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$P(\phi)_1$-process for the spin-boson model and a functional central limit theorem for associated additive functionals

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Abstract

We construct a random process with stationary increments associated to the Hamiltonian of the spin-boson model consisting of a component describing the spin and a component given by a Schwartz distribution-valued Ornstein-Uhlenbeck process describing the boson field. We use a path integral representation of the Hamiltonian to prove a functional central limit theorem for additive functionals, and derive explicit expressions of the diffusion constant for specific functionals.

Key-words: spin-boson model, ground state, functional central limit theorem

2010 MS Classification: 47D07, 47D99, 60J75, 81Q10

1 Introduction

The spin-boson model describes an atom with two energy levels and a scalar boson field coupled linearly. Its Hamiltonian can be defined as a self-adjoint operator acting on a Hilbert space, given by

$$H = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \alpha\sigma_x \otimes \phi_{\mathrm{b}}(h), \tag{1.1}$$

where σ_x, σ_z are two of the three 2×2 Pauli matrices describing the two-level atom, H_{f} is the free field Hamiltonian, $\phi_{\mathrm{b}}(h)$ is a field operator with a test function $\hat{h}$, and $\alpha \in \mathbb{R}$ is a coupling constant (see below for details). The spectrum of the spin-boson Hamiltonian has been studied in mathematical physics by various methods, such as operator analysis and stochastic analysis (functional integration), see [1, 3, 7, 9, 13].

A functional integral representation of e^{-tH} can be constructed by using a two-component random process consisting of a Poisson-driven random process describing the spin, and an

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infinite dimensional Ornstein-Uhlenbeck process describing the boson field. An advantage of this description is that it gives a non-perturbative approach in the spectral analysis of H, e.g., the existence of a ground state of H and its uniqueness can be derived for arbitrary values of coupling constants, and its qualitative properties can also be investigated in detail [7].

In this paper we construct a $P(\phi)_1$-process associated with the spin-boson model, and derive a functional central limit theorem (FCLT) of additive functionals of the properly scaled process. This corresponds then to the invariance principle in the spirit of Donsker in stochastic analysis [4], and describes a Gaussian behavior in a particular sense of various functionals related to H. As far as we are aware, this type of investigation in quantum field theory has been initiated by Betz and Spohn in [2], where the authors prove a FCLT for the classical Nelson model. Their approach relies on a method developed by Kipnis and Varadhan [10], where a martingale constructed for a reversible stationary process is used. In [5] we have further considered this problem for not only the classical but also for the relativistic Nelson model. In order to carry this out for the classical and relativistic Nelson models, an external potential must be imposed in order to create a ground state. Such a potential however destroys the stationarity of the increments of the random process appearing in the Feynman-Kac representation of the semigroup generated by the Nelson Hamiltonian. To produce a stationary process to work with, we constructed a $P(\phi)_1$-process obtained through the ground state transform of the Hamiltonian, indexed by the full time-line $\mathbb{R}$, which is a reversible stationary Markov process. This then allows to apply the Kipnis-Varadhan technique to prove the desired FCLT.

For the spin-boson model the situation is different from the Nelson model. A ground state of a spin-boson model exists whenever $\hat{h}/\omega \in L^2(\mathbb{R}^d)$, and it is unique. This is shown, for instance, in [7]. Although the random process associated with the spin-boson model is less regular than that of the Nelson model as here we have paths with jump discontinuities instead of Brownian motion, we have developed a framework in [7, 8] allowing to treat such less regular path measures.

The plan of the paper is as follows. After some preliminaries on the spin-boson model presented in Section 2, we construct a $P(\phi)_1$-process for the spin-boson Hamiltonian in Section 3. In Section 4 we obtain a functional central limit theorem for additive functionals of the scaled stochastic process, and derive explicit formulae on the variance for specific choices of functionals. In the Appendix we provide some remaining proofs.

2 Spin-boson model

2.1 Spin-boson model in Fock space

We define the spin-boson Hamiltonian as a self-adjoint operator on a Hilbert space. Let $\mathscr{F}_{\mathrm{b}} = \oplus_{n=0}^{\infty}(\otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^d))$ be the boson Fock space over $L^2(\mathbb{R}^d)$, where the subscript "sym" means symmetrized tensor product. The operator $H_{\mathrm{f}} = d\Gamma(\omega)$ is the free field Hamiltonian with dispersion relation $\omega(k) = |k|$, where $d\Gamma(\omega)$ denotes the second quantization of the operator ω on $L^2(\mathbb{R}^d)$. The creation and annihilation operators a^* and a satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \left(\bar{f}, g\right), \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)],$$

for all $f, g \in L^2(\mathbb{R}^d)$. We formally write $a^\sharp(f) = \int_{\mathbb{R}^d} a^\sharp(k) f(k) dk$ for $a^\sharp = a$ or a^*. The scalar field operator and its conjugate momentum on $\mathscr{F}_{\mathrm{b}}$ are defined, respectively, by

$$\phi_{\mathrm{b}}(h) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} a^*(k)\hat{h}(-k) + a(k)\hat{h}(k) dk \quad \text{and} \quad \Pi(h) = \frac{i}{\sqrt{2}} \int_{\mathbb{R}^d} a^*(k)\hat{h}(-k) - a(k)\hat{h}(k) dk, \tag{2.1}$$

where $h \in L^2(\mathbb{R}^d)$ and $\hat{h}$ is the Fourier transform of h. Consider the 2×2 Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With these components, the spin-boson Hamiltonian is defined on $\mathcal{H} = \mathbb{C}^2 \otimes \mathscr{F}_{\mathrm{b}}$ by the linear operator

$$H = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \alpha \sigma_x \otimes \phi_{\mathrm{b}}(h),$$

where $\alpha \in \mathbb{R}$ is a coupling constant. The Hamiltonian H can be transformed in a convenient form to study its spectrum in terms of path measures. Let $U = \exp\left(i\frac{\pi}{4}\sigma_y\right) \otimes \mathbb{1}$. Under this map H transforms as

$$\bar{H} = UHU^* = -\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \alpha \sigma_z \otimes \phi_{\mathrm{b}}(h),$$

and we see that $\bar{H}$ is a bounded self-adjoint operator on $D(\mathbb{1} \otimes H_{\mathrm{f}})$.

2.2 Spin-boson model in function space

Next we give the definition of the model in function space, and start with the state space for the spin. Consider $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Using the affine map $x \mapsto 2x - 1$, we arrive at the conventional spin variables $\{-1, +1\} \cong \mathbb{Z}_2$. Consider the Bernoulli measure

$$\nu(\sigma) = \frac{1}{2}(\delta_{-1}(\sigma) + \delta_{+1}(\sigma)), \quad \sigma \in \mathbb{Z}_2,$$

where $\delta_\pm$ is Dirac point measure with mass at $\sigma = \pm 1$. Let $(\mathbb{Z}_2, \mathcal{B}, \nu)$ be the probability space with σ-field $\mathcal{B} = \big\{\emptyset, \{-1\}, \{+1\}, \mathbb{Z}_2\big\}$. Define the space

$$L^2(\mathbb{Z}_2, \mathscr{F}_{\mathrm{b}}) = \Big\{F : \mathbb{Z}_2 \to \mathscr{F}_{\mathrm{b}} \,|\, \|F\|^2_{L^2(\mathbb{Z}_2, \mathscr{F}_{\mathrm{b}})} = \sum_{\sigma \in \mathbb{Z}_2} \|F(\sigma)\|^2_{\mathscr{F}_{\mathrm{b}}} < \infty\Big\},$$

which will be identified with $\mathcal{H}$ in what follows. The Hamiltonian $\bar{H}$ on this space is the self-adjoint operator

$$(\tilde{H}\Psi)(\sigma) = (H_{\mathrm{f}} + \alpha\sigma\phi_{\mathrm{b}}(h))\Psi(\sigma) + \Psi(-\sigma), \quad \sigma \in \mathbb{Z}_2.$$

To describe the free boson field in function space, next we define an infinite dimensional Ornstein-Uhlenbeck process. Let $\mathcal{M}$ be a Hilbert space over $\mathbb{R}$ with scalar product

$$(f,g)_{\mathcal{M}} = \int_{\mathbb{R}^d} \overline{\widehat{f}(k)}\widehat{g}(k)\frac{1}{2\omega(k)}dk.$$

Let A be a positive self-adjoint operator with Hilbert-Schmidt inverse on $\mathcal{M}$ such that $\sqrt{\omega}A^{-1}$ is bounded. Denote $C^\infty(A) = \cap_{n\in\mathbb{N}} D(A^n)$, and take

$$\mathcal{M}_n = \overline{C^\infty(A)}^{\|A^{n/2}\cdot\|_{\mathcal{M}}}.$$

We construct a triplet $\mathcal{M}_{+2} \subset \mathcal{M} \subset \mathcal{M}_{-2}$, where $\mathcal{M}_{+2}^* = \mathcal{M}_{-2}$. We set $Q = \mathcal{M}_{-2}$ and endow Q with its Borel σ-field $\mathcal{B}(Q)$, defining the measurable space $(Q, \mathcal{B}(Q))$.

Let $\mathscr{Y} = C(\mathbb{R}, Q)$ be the set of continuous functions on $\mathbb{R}$ with values in Q and denote its Borel σ-field by $\mathcal{B}(\mathscr{Y})$. We define a Q-valued Ornstein-Uhlenbeck process $(\xi_t)_{t\in\mathbb{R}}$, $\mathbb{R} \ni t \mapsto \xi_t \in Q$, on the probability space $(\mathscr{Y}, \mathcal{B}(\mathscr{Y}), \mathcal{G})$ with a probability measure $\mathcal{G}$ as follows. Let $\xi_t(f) = \langle\!\langle \xi_t, f\rangle\!\rangle$ for $f \in \mathcal{M}_{+2}$, where $\langle\!\langle ., .\rangle\!\rangle$ denotes the pairing between Q and $\mathcal{M}_{+2}$. For every $t \in \mathbb{R}$ and every $f \in \mathcal{M}_{+2}$, the random process $(\xi_t(f))_{t\in\mathbb{R}}$ is a Gaussian process with mean zero and covariance

$$\mathbb{E}_{\mathcal{G}}[\xi_t(f)\xi_s(g)] = \int_{\mathbb{R}^d} \overline{\widehat{f}(k)}\widehat{g}(k)\frac{e^{-|t-s|\omega(k)}}{2\omega(k)}dk. \tag{2.2}$$

By (2.2), every $\xi_t(f)$ can be uniquely extended to test functions $f \in \mathcal{M}$, which for simplicity we keep denoting in the same way. The stationary measure of $\mathcal{G}$ will be denoted by μ. Furthermore, $(\xi_t)_{t\in\mathbb{R}}$ is a stationary Markov process. Using the regular conditional probability measure $\mathcal{G}^\xi(\cdot) = \mathcal{G}(\cdot \mid \xi_0 = \xi)$, we have $\mathbb{E}_{\mathcal{G}}[\cdots] = \int_Q \mathbb{E}_{\mathcal{G}^\xi}[\cdots]d\mu$.

The connection between $\mathscr{F}_{\mathrm{b}}$ and $L^2(Q, d\mu)$ is given by the Wiener-Itô-Segal isomorphism $\theta : \mathscr{F}_{\mathrm{b}} \longrightarrow L^2(Q, d\mu)$ (for more details see [12]). The unitary equivalence of $L^2(\mathbb{Z}_2 \times Q, \nu \otimes \mu)$ and $\mathcal{H}$ is implemented by the unitary operator $U \otimes \theta : \mathcal{H} \longrightarrow L^2(\mathbb{Z}_2 \times Q, \nu \otimes \mu)$. Thus we have the identification $\mathcal{H} \cong L^2(\mathbb{Z}_2, L^2(Q)) \cong L^2(\mathbb{Z}_2 \times Q, \nu \otimes \mu)$. We define the scalar field operator and the free field Hamiltonian on $L^2(\mathbb{Z}_2 \times Q, \nu \otimes \mu)$ by $\widetilde{\phi}_{\mathrm{b}}(h) = \theta\phi_{\mathrm{b}}(h)\theta^{-1}$ and $\widetilde{H}_{\mathrm{f}} = \theta H_{\mathrm{f}}\theta^{-1}$, respectively. In what follows, we write H_{f} for $\widetilde{H}_{\mathrm{f}}$, and $\phi_{\mathrm{b}}(h)$ for $\widetilde{\phi}_{\mathrm{b}}(h)$. Then the spin-boson Hamiltonian H_{SB} is unitary equivalent to $\widetilde{H}$, i.e.,

$$H_{\mathrm{SB}} = \theta\widetilde{H}\theta^{-1} = -\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \alpha\sigma_z \otimes \phi_{\mathrm{b}}(h).$$

In what follows, we further use H_{SB}.

3 $P(\phi)_1$-process associated with the spin-boson Hamiltonian

In this section we construct a $P(\phi)_1$-process for H_{SB}. This has been done in [7, 8] for the Nelson model and the semi-relativistic Pauli-Fierz model before.

Proposition 3.1 *If $\hat{h}(-k) = \hat{h}(k)$ and $\hat{h}/\sqrt{\omega}, \hat{h}/\omega \in L^2(\mathbb{R}^d)$, then H_{SB} is a self-adjoint operator and it has a unique ground state φ_{g}. Furthermore, $\varphi_{\mathrm{g}} > 0$.*

Proof: See [7, Section 2]. □

For the remainder of the paper we assume that H_{SB} has a unique, strictly positive ground state φ_{g}.

Consider the ground state transform

$$U_{\mathrm{g}} : L^2(\mathbb{R}^d \times Q, \varphi_{\mathrm{g}}^2 d\nu \otimes d\mu) \to L^2(\mathbb{R}^d \times Q, d\nu \otimes d\mu), \quad U_{\mathrm{g}}\Phi = \varphi_{\mathrm{g}}\Phi.$$

Define the probability measure $d\mathbf{N}_0 = \varphi_{\mathrm{g}}^2(d\nu \otimes d\mu)$ on $\mathbb{Z}_2 \times Q$, and write

$$\mathscr{K} = L^2(\mathbb{Z}_2 \times Q, d\mathbf{N}_0).$$

Also, consider the self-adjoint operator

$$L_{\mathrm{SB}} = \frac{1}{\varphi_{\mathrm{g}}}(H_{\mathrm{SB}} - E)\varphi_{\mathrm{g}}$$

on $\mathscr{K}$ obtained under the ground state transform, where $E = \inf \sigma(H_{\mathrm{SB}})$ is the eigenvalue corresponding to φ_{g}.

Definition 3.2 **($P(\phi)_1$-process)** Let $(E, \mathscr{F}, P)$ be a probability space and K be a self-adjoint operator in $L^2(E, P)$, bounded from below. We say that an E-valued stochastic process $(Z_t)_{t\in\mathbb{R}}$ on a probability space $(\mathscr{Y}, \mathcal{B}, \mathcal{Q}^z)$ is a $P(\phi)_1$-process associated with $((E, \mathscr{F}, P), K)$ if conditions 1-4 below are satisfied:

1. $\mathcal{Q}^z(Z_0 = z) = 1$.
2. **(Reflection symmetry)** $(Z_t)_{t\geq 0}$ and $(Z_t)_{t\leq 0}$ are independent and $Z_t \stackrel{\mathrm{d}}{=} Z_{-t}$ for every $t \in \mathbb{R}$.
3. **(Markov property)** $(Z_t)_{t\geq 0}$ and $(Z_t)_{t\leq 0}$ are Markov processes with respect to the fields $\sigma(Z_s, 0 \leq s \leq t)$ and $\sigma(Z_s, t \leq s \leq 0)$, respectively.
4. **(Shift invariance)** Let $-\infty < t_0 \leq t_1 < ... \leq t_n < \infty$, $f_j \in L^\infty(E, dP)$, $j = 1, ..., n-1$ and $f_0, f_n \in L^2(E, P)$. Then for every $s \in \mathbb{R}$,

$$\begin{aligned}\int_E \mathbb{E}_{\mathcal{Q}^z}\left[\prod_{j=0}^n f_j(Z_{t_j})\right] dP(z) &= \int_E \mathbb{E}_{\mathcal{Q}^z}\left[\prod_{j=0}^n f_j(Z_{t_j+s})\right] dP(z) \\ &= (\mathbb{1}, f_0 e^{-(t_1-t_0)K} f_1 \cdots f_{n-1} e^{-(t_n-t_{n-1})K} f_n). \end{aligned} \tag{3.1}$$

Let $\mathcal{D} = D(\mathbb{R}, \mathbb{Z}_2 \times Q)$ be the space of càdlàg paths with values in $\mathbb{Z}_2 \times Q$. Next we construct a probability measure $\mathcal{N}^y$, $y = (\sigma, \xi) \in \mathbb{Z}_2 \times Q$, on the measurable space $(\mathcal{D}, \mathcal{B})$ such that the coordinate process $(Y_t)_{t\in\mathbb{R}}$ is a stationary Markov process generated by L_{SB}. This process is the so called $P(\phi)_1$-process associated with the spin-boson Hamiltonian H_{SB}.

Let $\Sigma' = \mathcal{B}(\mathbb{Z}_2) \otimes \mathcal{B}(Q)$ and define the family of set functions $\{\tilde{\mathcal{N}}_\Lambda : \Lambda \subset [0, \infty), \#\Lambda < \infty\}$ on $\times^{\#\Lambda}\Sigma'$ by

$$\tilde{\mathcal{N}}_\Lambda(A_0 \times \ldots \times A_n) = \left(\mathbb{1}_{A_0}, e^{-(t_1-t_0)L_{\mathrm{SB}}} \mathbb{1}_{A_1} e^{-(t_2-t_1)L_{\mathrm{SB}}} \mathbb{1}_{A_2} \cdots e^{-(t_n-t_{n-1})L_{\mathrm{SB}}} \mathbb{1}_{A_n}\right)_{\mathscr{K}}$$

for $\Lambda = \{t_0, t_1, \ldots, t_n\} \subset [0, \infty)$, $n \in \mathbb{N}$. They satisfy Kolmogorov's consistency condition

$$\tilde{\mathcal{N}}_{\{t_0,t_1,\ldots,t_{n+m}\}}\left((\times_{i=0}^{n} A_i) \times (\times_{i=n+1}^{n+m} \mathbb{Z}_2 \times Q)\right) = \tilde{\mathcal{N}}_{\{t_0,t_1,\ldots,t_n\}}\,(\times_{i=0}^{n} A_i)$$

for all $m, n \in \mathbb{N}$. With the projection map $\pi_\Lambda : (\mathbb{Z}_2 \times Q)^{[0,\infty)} \longrightarrow (\mathbb{Z}_2 \times Q)^{\Lambda}$ defined by $\pi_\Lambda : \omega \mapsto (\omega(t_0), \ldots, \omega(t_n))$ for $\Lambda = \{t_0, \ldots, t_n\}$, we see that

$$\mathfrak{A} = \left\{\omega \in (\mathbb{Z}_2 \times Q)^{[0,\infty)} : \pi_\Lambda(\omega) \in A,\, A \in (\mathcal{B}(\mathbb{Z}_2) \times \mathcal{B}(Q))^{\Lambda},\, \#\Lambda < \infty\right\}$$

is a finitely additive family of sets. Thus by the Kolmogorov extension theorem there exists a unique probability measure $\tilde{\mathcal{N}}$ on $((\mathbb{Z}_2 \times Q)^{[0,\infty)}, \sigma(\mathfrak{A}))$ such that for all $\Lambda = \{t_0, t_1, \ldots, t_n\} \subset [0, \infty)$, every $A_0, A_1, \ldots, A_n \in \Sigma'$, and every $n \in \mathbb{N}$,

$$\tilde{\mathcal{N}}\left(\pi_\Lambda^{-1}(A_0 \times \cdots \times A_n)\right) = \tilde{\mathcal{N}}_\Lambda(A_0 \times \cdots \times A_n) = \mathbb{E}_{\tilde{\mathcal{N}}}\left[\prod_{j=0}^{n} \mathbb{1}_{A_j}(\tilde{Y}_{t_j})\right]. \tag{3.2}$$

Here $(\tilde{Y}_t)_{t\geq 0}$ denotes the coordinate process on $\left((\mathbb{Z}_2 \times Q)^{[0,\infty)}, \sigma(\mathfrak{A}), \tilde{\mathcal{N}}\right)$. Let $\tilde{Y}_0 = y \in \mathbb{Z}_2 \times Q$, and denote $\mathcal{D}^+ = D([0,\infty), \mathbb{Z}_2 \times Q)$. Define a regular conditional probability measure on $(\mathcal{D}^+, \sigma(\mathfrak{A}))$ by $\tilde{\mathcal{N}}^y(\cdot) = \mathcal{N}(\cdot \,|\, \tilde{Y}_0 = y)$ for $y \in \mathbb{Z}_2 \times Q$.

Lemma 3.3 (Shift invariance) *If $F_0, \ldots, F_n \in \mathscr{K}$, $n \in \mathbb{N}$, then*

$$\mathbb{E}_{\tilde{\mathcal{N}}}\Big[\prod_{j=0}^{n} F_j(\tilde{Y}_{t_j})\Big] = \Big(F_0, e^{-(t_1-t_0)L_{\mathrm{SB}}} F_1 \cdots e^{-(t-t_{n-1})L_{\mathrm{SB}}} F_n\Big)_{\mathscr{K}} \tag{3.3}$$

for $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n = t$. Moreover $(\tilde{Y}_t)_{t\geq 0}$ is shift invariant under $\tilde{\mathcal{N}}$, i.e,

$$\mathbb{E}_{\tilde{\mathcal{N}}}\Big[\prod_{j=0}^{n} F_j(\tilde{Y}_{t_j})\Big] = \mathbb{E}_{\tilde{\mathcal{N}}}\Big[\prod_{j=0}^{n} F_j(\tilde{Y}_{t_j+s})\Big], \quad s \geq 0. \tag{3.4}$$

In particular, $\mathbb{E}_{\tilde{\mathcal{N}}}\Big[F(\tilde{Y}_t)\Big] = \big(\mathbb{1}, e^{-tL_{\mathrm{SB}}} F\big)_{\mathscr{K}} = (\mathbb{1}, F)_{\mathscr{K}} = \mathbb{E}_{\tilde{\mathcal{N}}}\Big[F(\tilde{Y}_0)\Big] = \int_{\mathbb{Z}_2\times Q} F(y)d\mathrm{N}_0(y)$.

Proof: (3.3) and (3.4) follow from (3.2) and a simple limiting argument. □

With the measure $\tilde{\mathcal{N}}$ we have the representation

$$(\Phi, e^{-tL_{\mathrm{SB}}}\Psi)_{\mathscr{K}} = \int_{\mathbb{Z}_2\times Q} \mathbb{E}_{\tilde{\mathcal{N}}^y}[\bar{\Phi}(\tilde{Y}_0)\Psi(\tilde{Y}_t)]d\mathrm{N}_0(y), \tag{3.5}$$

which implies that $\left(e^{-tL_{\mathrm{SB}}}\Phi\right)(\sigma, \xi) = \mathbb{E}_{\tilde{\mathcal{N}}^{(\sigma,\xi)}}[\Phi(\tilde{Y}_t)]$.

Lemma 3.4 (Markov property) *$(\tilde{Y}_t)_{t\geq 0}$ is a Markov process on $(\mathcal{D}^+, \sigma(\mathfrak{A}), \tilde{\mathcal{N}}^y)$ with respect to the natural filtration $\sigma(\tilde{Y}_t, 0 \leq s \leq t)$, and $e^{-tL_{\mathrm{SB}}}$ is its associated Markov semigroup.*

Proof: See Appendix. □

We extend $(\tilde{Y}_t)_{t\geq 0}$ to a Markov process indexed by the whole real line $\mathbb{R}$. Consider the product probability space $\left(\hat{\mathcal{D}^+}, \hat{\mathfrak{A}}, \hat{\tilde{\mathcal{N}}}^{(\sigma,\xi)}\right)$ with $\hat{\mathcal{D}^+} = \mathcal{D}^+ \times \mathcal{D}^+$, $\hat{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{A}$, and $\hat{\tilde{\mathcal{N}}}^{(\sigma,\xi)} = \tilde{\mathcal{N}}^{(\sigma,\xi)} \otimes \tilde{\mathcal{N}}^{(\sigma,\xi)}$. Let $(\hat{\tilde{Y}}_t)_{t\in\mathbb{R}}$ be a stochastic process on this product space, which is defined by $\hat{\tilde{Y}}_t(\omega) = \tilde{Y}_t(\omega_1)$ for $t \geq 0$, and $\hat{\tilde{Y}}_t(\omega) = \tilde{Y}_{-t}(\omega_2)$ for $t \leq 0$, with $\omega = (\omega_1, \omega_2) \in \hat{\mathcal{D}}^+$.

Lemma 3.5 *The following properties hold:*

(1) **(Reflection symmetry)** $\hat{\tilde{Y}}_0 = (\sigma, \xi)$*,* $\hat{\tilde{Y}}_t$ *and* $\hat{\tilde{Y}}_s$ *are independent for* $s < 0 < t$*, and* $\hat{\tilde{Y}}_t \stackrel{\mathrm{d}}{=} \hat{\tilde{Y}}_{-t}$ *for* $t \in \mathbb{R}$*.*

(2) **(Markov property)** $(\hat{\tilde{Y}}_t)_{t\geq 0}$ *(resp.* $(\hat{\tilde{Y}}_t)_{t\leq 0}$*) is a Markov process with respect to the filtration* $\sigma(\hat{\tilde{Y}}_s, 0 \leq s \leq t)$ *(resp.* $\sigma(\hat{\tilde{Y}}_s, t \leq s \leq 0)$*).*

(3) **(Shift invariance)** *For* $f_0, \ldots, f_n \in \mathscr{K}$ *and* $-t = t_0 \leq t_1 \leq \ldots \leq t_n = t$*, we have*

$$\mathbb{E}_{\hat{\tilde{\mathcal{N}}}_+}\left[\prod_{j=0}^{n} f_j(\hat{\tilde{Y}}_{t_j})\right] = \left(f_0, e^{-(t_1+t)L_{\mathrm{SB}}} f_1 \cdots e^{-(t-t_{n-1})L_{\mathrm{SB}}} f_n\right)_{\mathscr{K}}. \tag{3.6}$$

Proof: (1) is straightforward. (2) follows from Lemma 3.4. (3) follows from (3.2) and a simple limiting argument. □

Let $\hat{\tilde{Y}}_\cdot : (\hat{\mathcal{D}}, \hat{\mathfrak{A}}, \hat{\tilde{\mathcal{N}}}^{(\sigma,\xi)}) \to (\mathcal{D}, \mathcal{B})$ be a measurable function, and denote by $\mathcal{N}^{(y,\xi)} = \hat{\tilde{\mathcal{N}}}^{(\sigma,\xi)} \circ \hat{\tilde{Y}}_\cdot^{-1}$ the image measure on $(\mathcal{D}, \mathcal{B})$. Hence the coordinate process $(Y_t)_{t\in\mathbb{R}}$ on $(\mathcal{D}, \mathcal{B}, \mathcal{N})$ satisfies $Y_t \stackrel{\mathrm{d}}{=} \hat{\tilde{Y}}_t$. We conclude that the process $(Y_t)_{t\in\mathbb{R}}$ on $(\mathcal{D}, \mathcal{B}, \mathcal{N})$ is the $P(\phi)_1$-process associated with the pair $\left((\mathbb{Z}_2 \times Q, \mathcal{B}(\mathbb{Z}_2) \times Q, d\mathrm{N}_0), \mathrm{L}_{\mathrm{SB}}\right)$.

Recall that a stochastic process $(Z_t)_{t\in\mathbb{R}}$ is said to be reversible if $(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n})$ has the same distribution as $(Z_{\tau-t_1}, Z_{\tau-t_2}, \ldots, Z_{\tau-t_n})$, for all $t_1, \ldots, t_n, \tau \in \mathbb{R}$ and $n \in \mathbb{N}$.

Theorem 3.6 *The Markov process* $(Y_t)_{t\in\mathbb{R}}$ *is reversible and ergodic under* $\mathcal{N}$*.*

Proof: Let $\Phi, \Psi \in L^2(\mathbb{Z}_2 \times Q, d\mathbf{N}_0)$. Then by (3.5) we have

$$\mathbb{E}_{\mathcal{N}}[\Phi(Y_t)\Psi(Y_s)] = \left(\Phi, e^{-|t-s|L_{\mathrm{SB}}}\Psi\right)_{\mathscr{K}} = \left(\varphi_{\mathrm{g}}\Phi, e^{-|t-s|(H_{\mathrm{SB}}-E)}\Psi\varphi_{\mathrm{g}}\right)_{L^2(\mathbb{Z}_2\times Q, d\nu\otimes d\mu)}. \tag{3.7}$$

Hence $(Y_t)_{t\in\mathbb{R}}$ is a reversible Markov process under $\mathcal{N}$. For $t \in \mathbb{R}$, the semigroup e^{-tH} is positivity improving on $L^2(\mathbb{Z}_2 \times Q, \nu \otimes \mu)$; see also [7, Corollary 2.1]. Hence $(Y_t)_{t\in\mathbb{R}}$ is ergodic. □

4 Functional central limit theorem for H_{SB}

4.1 FCLT relative to the associated path measure

In this section we show a functional central limit theorem for additive functionals of the reversible Markov process $(Y_t)_{t\geq 0}$ of the form

$$(Z_t)_{t\geq 0} = \left(\int_0^t L_{\mathrm{SB}} F(Y_s) ds\right)_{t\geq 0}, \tag{4.1}$$

where $F = f \otimes g \in L^2(\mathbb{Z}_2) \otimes L^2(Q)$. For given F, define

$$\varrho(F) = \lim_{t\to\infty} \frac{1}{t}\sqrt{\mathbb{E}_{\mathcal{N}}[Z_t^2]}.$$

The aim of this section is to calculate explicitly the diffusion constant $\varrho^2(F)$ for additive functionals $(Z_t)_{t\geq 0}$ in the spin-boson model. Let

$$M_t = F(Y_t) - F(Y_0) + \int_0^t L_{\mathrm{SB}} F(Y_s) ds \quad \text{for } F \in D(L_{\mathrm{SB}}). \tag{4.2}$$

The following lemma can be proven in a similar manner as in [5].

Lemma 4.1 *$(M_t)_{t\geq 0}$ is a martingale with stationary increments under $\mathcal{N}$.*

Proof: See Appendix. □

We begin by proving a FCLT for the martingale $(M_t)_{t\geq 0}$ under $\mathcal{N}$.

Lemma 4.2 *Let $F = f\otimes g \in D(L_{\mathrm{SB}})$ be a non-constant function. Assume that $\mathbb{E}_{\mathcal{N}}[|F(Y_t)|^2] < \infty$ and $\mathbb{E}_{\mathcal{N}}[|(L_{\mathrm{SB}}F)(Y_t)|^2] < \infty$ for all $t \geq 0$. Then we have that*

$$\varrho^2(F) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\mathcal{N}}[M_t^2] = 2(F, L_{\mathrm{SB}}F) \quad \text{and} \quad \varrho^2(F) > 0. \tag{4.3}$$

Furthermore,

$$\varrho^2(F) = 2(F\varphi_{\mathrm{g}}, (F - F^{\pm})\varphi_{\mathrm{g}}^{\pm}) + 2(F\varphi_{\mathrm{g}}, [\mathbb{1} \otimes H_{\mathrm{f}}, F]\varphi_{\mathrm{g}}), \tag{4.4}$$

where $F^{\pm}(\sigma, \xi) = F(-\sigma, \xi)$ and $\varphi_{\mathrm{g}}^{\pm}(\sigma, \xi) = \varphi_{\mathrm{g}}(-\sigma, \xi)$ denote the functions under a spin-flip.

Proof: See Appendix for (4.3), we show here (4.4).

We see that $(\sigma_x f)(\sigma) = f(-\sigma)$, and $(\sigma_z f)(\sigma) = \sigma f(\sigma)$. We thus obtain that

$$\begin{aligned}(H_{\mathrm{SB}} - E) f \otimes g\varphi_{\mathrm{g}} &= \left(-\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \alpha\sigma_z \otimes \phi_{\mathrm{b}}(\hat{h}) - E\right) f \otimes g\varphi_{\mathrm{g}} \\ &= (f(\sigma) - f(-\sigma))\,\varphi_{\mathrm{g}}(-\sigma,\xi)g(\xi) + f(\sigma)H_{\mathrm{f}} g\varphi_{\mathrm{g}} - f(\sigma)gH_{\mathrm{f}}\varphi_{\mathrm{g}} \\ &= (f(\sigma) - f(-\sigma))\,\varphi_{\mathrm{g}}(-\sigma,\xi)g(\xi) + f(\sigma)[H_{\mathrm{f}}, g]\varphi_{\mathrm{g}}.\end{aligned}$$

Thus $L_{\mathrm{SB}}F(\sigma) = \frac{1}{\varphi_{\mathrm{g}}}\left((f(\sigma) - f(-\sigma))\,\varphi_{\mathrm{g}}(-\sigma,\xi)g(\xi) + f(\sigma)[H_{\mathrm{f}}, g]\varphi_{\mathrm{g}}\right)$. □

Next we make use of a general FCLT for martingales with stationary increments. Using assumption of Lemma 4.2 we obtain the following result.

Lemma 4.3 *Let $(B_t)_{t\geq 0}$ be standard Brownian motion and $F = f \otimes g \in D(L_{\mathrm{SB}})$ a non-constant function. Assume that*

$$\mathbb{E}_{\mathcal{N}}[|F(Y_t)|^2] < \infty \quad \textit{and} \quad \mathbb{E}_{\mathcal{N}}[|(L_{\mathrm{SB}}F)(Y_t)|^2] < \infty,$$

for all $t \geq 0$. Then

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} M_{st} = \varrho^2(F) B_t, \quad t \geq 0,$$

in distribution.

Proof: By Lemma 4.1 we have that $(M_t)_{t\geq 0}$ is a martingale with stationary increments under $\mathcal{N}$. Furthermore, $\varrho^2(F)$ is finite by (4.4). Thus by Lemma 4.4 below the claim follows, for a proof see [6, Section 5]. □

Lemma 4.4 (Helland) *Let $(m_t)_{t\in\mathbb{R}}$ be a martingale on a probability space $(\Omega, \mathcal{F}, P)$ such that $\beta^2 = \lim\limits_{t\to\infty} \frac{1}{t}\mathbb{E}_P[m_t^2] < \infty$, and assume that $(m_t)_{t\in\mathbb{R}}$ has stationary increments. Then $\lim\limits_{s\to\infty} \frac{1}{\sqrt{s}} m_{[st]} = \beta^2 B_t$, in distribution.*

Finally, under the assumption of Lemma 4.2 we are in the position to state the following result.

Corollary 4.5 (FCLT for spin-boson model) *Let $F = f \otimes g \in D(L_{\mathrm{SB}})$ be a non-constant function. Suppose that $\mathbb{E}_{\mathcal{N}}[|F(Y_t)|^2] < \infty$ and $\mathbb{E}_{\mathcal{N}}[|(L_{\mathrm{SB}}F)(Y_t)|^2] < \infty$ for all $t \geq 0$. Then the process $(Z_t)_{t\geq 0}$ given by (4.1) satisfies an FCLT with respect to $\mathcal{N}$, and the variance of the limit process is given by $\varrho^2(F)$ for all $F \in D(L_{\mathrm{SB}})$. That is, we have in distribution that*

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} Z_{st} = \varrho^2(F) B_t, \quad t \geq 0,$$

Proof: We have $\mathbb{E}_{\mathcal{N}}[L_{\mathrm{SB}}F(Y_t)] = (\varphi_{\mathrm{g}}, [(H_{\mathrm{SB}} - E)F]\varphi_{\mathrm{g}}) = ((H_{\mathrm{SB}} - E)\varphi_{\mathrm{g}}, F\varphi_{\mathrm{g}}) = 0$. Thus $\mathbb{E}_{\mathcal{N}}[\int_0^t L_{\mathrm{SB}}F(Y_s)ds] = 0$. By Theorem 3.6, $(Y_t)_{t\geq 0}$ is a reversible and ergodic Markov process under $\mathcal{N}$. Thus the assumptions of Proposition A.1 are satisfied and by [10, Corollary 1.9] the corollary follows, i.e., $(Z_t)_{t\geq 0}$ satisfies an FCLT with variance $\varrho^2(F)$ given in (4.4). □

4.2 Examples of the variance of the limit process

We conclude by giving some examples of particular interest for specific choices of the function $F \in D(L_{\mathrm{SB}})$, $F : \mathbb{Z}_2 \times Q \ni (\sigma, \xi) \mapsto F(\sigma, \xi) = (f \otimes g)(\sigma, \xi) = f(\sigma)g(\xi) \in \mathbb{C}$, allowing to compute the variance $\varrho^2(F)$ explicitly. In what follows we assume that $h \in L^2(\mathbb{R}^d)$ is any test function. The vector in $L^2(Q)$ associated with the conjugate momentum $\Pi(h)$ in $\mathscr{F}_{\mathrm{b}}$, will be denoted by the same symbol $\Pi(h)$, i.e., we have $[\xi(h), \Pi(h')] = \frac{1}{2}(h, h')$.

Example 4.6 Let $F(\sigma, \xi) = f(\sigma)$. Then $L_{\mathrm{SB}}f(\sigma) = \frac{1}{\varphi_{\mathrm{g}}}(f - f^{\pm})\varphi_{\mathrm{g}}^{\pm}$, and

$$\varrho^2(F) = 2\left(f\varphi_{\mathrm{g}}, (f - f^{\pm})\varphi_{\mathrm{g}}^{\pm}\right).$$

Choosing, in particular, $F(\sigma, \xi) = \sigma$, we obtain $\varrho^2(F) = 4\left(\varphi_{\mathrm{g}}, \varphi_{\mathrm{g}}^{\pm}\right)$.

Example 4.7 Let $F(\sigma,\xi)=\sigma\xi(h)$. We have

$$\begin{aligned}\varrho^2(F) &= 4\left(\sigma\varphi_{\rm g}\xi(h),\sigma\varphi_{\rm g}^{\pm}\right)+2\left(\sigma\varphi_{\rm g}\xi(h),\sigma[H_{\rm f},\xi(h)]\varphi_{\rm g}\right)\\ &= 4\left(\sigma\varphi_{\rm g}\xi(h),\sigma\varphi_{\rm g}^{\pm}\right)-2i\left(\sigma\varphi_{\rm g}\xi(h),i\sigma\Pi(\omega h)\varphi_{\rm g}\right).\end{aligned}$$

We calculate the second term above. Denote $X=-\left(\sigma\varphi_{\rm g}\xi(h),i\sigma\Pi(\omega h)\varphi_{\rm g}\right)$. We obtain

$$\begin{aligned}X &= -i\left(\sigma\varphi_{\rm g},\sigma\xi(h)\Pi(\omega h)\varphi_g\right)=-i\left(\sigma\varphi_{\rm g},\sigma\Pi(\omega h)\xi(h)\varphi_{\rm g}\right)+i\left(\sigma\varphi_{\rm g},\sigma\Pi(\omega h)\xi(h)\varphi_{\rm g}\right)\\ &= -i\left(\sigma\Pi^*(\omega h)\varphi_{\rm g},\sigma\xi(h)\varphi_{\rm g}\right)+\left(\sigma\varphi_{\rm g},(\omega h,h)\sigma\varphi_{\rm g}\right)=-\overline{X}+\left(\sigma\varphi_{\rm g},\sigma\varphi_{\rm g}\right)(\omega h,h).\end{aligned}$$

Hence $\varrho^2(F)=4\left(\varphi_{\rm g}\xi(h),\varphi_{\rm g}^{\pm}\right)+\|\sqrt{\omega}h\|^2$.

Example 4.8 Let $F(\sigma,\xi)=\sigma e^{i\xi(h)}$. We have

$$[H_{\rm f},e^{i\xi(h)}]=e^{i\xi(h)}\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2e^{i\xi(h)}.$$

A calculation gives

$$\varrho^2(F)=4\left(\varphi_{\rm g},\varphi_{\rm g}^{\pm}\right)+2\left(\varphi_{\rm g},\Pi(\omega h)\varphi_{\rm g}\right)+\|\sqrt{\omega}h\|^2 \tag{4.5}$$

A Appendix

Here we outline the proofs of Lemmas 3.4, 4.1 and 4.2 for a self-contained presentation; all the proofs are however a minor modification of those of [5].

A.1 Proof of Lemma 3.4

Let $p_t(y,A)=(e^{-tL_{\rm SB}}\mathbb{1}_A)(y)$ for $A\in\mathcal{B}(\mathbb{Z}_2\times Q)$. Notice that

$$p_t(y,A)=\mathbb{E}_{\tilde{\mathcal{N}}^y}[\mathbb{1}_A(\tilde{Y}_t)]=\mathbb{E}_{\tilde{\mathcal{N}}}[\mathbb{1}_A(\tilde{Y}_t)|\tilde{Y}_0=y]=\mathbb{E}_{\tilde{\mathcal{N}}}[\mathbb{1}_A(\tilde{Y}_t^y)].$$

Thus we have $\mathbb{E}_{\tilde{\mathcal{N}}}[\prod_{j=0}^n\mathbb{1}_{A_j}(\tilde{Y}_{t_j})]=\int_{(\mathbb{Z}_2\times Q)^n}\prod_{j=1}^n\mathbb{1}_{A_j}(\tilde{Y}_{t_j})\prod_{j=1}^n p_{t_j-t_{j-1}}(y_{j-1},d\mathrm{N}_0(y_j))$. We show that $p_t(y,A)$ is a probability transition kernel. By (3.5) we readily obtain that $e^{-tL_{\rm SB}}$ is positivity improving. For every function $F\in\mathscr{K}$ such that $0\le F\le\mathbb{1}$, we have

$$\left(e^{-tL_{\rm SB}}F\right)(y)=\mathbb{E}_{\tilde{\mathcal{N}}^y}[F(Y_t)]\le\mathbb{E}_{\tilde{\mathcal{N}}^y}[\mathbb{1}(\tilde{Y}_t)]=\mathbb{E}_{\tilde{\mathcal{N}}^y}[\mathbb{1}(\tilde{Y}_0)]=\mathbb{1}(y).$$

Hence $0\le e^{-tL_{\rm SB}}F\le\mathbb{1}$ and $e^{-tL_{\rm SB}}\mathbb{1}=\mathbb{1}$, and thus $p_t(y,\cdot)$ is a probability measure on $\mathbb{Z}_2\times Q$ with $p_t(y,\mathbb{Z}_2\times Q)=1$, for all $t\ge 0$. Secondly, $p_t(y,A)$ is measurable with respect to y. Third, by the semigroup property $e^{-sL_{\rm SB}}e^{-tL_{\rm SB}}\mathbb{1}_A=e^{-(s+t)L_{\rm SB}}\mathbb{1}_A$ and the Chapman-Kolmogorov identity follows, and hence $p_t(y,A)$ is a probability transition kernel. Thus $(\tilde{Y}_t)_{t\ge0}$ is a Markov process. $\square$

A.2 Proof of Lemma 4.1

By Lemma 3.4 we see that $(Y_t)_{t\geq 0}$ is a Markov process with semigroup $T_t = e^{-tL_{\rm SB}}$, $t \geq 0$. Hence we have $\mathbb{E}_{\mathcal{N}}[F(Y_t)|\mathcal{F}_s] = T_{t-s}F(Y_s)$, $t > s$. Since the function $t \mapsto T_tF$ is differentiable for $F \in D(L_{\rm SB})$, we obtain that $T_tF - F = -\int_0^t L_{\rm SB}T_sFds$ for all $t \geq 0$. We also have

$$\mathbb{E}_{\mathcal{N}}[M_t|\mathcal{F}_s] = M_s + \mathbb{E}_{\mathcal{N}}\left[F(Y_t) - F(X_s) + \int_s^t L_{\rm SB}F(Y_r)dr \,|\mathcal{F}_s\right] \quad \text{a.s.} \tag{1.1}$$

From the above it follows that the second term on the right hand side of (1.1) is zero. By the shift invariance property of $(Y_t)_{t\geq 0}$ it then follows that $(M_t)_{t\geq 0}$ is a martingale with stationary increments under $\mathcal{N}$. □

For the sake of a quick access, we also quote the following fundamental result. For a proof see [10, Theorem 1.8].

Proposition A.1 (Kipnis-Varadhan) *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, G)$ be a filtered probability space and (A, G_0) a measurable space. Let $(Y_t)_{t\geq 0}$ be an A-valued Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$. Assume that $(Y_t)_{t\geq 0}$ is a reversible and ergodic Markov process with respect to G. Let $F : A \to \mathbb{R}$ be an $L^2(G_0)$ function with $\int_A FdG_0 = 0$. Suppose in addition that $F \in D(L^{-1/2})$, where L is the generator of the process $(Y_t)_{t\geq 0}$. Let $R_t = \int_0^t F(Y_s)ds$. Then there exists a square integrable martingale $(N_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$, with stationary increments, such that $\lim\limits_{t\to\infty} \frac{1}{\sqrt{t}} \sup_{0\leq s\leq t} |R_s - N_s| = 0$ in probability under G, where $R_0 = N_0 = 0$. Moreover, $\lim\limits_{t\to\infty} \frac{1}{t}\mathbb{E}_G[|R_t - N_t|^2] = 0$.*

A.3 Proof of (4.3) of Lemma 4.2

By (4.2) we have

$$M_tZ_t = M_t\left(F(Y_0) - F(Y_t) + M_t\right) \tag{1.2}$$
$$= \left(F(Y_t) - F(Y_0) + Z_t\right)Z_t. \tag{1.3}$$

Consider equality (1.2). We have

$$|\mathbb{E}_{\mathcal{N}}[M_tZ_t]| \leqslant |\mathbb{E}_{\mathcal{N}}[M_tF(Y_0)]| + |\mathbb{E}_{\mathcal{N}}[M_tF(Y_t)]| + |\mathbb{E}_{\mathcal{N}}[M_t^2]|.$$

By using the shift invariance property of the process $(Y_t)_{t\geq 0}$ and Schwarz inequality, we have

$$|\mathbb{E}_{\mathcal{N}}[M_tF(Y_t)]| \leqslant \mathbb{E}_{\mathcal{N}}[F(Y_t)^2]^{\frac{1}{2}}\mathbb{E}_{\mathcal{N}}[M_t^2]^{\frac{1}{2}} = \mathbb{E}_{\mathcal{N}}[F(Y_0)^2]^{\frac{1}{2}}\mathbb{E}_{\mathcal{N}}[M_t^2]^{\frac{1}{2}}.$$

Thus we obtain

$$\lim_{t\to\infty}\frac{1}{t}|\mathbb{E}_{\mathcal{N}}[M_tF(Y_t)]| = 0.$$

In the same way, we have

$$\lim_{t\to\infty}\frac{1}{t}|\mathbb{E}_{\mathcal{N}}[M_tF(Y_0)]| = 0,$$

and hence

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[M_tZ_t]=\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[M_t^2].$$

Now consider equality (1.3). By the same argument, we see that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[M_tZ_t]=\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[Z_t^2].$$

Hence we deduce that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[Z_t^2]=\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[M_t^2].$$

Let $T_t=e^{-tL_{\mathrm{SB}}}$. A calculation gives

$$\begin{aligned}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[Z_t^2]&=\frac{1}{t}\mathbb{E}_{\mathcal{N}}\left[\int_0^t ds\int_0^t L_{\mathrm{SB}}F(Y_r)L_{\mathrm{SB}}F(Y_s)dr\right]=\frac{1}{t}\int_0^t ds\int_0^t \mathbb{E}_{\mathcal{N}}[L_{\mathrm{SB}}F(Y_0)L_{\mathrm{SB}}F(Y_{|r-s|})]dr\\&=\frac{1}{t}\int_0^t ds\int_0^t \left(T_{|s-r|}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}dr=\frac{2}{t}\int_{0\le r\le s\le t}\left(T_{s-r}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}drds\\&=\frac{2}{t}\int_{0\le r\le s\le t}\left(T_{r}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}drds=2\int_0^t(1-\frac{r}{t})\left(T_{r}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}dr.\end{aligned}$$

Since $\mathcal{N}$ is reversible, i.e., L_{SB} is a self-adjoint operator, we see that

$$\left(T_{r}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}=\left(T_{\frac{r}{2}}L_{\mathrm{SB}}F,T_{\frac{r}{2}}L_{\mathrm{SB}}F\right)_{\mathscr{K}}$$

is positive and the function $t\mapsto(1-\frac{r}{t})\left(T_{r}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}$ is increasing. Then by the monotone convergence theorem

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{N}}[Z_t^2]=2\int_0^\infty\left(T_{r}L_{\mathrm{SB}}F,L_{\mathrm{SB}}F\right)_{\mathscr{K}}dr=2\left(F,L_{\mathrm{SB}}F\right).$$

To prove that $\varrho^2(F)=2\left(\sqrt{L_{\mathrm{SB}}}F,\sqrt{L_{\mathrm{SB}}}F\right)>0$, assume, to the contrary, that $\varrho^2(F)=0$, i.e., $\sqrt{L_{\mathrm{SB}}}F=0$. Since $(Y_t)_{t\in\mathbb{R}}$ is an ergodic process and $\sqrt{L_{\mathrm{SB}}}F=0$ implies $L_{\mathrm{SB}}F=0$, equality $\varrho^2(F)=0$ implies that F is constant, which is impossible. Hence we conclude that $\varrho^2(F)>0$.

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