appears that the periodic measure and its geometric convergence for this system has not been previously noted in literature. Where the classical Ornstein-Uhlenbeck process is mean-reverting, the periodically forced Ornstein-Uhlenbeck processes a time-periodic mean reversion property. In applications, these properties are desirable for processes with underlying periodicity or seasonality. For instance, electricity prices in economics [BKM07, LS02] and daily temperature [BS07] were modelled by periodic Ornstein-Uhlenbeck processes. In neuroscience, the authors of [IDL14] performed statistical inference of biological neurons modelled by Ornstein-Uhlenbeck proceses with periodic forcing.

Example 3.10. Consider the following multidimensional Ornstein-Uhlenbeck equation

$$dX_t = (S(t) - AX_t)\, dt + \sigma dW_t, \tag{3.22}$$

where $A = M^{-1}DM \in \mathbb{R}^{d\times d}$ for some $M \in GL(\mathbb{R}^d)$ and $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_d)$ is a diagonal matrix with positive eigenvalues $\{\lambda_n\}_{n=1}^d$, $\sigma \in GL(\mathbb{R}^d)$ and $S : \mathbb{R}^+ \to \mathbb{R}^d$ be a T-periodic continuously differentiable function.

By applying Itô's formula on $e^{tA}X_t$ or by a variation of constants formula, we have

$$X_t = e^{-(t-s)A}X_s + \int_s^t e^{-(t-r)A}S(r)dr + \int_s^t e^{-(t-r)A}\sigma dW_r \quad t \geq s. \tag{3.23}$$

Observe that $\xi(t) := \int_{-\infty}^t e^{-A(t-r)}S(r)dr$ satisfies $\partial_r(e^{Ar}\xi) = e^{Ar}S(r)$ and is continuous and T-periodic. Then

$$J(s,t) := \int_s^t e^{-(t-r)A}S(r)dr = \xi(t) - e^{-(t-s)A}\xi(s). \tag{3.24}$$

By the T-periodicity of ξ, it is clear $\lim_{t\to\infty} J(s,t)$ does not converge. Instead, it converges along integral multiples of the period in the following way: let Id be the identity matrix on $\mathbb{R}^d$ and define

$$\xi_n(s) := J(s, s+nT) = (Id - e^{-nTA})\xi(s), \quad n \in \mathbb{N}.$$

Then $\xi(s) = \lim_{n\to\infty} \xi_n(s)$. We shall see $\xi(s)$ as the "long term periodic mean". From (3.23), it is easy to see that X_t is normally distributed. Specifically, we can compute

$$\mathbb{E}^{s,x}[X_t] = e^{-(t-s)A}x + J(s,t).$$

Since $A = M^{-1}DM$, then $e^{-(t-r)A} = M^{-1}e^{-(t-r)D}M$. Denoting $N = M\sigma$, component-wise, we have $(e^{-(t-r)D}NdW_r)_i = e^{-(t-r)\lambda_i}\sum_{k=1}^d N_{ik}dW_r^k$. Hence by independence of Brownian motion and properties of Itô's inner-product, we have

$$\begin{aligned}
C_{ij}(s,t) &:= \mathbb{E}^{s,x}\left[\left(\int_s^t (e^{-(t-r)D} N dW_r)_i\right)\left(\int_s^t (e^{-(t-r)D} N dW_r)_j\right)\right] \\
&= \sum_{k,k'=1}^{d} N_{ik} N_{jk'} \mathbb{E}^{s,x}\left[\left(\int_s^t e^{-(t-r)\lambda_i} dW_r^k\right)\left(\int_s^t e^{-(t-r)\lambda_j} dW_r^{k'}\right)\right] \\
&= \sum_{k=1}^{d} N_{ik} N_{jk} \mathbb{E}^{s,x}\left[\int_s^t e^{-(t-r)(\lambda_i+\lambda_j)} dr\right] \\
&= \frac{(M\sigma\sigma^T M^T)_{ij}}{\lambda_i+\lambda_j}\left(1 - e^{-(t-s)(\lambda_i+\lambda_j)}\right).
\end{aligned}$$

Hence covariance matrix

$$\begin{aligned}
\mathrm{Cov}(X_t|X_s = x) &:= \mathbb{E}^{s,x}[X_t X_t^T] - \mathbb{E}^{s,x}[X_t]\mathbb{E}^{s,x}[X_t^T] \\
&= M^{-1} C(s,t) M,
\end{aligned}$$

where $C(s,t)$ has entries $C_{ij}(s,t)$ as defined above. Thus, denoting $\mathcal{N}$ for the multivariate normal distribution, the Markov transition kernel of (3.22) is given by

$$P(s,t,x,\cdot) = \mathcal{N}\left(e^{-(t-s)A}x + J(s,t), M^{-1}C(s,t)M\right)(\cdot), \tag{3.25}$$

Since $\lim_{t\to\infty} J(s,t)$ does not converges (for any fixed s), (3.25) does not converge. This implies there does not exist a limiting invariant measure for this periodically forced Ornstein-Uhlenbeck process. This contrasts with the classical Ornstein-Uhlenbeck process (i.e. $S(t) = \text{const}$), where one often take $t \to \infty$ to yield a (unique) limiting invariant measure. On the other hand, for every fixed s, along integral multiple of the period i.e. $t = s + nT$, one has directly from (3.25)

$$\begin{aligned}
P(s, s+nT, x, \cdot) &= \mathcal{N}\left(e^{-nTA}x + \xi_n(s), M^{-1}C(s, s+nT)M\right)(\cdot) \\
&\to \mathcal{N}\left(\xi(s), M^{-1}CM\right)(\cdot) =: \rho_s(\cdot),
\end{aligned} \tag{3.26}$$

as $n \to \infty$, where C is the matrix with entries $C_{ij} = \frac{(M\sigma\sigma^T M^T)_{ij}}{\lambda_i+\lambda_j}$. That is to say that the long-time behaviour is characterised by ρ_s for every fixed $s \geq 0$. Since ξ is T-periodic, ρ is also T-periodic. Moreover, it easy to explicitly verify that ρ is periodic measure of the system, see Appendix C for detailed steps.

We take a brief moment and note that for every $s \in \mathbb{R}^+$, $\text{supp}(\rho_s) = \mathbb{R}^d$ (see Remark 3.4) and that the periodic measure (3.26) is unique.

The above calculations gives the existence and uniqueness of a periodic measure. However, it does not immediately give a convergence rate. For simplicity, we show the convergence rate for the one-dimensional case. We first recall that the Kullback-Leibler divergence, $D_{KL}(\cdot||\cdot)$, is pre-metric on $\mathcal{P}(\mathbb{R}^d)$. Let $P, Q \in \mathcal{P}(\mathbb{R}^d)$ with densities $p, q \in L^1(\mathbb{R}^d)$ respective, the Kullback-Leibler divergence can be defined by

$$D_{KL}(P||Q) := \int_{\mathbb{R}^d} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx.$$

For vectors $\mu_i \in \mathbb{R}^d$ and matrices $\sigma_i \in GL(\mathbb{R}^d)$ where $i = 1, 2$, we have specifically the following explicit expression for normal densities.

$$D_{KL}(\mathcal{N}(\mu_1, \sigma_1)||\mathcal{N}(\mu_2, \sigma_2)) = \frac{1}{2}\left(\text{Tr}(\sigma_2^{-1}\sigma_1) + (\mu_2 - \mu_1)^T \sigma_2^{-1}(\mu_2 - \mu_1) - d + \ln\left(\frac{\det(\sigma_2)}{\det(\sigma_1)}\right)\right).$$

Moreover, Pinsker's inequality gives the following upper bound on the total variation norm

$$\|P - Q\|_{TV}^2 \leq \frac{1}{2} D_{KL}(P||Q), \quad P, Q \in \mathcal{P}(E).$$

We recall the elementary identity $\ln(1-y) = -\sum_{k=1}^{\infty} \frac{y^k}{k}$ for any fixed $y \in (-1, 1)$. Hence, the following elementary inequality holds by a geometric sum

$$-(y + \ln(1-y)) = \sum_{k=2}^{\infty} \frac{y^k}{k} \leq \frac{y}{2} \sum_{k=1}^{\infty} y^k \leq \frac{y}{2} \frac{1}{1-y}, \quad y \in (0, 1).$$

Now, since both ρ_{s+t} and $P(s, s, +t, x, \cdot)$ are normally distributed, by Pinsker's inequality and (3.24), for all $t \geq \delta$ and for any fixed $\delta > 0$, $x \in \mathbb{R}^d$

$$\begin{aligned}
&\|P(s,s+t,x,\cdot) - \rho_{s+t}\|_{TV}^2 \\
&\leq \frac{1}{2} D_{KL}(P(s,s+t,x,\cdot)||\rho_{s+t}) \\
&= \frac{1}{4}\left(1 - e^{-2tA} + \frac{(\xi(s+t) - e^{-tA}x - J(s,s+t))^2}{\sigma^2/2\alpha} - 1 - \ln(1 - e^{-2tA})\right) \\
&= \frac{1}{4}\left(\frac{e^{-2tA}(\xi(s) - x)^2}{\sigma^2/2\alpha} + \frac{1}{2}\frac{e^{-2tA}}{1 - e^{-2tA}}\right) \\
&\leq \frac{e^{-2tA}}{\sigma^2}\frac{A}{2}\left((\xi(s) - x)^2 + \frac{\sigma^2}{4A}\frac{1}{1 - e^{-2\delta A}}\right).
\end{aligned}$$

Deducing indeed the convergence is geometric. We go a little further solely to align with Theorem 3.6. For every fixed $s \in [0,T)$ and for any fixed $\gamma > 0$, there exists a constant $r_s = r_s(\gamma) > 0$ such that

$$(x - \xi(s))^2 - (1+\gamma)x^2 = -2\xi(s)x + \xi^2(s) - \gamma x^2 \leq r_s.$$

Hence $(x-\xi(s))^2 \leq (1+\gamma)x^2 + r_s$. Define $R_s := \max\left\{1+\gamma, r_s + \frac{\sigma^2}{4A}\frac{1}{1-e^{-2A\delta}}\right\} > 1$, then

$$\|P(s,s+t,x,\cdot) - \rho_{s+t}\|_{TV}^2 \leq \frac{e^{-2At}}{\sigma^2}\frac{A}{2}R_s\left(x^2+1\right) \leq e^{-2At}\left(\sqrt{\frac{\alpha}{2}}\frac{R_s}{\sigma}\right)^2\left(x^2+1\right)^2,$$

where we trivially squared the last two terms. Specifically by letting $t = nT$, we have geometric ergodicity of the grid chain

$$\|P(s,s+nT,x,\cdot) - \rho_s\|_{TV}^2 \leq e^{-2nTA}\left(\sqrt{\frac{A}{2}}\frac{R_s}{\sigma}\right)^2\left(x^2+1\right)^2, \quad n \in \mathbb{N}.$$

For computationally inclined readers, we give explicit formula for ξ_t in the one dimensional case. Multidimensional case can be computed similarly. By Fourier Series, for any $S \in L^2[0,T]$, S can be represented by

$$S(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi}{T}t - n\pi\right) + B_n \sin\left(\frac{2n\pi}{T}t - n\pi\right),$$

with the usual Fourier coefficients for $n \in \mathbb{N}\backslash\{0\}$

$$A_n = \frac{2}{T}\int_0^T S(t)\cos\left(\frac{2n\pi}{T}t - n\pi\right)dt, \quad B_n = \frac{2}{T}\int_0^T S(t)\sin\left(\frac{2n\pi}{T}t - n\pi\right)dt.$$

It is trivial to see $\xi_0 := \frac{1}{A}\frac{A_0}{2}$ satisfies $\partial_t(e^{At}\xi_0) = \frac{A_0}{2}e^{tA}$. Similarly,

$$\xi_n^{\cos}(t) := \frac{1}{A}\frac{T^2\cos\left(n\pi - \frac{2n\pi}{T}t\right) - 2n\pi T\sin\left(n\pi - \frac{2n\pi}{T}t\right)}{4\pi^2 n^2 + T^2}$$

satisfies $\partial_t(e^{tA}\xi_n^{\cos}(t)) = e^{tA}\cos\left(\frac{2n\pi}{T}t - n\pi\right)$ and

$$\xi_n^{\sin}(t) := -\frac{1}{A}\frac{T^2\sin\left(n\pi - \frac{2n\pi}{T}t\right) + 2n\pi T\cos\left(n\pi - \frac{2n\pi}{T}t\right)}{4\pi^2 n^2 + T^2}$$

satisfies $\partial_t(e^{tA}\xi_n^{\sin}(t)) = e^{tA}\sin\left(\frac{2n\pi}{T}T - n\pi\right)$. Clearly $\xi_n^{\cos}$ and $\xi_n^{\sin}$ are both T-periodic and

$$\xi(t) := \xi_0 + \sum_{i=1}^{\infty} A_n\xi_n^{\cos}(t) + B_n\xi_n^{\sin}(t)$$

is the desired T-periodic continuous (hence) bounded function satisfying $\partial_t(e^{tA}\xi) = e^{tA}S$.

Example 3.11. The stochastic overdamped Duffing Oscillator has many physical applications including being a mathematical model to explain the physical phenomena of stochastic resonance in climate dynamics modelling of ice age [BPSV82, Nic82, Jun93] as detailed in the introduction. The Duffing Oscillator is given by

$$dX_t = \left[-X_t^3 + X_t + A\cos(\omega t)\right]dt + \sigma dW_t, \tag{3.27}$$

where $A, \omega \in \mathbb{R}$ and $\sigma \neq 0$ are (typically small) parameters. In the Benzi-Parisi-Sutera-Vulpiani climate change stochastic resonance model, $\omega = 2\pi/10^5$ and the two stable equilibrium climates are distanced by $10K$. The stochastic differential equation (3.27) is a normalised equation of the Benzi-Parisi-Sutera-Vulpiani model. According to Corollary 3.8, there exists a unique periodic measure which is geometric ergodic.

Remark 3.12. Through the theory of non-autonomous RDS, [CLRS17] gave the existence and uniqueness of the periodic measure for (3.27) in one dimension. Note that Theorem 3.9 goes further than [CLRS17] to deduce that the convergence is actually geometric. Moreover, Theorem 3.9 gives the other types of converges as presented in Theorem 2.18. To our knowledge, this thesis contains the first proof of the geometric ergodicity of the stochastic overdamped Duffing Oscillator. We note also that the approach we have taken applies to multidimensional state spaces and is completely different to that of [CLRS17]. We expect our approach can be extended to the infinite dimensional setting of SPDEs, this is discussed briefly more in Section 6.

3.4 Gradient SDEs

In this section, we give results for the existence and uniqueness of geometric periodic measures for stochastic T-periodic gradient systems. These are SDEs of the form

$$dX_t = -\nabla V(t, X_t)dt + \sigma(X_t)dW_t, \tag{3.28}$$

where $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ is T-periodic, $\nabla = (\partial_1, \cdots, \partial_d)$ is the spatial gradient operator, W_t denotes a d-dimensional Brownian motion and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. Note that the T-periodicity of V implies the T-periodicity of ∇V, hence the gradient SDE (3.28) is T-periodic.

Gradient systems arise naturally in physical applications, where V is referred to as the potential function [Gar09, MSH02, Pav14]. Indeed examples of T-periodic gradient systems, includes the periodically forced Ornstein-Uhlenbeck from Example 3.10 derived from $V(t,x) = \frac{\alpha}{2}\left(x - \frac{S(t)}{\alpha}\right)^2$ and the Duffing Oscillator from Example 3.11 derived from double-well potential $V(t,x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + A\cos(\omega t)x$. In fact, it is easy to verify that Theorem 3.9 is a special case of gradient SDEs derived from potential $V(t,x) = \sum_{i=1}^d \sum_{k=1}^{2p_i} \frac{S_k^i(t)}{k+1} x_i^{k+1}$. While these examples are weakly dissipative, where the Euclidean norm is a suitable Foster-Lyapunov function satisfying (3.16). In general, finding a Foster-Lyapunov function satisfying (3.16) for a given SDE is generally non-trivial (if at all possible), particularly in higher dimensions. A mathematical advantage of gradient systems is that V itself is a natural choice of Foster-Lyapunov function to satisfy (3.16). This is apparent by observing the generator of (3.28) is given by

$$\mathcal{L}(t)V(t,x) = \partial_t V(t,x) - \|\nabla V(t,x)\|^2 + \frac{1}{2}\sum_{i,j=1}^d \left(\sigma\sigma^T(x)\right)_{ij} \partial_{ij}^2 V(t,x), \tag{3.29}$$

and exploiting the normed gradient term.

We contrast briefly with its autonomous counterpart. Let $U = U(x)$ be a norm-like potential and consider the (autonomous) gradient SDE

$$dX_t = -\nabla U(X_t)dt + \sigma dW_t, \tag{3.30}$$

where $\sigma \in \mathbb{R}^+\backslash\{0\}$ and W_t is a d-dimensional Brownian motion. It is well-known [MV99, Gar09, Pav14] the invariant measure of this SDE has a particularly simple form and is given by (upon normalisation)

$$\pi(\Gamma) = \int_\Gamma \exp\left(-\frac{2U(x)}{\sigma^2}\right) dx, \quad \Gamma \in \mathcal{B}.$$

On the other hand, due to the intricate interplay between stochasticity and periodicity, periodic measures (with a minimal positive period) does not have such simple expression i.e.

$$\rho_s(\Gamma) \neq \int_\Gamma \exp\left(-\frac{2V(s,x)}{\sigma^2}\right) dx$$

in general. Indeed the periodic measure (3.26) from Example 3.10 does not take this simple form.

The following corollary of Theorem 3.6 is generally simple to verify to yield gradient SDEs with a geometric periodic measure.

Corollary 3.13. *Assume σ satisfy (3.6), (3.7) and (3.8). Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ be a norm-like function such that for all $n \in \mathbb{N}$*

$$\partial^\alpha V \quad \text{bounded on } \mathbb{R}^+ \times B_n, \alpha \in \mathbb{N}^{d+1}, |\alpha| \in \{1, d+1\},$$

and (3.16) holds, where $\mathcal{L}(t)$ is given by (3.29). Then the results of Theorem 2.18 holds for SDE (3.28).

While Corollary 3.13 covers all the examples considered thus far, it applies to a wider class of SDEs than that of weakly dissipative systems. In the next proposition, we use Corollary 3.13 to extend the case of Theorem 3.9 when $p_i = \text{const}$ for all i and allowing for products of the spatial variables. It does not aim to be most general however suffices a range of applications. We shall employ more multi-index notation: for spatial variables $x = (x_1, \cdots, x_d)$ and multi-index $\alpha \in \mathbb{N}^d$, define $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For $\alpha, \beta \in \mathbb{N}^d$, we have the partial ordering $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for each $1 \leq i \leq d$. We define the standard tuple basis $e_i = (0, \cdots, 1, \cdots, 0)$ where the 1 appears on the i'th index. For fixed $\beta \in \mathbb{N}^d$, we define $\sum_{\alpha \geq \beta}^{N} C_\alpha := \sum_{\alpha \geq \beta}^{|\alpha| \leq N} C_\alpha$. Recall standard asymptotic notation where for functions $f_1, f_2, g : \mathbb{R}^d \to \mathbb{R}$, we write $\max\{f_1, f_2\} = o(g)$ if $\lim_{\|x\| \to \infty} \frac{\max\{|f_1(x)|, |f_2(x)|\}}{g(x)} = 0$. This implies that for any $\epsilon > 0$, there exists $R > 0$ such that

$$\max\{|f_1(x)|, |f_2(x)|\} \leq \epsilon |g(x)|, \quad x \in B_R^c. \tag{3.31}$$

Proposition 3.14. *Assume σ satisfy (3.6), (3.7) and (3.8). Let $\{S_\alpha(t)\}_{\alpha \in \mathbb{N}^d}$ be continuously differentiable T-periodic functions and $\{S_i\}_{i=1}^d$ are strictly positive constants. Then the gradient system (3.28) with potential*

$$V(t, x) = \sum_{i=1}^{d} S_i x_i^p + \sum_{|\alpha|=0}^{p-1} S_\alpha(t) x^\alpha, \quad p \in 2\mathbb{N} := \{2, 4, ..., \},$$

satisfies Corollary 3.13 hence the results of Theorem 2.18 holds.

Proof. We compute

$$\begin{cases} \partial_t V = \sum_{|\alpha|=0}^{p-1} \dot{S}_\alpha x^\alpha, \\ \partial_i V = S_i p x_i^{p-1} + \sum_{\alpha \geq e_i}^{p-1} \alpha_i S_\alpha x^{\alpha - e_i}, \\ \partial_{ii}^2 V = p(p-1) S_i x_i^{2p-2} + \sum_{\alpha \geq 2e_i}^{p-1} \alpha_i(\alpha_i - 1) S_\alpha x^{(\alpha - 2e_i)}, \\ \partial_{ij}^2 V = \sum_{\alpha \geq e_i + e_j}^{p-1} S_\alpha \alpha_i \alpha_j x^{\alpha - e_i - e_j}, & i \neq j. \end{cases}$$

So

$$\|\nabla V\|^2 = \sum_{i=1}^{d} (\partial_i V)^2 = \sum_{i=1}^{d} \left[S_i^2 p^2 x_i^{2p-2} + 2 S_i p \sum_{\alpha \geq e_i}^{p-1} \alpha_i S_\alpha x^{\alpha + (p-2)e_i} + \left(\sum_{\alpha \geq e_i}^{p-1} \alpha_i S_\alpha x^{\alpha - e_i} \right)^2 \right].$$

Note that V,$\partial_t V$, $\partial_{ij}^2 V$ and $\left(\|\nabla V\|^2 - \sum_{i=1}^d S_i^2 p^2 x_i^{2p-2} \right)$ has maximum order p, $p-1$, $p-3$ and $2p-3$ respectively. Our assumptions ensures that $\max_{\alpha \in \mathbb{N}^d} (\sup_{t \in \mathbb{R}} |S_\alpha(t)|) < \infty$ and $\max_{i,j} \sup_{x \in \mathbb{R}^d} (\sigma \sigma^T)_{ij}(x) < \infty$. Since higher even powers dominates lower powers i.e. $x^\alpha = o(\sum_{i=1}^d c_i x_i^{2n})$ where $c_i > 0$ and

$|\alpha| < 2n$ where $n \in \mathbb{N}$, we have for any $\lambda > 0$

$$\max\left\{\lambda V, \partial_t V, \partial_{ij}^2 V, \left(\|\nabla V\|^2 - \sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right)\right\} = o\left(\sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right), \quad 2 < p \in 2\mathbb{N}.$$

Then for $2 < p \in 2\mathbb{N}$, by (3.31), for any $\epsilon \in (0, \frac{1}{4})$, there exists $R > 0$ such that

$$\begin{aligned}&\mathcal{L}(t)V + \lambda V \\ &\le |\partial_t V| - \|\nabla V\|^2 + \frac{1}{2}|\sum_{i,j=1}^d a_{ij} V| + \lambda V \le (4\epsilon - 1)\left(\sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}\right) \le 0, \quad x \in B_R^c.\end{aligned}$$

By continuity, $\mathcal{L}(t)V + \lambda V$ is bounded on B_R. Hence (3.16) is satisfied. For $p = 2$ where V and $\sum_{i=1}^d S_i^2 p^2 x_i^{2p-2}$ are of the same order, the same calculations holds provided one restricts $0 < \lambda < 4 \min_i S_i^2$. □

3.5 Periodically Forced SDEs

In physics literature, "periodically forced" or "periodically driven" generally refers to the addition of a periodic term on the drift which otherwise be autonomous i.e. $b(t, x) = b_0(x) + S(t)$ for some periodic function S and drift b_0 independent of t. Particular instances of Proposition 3.14 include periodically-forced systems such Example 3.10 and Example 3.11. Periodically forced gradient SDEs have a wide range of physical applications including the examples already considered. More generally, periodically forced systems have been applied to modulated Josephson-junctions systems, superionic conductors, excited chicken hearts to the dithered ring lasers as well as other laser systems [ZMJ90, Jun93]. For further discussions on periodically forced stochastic systems, we refer readers to the monograph [Jun93] for theory and applications.

Examples so far are systems with polynomial potentials. While polynomial approximation of potentials (by Weierstrass approximation theorem for instance) can be effective for practical reasons, we consider periodically forced gradient systems that need not be derived from a polynomial potential.

Consider again the autonomous gradient SDE on $\mathbb{R}^d$ (3.30),where σ satisfy (3.6), (3.7) and (3.8) and $U \in C^2(\mathbb{R}^d, \mathbb{R}^+)$ satisfies the (autonomous) geometric drift condition

$$LU \le C - \lambda U \quad \text{on } \mathbb{R}^d, \tag{3.32}$$

where $C \geq 0, \lambda > 0$ are constants and L is the infinitesimal generator of (3.30) given by

$$Lf(x) = -\langle \nabla U(x), \nabla f(x) \rangle + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T(x))_{ij} \partial_{ij}^2 f(x), \quad f \in C^2(\mathbb{R}^d).$$

This classical geometric drift condition yields the existence, uniqueness and ergodicity of an invariant measure. The context of the next lemma sufficiently yields a geometric periodic measure when the autonomous gradient system is periodically forced. Essentially, the autonomous system retains its stability up to replacing its invariant measure for a periodic measure with a minimal positive period. Note that we do not impose any particular form imposed on the potential, hence more general than polynomials. We note that the assumptions are easily satisfied for many practical systems.

Proposition 3.15. *Let $U \in C^2(\mathbb{R}^d, \mathbb{R}^+)$ be a norm-like potential satisfying (3.32) and that for any $c_1, c_2 > 0$, there exists a compact set $K = K(c_1, c_2) \in \mathcal{B}(\mathbb{R}^d)$ such that*

$$c_1\|x\| \leq c_2 U(x) \quad x \in K^c.$$

Then for any T-periodic ($T > 0$) continuously differentiable function $S : \mathbb{R}^+ \to \mathbb{R}^d$, the periodically forced gradient SDE

$$dX_t = -\left[\nabla U(X_t) + S(t)\right] dt + \sigma(X_t) dW_t$$

possesses a unique geometric periodic measure with a minimal positive period.

Proof. By Theorem 3.6, we verify $V(t,x) = U(x) - \langle S(t), x \rangle$ satisfies (3.16). By the assumptions on U and S, it is clear that $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ is a T-periodic norm-like potential satisfying the regularity assumptions of Corollary 3.13. Since $\partial_{ij}^2 V = \partial_{ij}^2 U$, we compute that

$$\begin{aligned}
\mathcal{L}(t)V &= -\langle \dot{S}, x \rangle - \langle \nabla U(X_t) + S(t), \nabla U(X_t) - S(t) \rangle + \sum_{i,j=1}^{d} (\sigma\sigma^T(x))_{ij} \partial_{ij}^2 V \\
&= -\langle \dot{S}, x \rangle - \|\nabla U\|^2 + \|S\|^2 + \sum_{i,j=1}^{d} (\sigma\sigma^T(x))_{ij} \partial_{ij}^2 U \\
&= \|S\|^2 - \langle \dot{S}, x \rangle + LU.
\end{aligned}$$

As U satisfies the geometric drift condition, by picking any fixed $\lambda^- \in (0,\lambda)$, we have

$$\begin{aligned}
\mathcal{L}(t)V &\leq \|S\|^2 - \langle \dot{S}, x\rangle + C - \lambda U \\
&= \|S\|^2 - \langle \dot{S}, x\rangle + C - (\lambda-\lambda^-)U - \lambda^- U + (\lambda-\lambda^-)\langle S, x\rangle - (\lambda-\lambda^-)\langle S, x\rangle \\
&= \|S\|^2 - \langle \dot{S} + (\lambda-\lambda^-)S, x\rangle + C - (\lambda-\lambda^-)V - \lambda^- U \\
&\leq \|S\|^2 + \|\dot{S} + (\lambda-\lambda^-)S\|_\infty \|x\| + C - (\lambda-\lambda^-)V - \lambda^- U,
\end{aligned}$$

where $\|\dot{S} + (\lambda-\lambda^-)S\|_\infty := \sup_{s\in[0,T]} \|\dot{S}(s) + (\lambda-\lambda^-)S(s)\| < \infty$ as S and $\dot{S}$ are bounded. Then, by assumption with $c_1 = \|\dot{S} + (\lambda-\lambda^-)S\|_\infty$ and $c_2 = \lambda^-$, we have a compact set $\mathcal{B}(\mathbb{R}^d)$ such that

$$c := \sup_{x\in K} \Big(\|\dot{S} + (\lambda-\lambda^-)S\|_\infty \|x\| - \lambda^- U \Big) < \infty.$$

Hence $\mathcal{L}(t)V \leq (C + c + \|S\|^2) - (\lambda-\lambda^-)\, V$ i.e. the geometric drift condition (3.16) is satisfied. Since S has a minimal positive period $T > 0$, by Proposition 2.15 the periodic measure will have a minimal positive period. □

3.6 Langevin Dynamics

Langevin equations originated to model noisy molecular systems and many other physical phenomena. As such, we expect applications to the physical sciences. In fact, we shall see it extends easily from stochastic gradient systems in an "overdamped" limit and applies immediately to the stochastic periodically-forced harmonic oscillator. We refer the reader to [Zwa01, Pav14] for further applications, details and derivations of Langevin equations. Akin to earlier sections, we give sufficient conditions for the existence, uniqueness and geometric convergence of a periodic measure for T-periodic Langevin equations. We study Langevin equations of the form

$$m d\dot{q}_t = (F(t,q_t) - \gamma \dot{q}_t)\, dt + \sigma dW_t, \tag{3.33}$$

with position $q_t \in \mathbb{R}^d$, velocity $\dot{q}_t \in \mathbb{R}^d$, acceleration $\ddot{q}_t \in \mathbb{R}^d$, constant mass $m > 0$, time-dependent force $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, d-dimensional Brownian motion W_t and constant matrix $\sigma \in GL(\mathbb{R}^d)$. For $\gamma \geq 0$, $\gamma\dot{q}_t$ is understood as the frictional force of the system. The proportional constant γ is referred as the damping constant. Without loss of generality, we take mass to be unit i.e.

$m = 1$.

Denote momentum $p_t = \dot{q}_t$, then (3.33) can be rewritten as a system of first order SDEs

$$\begin{cases} dq_t = p_t dt, \\ dp_t = \left(-\gamma p_t + F(t, q_t)\right) dt + \sigma dW_t. \end{cases} \tag{3.34}$$

To avoid possible confusion, we say Langevin equation (3.34) is T-periodic if F is T-periodic in time.

In phase space coordinates $X_t = (q_t, p_t) \in \mathbb{R}^{2d}$ this can be rewritten as

$$dX_t = b(t, X_t)dt + \Sigma d\mathcal{W}_t, \tag{3.35}$$

where

$$b(t,x) = b(t,q,p) = \begin{pmatrix} p \\ -\gamma p + F(t,q) \end{pmatrix} \in \mathbb{R}^{2d}, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} \in \mathbb{R}^{2d\times 2d}, \quad \mathcal{W}_t = \begin{pmatrix} 0 \\ W_t \end{pmatrix}. \tag{3.36}$$

On a physical level, observe that the noise is degenerate in that the noise affects q_t only through p_t. Formally, Langevin SDE (3.35) is degenerate since $\Sigma \notin GL(\mathbb{R}^{2d})$. Resultantly, Theorem 3.2 is not immediately applicable. In this thesis, we study Langevin dynamics with additive noise and leave the multiplicative noise case for future works.

Written in phase space coordinates, by classical arguments, (3.33) has unique solution provided b and σ are Lipschitz or locally Lipschitz and a Lyapunov function satisfying the regualrity condition (3.5). Labelling $x = (q,p) = (x_1, .., x_{2d})$, the infinitesimal generator is given by

$$\mathcal{L}(t)f(t,x) = \partial_t f + \langle p, \nabla_q f\rangle + \langle -\gamma p + F, \nabla_p f\rangle + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}\partial^2_{p_i p_j} f, \quad f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^{2d}), \tag{3.37}$$

where $\nabla_q := (\partial_{q_1}, \cdots, \partial_{q_d})^T$ and similarly $\nabla_p := (\partial_{p_1}, \cdots, \partial_{p_d})^T$.

Remark 3.16. We remark that in physical applications concerning small particles, the mass is typically small. This suggest the inertia term $m\ddot{q}_t$ can be neglected. Hence, informally, the dynamics (3.33) can be well-approximated by

$$0 = F(t, q_t) - \gamma \dot{q}_t + \sigma dW_t.$$

i.e. reduced to SDEs studied earlier in this section. Suggesting that Langevin equations may be studied with multiplicative noise in the context of small particles. A particular source of interesting dynamics and applications is the case when $F(t,q) = -\nabla_q V(t,q)$ for some potential $V(t,q)$ and so the Langevin equations are gradient systems (provided $\gamma > 0$). Such systems without inertia are called overdamped Langevin dynamics.

With the inapplicability of Theorem 3.2, we instead use the following lemma for the irreducibility of non-autonomous Langevin equation with additive noise. The lemma can be seen as the non-autonomous counterpart to the one seen in [MSH02], we provide a proof for completeness. We shall be explicit with norms for the proof for clarity.

Lemma 3.17. *Consider T-periodic Langevin equation (3.34) with locally Lipschitz F. Assume there exists a norm-like function V satisfying (3.5). Then the Markov transition kernel satisfies $P(s,t,x,\Gamma) > 0$ for any $s < t < \infty, x \in \mathbb{R}^{2d}$ and non-empty open $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$.*

Proof. It suffices to show $P(s,t,x,B_\delta(y)) > 0$ for any $x, y \in \mathbb{R}^{2d}$ and any $\delta > 0$. This is clear because one can take $y \in \Gamma$ and choose sufficiently small $\delta > 0$ such that $B_\delta(y) \subseteq \Gamma$. We begin by choosing any smooth function ξ such that

$$\begin{cases} (\xi(s), \dot{\xi}(s))^T = x \\ (\xi(t), \dot{\xi}(t))^T = y \end{cases}$$

Many such ξ,$\dot{\xi}$ exists e.g. linear function between the two points or more generally, a polynomial interpolation. By the invertibility of σ and the second order ODE to define w

$$\begin{cases} \frac{dw_t}{dt} = \sigma^{-1}\left(\frac{d\xi_t^2}{dt^2} + \gamma\frac{d\xi_t}{dt} - F(t,\xi_t)\right), \\ w_s = 0. \end{cases}$$

Observe that w is as smooth as F (since σ^{-1} commutes with the time-derivatives and ξ is smooth). Hence w solves the associated control problem

$$\begin{cases} \frac{d^2\xi_t}{dt^2} = -\gamma\frac{d\xi_t}{dt} + F(t,\xi_t) + \sigma\frac{dw_t}{dt}, \\ (\xi(s), \dot{\xi}(s))^T = x, \\ (\xi(t), \dot{\xi}(t))^T = y. \end{cases}$$

Let $\zeta_t = (\xi_s, \dot{\xi}_s)^T$ and $\omega_t = (0, w_t)^T$, then in terms of phase space coordinates, we have

$$\begin{cases} X_t = x + \int_s^t b(r, X_r)dr + \Sigma\mathcal{W}_t, \\ \zeta_t = x + \int_s^t b(r, \zeta_r)dr + \Sigma\omega_t, \end{cases}$$

where b as given by (3.36). By Lemma A.1 (of Appendix A), X_t is bounded on any bounded interval. Hence by the locally Lipschitz of F, using Cauchy-Schwartz and denoting we have for $x_i = (q_i, p_i)$,

$$\begin{aligned}
&\|b(r,x_1) - b(r,x_2)\|^2_{\mathbb{R}^{2d}} \\
&= \left\| \begin{pmatrix} p_1 - p_2 \\ -\gamma(p_1 - p_2) + F(r,q_1) - F(r,q_2) \end{pmatrix} \right\|^2_{\mathbb{R}^{2d}} \\
&= \left(\gamma^2 + 1\right) \|p_1 - p_2\|^2_{\mathbb{R}^d} + \|F(r,q_1) - F(r,q_2)\|^2_{\mathbb{R}^d} - 2\gamma \langle p_1 - p_2, F(r,q_1) - F(r,q_2)\rangle_{\mathbb{R}^d} \\
&\le \left(\gamma^2 + 1\right) \|p_1 - p_2\|^2_{\mathbb{R}^d} + L^2 \|q_1 - q_2\|^2_{\mathbb{R}^d} + 2\gamma \|p_1 - p_2\|_{\mathbb{R}^d} \|F(r,q_1) - F(r,q_2)\|_{\mathbb{R}^d} \\
&\le \left(\gamma^2 + 1\right) \|p_1 - p_2\|^2_{\mathbb{R}^d} + L^2 \|q_1 - q_2\|^2_{\mathbb{R}^d} + 2\gamma L \|p_1 - p_2\|_{\mathbb{R}^d} \|q_1 - q_2\|_{\mathbb{R}^d} \\
&\le \|p_1 - p_2\|^2_{\mathbb{R}^d} + (\gamma \|p_1 - p_2\|_{\mathbb{R}^d} + L \|q_1 - q_2\|_{\mathbb{R}^d})^2 \\
&\le (c+1) \|p_1 - p_2\|^2_{\mathbb{R}^d} + c \|q_1 - q_2\|^2_{\mathbb{R}^d} \\
&\le M^2 \|x_1 - x_2\|^2_{\mathbb{R}^{2d}} ,
\end{aligned}$$

where it was used that there exists a constant $c = c(\gamma, L) \ge \max\{\gamma, L\}$ such that $(\gamma x + Ly)^2 \le c(x^2 + y^2)$ for all $x, y \in \mathbb{R}$. Hence b is locally Lipschitz with constant $M := \sqrt{c+1}$. Thus by Gronwall's inequality,

$$\begin{aligned}
\|X_t - \zeta_t\|_{\mathbb{R}^{2d}} &\le \int_s^t \|b(r, X_r) - b(r, \zeta_r)\|_{\mathbb{R}^{2d}}\, dr + \|\Sigma\|_2 \|\mathcal{W}_t - \omega_t\|_{\mathbb{R}^{2d}} \\
&\le \int_s^t M \|X_r - \zeta_r\|_{\mathbb{R}^{2d}}\, dr + \|\sigma\|_2 \|W_t - w_t\|_{\mathbb{R}^d} \\
&\le \|\sigma\|_2 \|W_t - w_t\|_{\mathbb{R}^d}\, e^{M(t-s)}.
\end{aligned}$$

Now, by (a generalised) Lévy's forgery theorem [Fre83] or see [MSH02] and Theorem 4.20 of [Str82], we have for any fixed $\epsilon > 0$

$$\mathbb{P}\left(\sup_{s \le r \le t} \|W_r - w_r\|_{\mathbb{R}^d} \le \epsilon \right) > 0.$$

i.e. the Brownian motion approximates any continuous function under the uniform norm with positive probability. Hence, for sufficiently small $\epsilon = \epsilon(\delta)$, one has that $\mathbb{P}\left(\|X_t - \xi_t\|_{\mathbb{R}^{2d}} \leq \delta\right) > 0$ hence $P(s,t,x,B_\delta(y)) > 0$. $\square$

We now state and prove the following Langevin counterpart of Theorem 3.6.

Theorem 3.18. *Consider T-periodic Langevin equation (3.35) with F satisfying (3.9) (in place of b). Assume there exists a norm-like function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^{2d}, \mathbb{R}^+)$ satisfying (3.16) where $\mathcal{L}$ is given by (3.37). Then there exists a unique geometric periodic measure $\rho : \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^{2d})$ satisfying the convergences from Theorem 2.18.*

Proof. Let σ_i denote the i'th column of σ then $\Sigma_i = (0,\sigma_i)^T$ denote the i'th column of Σ. Denoting $Id \in \mathbb{R}^{d\times d}$ to be the identity matrix, observe that Lie bracket

$$[\Sigma_i, b] = (Db)\Sigma_i = \begin{pmatrix} 0 & Id \\ -d^2F(t,q) & -\gamma Id \end{pmatrix}\begin{pmatrix} 0 \\ \sigma_i \end{pmatrix} = \begin{pmatrix} \sigma_i \\ -\gamma\sigma_i \end{pmatrix}.$$

Since $\sigma \in GL(\mathbb{R}^d)$, the columns σ_i are linear independent. Hence the Lie algebra generated by Σ_i and b spans $\mathbb{R}^{2d}$. By the assumptions on F, b satisfies (3.9). Hence together with Foster-Lyapunov function V, there exists a smooth density $p(s,t,x,y)$ with respect to Λ by Theorem 1 of [HLT17] is satisfied. F is locally Lipschitz hence with Lemma 3.17, Theorem 2.20 holds. Hence the assumptions of Theorem 2.18 are satisfied. $\square$

3.7 Density of Periodic Measures

The Fokker-Planck equation is a well-known second-order linear parabolic PDE that describes the time evolution of the probability density function associated to SDEs and moreover the stationary solution to the Fokker-Planck equation is the density of invariant measures see e.g. [BKRS15, Ris96, Has12, Pav14, Gar09, Zwa01]. The existence and uniqueness of Fokker-Planck equation have been studied in many settings including irregular coefficients and time-dependent coefficients [LL08, BKRS15, RZ10, DR12].

In the context of periodic measures of SDEs, like invariant measure densities, it is interesting and important to know when periodic measures possess a density with respect to Lebesgue measure. In this section, we show also that the periodic measure density is the time-periodic solution of the Fokker-Planck PDE. We show that this is a necessarily and sufficient relation. Evidently then, solving the Fokker-Planck for time-periodic solutions opens another door to attain periodic measures and its density as well as their properties via PDE methods. For concreteness, we provide an explicit example for the periodically forced Ornstein-Uhlenbeck process where its periodic measure density is given and shown to satisfy the Fokker-Planck equation.

Owing to the physical intuition of the Fokker-Planck equation, we note that time-periodic solution of the Fokker-Planck has been studied previously [Jun89, CHLY17, JQSY19], however its relationship to periodic measure is formally established here. Furthermore, on a more fundamental level, since the Fokker-Planck is a parabolic PDE, this result establishes our first time-periodic Feynman-Kac duality for time-periodic SDEs. The Feynman-Kac duality presented in this thesis differs to the one in classical literature where one seeks a time-periodic solution of a parabolic PDE rather than a terminal-valued problem.

In previous sections, we have predominantly been focused on initial state, here we change our perspective to the forward spatial variable. As such, at the risk of confusion, we interchange the roles of x and y i.e. we take $y \in \mathbb{R}^d$ to be the initial state and $x \in \mathbb{R}^d$ to be the forward spatial variable.

We first show that for time-periodic SDEs, if the periodic measure Markov transition density exists, then the periodic measure density exists and provide a relation for it. It was seen in Theorem3.3 the sufficient conditions to imply the existence of the density $p(s,t,y,x)$ of the two-parameter Markov transition kernel, $P(s,t,y,\cdot)$ and the existence of a periodic measure. Let $(\rho_t)_{t\in\mathbb{R}^+}$ be a family of probability measures satisfying $\rho_t = P^*(s,t)\rho_s$ for $s \leq t$, then by Fubini's theorem yields

$$\begin{aligned} \rho_t(\Gamma) &= \int_{\mathbb{R}^d} P(s,t,y,\Gamma)\rho_s(dy) \\ &= \int_\Gamma \left(\int_{\mathbb{R}^d} p(s,t,y,x)\rho_s(dy) \right) dx, \quad \Gamma \in \mathcal{B}(\mathbb{R}^d), s \leq t. \end{aligned} \tag{3.38}$$

If ρ is a periodic measure, then

$$\rho_t(\Gamma) = \rho_{t+T}(\Gamma) = \int_\Gamma \left(\int_{\mathbb{R}^d} p(t,t+T,y,x)\rho_t(dy) \right) dx,$$

that is to say $q(t,x) = \int_{\mathbb{R}^d} p(t,t+T,y,x)\rho_t(dy)$ is the periodic measure density of ρ_t as observed in [FLZ19].

Given the existence of q, it is clear by (3.38) that q satisfies

$$q(t,x) = \int_{\mathbb{R}^d} p(s,t,y,x)q(s,y)dy, \quad s \leq t, \tag{3.39}$$

indeed this property holds for any family of measures with densities. It is well-known that q satisfies the following Fokker-Planck equation

$$\begin{cases} \partial_t q = L^*(t)q, \\ \lim_{t\downarrow s} q(t,\cdot) = q(s,\cdot). \end{cases}$$

where $L^*(t)$ is the Fokker-Planck operator given by

$$L^*(t)q = -\sum_{i=1}^{d} \partial_{x_i}(b_i(t,x)q) + \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{x_i x_j}\left(\left(\sigma\sigma^T(t,x)\right)_{ij} q\right). \tag{3.40}$$

In this section, we will always assume b_i are continuously x-differentiable, σ_{ij} are twice continuously x-differentiable and the operator $L^*(t)$ is uniformly elliptic i.e. there exists $\lambda > 0$ such that $\langle \xi, \sigma\sigma^T(t,x)\xi\rangle \geq \lambda\|\xi\|^2$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$.

In the following, we shall use the notation $X \sim q$ to denote that the random variable X is distributed by probability density $q \in L^1(\mathbb{R}^d)$. For random variables X^0 and X^1, we write $X^0 \sim X^1$ if they have the same distribution. We state and prove the following useful lemma.

Lemma 3.19. *Assume $(X_t^0)_{t\geq s}, (X_t^1)_{t\geq s+T}$ are two processes satisfying the T-periodic SDE (3.1). If $X_s^0 \sim X_{s+T}^1$ then $X_{s+t}^0 \sim X_{s+T+t}^1$ for all $t \geq 0$.*

Proof. For concreteness, let $X_s^0 \sim X_{s+T}^1 \sim q \in L^1(\mathbb{R}^d)$ and $p^0(s+t,\cdot)$ denote the distribution of X_{s+t}^0 and similarly $p^1(s+T+t,\cdot)$ for X_{s+T+t}^1. Then p^k satisfies the Fokker-Planck equation i.e. for $k=0,1$ and $t \geq 0$

$$\begin{cases} \partial_t p^k(t+kT,x) = L^*(t+kT)p^k(t+kT,x), \\ p^k(s+kT,\cdot) = q. \end{cases}$$

It is clear that $L^*(t) = L^*(t+T)$ by the T-periodic coefficients. By the linearity of the Fokker-Planck operator, it is easy to see that $\hat{p}(t,\cdot) := p^0(s+t,\cdot) - p^1(s+t+T,\cdot)$ satisfies

$$\begin{cases} \partial_t \hat{p} = L^*(t)\hat{p} \quad t \geq 0, \\ \hat{p}(0, \cdot) = 0. \end{cases}$$

Then an application of parabolic maximum principle or otherwise yields that $\hat{p}(t, \cdot) = 0$ for all $t \geq 0$ is the only physical solution. Hence concluding $p^0(s + t, \cdot) = p^1(s + T + t, \cdot)$ for all $t \geq 0$. □

With Lemma 3.19, we are now ready to state the main result of this section.

Theorem 3.20. *Consider T-periodic SDE (3.1) with $b_i \in C^{1,1}(\mathbb{R}^+ \times \mathbb{R}^d)$ and $\sigma_{ij} \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$. For $q \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ define $\rho : \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^d)$ by*

$$\rho_t(\Gamma) = \frac{1}{\|q(t, \cdot)\|_{L^1(\mathbb{R}^d)}} \int_\Gamma q(t, x)dx, \quad t \geq 0, \Gamma \in \mathcal{B}(\mathbb{R}^d). \tag{3.41}$$

Then ρ is a T-periodic measure if and only if

$$\begin{cases} \partial_t q = L^*(t)q, \\ q(0, \cdot) = q(T, \cdot). \end{cases} \tag{3.42}$$

Hence, if there exists a unique solution to (3.42), then there is a unique periodic measure with density q.

Proof. For notational convenience and without loss of generality, we let $q(t, x)$ be normalised. Assume ρ is a T-periodic measure with density q, then by definition, $\rho_t = \rho_{t+T}$ for all $t \geq 0$ i.e.

$$\int_\Gamma q(t, x)dx = \rho_t(\Gamma) = \rho_{t+T}(\Gamma) = \int_\Gamma q(t + T, x)dx, \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

As this holds for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, it follows that $q(t, \cdot) = q(t+T, \cdot)$. On the other hand, it is well known that $p(s, t, y, x)$ satisfies the Fokker-Planck equation

$$\partial_t p(s, t, y, x) = L^*(t)p(s, t, y, x).$$

Hence taking derivative with respect to t on both sides of (3.39), we have

$$\begin{aligned}
\partial_t q(t,x) &= \int_{\mathbb{R}^d} \partial_t p(s,t,y,x) q(s,y) dy \\
&= \int_{\mathbb{R}^d} L^*(t) p(s,t,y,x) q(s,y) dy \\
&= \int_{\mathbb{R}^d} -\sum_{i=1}^d \partial_{x_i} (b_i(t,x) p(s,t,y,x)) q(s,y) dy \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \left(\left(\sigma\sigma^T(t,x)\right)_{ij} p(s,t,y,x) \right) q(s,y) dy \\
&:= A + B.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
A &= -\sum_{i=1}^d \int_{\mathbb{R}^d} \left[\partial_{x_i}(b_i(t,x)) p(s,t,y,x) + b_i(t,x) \partial_{x_i}(p(s,t,y,x)) \right] q(s,y) dy \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t,x)) \int_{\mathbb{R}^d} p(s,t,y,x) q(s,y) dy - \sum_{i=1}^d b_i(t,x) \partial_{x_i} \int_{\mathbb{R}^d} p(s,t,y,x) q(s,y) dy \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t,x)) q(t,x) - \sum_{i=1}^d b_i(t,x) \partial_{x_i} q(t,x) \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t,x) q(t,x)).
\end{aligned}$$

Similarly, for the second term, we have

$$B = \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \left(\left(\sigma\sigma^T(t,x)\right)_{ij} q(t,x) \right).$$

Therefore, the density function $q(\cdot,\cdot)$ satisfies

$$\begin{cases} \partial_t q = L^*(t) q, \\ q(t,\cdot) = q(t+T,\cdot) \quad \text{for all } t \geq 0. \end{cases}$$

By Lemma 3.19, it suffices that this PDE holds specifically for $t = 0$ hence we have (3.42).

To prove the converse, suppose (3.42) holds. Then Lemma 3.19 yields that $q(t,\cdot) = q(t+T,\cdot)$ for all $t \geq 0$. Thus by the construction (3.41), ρ is

T-periodic. By (3.39) and Fubini's theorem, it is clear that for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned} P^*(s,t)\rho_s(\Gamma) &:= \int_{\mathbb{R}^d} P(s,t,y,\Gamma)\rho_s(dy) \\ &= \int_{\mathbb{R}^d} \left[\int_\Gamma p(s,t,y,x)dx \right] q(s,y)dy \\ &= \int_\Gamma q(t,x)dx \\ &= \rho_t(\Gamma), \end{aligned}$$

concluding that ρ is a T-periodic measure. $\square$

There is an "alternative" way to arrive the PDE of Theorem 3.20 as seen in [Jun89]. By considering lifted coordinates (t, X_t), one can consider stationary solutions of the lifted Fokker-Planck operator $\mathcal{L}^*$ i.e. $q(t,x)$ satisfying

$$\mathcal{L}^*(t)q := -\partial_t q(t,x) + L^*(t)q(t,x) = 0. \tag{3.43}$$

This is equivalent to (3.42) upon rearranging. However, this approach does not naturally imposes any boundary conditions, hence is not sufficient for q to be the density of the periodic measure. Theorem 3.20 shows that the (periodic) boundary conditions is necessary. While [Jun89] imposes the periodic boundaries, the reasoning does not seem apparent. We shall show in the example below that, despite $L^*(t)$ is T-periodic, a solution to the PDE need not be periodic. In fact, relaxing the boundary condition can lead to infinitely many solutions.

Example 3.21. The one-dimensional periodically-forced Ornstein-Uhlenbeck process of Example 3.10 has its Fokker-Planck operator given explicitly by

$$\begin{aligned} \mathcal{L}^*(t)q &= -\partial_t q - \partial_x((S(t) - \alpha x)q) + \frac{\sigma^2}{2}\partial_x^2 q \\ &= -\partial_t q - S(t)\partial_x q + \alpha q + \alpha x \partial_x q + \frac{\sigma^2}{2}\partial_x^2 q \end{aligned}$$

and the periodic measure is $\rho_t = \mathcal{N}\left(\xi(t), \frac{\sigma^2}{2\alpha}\right)$, where $\xi(t) = e^{-\alpha t}\int_{-\infty}^t e^{\alpha r}S(r)dr$. Here, the density of the periodic measure is given by

$$q(t,x) = \frac{1}{\sqrt{\pi\sigma^2/\alpha}} \exp\left(-\frac{(x-\xi(s))^2}{\sigma^2/\alpha} \right).$$

We compute

$$\dot{\xi} = -\alpha\xi + S(t), \quad \partial_t q = 2\frac{\alpha}{\sigma^2}\dot{\xi}(x-\xi)q, \quad \partial_x q = -2\frac{\alpha}{\sigma^2}(x-\xi)q,$$

and

$$\partial_x^2 q = -2\frac{\alpha}{\sigma^2}\left[\partial_x(xq) - \xi\partial_x q\right] = -2\frac{\alpha}{\sigma^2}\left[1 - 2\frac{\alpha}{\sigma^2}(x-\xi)^2\right]q.$$

Hence, substituting directly into (3.43),

$$\begin{aligned}
\frac{\mathcal{L}^*(t)q}{q} &= -2\frac{\alpha}{\sigma^2}\dot{\xi}(x-\xi) + 2\frac{\alpha}{\sigma^2}S(t)(x-\xi) + \alpha - 2\frac{\alpha^2}{\sigma^2}x(x-\xi) - \alpha\left[1 - 2\frac{\alpha}{\sigma^2}(x-\xi)^2\right] \\
&= 2\frac{\alpha^2}{\sigma^2}\xi(x-\xi) - 2\frac{\alpha^2}{\sigma^2}x(x-\xi) + 2\frac{\alpha^2}{\sigma^2}(x-\xi)^2 \\
&= 0.
\end{aligned}$$

Thus indeed the q satisfies (3.42).

We show that the periodic condition of (3.42) cannot simply be dropped because of periodic coefficients. From (3.25), the transition density

$$p(s,t,y,x) = \frac{1}{\sqrt{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha(t-s)})}} \exp\left(-\frac{(x - e^{-\alpha(t-s)}y - J(s,t)}{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha(t-s)})}\right)$$

satisfies

$$-\partial_t p(t,x) + L^*(t)p(t,x) = 0,$$

for every fixed initial time s and point y. However, p is not periodic as J is not periodic. Since there is a non-periodic solution for every $y \in \mathbb{R}$, there are, in fact, infinite number of solutions to the PDE if one relaxes the demand of periodicity. As a consequence of Theorem 2.18, it is expected that the transition density converges geometrically to a periodic solution. Indeed in general, we do not expect this convergence.

4 Expected Exit Time and Duration

4.1 Definitions and Notations

As motivated in the introduction, we now turn our attention studying the expected exit time. Consider a stochastic process $(X_t)_{t\geq s}$ on $\mathbb{R}^d$ with continuous sample-paths and an open non-empty (possibly unbounded) domain $D \subset \mathbb{R}^d$ with boundary ∂D. Without loss of generality, we assume throughout this thesis that D is connected. Indeed if D is disconnected, one can solve separately on each connected subset. We define the first exit time from the domain D (or first passage time or first hitting time to the boundary) by

$$\eta_D(s,x) := \inf_{t\geq s}\{X_t \notin D | X_s = x\} = \inf_{t\geq s}\{X_t \in \partial D | X_s = x\}, \tag{4.1}$$

where $x \in D$ and the equality holds by sample-path continuity. We let $\eta_D(s,x) = \infty$ if X_t never exits D. While the absolute time in (4.1) is important, it is mathematically convenient and practically useful to study instead the exit duration

$$\tau_D(s,x) := \eta_D(s,x) - s \tag{4.2}$$

directly. As D is generally fixed, where unambiguous, we omit the subscript D i.e. $\eta(s,x) = \eta_D(s,x)$ and $\tau(s,x) = \tau_D(s,x)$. By Début theorem, $\eta(s,x)$ is both a hitting time and a stopping time. In general, $\tau(s,x)$ is not. Thus some proofs and computations will be first done for $\eta(s,x)$, then related to $\tau(s,x)$ via (4.2). To avoid possible confusion, we note that in the current section, Section 4, η will always refer to exit time rather than the local Doeblin constant of (2.5).

In this thesis, we are interested in expectations of these quantities i.e.

$$\bar{\eta}(s,x) := \mathbb{E}[\eta(s,x)], \quad \bar{\tau}(s,x) := \mathbb{E}[\tau(s,x)]. \tag{4.3}$$

In conventional notation, one typically writes $\bar{\eta} = \mathbb{E}^{s,x}[\eta]$ and $\bar{\tau} = \mathbb{E}^{s,x}[\tau]$. For subsequent proofs, it is often more convenient that we keep the explicit dependence on the random variables.

In this thesis, we are specifically interested in the expected exit and duration time for T-periodic non-degenerate SDEs on $\mathbb{R}^d$ of the form

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & t \geq s, \\ X_s = x, & x \in D, \end{cases} \tag{4.4}$$

where akin to again W_t is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $b \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^{d\times d})$ are T-periodic such that a unique solution $X_t = X_t^{s,x}$ exist. Unlike (3.1), note that $x \in D$ i.e. the process is required to start inside the domain D.

It is well-known that for autonomous SDEs, the expected exit time and expected duration coincide [Gar09, Pav14, Zwa01]. Denoting both the expected exit and duration time by $\bar{\tau}(x)$, it is moreover known that $\bar{\tau}(x)$ satisfies the following second-order elliptic PDE with vanishing boundaries [Has12, Gar09, Pav14, Zwa01, Ris96]

$$\begin{cases} L\bar{\tau} = -1, & \text{in } D, \\ \bar{\tau} = 0 & \text{on } \partial D, \end{cases} \tag{4.5}$$

where

$$Lf = \sum_{i=1}^{d} b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}(x)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d), \tag{4.6}$$

is the usual infinitesimal stochastic generator. For non-autonomous SDEs however, due to the explicit dependence on time, expected exit time and expected duration no longer coincide. That is, $\bar{\tau}(s, x)$ generally depends on both initial time and initial state. Let us consider the stochastic infinitesimal generator of (4.4) for any fixed time $s \in \mathbb{R}^+$ by

$$L(s)f(x) = \sum_{i=1}^{d} b^i(s, x)\partial_i f(x) + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(s, x)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d). \tag{4.7}$$

For our discussion of expected duration in this non-autonomous case, for interpretability, it is preferred to consider $L(s)$ rather than $\mathcal{L}(s)$ given by (3.2). It is important to note that for non-autonomous SDEs, $\bar{\tau}(s, x)$ does not satisfies (4.5) even if L is replaced by $L(s)$.

4.2 Derivation of the Expected Duration PDE

In this section, we give the rigorous derivation for the expected duration PDE. It appears that in numerous existing literature, the expected exit time is implicitly assumed to be almost surely finite. Particularly for degenerate noise, it may well be that the exit time is infinite with positive probability or indeed almost surely. Utilising asymptotic stability of diffusion processes, it is easy to construct examples where the process never leaves a point or domain. Indeed (2.12) in one such example, where if the process starting at the origin, it does not leave. For initial condition outside the origin, it is easy to show that the process tends towards the origin hence not leave when it enters. We refer readers to [Mao07] for similar examples and the wider theory. In the following lemma, we give verifiable conditions to imply irreducibility and show further that η is almost surely finite with finite first and second moments.

Lemma 4.1. *Let $D \subset \mathbb{R}^d$ be a non-empty open bounded set. Assume that the T-periodic SDE (4.4) satisfies (3.5), (3.9), (3.6), (3.8) and (3.7). Then $\eta(s,x)$ is finite almost surely for all $(s,x) \in \mathbb{R}^+ \times D$. Moreover, $\eta(s,x)$ has finite first and second moments.*

Proof. By Theorem 3.2, conditions (3.5), (3.6) and (3.7) implies P is irreducible i.e. $P(s,t,x,\Gamma) > 0$ for all $x \in \mathbb{R}^d$, $0 \leq s < t < \infty$ and non-empty open set $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. Then for any fixed $s \in \mathbb{R}^+$, for all $x \in \mathbb{R}^d$, it follows that there exists an $\epsilon(x) = \epsilon(s,x,D) \in (0,1)$ such that

$$\epsilon(x) = Ps, s+T, x, D).$$

In particular, $P(s,s+T,x,D^c) > 0$ for any $x \in D$. Recall that when conditions (3.7), (3.8), (3.9) and (3.5) hold, then the results of [HLT17] implies P possesses a smooth density. This implies that P is strong Feller i.e. $P(s,t,\cdot,\Gamma)$ is continuous for all $s < t$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. Then it follows from the boundedness of D that the probability of staying within D in one period is at most

$$\epsilon := \sup_{x \in \bar{D}} \epsilon(x) > 0.$$

Since $D \subset \mathbb{R}^d$ is bounded, $\epsilon < 1$. By (2.3), $Z^{s,x} = (Z_n^{s,x}) := (X_{s+nT})_{n\in\mathbb{N}}$ is time-homogeneous Markov chain with one-step Markovian transition $P(s,s+T,x,\cdot)$.

Define the exit time

$$\eta_Z = \eta_Z(s,x) := \min\{n \in \mathbb{N} : Z_n^{s,x} \notin D\}.$$

By sample-path continuity of X_t, it is clear that $X_t \notin D$ for at least one $t \in [s + (\eta_Z - 1)T, s + \eta_Z T]$. Hence $\eta(s,x) \leq s + \eta_Z(s,x) \cdot T$, in particular, we have

$$\{\eta_Z(s,x) \geq n\} \subset \{\eta(s,x) \geq s + nT\}.$$

Hence if $\mathbb{P}(\eta_Z < \infty) = 1$ then $\mathbb{P}(\eta < \infty) = 1$ i.e. if Z_n^s leaves D in almost surely finite time then X_t does also. For any $n \in \mathbb{N}$, it is easy to see that

$$\{\eta_Z = n\} = \{Z_n^s \in D^c\} \cap \bigcap_{m=1}^{n-1} \{Z_m^s \in D\}.$$

Since $Z_0^s = x \in D$, by elementary time-homogeneous Markov chain properties,

$$\begin{aligned}
\mathbb{P}(\eta_Z = n) &= \mathbb{P}(Z_n^s \in D^c | \bigcap_{m=0}^{n-1} \{Z_m^s \in D\}) \mathbb{P}(\bigcap_{m=0}^{n-1} \{Z_m^s \in D\}) \\
&= \mathbb{P}(Z_n^s \in D^c | Z_{n-1}^s \in D) \prod_{m=1}^{n-1} \mathbb{P}(Z_m^s \in D | Z_{m-1}^s \in D) \\
&\leq \epsilon^{n-1}.
\end{aligned} \tag{4.8}$$

This concludes that η is almost surely finite. Via (4.8), it is elementary to show that τ has finite first and second moments:

$$\mathbb{E}[\tau(s,x)] \leq T\mathbb{E}[\eta_Z(s,x)] = T \sum_{n=0}^{\infty} n\mathbb{P}(\eta_Z = n) \leq T \sum_{n=0}^{\infty} \frac{d}{d\epsilon}\epsilon^n = T\frac{d}{d\epsilon}\frac{1}{1-\epsilon} = \frac{T}{(1-\epsilon)^2} < \infty.$$

Similarly,

$$\mathbb{E}[\tau^2(s,x)] \leq T^2\mathbb{E}[\eta_Z^2(s,x)] = T^2 \sum_{n=0}^{\infty} \left[\frac{d^2}{d\epsilon^2}\epsilon^{n+1} - \frac{d}{d\epsilon}\epsilon^n\right] = \frac{T^2(1+\epsilon)}{(1-\epsilon)^3} < \infty.$$

It follows that η has finite first and second moments. $\square$

Remark 4.2. Observe that Lemma 4.1 holds more generally provided that P is irreducible and strong Feller.

Remark 4.3. It should be clear that Lemma 4.1 can be adapted to hold in the more general (not-necessarily T-periodic) non-autonomous case. Namely by picking any fixed $T > 0$, define $\epsilon_n := \sup_{x\in\overline{D}} P(s+(n-1)T, s+nT, x, D)$, then the same calculations via properties of the two-parameter Markov kernel yields $\mathbb{P}(\eta_Z = n) \le \epsilon^n$, where $\epsilon := \max_{n\in\mathbb{N}} \epsilon_n$.

To study the exit problem, we study the evolution of the probability density of the process evolving in the domain D and impose absorbing boundaries [Ris96, Gar09, Pav14]. Specifically, let $p_D(s,t,x,y)$ denote the probability density of the process starting at x at time s to y at time t that gets absorbed on ∂D. Then the density p_D satisfies the following Fokker-Planck equation

$$\begin{cases} \partial_t p_D(s,t,x,y) = L^*(t) p_D(s,t,x,y), & \\ p_D(s,s,x,y) = \delta_x(y), & x \in D, \\ p_D(s,t,x,y) = 0, & \text{if } y \in \partial D, t \ge s, \end{cases} \tag{4.9}$$

where we recall that $L^*(t)$ is given by (3.40) and acts on forward variable y. To discuss the solvability of (4.9) and subsequent PDEs, we lay out typical PDE notations and conditions.

We fix some standard nomenclature and notation. For the open domain $D \subset \mathbb{R}^d$ and open interval $I \subset \mathbb{R}^+$. We define their Cartesian product by $D_I := I \times D$. When $I = (0,T)$, we define $D_T := (0,T) \times D$. To discuss regularity of the coefficients, we say a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be locally θ_x-Hölder for some $\theta_x \in (0,1]$ if for any compact set $K \subset \mathcal{B}(\mathbb{R}^d)$ there exists a constant $C = C(K)$ such that

$$|f(x) - f(y)| \le C\|x-y\|^{\theta_x}, \quad x, y \in K.$$

If $\theta_x = 1$, then f is said to be locally Lipschitz. If $K = \mathbb{R}^d$ then f is said to be (globally) θ_x-Hölder and Lipschitz if $\theta_x = 1$. The collection of all such functions is denoted $C^{\theta_x}(D)$. Similarly, for $\theta_t, \theta_x \in (0,1]$, $C^{\theta_t,\theta_x}(I\times D)$ contains all functions f such that there exists a constant C

$$|f(t,x) - f(s,y)| \le C(\|x-y\|^{\theta_x} + |t-s|^{\theta_t}), \quad x, y \in D, s, t \in I.$$

Let $k_t, k_x \in \mathbb{N}$, we denote by $C^{k_t,k_x}(I \times D)$ to be the space of continuously k_t-differentiable functions in t and continuously k_x-differentiable function in x. For $\theta_t, \theta_x \in (0,1]$, we let $C^{k_t+\theta_t,k_x+\theta_t}(I \times D)$ denote the space of $C^{k_t,k_x}(I \times D)$

functions in which the k_t'th t-derivative and k_x'th x-derivatives are θ_t and θ_x are Hölder respectively.

We can now write the main conditions required for the well-posedness for the expected duration PDE. Observe that the conditions are weaker than the conditions required by Lemma 4.1.

Condition A1: For some $\theta \in (0, 1]$,

(i) Domain $D \in \mathcal{B}(\mathbb{R}^d)$ is non-empty and open with boundary $\partial D \in C^\theta(\mathbb{R}^{d-1})$.

(ii) The coefficients $a^{ij}, b^i \in C^{\frac{\theta}{2},\theta}(\bar{D}_T)$.

(iii) The matrix $a(s, x) = (a^{ij}(s, x))$ is uniformly elliptic i.e. there exists $\alpha > 0$ such that

$$\langle a(s, x)\xi, \xi\rangle_{\mathbb{R}^d} \geq \alpha \|\xi\|^2_{\mathbb{R}^d}, \quad (s, x) \in D_T, \xi \in \mathbb{R}^d. \tag{4.10}$$

Particularly for adjoint operator $L^*(t)$ where more differentiability is required, we consider further

Condition A2: For some $\theta \in (0, 1]$, Condition A1 holds and moreover $a^{ij}, b^i \in C^{1+\theta,2+\theta}(\bar{D}_T)$ and $\partial D \in C^{2+\theta}(\mathbb{R}^{d-1})$.

It is well-known that if Condition A2 holds, then there exists a unique solution $p_D(s, \cdot, x, \cdot) \in C^{1,2}(D_T)$ to (4.9). Moreover, $p_D(s, t, x, y)$ is jointly continuous in (x, y). For details, we refer readers to Section 7, Chapter 3 in [Fri64]. The following lemma and its proof are similar to the one presented in [Gar09, Pav14, Ris96] when the coefficients are time-independent. We prove for the time-dependent coefficients case. For clarity of the key ingredients of the following lemma, we assume η to have finite second moment rather than the conditions assumed in Lemma 4.1.

Lemma 4.4. *Assume that Condition A2 holds for SDE (4.4). Assume further that η has finite second moment. Then*

$$\bar{\tau}(s, x) = \int_s^\infty \int_D p_D(s, t, x, y)dydt, \tag{4.11}$$

where $p_D(s, \cdot, x, \cdot)$ is the unique solution to (4.9).

Proof. Let $G(s, t, x)$ be the probability that the process starting at x at time s is still within D at time $t \geq s$. In the derivation below, we treat (s, x) as fixed parameters so that G is only a function of t. By the absorbing boundary conditions of p_D, we have

$$G(s,t,x) = \int_D p_D(s,t,x,y)dy. \tag{4.12}$$

On the other hand,

$$G(s,t,x) = \mathbb{P}(\eta(s,x) > t) = 1 - \mathbb{P}(\eta(s,x) \leq t).$$

Then, since p_D is t-differentiable, by (4.12), it is clear that a density $p_\eta(s,t,x)$ exists for $\eta(s,x)$ given by

$$p_\eta(s,t,x) = -\partial_t G(s,t,x). \tag{4.13}$$

Note that if $x \in D$ then $G(s,s,x) = 1$. Note further that by Chebyshev's inequality,

$$G(s,t,x) = \mathbb{P}(\eta(s,x) > t) \leq \frac{1}{t^2}\mathbb{E}\left[\eta^2(s,x)\right], \quad t > 0.$$

Since $G \geq 0$, it follows that $\lim_{t\to\infty} tG(s,t,x) = 0$, hence the following holds by an integration by parts

$$\begin{aligned}
\bar{\eta}(s,x) &= \int_s^\infty tp_\eta(s,t,x)dt \\
&= -\int_s^\infty t\partial_t G(s,t,x)dt \\
&= -tG(s,t,x)|_{t=s}^\infty + \int_s^\infty G(s,t,x)dt \\
&= s + \int_s^\infty G(s,t,x)dt.
\end{aligned}$$

Hence

$$\bar{\tau}(s,x) = \int_s^\infty G(s,t,x)dt. \tag{4.14}$$

The result follows by (4.12). □

While finite first moment of η was not explicitly used in Lemma 4.4, we note that it is of course finite since it has finite second moment and applying Hölder's inequality. It is then obvious then that (4.14) is finite.

For T-periodic SDEs, we show in the next lemma, that the expected duration $\bar{\tau}$ is also T-periodic. While this holds in expectation, the same cannot be said of the sample-path realisations of τ. This is essentially because the noise realisation is not periodic! In the context of random dynamical systems, this can be proven rigorously. Indeed, if ω denotes the noise realisation and θ_t to be the Wiener shift, then one has $\tau(s,x,\omega) = \tau(s+T,x,\theta_T\omega)$, see [FZ16] for further details.

Lemma 4.5. *Assume that Condition A2 holds for T-periodic SDE (4.4). Assume further that η has finite second moment. Then $\bar{\tau}$ is also T-periodic.*

Proof. By Lemma 3.19 and (4.11), we have

$$\begin{aligned}\bar{\tau}(s,x) &= \int_s^\infty \int_D p_D(s,r,x,y)dydr \\ &= \int_s^\infty \int_D p_D(s+T,r+T,x,y)dydr \\ &= \int_{s+T}^\infty \int_D p_D(s+T,r,x,y)dydr \\ &= \bar{\tau}(s+T,x).\end{aligned}$$

For the following theorem, we recall Kolmogorov's backward equation

$$\partial_s p(s,t,x,y) + L(s)p(s,t,x,y) = 0, \tag{4.15}$$

where $L(s)$ acts on x variable. □

We are now ready to derive the PDE in which $\bar{\tau}(s,x)$ satisfies. When the SDE is T-periodic, we show $\bar{\tau}(s,x)$ is the T-periodic solution of a second-order linear parabolic PDE. This contrasts with the autonomous case where the expected exit time satisfies the second-order linear elliptic PDE (4.5). To our knowledge the derived PDE and particularly its interpretation is new in literature. We note further that the following theorem establishes a Feynman-Kac duality for time-periodic SDEs for the expected duration.

Theorem 4.6. *Assume T-periodic SDE (4.4) satisfies the same conditions as Lemma 4.1. Then the expected duration $\bar{\tau}$ is the periodic solution of the following parabolic partial differential equation of backward type*

$$\begin{cases} \partial_s \bar{\tau}(s,x) + L(s)\bar{\tau}(s,x) = -1, & in\ D_T, \\ \bar{\tau} = 0, & on\ [0,T] \times \partial D, \\ \bar{\tau}(0,\cdot) = \bar{\tau}(T,\cdot). & on\ D. \end{cases} \tag{4.16}$$

Proof. By Lemma 4.1, η has finite second moment and Condition A2 holds. Hence Lemma 4.4 holds. Thus, by (4.14), observe that for any $\delta > 0$,

$$\begin{aligned} &\bar{\tau}(s+\delta,x) - \bar{\tau}(s,x) \\ &= \int_{s+\delta}^{\infty} G(s+\delta,t,x)dt - \int_{s}^{\infty} G(s,t,x)dt \\ &= \int_{s+\delta}^{\infty} G(s+\delta,t,x)dt - \int_{s+\delta}^{\infty} G(s,t,x)dt + \int_{s+\delta}^{\infty} G(s,t,x)dt - \int_{s}^{\infty} G(s,t,x)dt \\ &= \int_{s+\delta}^{\infty} \left(G(s+\delta,t,x) - G(s,t,x)\right) dt - \mathcal{G}(s+\delta), \end{aligned}$$

where for clarity, $\mathcal{G}(r) := \int_s^r G(s,t,x)dt$. It follows by the fundamental theorem of calculus that

$$\begin{aligned} \partial_s \bar{\tau}(s,x) &= \int_s^{\infty} \partial_s G(s,t,x)dt - \mathcal{G}'(s) \\ &= \int_s^{\infty} \int_D \partial_s p_D(s,t,x,y)dydt - G(s,s,x) \\ &= \int_s^{\infty} \int_D \partial_s p_D(s,t,x,y)dydt - 1, \end{aligned}$$

where recall that G is expressed by (4.12) and $G(s,s,x) = 1$ since $x \in D$. Acting $L(s)$ on $\bar{\tau}$ by (4.11) and (4.15), we have

$$L(s)\bar{\tau}(s,x) = \int_s^{\infty} \int_D L(s)p_D(s,t,x,y)dydt = -\int_s^{\infty} \int_D \partial_s p_D(s,t,x,y)dydt.$$

Summing these quantities yields

$$(\partial_s + L(s))\bar{\tau}(s,x) = -1. \tag{4.17}$$

For T-periodic systems, Lemma 4.5 requires that $\bar{\tau}(s,\cdot) = \bar{\tau}(s+T,\cdot)$ for all $s \in \mathbb{R}^+$ hence by Lemma 4.5, this is sufficient by imposing $\bar{\tau}(0,\cdot) = \bar{\tau}(T,\cdot)$ and the result follows that $\bar{\tau}$ satisfies (4.16). □

Remark 4.7. In the proof of Theorem 4.6, note that T-periodicity was not assumed until (4.17). This suggests that for general non-autonomous (not necessarily periodic) SDEs, $\bar{\tau}$ will still satisfy (4.17). However, as (4.17) is a parabolic PDE, in the absence of initial (or terminal) conditions, PDE (4.17) alone is generally ill-posed. It is clear that the initial condition is a part of the unknown. Indeed, if $\bar{\tau}(0,\cdot)$ is known, then this implies we already know the expected exit time when the system starts at time $s = 0$. This issue is partially resolved for time periodic SDEs as the initial and terminal conditions coincide, albeit unknown, by Lemma 4.5.

Remark 4.8. It should be clear that for coefficients with non-trivial time-dependence, the parabolic PDE (4.16) would generally imply that $\bar{\tau}(s,x) - \bar{\tau}(s',x) \neq (s - s')$ for $s \neq s'$. That is, the difference in initial starting time does not imply the same difference in expected time. This reinforce that initial time generally plays a non-trivial role in the expected duration.

Remark 4.9. In the forthcoming section, we will discuss the well-posedness of (4.16) under typical PDE conditions. We will see however the PDE conditions are generally too weak for the formal derivation of (4.16) i.e. does not generally satisfy the conditions listed in Theorem 4.6. This is expected since weak solutions of parabolic PDEs on bounded domains can be often attained with relatively weak conditions. On the other hand, from a SDE perspective, it is not known as a priori whether the process would exit in finite time (almost surely) let alone finite expectation. Therefore with the stronger SDE conditions, Lemma 4.4 considered the process on the entire $\mathbb{R}^d$ and proves the expected exit time has finite expectation.

As mentioned in the introduction, numerically solving PDE (4.16) can be an appealing alternative to stochastic simulations of the expected hitting time. We note further that solving (4.16) solves the expected hitting time for all initial starting point. On the other hand, direct simulation would (naively) require many simulations for each starting point and time.

Assuming a priori that the expected exit time is finite, then we can derive a simpler alternative proof of Theorem 4.6 via Dynkin's formula. This reassures that Theorem 4.6 is correct.

Proposition 4.10. *Assume η associated to T-periodic (4.4) has finite expectation. Then Theorem 4.6 holds.*

Proof. Since η is a stopping time and has finite expectation, by Itô's and

Dynkin's formula, then

$$\mathbb{E}^{s,x}\left[\varphi(\eta, X_\eta)\right] = \varphi(s,x) + \mathbb{E}^{s,x}\left[\int_s^\eta (\partial_t + L(t))\varphi(t, X_t)dt\right], \quad \varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d).$$

Remark 4.7 implies that there does not generally exist a $u \in C^{1,2}(D)$ such that $(\partial_s + L(s))u(s,x) = -1$ and vanishes on ∂D until we impose T-periodicity of u. Furthermore and subtly, it is straightforward to show that, due to the T-periodic coefficients of (4.4), u cannot be T'-periodic for $0 < T' < T$. Therefore if such a u exists, we have by (4.2)

$$0 = \mathbb{E}^{s,x}\left[u(\eta, X_\eta)\right] = u(s,x) + \mathbb{E}^{s,x}\left[\int_s^\eta -1dt\right] = u(s,x) - \bar{\tau}(s,x).$$

i.e. $u(s,x) = \bar{\tau}(s,x)$ and so the results follows. □

4.3 Well-Posedness of the Expected Duration PDE

4.3.1 Fixed Point of an Initial Value Problem

In this section, utilising classical results for the well-posedness of initial-valued parabolic PDEs, we will show the existence of a unique solution to the expected duration PDE (4.16) for the associated T-periodic SDE. As mentioned in Remark 4.9, we solve (4.16) with typical PDE conditions rather than the stronger conditions required for the SDE to justify the rigorous derivation of the PDE. This has the advantage of a clearer exposition and key elements to solve the PDE.

In this subsection, we associate (4.16) with an initial-value boundary PDE problem and show that (4.16) can be rewritten as a fixed point problem. We note however that (4.16), as an initial valued problem, is a backward parabolic equation. Such equations are known to be generally ill-posed in typical PDE spaces. By reversing the time, we introduce a minus sign thus PDE is uniformly elliptic and hence more readily solvable in typical function spaces.

We give a general uniqueness and existence result via a spectral result of [Hes91] in $L^p(D)$. Specifically on $L^2(D)$, we show that if the associated bilinear form is coercive then one can apply a Banach fixed point argument to deduce the existence and uniqueness. This yields a practical way to numerically compute the desired solution.

To discuss the well-posedness of (4.16), we recall some standard function spaces. For any $1 \leq p < \infty$, we denote the Banach space $L^p(D)$ to be the space of functions $f : D \to \mathbb{R}$ such that its norm $\|f\|_{L^p(D)} := \left(\int_D |f(x)|^p dx\right)^{1/p} < \infty$. For tuple $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$, define $|\beta| = \sum_{i=1}^d \beta_i$ then for sufficiently regular function f, define $\partial^\beta f := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d} f$. Then for $k \in \mathbb{N}$, we define as usual the Sobolev space $W^{k,p}(D)$ to contain all functions f in which its norm $\|f\|_{W^{k,p}(D)} := \left(\sum_{|\beta|\leq k} \|\partial^\beta f\|^p_{L^p(D)}\right)^{1/p} < \infty$. We let $W_0^{k,p}(D) = \{f \in W^{k,p}(D) | f = 0 \quad \text{on } \partial D\}$. For $p = 2$, $L^2(D)$ and $H_0^k(D) := W_0^{k,2}(D)$ are Hilbert spaces with inner-product $\langle f, g\rangle_{L^2(D)} := \left(\int_D f(x)g(x)dx\right)^{1/2}$ and $\langle f, g\rangle_{H_0^k(D)} := \sum_{|\beta|\leq k} \sum_{i=1}^d \langle \partial^\beta f, \partial^\beta g\rangle_{L^2(D)}$ respectively. Occasionally, we let $(H, \|\cdot\|_H)$ denote a generic Hilbert space. To avoid any possible confusion, we will be verbose with the norms and inner-products.

We begin by fixing $1 < p < \infty$ and define the time-reversed uniformly elliptic operator associated to (4.7) by

$$L_R(s) := L(T-s) = \sum_{i=1}^d b^i(T-s,x)\partial_i + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(T-s,x)\partial^2_{ij}, \quad s \in [0,T], \tag{4.18}$$

on $L^p(D)$. Note that $\mathcal{D}(L_R(s)) = W^{2,p}(D) \cap W_0^{1,p}(D) \subset L^p(D)$ for all $s \in (0,T)$. As mentioned, the initial boundary value problem (IBVP) associated to (4.16) is a backward hence ill-posed in $L^p(D)$. Suppose that u satisfies (4.16), consider the the time-reversed solution $v(s,x) = u(T-s,x)$. Then v satisfies

$$\begin{cases} \partial_s v - L_R(s)v = f, & \text{in } D_T, \\ v = 0 & \text{on } [0,T] \times \partial D, \\ v(0,\cdot) = v(T,\cdot), & \end{cases} \tag{4.19}$$

where $f \equiv 1$. Clearly the solvability of (4.16) is equivalent to (4.19) up to time-reversal. Hence, for the rest of the thesis we focus on showing existence and uniqueness of a solution to (4.19).

Due to the general applicability of the methods presented in this thesis, where possible, we retain a general inhomogeneous function $f : [0,T] \to L^p(D)$. We expect that this generality benefits some readers for solving similar problems.

We the following IBVP associated to (4.19),

$$\begin{cases} (\partial_s - L_R(s))v = f, & \text{in } D_T, \\ v = 0, & \text{on [0,T]}\times\partial\text{D}, \\ v(0,\cdot) = v_0 & \text{on } D, \end{cases} \tag{4.20}$$

is a "forward" parabolic equation and is readily solvable. We say that v is a generalised solution of (4.20) if $v \in C([0,T], W^{2,p}(D) \cap W_0^{1,p}(D)$, its derivative $\frac{dv}{ds} \in C((0,T), L^p(D))$ exists and v satisfies (4.20) in $L^p(D)$ [Paz92, Ama95, DM92]. Consider also $\phi(s,x)$ satisfying the homogeneous PDE of (4.20) i.e.

$$\begin{cases} (\partial_s - L_R(s))\phi = 0, & \text{in } (r,T) \times D, \\ \phi = 0 & \text{on } [r,T] \times \partial D, \\ \phi(r,\cdot) = \phi_r, & \text{in } D. \end{cases} \tag{4.21}$$

Given $\phi_r \in L^p(D)$, (4.21) is well-posed, we can define the evolution operator

$$\Phi(r,s) : L^p(D) \to W^{2,p}(D) \cap W_0^{1,p}(D), \quad r \le s \le T, \tag{4.22}$$

by

$$\Phi(r,s)\phi_r := \phi(s). \tag{4.23}$$

It is known that $\Phi(s,r)$ satisfies the semigroup property $\Phi(r,r) = Id$ and $\Phi(r,s) = \Phi(s,t)\Phi(r,s)$ for $r \le s \le t$. We refer readers to [Paz92] for regularity properties of Φ. When (4.20) is well-posed, it is well-known that by a variation of constants or Duhamel's formula [Ama95, DM92, Paz92], the solution to inhomogeneous problem (4.20) satisfies

$$v(s) = \Phi(r,s)v_r + \int_r^s \Phi(r',s)f(r')dr'. \tag{4.24}$$

It is well-known that if Condition A1 and $f \in C^\gamma(0,T; L^p(D))$ for some $\gamma \in (0,1)$, then (4.20) is well-posed [Paz92, Ama95]. Furthermore, we can define the solution operator after one period $\mathcal{A} : L^p(D) \to W^{2,p}(D) \cap W_0^{1,p}(D)$ by

$$\mathcal{A}\varphi := \Phi(0,T)\varphi + \int_0^T \Phi(r,T)f(r)dr. \tag{4.25}$$

We discuss further conditions for regular solutions. Theorem 24.2 of [DM92] employed Schauder estimates and Sobolev embedding to show that if $p > d/2$,

$\partial D \in C^2(\mathbb{R}^{d-1})$ then the solution to IBVP (4.20) with initial condition $v_0 \in W_0^{2,p}(D)$ satisfies the following regularity

$$v \in C(\bar{D}_T) \cap C^{1+\frac{\theta}{2},2+\theta}((0,T] \times \bar{D}). \tag{4.26}$$

Furthermore, if $d < p < \infty$, by Sobolev embedding, then $v \in C^{\frac{1+\xi}{2},1+\xi}(\bar{D}_T) \cap C^{1+\frac{\theta}{2},2+\theta}((0,T] \times \bar{D})$ for some $\xi \in (0,1)$, see [Hes91]. This write our first existence and uniqueness result.

Proposition 4.11. *Assume Condition A1 holds. Assume that $d < p < \infty$, $\partial D \in C^2(\mathbb{R}^{d-1})$ and $f \in C^\gamma(0,T;L^p(D))$ for some $\gamma \in (0,1)$. Then there exists a unique regular solution satisfying (4.19). Moreover, if $f \neq 0$ then the solution is non-trivial.*

Proof. Since $f \in C^\gamma(0,T;L^p(D))$, by Condition A1, IBVP (4.20) is well-posed for any $v_0 \in L^p(D)$. Hence the evolution operator Φ defined by (4.23) is well-defined. To solve T-periodic PDE (4.19), by Duhamel's formula (4.24), one wishes to find existence and uniqueness of a $v_0 \in L^p(D)$ such that

$$v_0 = \mathcal{A}v_0. \tag{4.27}$$

For initial conditions in $W_0^{2,p}(D)$, by rearranging (4.25), this is equivalent to

$$(I - \Phi(0,T))v_0 = \int_0^T \Phi(r,T)f(r)dr, \tag{4.28}$$

where $\Phi(0,s) : W_0^{2,p}(D) \to W_0^{2,p}(D)$ and $I : W_0^{2,p}(D) \to W_0^{2,p}(D)$ is the identity operator. With the current conditions, via Krein-Rutman theorem, it was shown in [Hes91] that $\lambda = \rho(\Phi(0,T)) \in (0,1)$, where λ denotes the spectral radius of $\Phi(0,T)$. This implies that 1 is in the resolvent i.e. $(I - \Phi(0,T)) : W_0^{2,p}(D) \to W_0^{2,p}(D)$ is invertible. It follows that

$$v_0 = (I - \Phi(0,T))^{-1} \int_0^T \Phi(r,T)f(r)dr, \tag{4.29}$$

uniquely solves (4.27). By Sobolev embedding,

$$v(s,\cdot) = \Phi(0,s)v_0 + \int_0^s \Phi(r,s)f(r)dr, \quad s \in (0,T], \tag{4.30}$$

is a regular solution to (4.19). It is easy to see that (4.19) does not admit trivial solutions since (D is non-empty and) $v \equiv 0$ cannot satisfy (4.19) for $f \neq 0$. $\square$

As noted in [DM92], via the semigroup property, one can approximate $\Phi(0,T) \simeq \prod_{n=0}^{N-1} \Phi(t_n, t_{n+1})$ for $0 = t_0 < t_1 < ... < t_N = T$. Hence one can approximate the inverse in (4.29) by

$$(I - \Phi(0,T))^{-1} \simeq (I - \prod_{n=0}^{N-1} \Phi(t_n, t_{n+1}))^{-1}. \tag{4.31}$$

We note however computing (4.31) is generally computationally expensive.

We can gain more from (4.29). We recall the weak maximum principle: if the solution is regular and $f \geq 0$, then

$$\min_{(s,x)\in\bar{D}_T} v(s,x) = \min_{x\in\bar{D}} v_r(x) \tag{4.32}$$

holds. We have seen that, by (4.28), the existence and uniqueness of $v_0 \in L^2(D)$ satisfying (4.27) requires the invertibility of $I - \Phi(0,T)$. By von Neumann series, we have

$$(I - \Phi(0,T))^{-1} = \sum_{k=0}^{\infty} \Phi^k(0,T),$$

where $\Phi^k(0,T)$ denotes the composition of the operator $\Phi(0,T)$.

It is well-known that parabolic PDEs experience parabolic smoothing (see e.g. [Paz92, Eva10]) i.e. the solution of parabolic equations are as smooth as the coefficients and initial data. For example if $p > d/2$ and $f \in C^\gamma(0,T; W^{2,p}(D))$, then $\Phi(s,t)f$ is a regular solution by (4.26). Moreover, if $f \geq 0$, by the maximum principle, $\Phi(s,t)f \geq 0$ for all $0 \leq s \leq t \leq T$. It follows that $\mathcal{I} := \int_0^T \Phi(r,T)f(r)dr \geq 0$ and $\Phi^k(0,T)\mathcal{I} \geq 0$ for all $k \in \mathbb{N}$. Moreover, it follows from (4.29) that

$$v_0 = \sum_{k=0}^{\infty} \Phi^k(0,T)\mathcal{I} \geq 0, \tag{4.33}$$

i.e. the solution to (4.27) is non-negative. Furthermore, if the coefficients and f are smooth then condition $p > d/2$ can be dropped and the same conclusion holds with a smooth solution [Paz92]. In particular, since $1 \in C^\infty((0,T) \times D)$

is non-negative, this aligns with physical reality that expected duration time $\bar{\tau}(0,\cdot) = v_0$ indeed is non-negative.

To gain further insight into solving (4.19) from both a theoretically and computational viewpoint, we progress our study with Hilbert spaces i.e. $p = 2$ and forego some of the regularity gained from Sobolev embedding e.g. (4.26).

We start with a standard framework to deduce the existence and uniqueness of (4.19) on the Hilbert space $L^2(D)$. For convenience, we define the bilinear form $B_R : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ associated to $-L_R$ defined by

$$\begin{aligned} B_R[\varphi, \psi; s] = &- \sum_{i=1}^{d} \int_D \tilde{b}^i(T-s,x)\partial_i\varphi(x)\psi(x)dx, \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} \int_D a^{ij}(T-s,x)\partial_i\varphi(x)\partial_j\psi(x)dx \end{aligned} \tag{4.34}$$

where $\tilde{b}^i(s,x) = b^i(s,x) + \sum_{j=1}^d \partial_j a^{ij}(s,x)$ for each $1 \le i \le d$. We recall that a bilinear form $B : H \times H \to \mathbb{R}$ is coercive if there exists a constant $\alpha > 0$ such that

$$B[\varphi,\varphi] \ge \alpha \left\|\varphi\right\|_H^2, \quad \varphi \in H. \tag{4.35}$$

Assuming coercivity, we give the following existence and uniqueness theorem to (4.19).

Theorem 4.12. *Assume that $a^{ij}, b^i \in L^\infty(D_T)$ and $a(\cdot,\cdot)$ satisfies uniformly elliptic condition (4.10) and furthermore (4.34) is coercive for $s \in [0,T]$. Then for any $f \in L^2(0,T;L^2(D))$, there exists a unique solution $v \in C([0,T], H_0^1(D))$to (4.19). If $f \neq 0$, then the solution is non-trivial.*

Proof. It is well-known (e.g. [Eva10]) that there exists a unique weak solution v to the IBVP (4.20) i.e. $v \in C([r,T];L^2(D)) \cap L^2(r,T;H_0^1(D))$ such that $v(r) = v_r$, $\partial_s v \in L^2(r,T;H^{-1}(D))$ and for almost every $s \in [r,T]$,

$$\langle \partial_s v(s), \varphi\rangle_{H^{-1}(D)\times H_0^1(D)} + B_R[v,\varphi;s] = \langle f(s),\varphi\rangle_{H^{-1}(D)\times H_0^1(D)} \quad \varphi \in H_0^1(D), \tag{4.36}$$

where $\langle\cdot,\cdot\rangle_{H^{-1}(D)\times H_0^1(D)} : H^{-1}(D) \times H_0^1(D) \to \mathbb{R}$ denotes the duality pairing between $H^{-1}(D)$ and $H_0^1(D)$. To prove our result, it is sufficient to assume $f \in L^2(D)$. To cast (4.27) in terms of a self-mapping, consider $\bar{\Phi}(0,T) : L^2(D) \to L^2(D)$ as the operator $\Phi(0,T)$ with its range enlarged to $L^2(D)$ and define $\bar{\mathcal{A}} : L^2(D) \to L^2(D)$ by

$$\bar{\mathcal{A}}\varphi := \bar{\Phi}(0,T)\varphi + \int_0^T \bar{\Phi}(r,T)f(r)dr. \tag{4.37}$$

We show there exists a unique fixed point of operator $\bar{\mathcal{A}}$. By Banach fixed point theorem, it suffices to show $\bar{\mathcal{A}}$ is a contraction on $L^2(D)$. Observe that this is sufficient provided $\bar{\Phi}(0,T)$ is a contraction mapping on $L^2(D)$ since

$$\left\|\bar{\mathcal{A}}\varphi - \bar{\mathcal{A}}\psi\right\|_{L^2(D)} = \left\|\bar{\Phi}(0,T)(\varphi-\psi)\right\|_{L^2(D)} \leq \|\bar{\Phi}(0,T)\| \, \|\varphi-\psi\|_{L^2(D)}, \quad \varphi,\psi \in L^2(D).$$

In fact, we show that $\bar{\Phi}(0,s)$ is a contraction for any $s > 0$. Let $\phi(s)$ be the homogeneous solution of (4.21), then from (4.36), one has by coercivity

$$\begin{aligned} 0 &= \langle \partial_s \phi, \phi(s)\rangle + B_R[\phi(s),\phi(s);s] \\ &\geq \frac{1}{2}\frac{d}{ds}\|\phi(s)\|^2_{L^2(D)} + \alpha\|\phi(s)\|^2_{H_0^1(D)} \\ &\geq \frac{1}{2}\frac{d}{ds}\|\phi(s)\|^2_{L^2(D)} + \alpha\|\phi(s)\|^2_{L^2(D)}. \end{aligned}$$

Gronwall's inequality then yields

$$\|\phi(s)\|^2_{L^2(D)} \leq e^{-2\alpha s}\|\phi_0\|^2_{L^2(D)}, \quad s \geq 0.$$

Hence indeed

$$\left\|\bar{\Phi}(0,s)\right\| := \sup_{\phi_0 \in L^2(D)} \frac{\left\|\bar{\Phi}(0,s)\phi_0\right\|_{L^2(D)}}{\|\phi_0\|_{L^2(D)}} \leq e^{-\alpha s} < 1, \quad s > 0. \tag{4.38}$$

i.e. $\bar{\Phi}(0,s)$ is a contraction on $L^2(D)$. Therefore there exists a unique $v_0 \in L^2(D)$ satisfying (4.37). Since $\mathcal{A} : L^2(D) \to H_0^1(D) \subsetneq L^2(D)$, then by the right hand side of (4.27), it is easy to deduce that $v_0 \in H_0^1(D)$. Define v by (4.30), then $v \in C([0,T], H_0^1(D))$ is the unique solution to (4.19). Lastly, if $0 \neq f \in L^2(D)$ then v is non-trivial. $\square$

We make three comments on Theorem 4.12. Theorem 4.12 offers not only a theoretical existence and uniqueness result on the solution to (4.19), by Banach fixed point, Theorem 4.12 immediately offers an iterative numerical approach to the solution. To numerically computing the next Banach fixed point iterate, one only requires to solve a IBVP for the parabolic PDE. Compared to (4.31), there are well-established numerical schemes for parabolic PDEs with known order of convergences. Observe that coercivity is actually stronger than required. In the proof of Theorem 4.12, it is sufficient that $B[\varphi, \varphi; s] \geq \alpha \|\varphi\|^2_{L^2(D)}$. Finally, by Sobolev embedding, one can deduce the weak solution of Corollary 4.13 is a strong solution if $d = 1$.

In light of Remark 4.9, we apply Theorem 4.12 specifically to expected duration $\bar{\tau}$ of SDEs.

Corollary 4.13. *Let $D \subset \mathbb{R}^d$ be a non-empty open bounded set. Assume that the T-periodic SDE (4.4) satisfies (3.5), (3.9), (3.6), (3.8) and (3.7). Assume further that (4.34) is coercive. Then there exists a unique non-negative non-trivial $\bar{\tau} \in C([0,T], H^1_0(D))$ satisfying (4.16).*

Proof. The current conditions and $1 = f \in L^2(0,T; L^2(D))$ satisfy the assumptions of Theorem 4.12. Hence there exists a unique non-negative non-trivial solution $v \in C([0,T], H^1_0(D))$ to (4.19). Since (4.16) is equivalent to (4.19) by time-reversal, by (4.24), it follows that

$$\bar{\tau}(s, \cdot) := v(T-s, \cdot) = \Phi(0, T-s)v_0 + \int_0^{T-s} \Phi(r, T-s)1dr, \quad s \in [0,T],$$

satisfies (4.16). □

We give an example where coercivity is shown. We consider the example of a one-dimensional Brownian motion with periodic drift.

Example 4.14. Let $S \in C^1(\mathbb{R}^+)$ be a T-periodic function and $\sigma \neq 0$ and consider the one-dimensional T-periodic SDE

$$dX_t = S(t)dt + \sigma dW_t,$$

on some bounded interval D. Clearly Condition A2 is satisfied. By Corollary 4.13, it is sufficient to show the associated (time-reversed) bilinear form

$$B_R[\varphi, \psi; s] = -\int_D S(T-s)\partial_x\varphi(x)\psi(x)dx + \frac{\sigma^2}{2}\int_D \partial_x\varphi(x)\partial_x\psi(x)dx, \quad \varphi, \psi \in H^1_0(D),$$

is coercive. This is obvious by an integration by parts with vanishing boundaries and applying the Poincaré inequality

$$\begin{aligned}B_R[\varphi,\varphi;s] &= -\frac{S(T-s)}{2}\int_D \partial_x(\varphi^2(x))dx + \frac{\sigma^2}{2}\|\partial_x\varphi\|^2_{L^2(D)}\\ &\geq -\frac{S(T-s)}{2}\varphi^2(x)|_{\partial D} + \frac{\sigma^2}{4}\|\partial_x\varphi\|^2_{L^2(D)} + \frac{\sigma^2}{4C_D}\|\varphi\|^2_{L^2(D)}\\ &\geq \alpha\|\varphi\|^2_{H^1_0(D)},\end{aligned}$$

where C_D denotes the Poincaré constant for the domain D such that $\|\varphi\|^2_{L^2(D)} \leq C_D\|\partial_x\varphi\|^2_{L^2(D)}$ and $\alpha = \min(\frac{\sigma^2}{4}, \frac{\sigma^2}{4C_D}) > 0$. Hence by Theorem 4.12, there exists a unique solution to (4.16).

4.3.2 Convex Optimisation

In Section 4.3.1, we showed that if the bilinear form associated to the PDE is coercive, then Theorem 4.12 yields a unique weak solution to (4.19). However, in general, coercivity of the associated bilinear form can be difficult to verify. Furthermore, it is not immediate whether the weak solution is indeed a classical solution. In this section, we now seek to solve (4.19) by casting it as an convex optimisation problem with a natural cost functional. Convex optimisation has been a standard method to study solutions of elliptic PDEs.

In this convex optimisation framework, we show that the unique minimiser of the cost functional is a solution to (4.19). In this approach, coercivity of the functional is holds almost immediately, provided the maximum principle holds. Since the maximum principle holds, it follows that the solution is a classical/strong solution as opposed to the weak solution given in Theorem 4.12.

Casting the optimisation problem on a Hilbert space, we will derive equations for the gradient of the cost functional. Therefore, one can readily compute the minimiser numerically by standard gradient methods.

We begin with a standard convex optimisation result on Hilbert spaces. Let $(H, \|\cdot\|_H)$ be a Hilbert space, $\mathscr{C} \subseteq H$ be a closed convex subset and $F : H \to \mathbb{R}$ be a functional. The functional F is said to be coercive over $\mathscr{C}$ if

$$F(\varphi) \to \infty, \quad \text{as } \|\varphi\|_H \to \infty, \quad \varphi \in \mathscr{C}.$$

The functional F is Gateaux differentiable at $\varphi \in H$ if for any $\phi_0 \in H$, the directional derivative of F at φ in the direction ϕ_0, denoted by $DF(\varphi)(\phi_0)$, given by

$$DF(\varphi)(\phi_0) = \lim_{\epsilon \to 0} \frac{F(\varphi + \epsilon\phi_0) - F(\varphi)}{\epsilon} \tag{4.39}$$

exists. The gradient $\frac{\delta F}{\delta \varphi}$ is obtained by Riesz representation theorem such that

$$\langle \frac{\delta F}{\delta \varphi}, \phi_0 \rangle = DF(\varphi)(\phi_0), \quad \phi_0 \in H. \tag{4.40}$$

We shall use the following standard result from convex optimisation theory (see e.g. [ET99, Tro10]).

Lemma 4.15. *Let H be a Hilbert space and $\mathscr{C} \subseteq H$ be a closed convex set. Let $F : H \to \mathbb{R}$ be a functional such that F is is convex and coercive over $\mathscr{C}$. Assume further that F is a (lower semi)continuous and bounded from below. Then there exists at least one $v_0 \in \mathscr{C}$ such that $F(v_0) = \inf_{\varphi \in \mathscr{C}} F(\varphi)$. If F is Gateaux differentiable, then for any such v_0, $DF(v_0)(\cdot) = 0$. If F is strictly convex then v_0 is unique.*

We now focus specifically on using Lemma 4.15 to solve (4.19). Recall that if Condition A1 holds then (4.20) is well-posed. We then associate to (4.20) the natural cost functional $F : L^2(D) \to \mathbb{R}$ defined by

$$F(\varphi) = \frac{1}{2} \int_D [(\mathcal{A}\varphi)(x) - \varphi(x)]^2 dx, \tag{4.41}$$

where $\mathcal{A}$ is given by (4.25). This functional is natural to our periodic problem because if there exists $v_0 \in L^2(D)$ which minimises the functional to zero, it is a solution to (4.19) i.e.

$$F(v_0) = 0 \quad \iff \quad \mathcal{A}v_0 = v_0,$$

i.e. v_0 solves (4.27) and therefore is a (possibly weak) solution to (4.19).

Optimisation briefly aside, we recommend using the cost functional F to quantify the convergence of the Banach iterates of Theorem 4.12.

In order to apply Lemma 4.15 on F, we recall some properties associated to linear parabolic PDEs. Suppose that (4.20) is well-posed. Since PDE (4.20) is linear, by the superposition principle, $\Phi(\cdot, \cdot)$ is a linear operator i.e. $\Phi(s,t)[\lambda_1\phi_1 + \lambda_2\phi_2] = \lambda_1\Phi(s,t)\phi_1 + \lambda_2\Phi(s,t)\phi_2$. However, due to the inhomogeneous term, observe that $\mathcal{A}$ is not linear. Instead, if $\lambda_1, \lambda_2 \geq 0$ such that

$\lambda_1 + \lambda_2 = 1$ then

$$\begin{aligned}\mathcal{A}(\lambda_1\varphi + \lambda_2\psi) &= \Phi(0,T)[\lambda_1\varphi + \lambda_2\psi] + (\lambda_1 + \lambda_2)\int_0^T \Phi(r,T)f(r)dr \\ &= \lambda_1\mathcal{A}\varphi + \lambda_2\mathcal{A}\psi. \end{aligned}\tag{4.42}$$

Since $\bar{\tau}$ is non-negative, we consider

$$\mathscr{C}(D) := \{\varphi \in H_0^2(D) | \varphi \geq 0\}.$$

It is easy to verify that $\mathscr{C}(D)$ is a closed convex Hilbert subspace of $L^2(D)$.

Theorem 4.16. *Let Condition A1 hold, $f \in C^\gamma(0,T;L^2(D))$ for some $\gamma \in (0,1)$, $f \geq 0$ and $d \leq 3$. Let $F : \mathscr{C}(D) \subset L^2(D) \to \mathbb{R}$ be defined by (4.41) . Then there exists a unique $v_0 \in \mathscr{C}(D)$ minimising F.*

Proof. Since Condition A1 holds, the IBVP (4.20) is well-posed. Hence the operators $\mathcal{A}$ and $\Phi(s,t)$ and thus F are all well-defined. We show that F satisfies the assumptions of Lemma 4.15. Obviously $F \geq 0$ and hence bounded from below. By the well-posedness of (4.20), it is clear that $\varphi \to \mathcal{A}\varphi$ and moreover $\varphi \to \mathcal{A}\varphi - \varphi$ are continuous from $L^2(D)$ to $L^2(D)$. By the L^2-norm continuity, it follows that F is continuous. Utilising the strong convexity of the quadratic function and (4.42), we show the strong convexity of F: for any $\lambda \in (0,1)$ and $\varphi, \psi \in \mathscr{C}(D)$, we have that

$$\begin{aligned}F(\lambda\varphi + (1-\lambda)\psi) &= \frac{1}{2}\int_D [\mathcal{A}(\lambda\varphi + (1-\lambda)\psi) - (\lambda\varphi + (1-\lambda)\psi)]^2 dx \\ &= \frac{1}{2}\int_D [\lambda\,(\mathcal{A}\varphi - \varphi) + (1-\lambda)(\mathcal{A}\psi - \psi)]^2 dx \\ &< \frac{1}{2}\int_D \left[\lambda\,(\mathcal{A}\varphi - \varphi)^2 + (1-\lambda)(\mathcal{A}\psi - \psi)^2\right] dx \\ &= F(\varphi) + (1-\lambda)F(\psi).\end{aligned}$$

Since $d \leq 3$, by Sobolev embedding and Schauder estimates, it follows from (4.26) that for any $\varphi \in \mathscr{C}(D) \subset H^2(D)$ and $f \geq 0$, the solution to (4.20) with initial condition φ is regular. Therefore, the maximum principle (4.32) applies. Hence together with the homogeneous Dirichlet boundary conditions, it follows that $\mathcal{A}\varphi \geq 0$. Therefore, for any $x \in D$ and $\epsilon \in (0,1)$, Young's inequality yields that

$$\begin{aligned}
F(\varphi) &= \frac{1}{2}\int_D ((\mathcal{A}\varphi)^2 - 2(\mathcal{A}\varphi)\varphi + \varphi^2)dx \\
&\geq \frac{1}{2}\int_D ((1-\epsilon)\varphi^2 + (1-\epsilon^{-1})(\mathcal{A}\varphi)^2)dx \\
&= \frac{1-\epsilon}{2}\|\varphi\|^2_{L^2(D)} + \frac{1-\epsilon^{-1}}{2}\|\mathcal{A}\varphi\|^2_{L^2(D)}.
\end{aligned}$$

Hence it follows that $F(\varphi) \to \infty$ as $\|\varphi\|_{L^2(D)} \to \infty$. Then Lemma 4.15 yields a unique solution $v_0 \in \mathscr{C}(D)$ minimising F. $\square$

In the following proposition, we derive an expression for the directional derivative $DF(\varphi)(\phi_0)$. While it is then straightforward to apply the maximum principle to show that $DF(\varphi)(\cdot)$ is a linear continuous operator to deduce existence and uniqueness of the gradient (via Riesz representation), we employ numerical analysts' adjoint state method (see e.g. [GP00, CLPS03, SFP14, Ple06]) to attain an expression for the gradient directly. From a numerical perspective, the gradient allows us to apply gradient methods to iteratively minimise F. Numerically, we note that it is not necessary to use adjoint state method to compute the gradient. However, it is well-known that adjoint state method is (generally) computationally efficient see e.g. [SFP14]. However, compared to Banch fixed point iterates of Theorem 4.12, the adjoint state method is computationally less efficient because a pair of IBVP, rather than one, is required to be solved .

To employ the adjoint state method, we recall that $L^*(s)$ is the Fokker-Planck operator given by (3.40). If Condition A2 holds, then

$$\begin{cases}
\partial_s w = L^*(s)w & \text{in } D_T, \\
w = 0 & \text{on } [0,T]\times\partial D\ , \\
w(0,\cdot) = w_0, & \text{on } D.
\end{cases} \tag{4.43}$$

is well-posed for any $w_0 \in L^2(D)$. Hence, akin to (4.21) and (4.22), we define the evolution operator W for (4.43) i.e.

$$w(s,\cdot) = W(0,s)w_0, \quad s \geq 0, \tag{4.44}$$

where $W(0,s) : L^2(D) \to H^2(D) \cap H^1_0(D)$.

The following proposition was inspired by [AW10, BGP98] in employing the adjoint state method for periodic solutions of the Benjamin-Ono and autonomous evolution equations respectively.

Proposition 4.17. *Let Condition A2 hold and F be defined by (4.41). Then for any $\varphi \in L^2(D)$, we have the expressions for the directional derivative*

$$DF(\varphi)(\phi_0) = \int_D (\mathcal{A}\varphi - \varphi)(\Phi(0,T)\phi_0 - \phi_0)dx, \quad \phi_0 \in L^2(D), \tag{4.45}$$

and the gradient

$$\frac{\delta F}{\delta \varphi} = W(0,T)w_0 - w_0, \tag{4.46}$$

with initial condition $w_0 = \mathcal{A}\varphi - \varphi$.

Proof. Utilising the linearity properties of $\mathcal{A}$ and Φ, from (4.25) and (4.39), we have

$$\begin{aligned} DF(\varphi)(\phi_0) &= \lim_{\epsilon\to 0} \frac{1}{2}\int_D \frac{(\mathcal{A}(\varphi+\epsilon\phi_0) - (\varphi+\epsilon\phi_0))^2 - (\mathcal{A}\varphi-\varphi)^2}{\epsilon}dx \\ &= \lim_{\epsilon\to 0} \frac{1}{2}\int_D \frac{((\mathcal{A}\varphi+\epsilon\Phi(0,T)\phi_0) - (\varphi+\epsilon\phi_0))^2 - (\mathcal{A}\varphi-\varphi)^2}{\epsilon}dx \\ &= \lim_{\epsilon\to 0} \frac{1}{2}\int_D \frac{((\mathcal{A}\varphi-\varphi) + \epsilon(\Phi(0,T)\phi_0 - \phi_0))^2 - (\mathcal{A}\varphi-\varphi)^2}{\epsilon}dx \\ &= \lim_{\epsilon\to 0} \frac{1}{2}\int_D 2(\mathcal{A}\varphi-\varphi)(\Phi(0,T)\phi_0-\phi_0) + \epsilon(\Phi(0,T)\phi_0-\phi_0)^2 dx. \end{aligned}$$

Hence (4.45) follows in the limit. We now wish to find $\frac{\delta F}{\delta\varphi} \in L^2(D)$ such that

$$DF(\varphi)(\phi_0) = \langle \frac{\delta F}{\delta \varphi}, \phi_0 \rangle = \int_D \frac{\delta F}{\delta\varphi}(x)\phi_0(x)dx, \quad \phi_0 \in L^2(D).$$

To compute the gradient, consider $w_R(s,x)$ satisfying the adjoint equation of PDE (4.21)

$$\begin{cases} -\partial_s w_R = L_R^*(s) w_R & \text{in } D_T, \\ w_R = 0, & \text{on } [0,T]\times\partial D, \\ w_R(T,\cdot) = w_T, & \text{on } D. \end{cases} \tag{4.47}$$

for some terminal condition $w_T \in L^2(D)$. Note that (4.47) is a backward equation. However, as terminal conditions are provided, (4.47) is well-posed provided Condition A2 are satisfied. This contrasts to the backward equation associated to (4.16) as a IBVP with initial conditions. For clarity, we introduce $w(s,\cdot) = w_R(T-s,\cdot)$. Then it is clear that w satisfies (4.43), since $L_R^*(T-s) = L^*(T-(T-s)) = L^*(s)$ by (4.18). In this form, it is clear that (4.43) and equivalently (4.47) are well-posed provided Condition A2 is satisfied. With the repeated time-reversal, $w(s,x)$ is understood to solve the Fokker-Planck equation forward in time.

Let ϕ be the homogeneous solution satisfying (4.21) with initial conditions $\phi(0,\cdot) = \phi_0$. We multiply ϕ by w_R and integrating over D_T. Integrating by parts in time for the first integrand and space for the second integrand, we have by (4.47) and its Dirichlet boundary conditions

$$\begin{aligned}
0 &= \int_0^T \int_D (\partial_s \phi(s,x) - L_R(s)\phi(s,x)) \cdot w_R(s,x) dx ds \\
&= \int_D \left[\phi(s,x) w_R(s,x)\right]_{s=0}^T dx - \int_0^T \int_D \phi(s,x) \partial_s w_R(s,x) dx ds - \int_0^T \int_D \phi(s,x) L_R^*(s) w_R(s,x) dx ds \\
&= \int_D \phi(T,x) w_R(T,x) dx - \int_D \phi_0(x) w_R(0,x) dx.
\end{aligned}$$

That is, in terms of w and Φ,

$$\int_D \Phi(0,T)\phi_0(x) \cdot w(0,x) dx = \int_D \phi_0(x) w(T,x) dx. \tag{4.48}$$

Impose the initial condition,

$$w(0,\cdot) = w_T = \mathcal{A}\varphi - \varphi. \tag{4.49}$$

Then it follows from (4.45), (4.48) and (4.49) that

$$\begin{aligned}
DF(\varphi)(\phi_0) &= \int_D w_0(x) \Phi(0,T) \phi_0(x) dx - \int_D w_0(x) \phi_0(x) dx \\
&= \int_D \left[w(T,x) - w_0(x)\right] \phi_0(x) dx.
\end{aligned} \tag{4.50}$$

Since ϕ_0 was arbitrary, we attain (4.46). □

We note that while Lemma 4.15 yields a unique minimiser, it was not immediate whether F was minimised to zero. In the following theorem, we show indeed that the unique minimiser of F indeed minimises F to zero.

Theorem 4.18. *Let Condition A2 hold and $d \leq 3$, $f \in C^\gamma(0,T;H^2(D))$ for* *some* $\gamma \in (0,1)$ *and* $f \geq 0$. *Then* $v_0 \in \mathscr{C}(D)$ *obtained in Theorem 4.16 is the unique function in* $L^2(D)$ *satisfying (4.27). Moreover there exists a unique* $v \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ *satisfying (4.19)*

Proof. By Theorem 4.16, the functional F has a unique minimiser $v_0 \in \mathscr{C}(D)$. By Lemma 4.15 and (4.50), it follows that

$$DF(v_0)(\phi) = \int_D (w(T,x) - w_0(x))\phi_0(x)dx = 0, \quad \phi_0 \in L^2(D).$$

By the fundamental lemma of calculus of variations, we have by (4.44)

$$0 \equiv w(T,\cdot) - w_0(\cdot) = W(0,T)w_0 - w_0,$$

i.e. w_0 is a fixed point of $W(0,T)$. Clearly $w_0 \equiv 0$ is a fixed point of $W(0,T)$. Let $w_0 \in H^2(D)$ be another fixed point solution to $W(0,T)$ and define $w(s,\cdot)$ by (4.44). With $d \leq 3$, by (4.26), it follows that $w \in C(\bar{D}_T) \cap C^{1+\frac{\theta}{2},2+\theta}((0,T] \times \bar{D})$. In fact, since D is bounded, $w(s,\cdot) \in L^\infty(D)$ for $s \in [0,T]$. Note that p_D of (4.9) is a fundamental solution of (4.43), hence since w_0 is a fixed point of $W(0,T)$, it follows that

$$w_0(x) = \int_D p_D(0,T,x,y)w_0(y)dy, \quad x \in D.$$

Due to the absorbing boundaries of (4.43), note that for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} P_D(s,t,x,\Gamma) &:= \int_{D\cap\Gamma} p_D(s,t,x,y)dy \\ &= \mathbb{P}(\{X_t^{s,x} \in D \cap \Gamma\} \cap \bigcap_{r=s}^{t} \{X_r^{s,x} \in D\}) \\ &\leq \mathbb{P}(\{X_t^{s,x} \in D \cap \Gamma\}) \\ &= P(s,t,x,D). \end{aligned}$$

Hence with $\epsilon \in (0,1)$ from Lemma 4.1, it follows that

$$\|w_0\|_\infty \leq \|w_0\|_\infty \int_D p(0,T,x,y)dy \leq \epsilon \|w_0\|_\infty.$$

Thereby deducing $0 \in H^2(D)$ is the only fixed point of $W(0,T)$. herefore, from (4.49), $v_0 \in \mathscr{C}(D)$ is the unique minimiser such that $\mathcal{A}v_0 = v_0$ and $F(v_0) = 0$.

It follows then from (4.26) that

$$v(s,x) := \Phi(0,s)v_0(x) + \int_0^s \Phi(r,s)f(r)dr \in C(\bar{D}_T) \cap C^{1+\frac{\theta}{2},2+\theta}((0,T] \times \bar{D})$$

satisfies (4.19). We show that v_0 is the unique fixed point of $\mathcal{A}$ in the entire $L^2(D)$. Indeed suppose there exists another solution $\tilde{v}_0 \in L^2(D)\backslash\mathscr{C}(D)$ to (4.27) such that $\tilde{v}_0 = \mathcal{A}\tilde{v}_0$. By (4.26), $\mathcal{A}\tilde{v}_0 \in C^{2+\theta}(\bar{D}) \subset H^2(\bar{D})$ and satisfies the boundary conditions i.e. $\tilde{v}_0 \in H_0^2(D)$. Since $H^2(D) \ni f \geq 0$, it follows from (4.33) that $\tilde{v}_0 \geq 0$ i.e. $\tilde{v}_0 \in \mathscr{C}(D)$. Since uniqueness already holds in $\mathscr{C}(D)$, we conclude the uniqueness of v_0 satisfying (4.27) extends to $L^2(D)$. □

Akin to Corollary 4.13, we apply Theorem 4.18 specifically to (4.16).

Corollary 4.19. *Let $d \leq 3$ and $D \subset \mathbb{R}^d$ be a non-empty open bounded set. Assume that the T-periodic SDE (4.4) satisfies (3.5), (3.9), (3.6), (3.8) and (3.7). Then there exists a unique non-negative non-trivial $\bar{\tau} \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ satisfying (4.16).*

Proof. Since D is bounded, obviously $1 \equiv f \in C^\gamma(0,T;H^2(D))$ is non-negative. Then by Theorem 4.18, there exists a unique non-trivial non-negative solution $v \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ satisfying (4.19). Then by time-reversal $\bar{\tau}(s,\cdot) := v(T-s,\cdot)$ satisfies (4.16). □

We end this section by drawing some differences Theorem 4.12 and Theorem 4.18. While the conditions of Theorem 4.18 is straightforward to verify, the dimensionality is capped to $d \leq 3$. This contrasts with Theorem 4.12, where coercivity may be difficult to verify however applies to any (finite) dimensions. The dimension difference between the theorem has regularity implications. Indeed Theorem 4.18 yields classical solutions, while on the other hand, Theorem 4.12 yields weak solutions which are indeed strong provided $d = 1$. We comment that Theorem 4.18 has $d \leq 3$ rather than $d \leq 1$ because the convex subset (of initial conditions) of the cost functional is a subset of $H^2(D)$ hence sufficiently regular for (4.26) to hold. On the other hand, for Theorem 4.12, we can only deduce $H_0^1(D)$. It is worth noting that Theorem 4.18 can be extended to $d > 3$ by demanding more regularity of $\mathscr{C}(D)$ and applying Sobolev embedding, however, this comes at the cost that we cannot conclude uniqueness in the entire $L^2(D)$ as previously demonstrated. From a numerical perspective, computing Banach iterates for Theorem 4.12 is faster than gradient descent methods given by Theorem 4.18, this is because one has to solve the IBVP (4.43) also in order to attain (4.46).

5 Applications

5.1 Weakly Dissipative Systems

As discussed in the introduction, expected exit times have a range of applications to model certain events to occur. Depending on context, the problem are typically phrased as the stochastic process hitting a barrier or a threshold. While many physical problems have naturally bounded domains, many applications have naturally unbounded domains. For example, $D = (0, \infty)$ is a typical unbounded domain for the population of a species or a wealth process and exit from D implies extinction and bankruptcy respectively. Unbounded domain brings technical difficulties to the derivation of the expected duration PDE and indeed its solution via PDE methods both theoretically and numerically.

However, unbounded domain brings various technical difficulties for the expected duration PDE. Particularly from a computational viewpoint, any numerical PDE scheme requires a finite domain. In the following remark, we show that the recurrency condition (5.1) below is sufficient to approximate the expectation duration by a finite domain rather than the unbounded domain.

Remark 5.1. Suppose that there exists a radius $r_* > 0$ and $\epsilon > 0$ such that the coefficients of SDE (4.4) satisfies

$$2\langle b(t,x), x\rangle + \|\sigma\|_2^2 \leq -\epsilon \quad \text{on } \mathbb{R}^+ \times B_{r_*}^c. \tag{5.1}$$

Note that if b is continuous, then there exists a constant $M \geq -\epsilon$ such that $2\langle b(t,x), x\rangle + \|\sigma\|_2 \leq M$. Let D be an unbounded domain that is bounded in at least one direction hence $r_D := \inf_{y\in\partial D}\|y\|$ is finite. We suppose initial condition $x \in B_{r_I} \cap D$ for some fixed $r_I \geq 0$.

For any fixed $R_* > \max\{r_*, r_D, r_I\}$, define $\tilde{D} := D \cap B_{R_*}$. Note that $\tilde{D}$ is an open bounded domain with boundary $\partial\tilde{D} = \partial\tilde{D}_1 \cup \partial\tilde{D}_2$, where $\partial\tilde{D}_1 := \partial D \cap B_{R_*}$ and $\partial\tilde{D}_2 := D \cap \partial B_{R_*}$ are the subset of original boundary and "artificial" boundary to "close up" the original boundaries respectively. Observe that $\|x\| \in (r_D, R_*)$ for $x \in \partial\tilde{D}_1$ and $\|x\| = R_*$ for $x \in \partial\tilde{D}_2$.

For $\eta_{\tilde{D}}$ as defined by (4.1) for the domain $\tilde{D}$, by Itô's formula, we have

$$\|X_{t\wedge\eta_{\tilde{D}}}\|^2 = \|x\|^2 + \int_s^{t\wedge\eta_{\tilde{D}}} \left(2\langle b(r,X_r),X\rangle + \|\sigma\|_2^2\right) dr + \int_s^{t\wedge\eta_{\tilde{D}}} \langle X_r, \sigma dW_r\rangle.$$

Under the assumption (5.1), Corollary 3.2 of [Has12] implies that $\mathbb{E}^{s,x}(\eta_{\tilde{D}}-s) \leq \frac{\|x\|^2}{\epsilon}$. Since $X_{r\wedge\eta_{\tilde{D}}}$ is bounded for $r \in [s,t]$, it follows that

$$\mathbb{E}^{s,x}\|X_{t\wedge\eta_{\tilde{D}}}\|^2 \leq \|x\|^2 + M\mathbb{E}^{s,x}[\eta_{\tilde{D}} - s] \leq (1+\frac{1}{\epsilon})\|x\|^2.$$

By Markov's inequality, it follows that $\mathbb{P}(\|X_{\eta_{\tilde{D}}}\|^2 \geq R_*^2) \leq R_*^{-2}(1+\frac{1}{\epsilon})\|x\|^2 \leq R_*^{-2}(1+\frac{1}{\epsilon})r_I^2 \to 0$ as $R_* \to \infty$. This implies that for sufficiently large R_*, the process is exits $\tilde{D}$ via $\partial\tilde{D}_1$ rather than $\partial\tilde{D}_2$, thus

$$\bar{\tau}_D|_{\tilde{D}}(s,\cdot) \simeq \bar{\tau}_{\tilde{D}}(s,\cdot),$$

where $\bar{\tau}_D|_{\tilde{D}}$ denotes $\bar{\tau}_D$ restricted to $\tilde{D}$. In practice, $R_* = 2\max\{r_*, r_D, r_I\}$ is sufficient for weakly dissipative SDEs. We consider two examples and assume for simplicity that $r_I = r_*$.

It was shown in [FZZ19] that the periodic Ornstein-Uhlenbeck process possesses a geometric periodic measure [FZ16], furthermore it has a periodic mean reversion property akin to its classical counterpart. In applications, these properties are desirable for processes with underlying periodicity or seasonality. Indeed electricity prices in economics [BKM07, LS02] and daily temperature [BS07] were modelled by periodic Ornstein-Uhlenbeck processes. In [IDL14], the authors performed statistical inference of biological neurons modelled by Ornstein-Uhlenbeck proceses with periodic forcing. In such models, one may be interested in the expected time in which a threshold is reached. For model parameter estimation, we refer readers to [DFK10].

Example 5.2. Consider the periodically forced multi-dimensional Ornstein-Uhlenbeck process

$$dX_t = (S(t) - AX_t)\,dt + \sigma dW_t,$$

where $S \in C(\mathbb{R}^+, \mathbb{R}^d)$ is T-periodic and $\sigma, A \in \mathbb{R}^{d\times d}$ with A positive definite i.e. there exists a constant $\alpha > 0$ such that $\langle Ax, x\rangle \geq \alpha\|x\|^2$ for all $x \in \mathbb{R}^d$. Denote $\|S\|_\infty = \sup_{t\in[0,T]}\|S(t)\|$. By Cauchy-Schwarz inequality and Young's inequality, it follows that

$$2\langle S(t) - Ax, x\rangle \leq 2\|S\|_\infty \|x\| - 2\alpha\|x\|^2 \leq \frac{\|S\|_\infty^2}{\alpha} - \alpha\|x\|^2,$$

i.e. weakly dissipative with coefficients $c = \frac{\|S\|_\infty^2}{\alpha}$ and $\lambda = \alpha$. Then

$$r_*^2 = \frac{1}{\alpha}\left(\frac{\|S\|_\infty^2}{\alpha} + \|\sigma\|_2\right).$$

Remark 5.1 suggests one can reasonably approximate the unbounded domain $D = (0,\infty)^d$ by the bounded domain $\tilde{D} = (0, 2r_*)^d$.

Example 5.3. Consider the stochastic overdamped Duffing oscillator (3.27). Theorem 3.9 shows that (3.27) satisfies the weakly dissipative condition with any $\lambda \in (0,2)$ and

$$c = \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} f(t,x) + \lambda \sup_{x\in\mathbb{R}}(x^2 - x^4),$$

where

$$\begin{aligned} f(t,x) &= 2\langle b(t,x), x\rangle + \lambda x^4 \\ &= 2x^2 - (2-\lambda)x^4 + 2A\cos(\omega t)x. \end{aligned}$$

By elementary calculus, we compute $\sup_{x\in\mathbb{R}}(x^2 - x^4) = \frac{1}{4}$ and $\sup_{x\in\mathbb{R}} f(\frac{\pi}{2}, x) = \frac{1}{2-\lambda}$. Hence for $x \in [-1,1]$,

$$f(t,x) \leq x^2 - (1-\lambda)x^4 + 2|A| \leq \frac{1}{2-\lambda} + 2|A|. \tag{5.2}$$

For simplicity, let $\lambda = 1$, then for $\epsilon > 0$, observe that

$$\begin{aligned} \partial_x f(t, 1+\epsilon) &\leq 4(1+\epsilon) - 4(1+\epsilon)^3 + 2|A| \\ &\leq -4\epsilon(2 + 3\epsilon + \epsilon^2) \\ &\leq -8\epsilon + 2|A|. \end{aligned}$$

Hence it follows that if $\epsilon \geq \frac{|A|}{4}$ then $\partial_x f(t, 1+\epsilon) < 0$. If A is small, then it follows that (5.2) is in a small neighbourhood of the maximum of f. Therefore, (3.27) satisfies the weakly dissipative condition for any fixed $\lambda \in (0,2)$ and

$c = \frac{1}{2-\lambda} + 2|A| + \frac{\lambda}{4}$ for small A, hence

$$r_*^2 = \frac{1}{\lambda}\left(\frac{1}{2-\lambda} + 2|A| + \|\sigma\|_2\right) + \frac{1}{4}.$$

For concreteness, suppose that $A = 0.12$, $\sigma = 2.85$ and $\lambda = 1$, then $r_* = \sqrt{1.57} = 1.25$ (2 dp). The heuristic suggests that the process exiting $D = (-1, \infty)$ can be approximated by the bounded domain $\tilde{D} = (-1, 2r_*)$.

Via Monte Carlo simulations, we numerically demonstrate the Remark 5.1 for (3.27) to estimate $\bar{\tau}_{\tilde{D}}(0, x)$ for different bounded domains $\tilde{D}$. We partition $\tilde{D}$ into $N_x^{\text{sde}} = 100$ uniform initial conditions. For each fixed initial condition $x \in \tilde{D}$, we employ Euler–Maruyama method with time intervals of $\Delta t = 5{\cdot}10^{-3}$ to generate 1000 sample-paths of X_t until it exits $\tilde{D}$. We record and average the sample-path exit time to yield an estimate for $\bar{\tau}_{\tilde{D}}(0, x)$. Figure 1 shows that the estimation of $\bar{\tau}_{\tilde{D}}$ are "stable" for bounded domain $\tilde{D} = (-1, 2)$ and larger. Where the differences between these curves can be explained by the randomness of Monte Carlo simulations and sample size. On the other hand, the estimation of $\bar{\tau}_{\tilde{D}}$ differs significantly for $\tilde{D} = (-1, 1.5)$. Physically, this is interpreted as the artificially boundary $R_* = 1.5$ set too low and many sample-paths leaves via this artificial boundary rather than via -1.

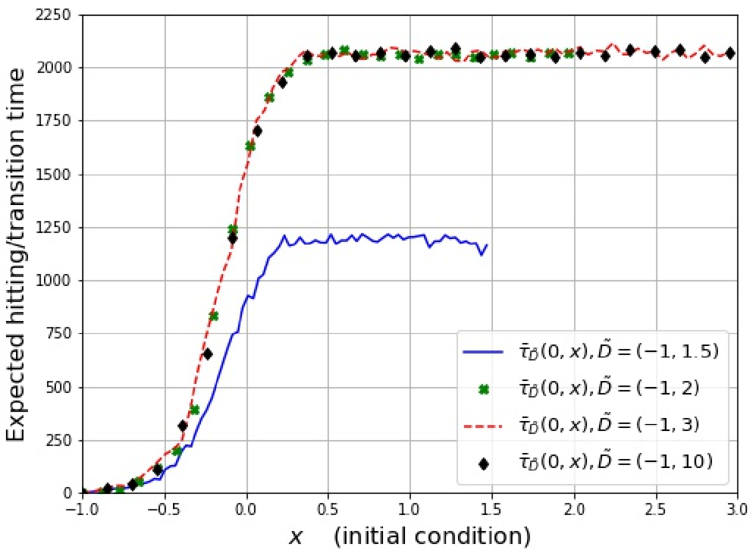

Figure 1: Monte Carlo estimation of $\bar{\tau}_{\tilde{D}}(0, x)$ with different $\tilde{D}$, plotted for $x \in (-1, 3) \cap \tilde{D}$.

Finally, for subsequent analysis, while $\tilde{D} = (-1, 2)$ is sufficient, we shall will reduce D to $\tilde{D} = (-1, 3)$. We pick this larger domain to accommodate when we use $\sigma = 1$ where $r_* = 1.58$ (2 dp).

5.2 Stochastic Resonance

5.2.1 Background

In this section we apply the theory developed for both the periodic measures and expected exit time to specifically study the physical phenomena of stochastic resonance. As described in the introduction, stochastic resonance is the physical phenomena to explain the periodic transitions between metastable states and is typically modelled by a double-well potential SDE with periodic forcing.

Section 3.3 shown that time-periodic weakly dissipative SDEs, which includes double-well potential SDEs, possesses a unique geometric periodic measure. The existence and uniqueness of geometric periodic measure of double-well potential SDEs implies that transitions between the metastable states occurs as well as asymptotic periodic behaviour. Note however this does not necessarily imply that the transitions between the wells is periodic.

In the absence of the periodic forcing i.e. the autonomous case, transitions between the wells occurs when the noise is sufficiently large for the process to surmount the well and reach the other well. In this autonomous case, the double well potential does not evolve in time. With periodic forcing, the double-well is no longer static. The wells move asymmetrically that is, when one well goes up, the other goes down. In contrast to the static double well, transitions are much more likely to occur when the well is at or near the highest position and the other well is near or is at its lowest position.

Although the double well potential oscillates, as noted, this does not immediately imply the transitions occur periodically. This is a reflection that despite the well being at its peak, transition is relatively more likely, it is not guaranteed. If the noise intensity is too small, transition well's peak may not occur with large enough probability. On the other hand, if the noise intensity is too large, transitions may occur even when the process at the well's trough rather than peak. Hence, the system is said to exhibit the phenomena of stochastic resonance when the noise intensity is fine tuned so that transition occur at the well's peaks and does not occur at the well's trough with large

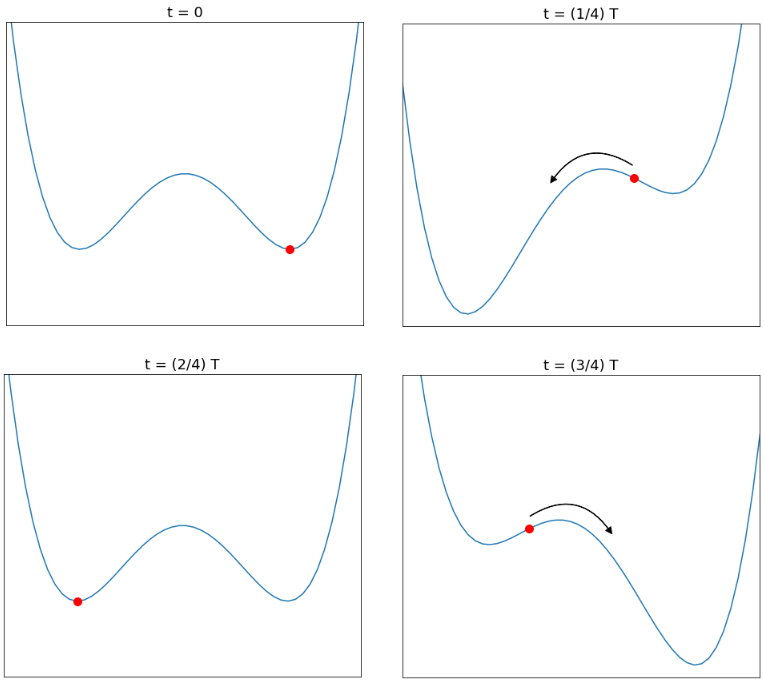

Figure 2: Visual portrayal of a double well SDE exhibiting stochastic resonance, where depicts transitions between the wells after a half period at the well's peak.

probability. From a transition time perspective, the system exhibits stochastic resonance when the expected transition time between the metastable states is (roughly) half the period [CLRS17]. We portray this heuristic visually in Figure 2.

For concreteness, we consider specifically the stochastic overdamped Duffing Oscillator (3.27) as our model of stochastic resonance. This is a typical model in literature [BPSV82, BPSV81, BPSV83, CLRS17, GHJM98, HI05, HIP05]. As noted (3.27) is a gradient SDE

$$dX_t = -\partial_x V(t, X_t)dt + \sigma dW_t,$$

derived from the time-periodic double-well potential $V \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ given by

$$V(t, x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 - Ax\cos(\omega t).$$

In the absence of the periodic forcing ($A = 0$), V has two local minima at $x = \pm 1$ which are the metastable states and has a local maximum at $x = 0$, the unstable state. We consider the left and right well to be the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively. Although the local minima change over time, by the nature of the problem, we shall normalise the problem to have $x = -1, +1$ as the bottom of the left and right well respectively.

Currently, there does not appear to be a standard nor rigorous definition of stochastic resonance [HI05, JH07]. In the context of this thesis, a working definition is that the stochastic system is in stochastic resonance if the noise intensity is tuned optimally such that the expected transition time between the metastable states is (approximately) half the period [CLRS17]. For the stochastic resonance problem, let $D = (-1, \infty)$ and consider

$$\tau_\sigma(s, x) = \inf_{t \geq s}\{X_t = -1 | X_s = x\} - s = \inf_{t \geq s}\{X_t \in \partial D | X_s = x\} - s, \quad x \in D.$$

For convenience, let $\tau_\sigma(x) := \tau(0, x)$, then $\tau_\sigma(x)$ is interpreted as the sample-path exit time from initial point x to the bottom of the left-well. In the context of stochastic resonance, we keep the explicit σ dependence and refer to the exit time as transition time (between the metastable states).

5.2.2 Estimating by Monte Carlo and PDE

We first demonstrate the validity of solving (4.19) for the expected duration for the Duffing Oscillator (3.27). Following Example 5.3, we choose the same parameters $A = 0.12$, $\omega = 10^{-3}$ and $\sigma = 0.285$. The same parameters was considered in [CLRS17]. As Example 5.3 and Figure 1 demonstrated, we reduce the unbounded domain to the bounded domain $\tilde{D} = (-1, 3)$. We then estimate $\bar{\tau}_{0.285}$ by three approaches. In this demonstration, we let $\bar{\tau}^{\text{sde}}_{0.285}$, $\bar{\tau}^{\text{bfp}}_{0.285}$ and $\bar{\tau}^{\text{grad}}_{0.285}$ to respectively represent the Monte Carlo simulation from Example 5.3, Banach fixed point iteration (Theorem 4.12) and gradient descent iteration via convex optimisation (Theorem 4.18) approximations to $\bar{\tau}_{0.285}$. Figure 3 shows these approximations.

For Figure 3, we re-use $\bar{\tau}^{\text{sde}}_{0.285}$ from Example 5.3 for the domain $\tilde{D} = (-1, 3)$. To compute $\bar{\tau}^{\text{bfp}}_{0.285}$ and $\bar{\tau}^{\text{grad}}_{0.285}$ via PDE methods, we partition $\tilde{D}$ and the time interval $[0, T]$ into $N^{\text{pde}}_x = 500$ and $N^{\text{pde}}_t = \lfloor 2T \rfloor$ uniform points respectively. We implement a backward Euler finite different method to evolve IBVPs (4.21) and (4.43). This yields the gradient (4.46) to compute the gradient descent iterates $v_{n+1} = v_n - \gamma_n \frac{\delta F}{\delta v_n}$. Due to the strict convexity of F, the rate of descent γ_n can be chosen adaptively and large provided F decreases. We

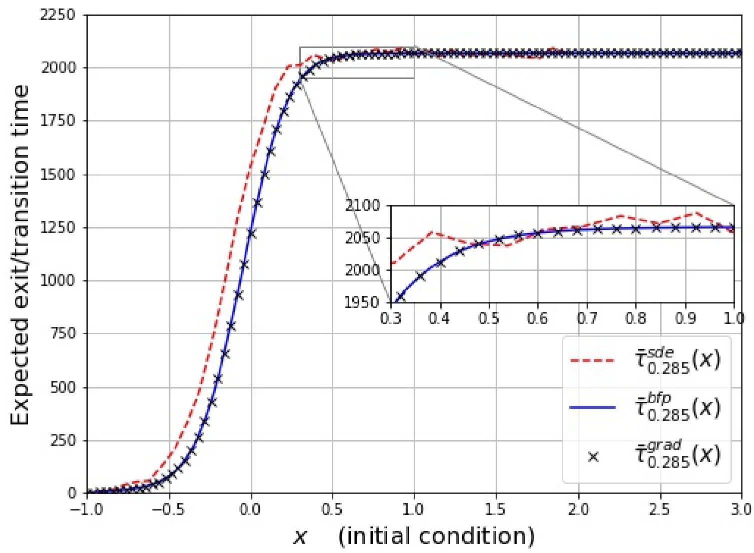

Figure 3: Numerical approximation of the expected transition time $\bar{\tau}_{0.285}(x)$ by $\bar{\tau}^{\text{sde}}_{0.285}(x)$, $\bar{\tau}^{\text{bfp}}_{0.285}(x)$ and $\bar{\tau}^{\text{grad}}_{0.285}(x)$ to SDE (3.27) with parameters $A = 0.12$, $\omega = 0.001$, $\sigma = 0.285$, $s = 0$, $\tilde{D} = (-1, 3)$.

continue both the Banach fixed point and gradient iterates schemes until (the numerical approximation of) $F(v_n) \leq 10^{-5}$.

Figure 3 shows that $\bar{\tau}^{\text{bfp}}_{0.285}$ and $\bar{\tau}^{\text{grad}}_{0.285}$ are closely approximate each other and in turn both visually approximate $\bar{\tau}^{\text{sde}}_{0.285}$ well, particularly for initial conditions starting in the right well. In the absence of an analytic formulae of $\bar{\tau}_{0.285}$, we assume $\bar{\tau}^{\text{sde}}_{0.285}$ is the "true" solution and numerically estimated the relative error by $\frac{\|\bar{\tau}^{\text{sde}}_{0.285} - \bar{\tau}^{\text{bfp}}_{0.285}\|_{L^2(\tilde{D})}}{\|\bar{\tau}^{\text{sde}}_{0.285}\|_{L^2(\tilde{D})}} = 0.57\%$ (2 dp). In particular, since our interest lies in expected transition time between the wells, we compute also $\frac{\|\bar{\tau}^{\text{sde}}_{0.285} - \bar{\tau}^{\text{bfp}}_{0.285}\|_{L^2(0,3)}}{\|\bar{\tau}^{\text{sde}}_{0.285}\|_{L^2(0,3)}} = 0.1\%$ (2 dp). The relative errors are very similar for $\bar{\tau}^{\text{grad}}_{0.285}$. The small relative error validates approximating $\bar{\tau}_{0.285}$ by numerically solving PDE (4.19) by either $\bar{\tau}^{\text{bfp}}_{0.285}$ or $\bar{\tau}^{\text{grad}}_{0.285}$ for the Duffing Oscillator. It may be particularly remarkable that the Banach fixed point iterates converges because it is not immediate whether the associated bilinear form is coercive.

5.2.3 Noise Intensity Fine Tuning

Following Section 5.2.2 where we demonstrated the validity of estimating the expected transition time by solving the PDE (4.19), in this section, we fine tune σ until the stochastic system exhibits stochastic resonance.

For completeness in discussing stochastic resonance, we consider also the transition from the left well to the right well. Specifically, consider the SDE

$$\begin{cases} dY_t = [Y_t - Y_t^3 + A\cos(\omega t)]\,dt + \sigma dW_t, & t \geq \frac{T}{2}, \\ Y_{\frac{T}{2}} = y, \end{cases}$$

and define

$$\tau_\sigma^{L\to R}(y) = \inf_{t\geq \frac{T}{2}} \{Y_t \in \partial D | Y_{\frac{T}{2}} = y\} - \frac{T}{2}, \quad y \in D_L,$$

where $D_L = (-\infty, 1)$ and noting that $s = \frac{T}{2}$. Clearly, by a change of variables, $\tilde{Y}_t = -Y_{t+\frac{T}{2}}$ and since $\cos(\omega(t+\frac{T}{2})) = -\cos(\omega t)$, we have that

$$\begin{cases} d\tilde{Y}_t = \left[\tilde{Y}_t - \tilde{Y}_t^3 + A\cos(\omega t)\right] dt - \sigma d\tilde{W}_t, \\ \tilde{Y}_0 = -y, \end{cases}$$

where $\tilde{W}_t = W_{t+\frac{T}{2}} - W_{\frac{T}{2}}$. It follows then that $\tau_\sigma^{L\to R}(y) = \inf_{t\geq 0}\{\tilde{Y}_t \in \partial D | \tilde{Y}_0 = -y\}$. Note that $\tilde{W}$ and W have the same distribution, hence

$$\bar{\tau}_\sigma(x) = \bar{\tau}_\sigma^{L\to R}(-x), \quad x \in D. \tag{5.3}$$

Indeed the same computation holds provided the drift is an odd function when $A = 0$.

Specifically for SDE (3.27) where $\omega = 0.001$, $T = 2000\pi$ is the period. Given (5.3), it is sufficient to cast the stochastic resonance problem as finding $\sigma_* \neq 0$ such that the transition time from the right well to the left i.e.

$$\bar{\tau}_{\sigma_*}(1) \simeq \frac{T}{2} = 1000\pi. \tag{5.4}$$

i.e. the expected transition time between the wells is half the period.

To fine tune for stochastic resonance, we repeat the same PDE computations with the same numerical parameters and methods (as for Figure 3), changing only σ and considering the expected transition time $\bar{\tau}_\sigma^{\mathrm{grad}}(1)$ is as a function of σ. We vary σ in the σ-domain $[0.2, 1]$. We partition this σ-domain into two subintervals $[0.2, 0.3]$ and $[0.3, 1]$ and uniformly partition them into 50

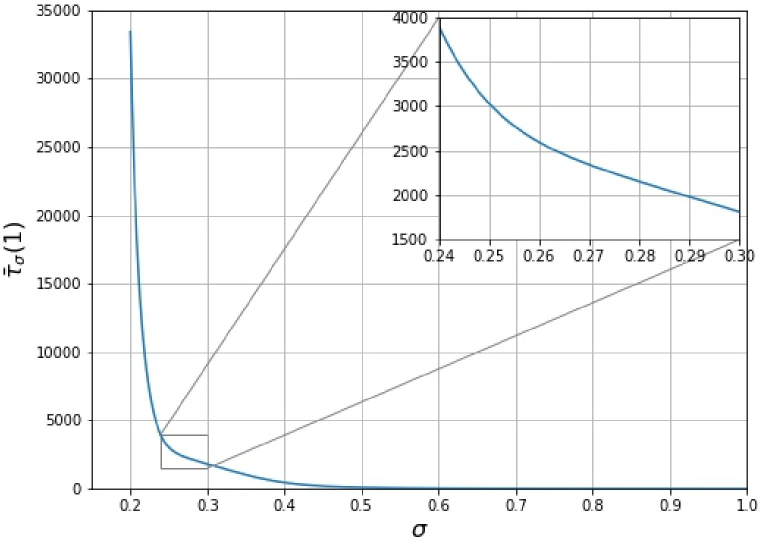

Figure 4: Plot of $\bar{\tau}_\sigma^{\mathrm{grad}}(1)$ for $\sigma \in [0.2, 1]$ for SDE (3.27) with parameters $A = 0.12$ and $\omega = 0.001$, $s = 0$ and $\tilde{D} = (-1, 3)$.

σ	0.245	0.2455	0.246	0.2465	0.247
$\bar{\tau}_\sigma^{\mathrm{grad}}(1)$	3388	3346	3306	3267	3230

σ	0.2475	0.248	0.2485	0.249	0.2495	0.25
$\bar{\tau}_\sigma^{\mathrm{grad}}(1)$	3194	3159	3125	3093	3061	3030

Table 1: Computation of $\bar{\tau}_\sigma^{\mathrm{grad}}(1)$ (4sf) on a finer partition of $\sigma \in [0.245, 0.25]$.

and 100 points respectively. As a function of σ, we plot the expected transition time $\bar{\tau}_\sigma^{\mathrm{grad}}(1)$ in Figure 4.

It can be seen from Figure 4 that (5.4) is satisfied for some $\sigma_* \in [0.245, 0.25]$. We compute further $\bar{\tau}_\sigma^{\mathrm{grad}}(1)$ on a finer partition of the interval $[0.245, 0.25]$ further and tabulate its numerical values in Table 1. Numerically, from Table 1, it can be seen that $\bar{\tau}_{0.2485}^{\mathrm{grad}} \simeq \frac{T}{2} = 1000\pi$ to the nearest $5 \cdot 10^{-4}$.

6 Future works

Generalisation to SPDEs Via Foster-Lyapunov functions and a coupling approach, Section 3 attained the existence and uniqueness of a periodic measure for SDEs with verifiable conditions. We expect the approach carrier forward to SPDEs. This expectation derives from the existing literature where invariant measures for SPDEs were attained via a coupling method similar akin to the one used in this thesis. For instance, in [Mat02, EMS01, KS01, KPS02], the respective authors utilised the coupling method to attain an invariant measure for the 2D SNS (two-dimensional stochastic Navier-Stokes equation). In fact, it is shown that the convergence of invariant measure for the 2D SNS equation is geometric in [HM08]. Other examples includes [Hai06] for a class of degenerate parabolic SPDEs including the complex Ginzburg-Landau equation and [Mat02] for dissipative SPDEs. We refer readers to [Mat03] where key aspects to attain invariant measures via the coupling method in the infinite dimensional setting was discussed. Moreover, in [DZ96], the authors deduced invariant measures for autonomous SPDEs via irreducibility and strong Feller property of the Markov semigroup without utilising coupling.

Noise non-degeneracy It was shown in Theorem 2.20 that the local Doeblin condition holds when the Markov transition probability is irreducible and has the strong Feller property. Theorem 3.2 was used to deduce irreducibility. In its proof, the diffusion matrix was assumed to have linear growth (3.6) and alongside its inverse to have bounded Frobenius norm. While this non-degenerate case studied in this thesis is already applicable in many physical problems such as the stochastic periodic double well potential problem in stochastic resonance, it is believed that this is a technical requirement. Specifically, the author believes that the diffusion condition can be relaxed to being just locally non-degenerate condition locally to the compact set that the local Doeblin condition is supposed to hold for. Since Hörmander's condition does not necessarily imply irreducibility (see Remark 2.2 of [Hai11]), (time-dependent) Hörmander's condition is generally insufficient. However, recall that Langevin equations of Section 3.6 posses a geometric periodic measure although the diffusion are degenerate and satisfying Hörmander's condition. This suggests the the general conditions is a balance between the inherent SDE structure and its degeneracy.

Markov Transition Density Conditions In results where the Markov transition density was required, it was noted that Hörmander's classical condition was insufficient in the time-periodic case that we have studied. To get the existence of the density, we relied upon Theorem 1 of [HLT17]. This theorem required that the SDE coefficients to be locally smooth and the diffusion coefficient to be time-independent - specifically (3.9) and (3.8). While these assumptions are sufficient for applications, they are likely to be a technical assumption that may be relaxed. In particular, it would be of interest to extend this to case where σ is T-periodic. Indeed we expect a range of physical systems the noise amplitude changes in time. For example the financial markets are generally quieter in the summer months as fewer market participants are available (due to the summer holidays) and retail markets experiences periodic sales for seasonal goods.

Subgeometric rates of convergence Another direction of generalisation is towards different rates of convergence (e.g. polynomial) for the periodic measure and possibly in different norm (e.g. f-norms). Indeed, it can be expected that in some physical applications, the rate of convergence is sub-geometric. For invariant measures, there are some works in polynomial convergence rates e.g.[Ver97, Ver00] for (autonomous) SDEs and [JR02] for (time-homogeneous) Markov chains.

Time-Periodic Feynman-Kac Duality Theorem 4.6 and Theorem 3.20 rigorously shown that the expected duration and the periodic measure density of a time-periodic SDE is the time-periodic solution of their respective second-order linear parabolic PDE. These are instance of time-periodic Feynman-Kac duality. The author anticipates other important quantities, or indeed a family of quantities, can related to time-periodic SDEs are time-periodic solutions to appropriate parabolic PDEs.

Equivalence of Expected Duration PDE and SDE In the derivation of the expected duration PDE of Section 4, it was shown rigorously that if expected exit time has finite moments then the PDE can be formally justified i.e. starting from the SDE perspective, a PDE is derived. One can ask the converse question akin to a priori estimates and weak solutions. Suppose the expected duration PDE (4.16) is well-posed, does that necessarily imply the expected duration of the SDE problem has finite first moment?

Lévy Processes In Section 3, we applied the abstract results of Section 2 to time-periodic SDEs with the influence of Brownian motion. An area of future research to extend apply results of Section 2 in the direction of time-periodic Lévy processes. We mention that Höpfner and Löcherbach studied ergodic properties of periodically forced Ornstein-Uhlenbeck process under the influence of Lévy noise [HL11]. Time-periodic financial models with jumps should see some benefits in this line of further research.

Convex Optimisation of Expected Duration PDE Lemma 4.15 applies not only to Hilbert spaces but also for reflexive Banach spaces such as $L^p(D)$, $1 < p < \infty$. The author expects reasonably straightforward to define a relevant cost functional to (4.19) in $L^p(D)$ and furthermore use Lemma 4.15 to show existence of a unique minimiser. However, since $L^p(D)$ is no longer Hilbert for $p \neq 2$, the gradient of the functional can only be defined on the dual space. Establishing a fruitful gradient method on the dual space is generally more difficult, particularly on infinite dimensional spaces. If the analogous results can be shown for $L^p(D)$, due to Sobolev embedding, the advantage is that it would then apply to higher dimensional systems.

Appendix

A Regular Solutions of SDEs

We note the following lemma for the existence of regular solution holds more generally on locally compact metric spaces with a similar construction [MT93].

Lemma A.1. *(Theorem 3.5 [Has12]) Assume that b and σ are locally Lipschitz and the regularity condition holds i.e. there exists a norm-like function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ and a constant $c > 0$ such that*

$$\mathcal{L}(t)V \leq cV.$$

Then there exists a unique (up to equivalence) solution X_t to (3.1) which is an almost surely continuous stochastic process and regular. Moreover, this process satisfies the inequality

$$\mathbb{E}[V(t, X_t)] \leq e^{c(t-r)}\mathbb{E}[V(r, X_r)], \quad s \leq r \leq t \tag{A.1}$$

if the expectation on the right exists.

Proof. As b and σ are locally Lipschitz assumption, one can construct Lipschitz functions b_n and σ_n on $\mathbb{R}^+ \times \mathbb{R}^d$ such that

$$\begin{cases} b_n(t,x) = b(t,x) & \text{on } \mathbb{R}^+ \times B_n, \\ \sigma_n(t,x) = \sigma(t,x) & \text{on } \mathbb{R}^+ \times B_n. \end{cases}$$

Then standard uniqueness and existence theorem, the SDE

$$\begin{cases} dX_t^n = b_n(t, X_t^n)dt + \sigma_n(t, X_t^n)dW_t, \\ X_s^n = x. \end{cases}$$

has a unique solution X_t^n for $s \leq t \leq t'$ for some $t' > s$ and each $n \in \mathbb{N}$. For $m \in \mathbb{N}$, define

$$\tau_n^m = \inf_{t \geq s}\{X_t^m \geq n\}.$$

It is intuitively clear (and provable see e.g. [Doo55]) that $\{\tau_n^m\}_{m \geq n}$ are identical. Hence we simply denote τ_n. Similarly,

$$\mathbb{P}\left\{\sup_{s\le t\le\tau_n}\|X_t^n - X_t^m\| > 0\right\} = 0, \quad m > n.$$

We define the process

$$\widetilde{X}_t := X_t^n, \quad t \le \tau_n$$

and show that it exists, unique (up to equivalence) and regular. By the regularity condition, we see that

$$W(t,x) = e^{-c(t-s)}V(t,x)$$

satisfies

$$LW = e^{-c(t-s)}(LV - cV) \le 0.$$

Hence, for stopping time $\tau_n(t) = \tau_n \wedge t$, Dynkin's formula yields

$$\mathbb{E}\left[e^{-c(t-s)}V(\tau_n(t), X_{\tau_n(t)})\right] - \mathbb{E}[V(s,X_s)] = \mathbb{E}\left[\int_s^{\tau_n(t)} LW(u,X_u)du\right] \le 0. \tag{A.2}$$

Since $\tau_n(t) \le t$ and $V \ge 0$, we have that

$$\mathbb{E}\left[V(\tau_n(t), \widetilde{X}_{\tau_n(t)})\right] \le e^{c(t-s)}\mathbb{E}[V(s,X_s)]. \tag{A.3}$$

Let I_A be the indicator function, then we also have

$$\mathbb{E}\left[V(\tau_n(t), \widetilde{X}_{\tau_n(t)})\right] \ge \mathbb{E}\left[I_{\{\tau_n\le t\}}V(\tau_n, \widetilde{X}_{\tau_n(t)})\right] \ge \inf_{u\ge s,\|x\|\ge n} V(u,x)\mathbb{P}(\tau_n \le t).$$

From (A.2),

$$\mathbb{P}(\tau_n \le t) \le \frac{e^{c(t-s)}\mathbb{E}[V(s,X_s)]}{\inf_{u\ge s,\|x\|\ge n} V(u,x)}.$$

Since V is norm-like, by taking $n \to \infty$ yields $\widetilde{X}_t$ is regular i.e. $\widetilde{X}_t$ is a solution to (3.1) for all $t \ge s$. This solution is unique up to equivalence: it follows from the definition of $\widetilde{X}_t$ and from the uniqueness of (3.1) in the domain B_n that for every pair of solution X_t and Y_t,

$$\mathbb{P}\left\{\sup_{s<t\le\tau_n}\|X_t - Y_t\| > 0\right\} = 0.$$

Since the solutions are regular, existence and uniqueness follows by letting

$n \to \infty$. The inequality (A.1) follows from (A.3) via Fatou's lemma. $\square$

B Hörmander's Condition

In this brief section, we remind ourselves of Hörmander's condition and its consequence to autonomous SDEs. We first recall that for two C^1 vector fields V, W on $\mathbb{R}^d$, we define their Lie bracket by

$$[V, W](x) = DV(x)W(x) - DW(x)V(x),$$

where DV denotes the Jacobian of V i.e. $(DV)_{ij} = \partial_j V_i$.

Let $\{A_i\}_{i=0}^m$ be (time-independent) vector fields on $\mathbb{R}^d$ and consider the autonomous SDE written in Stratonovich form with $\circ$

$$\begin{cases} dX_t = A_0(X_t)dt + \sum_{i=1}^m A_i(X_t) \circ dW_t^i, \quad t \geq 0 \\ X_0 = x. \end{cases} \tag{B.1}$$

Let A be a matrix with A_i as its columns, then to convert (B.1) to Itô form, define

$$\tilde{A}_0^i(x) = A_0^i(x) + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^d \frac{\partial A_{ij}}{\partial x_k} A_{kj}.$$

Then (B.1) in Itô form reads $dX_t = \tilde{A}_0(X_t)dt + \sum_{i=1}^m A_i(X_t)dW_t^i$.

We say that the SDE satisfies the parabolic Hörmander condition if for every $x \in \mathbb{R}^d$, the vectors

$$[A_{i_0}(x)], [A_{i_0}(x), A_{i_1}(x)], [[A_{i_0}(x), A_{i_1}(x)], A_{i_2}(x)], \ldots,$$

for $1 \leq i_0 \leq d$ and $0 \leq i_1, \ldots, i_m \leq d$ span $\mathbb{R}^d$. We note this condition is significantly weaker than demanding $(AA^T)(x)$ is invertible for all $x \in \mathbb{R}^d$.

Lemma B.1. *([Mal78, Hör85, RW00]) If the SDE (B.1) satisfies the parabolic Hörmander condition and possesses a regular solution (see Appendix A), then the solutions to (B.1) admits a smooth density with respect to the Lebesgue measure i.e. the Markov transition kernel exists and possess a smooth density.*

C Verifying Ornstein-Uhlenbeck Periodic Measure

For simplicity, we verify that the periodic measure of the Ornstein-Uhlenbeck process in 1-dimension. Specifically, we consider the 1-dimensional (3.22) from Example 3.10. Then it was claimed that $\rho_s = \mathcal{N}\left(\xi(s), \frac{\sigma^2}{2A}\right)$, where $\xi(t) := \int_{-\infty}^{t} e^{-A(t-r)}S(r)dr$, is the periodic measure. We explicitly verify this. The Markov transition kernel P in this example was computed as

$$P(s,t,x,\cdot) = \mathcal{N}(e^{-A(t-s)}x + J(s,t), \frac{\sigma^2}{2A}(1-e^{-2A(t-s)})(\cdot),$$

where

$$J(s,t) := \int_s^t e^{-A(t-r)}S(r)dr = \xi(t) - e^{-A(t-s)}\xi(s).$$

Since ξ is T-periodic then ρ is also T-periodic. Hence to verify ρ is the periodic measure, it suffices to verify $P^*(s,t)\rho_s = \rho_t$ for $s \leq t$. Indeed for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, by Fubini's theorem, we have

$$\begin{aligned}
&P^*(s,t)\rho_s(\Gamma)\\
=&\int_{\mathbb{R}} P(s,t,x,\Gamma)\rho_s(dx)\\
=&\frac{1}{\sqrt{\pi\sigma^2/A}}\int_{\mathbb{R}} \mathcal{N}(e^{-A(t-s)}x + J(s,t), \frac{\sigma^2}{2A}(1-e^{-2A(t-s)})(\Gamma)\exp\left(-\frac{(x-\xi(s))^2}{\sigma^2/2A}\right)dx\\
=&\frac{1}{\sqrt{\pi\sigma^2/A}\sqrt{\pi\left(\sigma^2/A\right)\left(1-e^{-2A(t-s)}\right)}}\\
&\times\int_\Gamma\int_{\mathbb{R}}\exp\left(-\frac{\left[y-(e^{-A(t-s)}x+J(s,t))\right]^2+(1-e^{-2A(t-s)})(x-\xi(s))^2}{\left(\sigma^2/2A\right)\left(1-e^{-2A(t-s)}\right)}\right)dxdy.
\end{aligned}$$

With $J(s,t) = \xi(t) - e^{-A(t-s)}\xi(s)$, we calculate that

$$\begin{aligned}
&\left[y-(e^{-A(t-s)}x+J(s,t))\right]^2\\
=&\left[y-\xi(t)+e^{-A(t-s)}(x-\xi(s))\right]^2\\
=&(y-\xi(t))^2+2(y-\xi(t))e^{-A(t-s)}(x-\xi(s))+e^{-2A(t-s)}(x-\xi(s))^2.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\left[y-(e^{-A(t-s)}x+J(s,t))\right]^2+(1-e^{-2A(t-s)})(x-\xi(s))^2\\
=&(y-\xi(t))^2+2(y-\xi(t))e^{-A(t-s)}(x-\xi(s))+(x-\xi(s))^2\\
=&(y-\xi(t))^2(1-e^{-2A(t-s)})+[(x-\xi(s))+e^{-A(t-s)}(y-\xi(t))]^2,
\end{aligned}
$$

where in the last line we added and subtracted $(y-\xi(t))^2e^{-2A(t-s)}$ and grouped appropriately. Then by the standard Gaussian integral identity $\int_{\mathbb{R}}\exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)dz=\sqrt{2\pi\sigma^2}$ for any fixed μ,

$$
\begin{aligned}
&P^*(s,t)\rho_s(\Gamma)\\
=&\frac{1}{\sqrt{\pi\sigma^2/2A}\sqrt{\pi\,(\sigma^2/A)\,(1-e^{-2A(t-s)})}}\\
&\times\int_\Gamma\int_{\mathbb{R}}\exp\left(-\frac{(y-\xi(t))^2(1-e^{-2A(t-s)})+[(x-\xi(s))+e^{-A(t-s)}(y-\xi(t)]^2}{(\sigma^2/2A)\,(1-e^{-2A(t-s)})}\right)dxdy\\
=&\frac{1}{\sqrt{\pi\sigma^2/2A}\sqrt{\pi\,(\sigma^2/A)\,(1-e^{-2A(t-s)})}}\\
&\times\int_\Gamma\exp\left(-\frac{(y-\xi(t))^2}{(\sigma^2/2A)}\right)\int_{\mathbb{R}}\exp\left(-\frac{[(x-\xi(s))+e^{-A(t-s)}(y-\xi(t)]^2}{(\sigma^2/2A)\,(1-e^{-2A(t-s)})}\right)dxdy\\
=&\frac{1}{\sqrt{\pi\sigma^2/2A}}\int_\Gamma\exp\left(-\frac{(y-\xi(t))^2}{(\sigma^2/2A)}\right)dy\\
=:&\rho_t(\Gamma)
\end{aligned}
$$

References

[ANMS99] V. S. Anishchenko, A. B. Neiman, F. Moss and L. Schimansky-Geier, Stochastic resonance: noise-enhanced order, *Physics-Uspekhi*, Vol 42 (1999), 7-36.

[Ama95] H. Amann, *Linear and Quasilinear Parabolic Problems Volume I: Abstract Linear Theory*, Birkhäuser Basel (1995).

[AW10] D. M. Ambrose and J. Wilkening, Computation of Time-Periodic Solutions of the Benjamin–Ono Equation, *Journal of Nonlinear Science*, Vol 20 (2010), 277-308.

[Arn98] L. Arnold, *Random Dynamical Systems*, Springer-Verlag Berlin Heidelberg New York (1998).

[Aro67] D. G. Aronson, Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.* 73 (1967), 890-896.

[BKM07] F. E. Benth, J. Kallsen and T. Meyer-Brandis, A Non-Gaussian Ornstein–Uhlenbeck Process for Electricity Spot Price Modelling and Derivatives Pricing, *Applied Mathematical Finance,* Vol 14 (2007), 153-169.

[BS07] F. E. Benth and J. Saltyte-Benth, The volatility of temperature and pricing of weather derivatives, *Quantitative Finance*, Vol 7(5) (2007), 553–561.

[BPSV81] R. Benzi, G. Parisi, A. Sutera and A. Vulpiani, The mechanism of stochastic resonance, *J. Phys. A*, Vol 14 (1981), 453-457.

[BPSV82] R. Benzi, G. Parisi, A. Sutera and A. Vulpiani, Stochastic resonance in climatic change, *Tellus*, Vol 34 (1982), 10-16.

[BPSV83] R. Benzi, G. Parisi, A. Sutera and A. Vulpiani, A theory of stochastic resonance in climatic change, *SIAM J. Appl. Math.*, 43 (1983), 563-578.

[Ber11] N. Berglund, Kramers' law: Validity, derivations and generalisations, *Markov Processes and Related Fields*, Vol 19(3) (2011).

[BR04] T. R. Bielecki and M. Rutkowski, *Credit Risk: Modeling, Valuation and Hedging*, Springer-Verlag Berlin Heidelberg (2004).

[BC76] F. Black and J. C. Cox, Valuing corporate securities, *J. Finance*, Vol 31 (1976), 351-367.

[BKRS15] V. I. Bogachev, N. V. Krylov, M. Röckner and S. V. Shaposhnikov, *Fokker–Planck–Kolmogorov Equations*, Mathematical Surveys and Monographs, American Mathematical Society (2015).

[Bre04] P. C. Bressloff, *Stochastic Processes in Cell Biology*, Springer International Publishing (2014).

[BGP98] M. O. Bristeau, R. Glowinski and J. Périaux, Controllability methods for the computation of time-periodic solutions; application to scattering, *J. Comput. Phys.* 147 (1998), 265–292.

[CLPS03] Y. Cao, S. Li, L. Petzold, and R. Serban, Adjoint sensitivity analysis for differential-algebraic equations: the adjoint DAE system and its numerical solution, *SIAM J. Sci. Comput. 24* (2003), 1076–1089.

[CH16] T. Caraballo and X. Han, *Applied Nonautonomous and Random Dynamical Systems*, Springer International Publishing (2016).

[CKS04] T. Caraballo, P. E. Kloeden and B. Schmalfuss, Exponentially stable stationary solutions for stochastic evolution equations and their perturbation. *Appl. Math. Optim*, Vol. 50 (2004), 183-207.

[CGM05] J. Casado-Pascuala, J. Gómez-Ordóñez and M. Morillo, Stochastic resonance: Theory and numerics, *Chaos*, Vol 15 (2005).

[CHLY17] F. Chen, Y. Han, Y. Li, X. Yang, Periodic solutions of Fokker–Planck equations, *Journal of Differential Equations*, Vol 263 (2017) 285-298.

[CLRS17] A. M. Cherubini, J. S W Lamb, M. Rasmussen and Y. Sato, A random dynamical systems perspective on stochastic resonance, *Nonlinearity*, Vol 30 (2017), 2835-2853.

[DR12] G. Da Prato and M. Röckner, Well posedness of Fokker-Planck equations for generators of time-inhomogeneous Markovian transition probabilities, *Atti Accad. Naz. Lin-cei Rend. Lincei Mat. Appl*, Vol 23 (2012), 361–376.

[DZ92] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press (1992).

[DZ96] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press (1996).

[DM92] D. Daners and P. K. Medina, *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Mathematics Series 279, Longman Scientific & Technical (1992).

[DFK10] H. Dehling, B. Franke and T. Kott, Drift estimation for a periodic mean reversion process, *Statistical Inference for Stochastic Processes*, 13(3) (2010).

[Doo55] J. L. Doob, Martingales and one dimensional diffusion, *Trans. Amer. Math. Soc.*, Vol 78 (1955) 168-208.

[EKMS00] W. E, K. Khanin, A. Mazel and Ya. Sinai, Invariant measures for Burgers equation with stochastic forcing, *Ann. of Math*, Vol.151 (2000), 877-960.

[EMS01] W. E, J. C. Mattingly and Y. Sinai, Gibbsian Dynamics and ergodicity for the stochastically forced Navier-Stokes equation, *Communications in Mathematical Physics*, Vol 224 (2001), 83-106.

[EM01] W. E and J. C. Mattingly, Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation, *Communications on Pure and Applied Mathematics*, Vol 54 (2001), 1386-1402.

[ET99] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, Society for Industrial and Applied Mathematics (1999).

[Eva10] L. C. Evans, *Partial Differential Equation*, American Mathematical Society, Second Edition (2010).

[FLZ17] C. R. Feng, Y. Liu and H. Z. Zhao, Numerical approximation of random periodic solutions of stochastic differential equations, *Zeitschrift für angewandte Mathematik und Physik*, Vol 68:119(2017), 1-32.

[FLZ19] C. R. Feng, Y. Liu and H. Z. Zhao, Weak convergence of numerical scheme of periodic measures for stochastic differential equations, in preparations.

[FWZ16] C. R. Feng, Y. Wu and H. Z. Zhao, Anticipating Random Periodic Solutions I. SDEs with Multiplicative Linear Noise, *Journal of Functional Analysis*, Vol 271 (2016), 365-417.

[FZ12] C.R. Feng and H. Z. Zhao, Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding, *J. Funct. Anal.*, Vol. 262 (2012), 4377-4422.

[FZ16] C. R. Feng and H. Z. Zhao, Random Periodic Processes, Periodic Measures and Ergodicity, *arXiv*: 1408.1897v5 (2016).

[FZZ19] C. R. Feng, H. Z. Zhao and J. Zhong, Expected exit time for periodic stochastic differential equations, in preparations.

[FZZ11] C. R. Feng, H. Z. Zhao and B. Zhou, Pathwise random periodic solutions of stochastic differential equations, *Journal of Differential Equations*, Vol 251 (2011), 119-149.

[Fre83] D. A. Freedman, *Brownian Motion and Diffusion*, Springer-Verlag New York (1983).

[Fre00] M. I. Freidlin, Quasi-deterministic approximation, metastability and stochastic resonance, *Phys D*, Vol 137 (2000), 333-352.

[FW98] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer, New York, Second Edition (1998).

[Fri64] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall (1964).

[GHJM98] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Stochastic resonance, *Reviews of Modern Physics*, Vol 70, (1998), 223-287.

[Gar09] C. Gardiner, *Stochastic Methods - A Handbook for the Natural and Social Sciences*, Springer-Verlag Berlin Heidelberg, Fourth edition (2009).

[GP00] M. Giles and N. Pierce, An introduction to the adjoint approach to design, *Flow Turbul. Combust.* 65 (2000), 393–415.

[Hai06] M. Hairer, Coupling stochastic PDEs, *XIVth International Congress on Mathematical Physics* (2006), 281-289.

[Hai11] M. Hairer, On Malliavins proof of Hörmander's theorem, *Bull. Sci. Math*, Vol 135 (2011), 650-666.

[HM08] M. Hairer and J. C. Mattingly, Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes Equations, *Annals of Probability*, Vol 36 (2008), 2050-2091.

[HM11] M. Hairer and J. C. Mattingly, Yet another look at Harris' ergodic theorem for Markov Chains, Seminar on Stochastic Analysis, *Random Fields and Applications VI Progress in Probability*, Springer, Basel, Vol 63 (2011), 109-117.

[Hal10] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, (2010).

[Has12] R. Z. Has'minskii, *Stochastic Stability of Differential Equations*, Springer, Second Edition (2012).

[Has76] K. Hasselmann, Stochastic climate models Part I. Theory, *Tellus*, Vol 28, (1976), 473-484.

[HIS76] J. D. Hays, John Imbrie and N. J. Shackleton, Variations in the earth's orbit: pacemaker of the ice ages, *Science*, Vol. 194 (1976), 1121-1132.

[HI05] S. Herrmann and P. Imkeller, The exit problem for diffusions with time-periodic drift and stochastic resonance, *The Annals of Applied Probability*, Vol 15 (2005), 39-68.

[HIP05] S. Herrmann, P. Imkeller and I. Pavlyukevich, Two Mathematical Approaches to Stochastic Resonance, *Deuschel JD, Greven A. (eds) Interacting Stochastic Systems*, Springer, Berlin, Heidelberg (2005)

[HIPP14] S. Herrmann, P. Imkeller, I. Pavlyukevich and D. Peithmann, *Stochastic resonance: A mathematical approach in the small noise limit*, American Mathematical Society (2013).

[Hes91] P. Hess, *Periodic-parabolic Boundary Value Problems and Positivity*, Longman Scientific & Technical (1991).

[HL11] R. Höpfner and E. Löcherbach, On ergodicity properties for time inhomogeneous Markov processes with T-periodic semigroup, *arXiv*: 1012.4916 (2011).

[HLT17] R. Höpfner, E. Löcherbach and M. Thieullen, Strongly degenerate time inhomogeneous SDEs: densities and support properties. Application to a Hodgkin-Huxley system with periodic input, *Bernoulli*, Vol 23 (2017), 2587-2616.

[Hör85] L. Hörmander, *The Analysis of Linear Partial Differential Operators I-IV*, Springer-Verlag Berlin Heidelberg (1985).

[IP01] P. Imkeller and I. Pavlyukevich, Stochastic resonance in two-state Markov chains, *Archiv der Mathematik*, Vol. 77 (2001), 107-115.

[IDL14] A. Iolov, S.Ditlevsen and A. Longtin, Fokker–Planck and Fortet Equation-Based Parameter Estimation for a Leaky Integrate-and-Fire Model with Sinusoidal and Stochastic Forcing, *The Journal of Mathematical Neuroscience*, Vol 4 (2014).

[JR02] S. F. Jarner and G. O. Roberts, Polynomial Convergence Rates of Markov Chains, *Ann. Appl. Probab.* Vol 12, (2002), 224-247.

[JQSY19] M. Ji, W. Qi, Z. Shen, Y. Yi, Existence of periodic probability solutions to Fokker-Planck equations with applications, *Journal of Functional Analysis* (2019)

[Jun89] P. Jung, Thermal activation in bistable systems under periodic forces, *Zeitschrift für Physik B Condensed Matter*, Vol 76 (1989), 521-535.

[Jun93] P. Jung, Periodically driven stochastic systems, *Physics Reports*, Vol 234 (1993), 175-295.

[JH07] P. Jung and P. Hanggi, Stochastic Nonlinear Dynamics Modulated by External Periodic Forces, *EPL (Europhysics Letters)* 8(6) (2007), 505-510.

[KR11] P. E. Kloeden and M. Rasmussen, *Nonautonomous Dynamical Systems*, American Mathematical Society (2011).

[Kra40] H. A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica*, Vol 7 (1940), 284-304.

[KPS02] S. Kuksin, A. Piatniski and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. II, *Communications in Mathematical Physics*, Vol 230 (2002), 81-85.

[KS01] S. Kuksin and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's I, *Communications in Mathematical Physics*, Vol 221 (2001), 351-366.

[LL08] C. Le Bris and P-L. Lions Existence and Uniqueness of Solutions to Fokker–Planck Type Equations with Irregular Coefficients, *Communications in Partial Differential Equations*, Vol 33 (2008), 1272-1317.

[LZ12] P. Lian and H. Z. Zhao, Pathwise properties of random mappings, In: *New Trends in Stochastic Analysis and Related Topics - volume in honour of Professor K.D. Elworthy*, edited by H.Z. Zhao and A. Truman, World Scientific, (2012) 227-300.

[LM18] L. S. Lima and L. L. B. Miranda, Price dynamics of the financial markets using the stochastic differential equation for a potential double well, *Physics A*, Vol 490 (2018), 828-833.

[Lin92] T. Lindvall, *Lectures on the Coupling Method*, Wiley, New York (1992).

[LS02] J. J. Lucia and E. Schwartz, Electricity prices and power derivatives: Evidence from the Nordic Power Exchange, *E.S. Review of Derivatives Research*, Vol 5 (2002).

[MS01] R. S. Maier and D. L. Stein , Noise-Activated Escape from a Sloshing Potential Well, *Phys Rev Lett*, Vol 86 (2001).

[Mal78] P. Malliavin, Stochastic Calculus of Variations and Hypoelliptic Operators, *Proc. Intern. Symp. SDE* (1978), 195-263.

[Mao07] X. Mao, *Stochastic Differential Equations and Applications*, Woodhead Publication, Second Editions (2007).

[MV99] P. Markowich and C. Villani, On the trend to equilibrium for the Fokker Planck equation an interplay between physics and functional analysis, *Matematica Contemporânea* (1999).

[Mat02] J. C. Mattingly, Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics, *Communications in Mathematical Physics*, Vol 230 (2002), 421-462.

[Mat03] J. C. Mattingly, On recent progress for the stochastic Navier-Stokes equations, *Journées équations aux dérivées partielles* (2003).

[MSH02] J.C Mattingly, A.M. Stuart and D. J. Higham, Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise, *Stochastic Processes and their Applications*, Vol 101 (2002), 185-232.

[MW89] B. McNamara and K. Wiesenfeld, Theory of stochastic resonance, *Phys. Rev. A*, Vol 39 (1989), 4854-4869.

[MT92] S. P. Meyn and R. L. Tweedie, Stability of Markovian processes I: Criteria for discrete-time chains, *Advances in Applied Probability*, Vol 24 (1992), 542-574.

[MT93] S. P. Meyn and R. L. Tweedie, Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time Processes, *Advances in Applied Probability*, Vol 25 (1993), 518-548.

[MT09] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Second Edition, Cambridge University Press (2009).

[Mil30] M. Milankovitch, Handbuch der Klimatologie 1, *Teil. Doppen and Geiger*, Berlin (1930).

[Nic82] C. Nicolis, Stochastic aspects of climatic transitions response to a periodic forcing, *Tellus*, Vol 34 (1982), 1-9.

[Nor98] J. R. Norris, *Markov Chains*, Cambridge University Press (1998).

[OO99] G. Ochs and V. I. Oseledets, Topological fixed point theorems do not hold for random dynamical systems, *Journal of Dynamics and Differential Equations*, Vol 11 (1999), 583-593.

[Pav14] G. A. Pavliotis, *Stochastic Processes and Applications*, Springer-Verlag New York (2014).

[Paz92] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Second Edition (1992).

[Ple06] R. Plessix, A review of the adjoint-state method for computing the gradient of a functional with geophysical applications, *Geophysical Journal International*, Vol 167(2) (2006), 495-503.

[RS79] L. M. Ricciardi and L. Sacerdote, The Ornstein-Uhlenbeck Process as a Model for Neuronal *Activity*, *Biological* Cybernetics, Vol 35 (1979), 1-9.

[Ris96] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications*, Springer-Verlag Berlin Heidelberg, Second Edition (1996).

[RZ10] M. Röckner and X. Zhang, Weak uniqueness of Fokker-Planck equations with degenerateand bounded coefficients, *R. Acad. Sci. Paris*, (2010), 435–438.

[RW00] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales: Volume 2, Ito Calculus*, Second Edition, Cambridge University Press (2000).

[Sat78] S. Sato, On the moments of the firing interval of the diffusion approximated model neuron, *Mathematical Biosciences*, Vol 39 (1978) 53-70.

[Sch98] B. Schmalfuss, A fixed point theorem and the random graph transformation, *Journal of Mathematical Analysis and Applications*, Vol. 225 (1998), 91-113.

[SFP14] B. Sengupta, K. J. Friston and W. D. Penny, Efficient gradient computation for dynamical models, *NeuroImage*, Vol 98 (2014), 521-527.

[Sin96] Ya. Sinai, Burgers system driven by a periodic stochastic flows. *In: Ito's stochastic calculus and probability theory*, Springer, Tokyo, (1996) 347-353.

[Str82] D. W. Stroock, Lectures on Topics in Stochastic Differential Equations, Tata Institute of Fundamental Research (1982)

[Str05] D. W. Stroock, *An Introduction to Markov Processes*, Springer-Verlag, Berlin Heidelberg (2005).

[SV06] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer-Verlag Berlin Heidelberg (2006).

[Tho00] H. Thorisson, *Coupling, Stationarity and Regeneration*, Springer, New York (2000).

[Tro10] F. Troltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, American Mathematical Soc. (2010).

[Ver97] A.Y. Veretennikov, On polynomial mixing bounds for stochastic differential equations, *Stochastic Processes Appl.*, 70(1) (1997) 115-127.

[Ver00] A.Y. Veretennikov, On polynomial mixing and convergence rate for stochastic difference and differential equations, *Theory of Probability & Its Applications*, Vol 4 (2000), 361–374.

[ZZ07] Q. Zhang and H. Zhao, Stationary solutions of SPDEs and infinite horizon BDSDEs, *Journal of Functional Analysis*, Vol. 252 (2007), 171-219.

[ZZ09] H.Z. Zhao and Z. H. Zheng, Random periodic solutions of random dynamical systems, *Journal of Differential Equations*, Vol 246 (2009), 2020-2038.

[ZMJ90] T. Zhou, F. Moss and P. Jung, Escape-time distributions of a periodically modulated bistable system with noise, *Phys. Rev. A*, Vol 42 (1990), 3161-3169.

[Zwa01] R. Zwanzig, *Nonequilibrium Statistical Mechanics*, Oxford University Press (2001).