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Integral Equations and Operator Theory



Ambrosetti–Prodi Type Results for Dirichlet Problems of Fractional Laplacian-Like Operators

Anup Biswas and József Lőrinczi

Abstract. We establish Ambrosetti–Prodi type results for viscosity and classical solutions of nonlinear Dirichlet problems for fractional Laplace and comparable operators. In the choice of nonlinearities we consider semi-linear and super-linear growth cases separately. We develop a new technique using a functional integration-based approach, which is more robust in the non-local context than a purely analytic treatment.

Mathematics Subject Classification. 35J60, 35J55, 58J55.

Keywords. Semi-linear nonlocal exterior value problem, Ambrosetti–Prodi problem, Viscosity solutions, Bifurcations, Fractional Schrödinger operator, Principal eigenvalues, Maximum principles.

1. Introduction and Statement of Results

In this paper our goal is to present a counterpart for the fractional Laplacian and operators comparable in a specific sense, of the classical Ambrosetti–Prodi problem studied for a class of elliptic differential operators with nonlinear terms. In contrast with topological and variational methods used in the classical context, we propose a new technique based on a path integration approach, which accommodates a large class of non-local operators going well beyond the fractional Laplacian, and also applies to viscosity solutions. This larger class is motivated by a number of applications including operators with Lévy jump kernels having a lighter than polynomial tail, however, an extension to this class requires a number of extra steps and concepts, which will be pursued in a future work. Apart from this, another general advantage of our approach seems to be that it is more robust than purely analytic techniques, dealing better with the difficulties resulting from boundary roughness. Our techniques and framework have been developed recently in [12,13], to which

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we now intend to add the new dimension of including nonlinearities. First we briefly recall the original problem, then state our results, and in the next section present the proofs.

Let $\mathrm{D} \subset \mathbb{R}^d$ be a bounded open domain with a $\mathcal{C}^{2,\alpha}(\mathrm{D})$ boundary, $\alpha \in (0,1)$, and consider the Dirichlet problem

$$\begin{cases} \Delta u + f(u) = g(x) & \text{in } \mathrm{D}, \\ u = 0 & \text{on } \partial\mathrm{D}, \end{cases} \tag{1.1}$$

where Δ is the Laplacian, $f \in \mathcal{C}^2(\mathbb{R})$, and $g \in \mathcal{C}^{0,\alpha}(\bar{D})$. In the pioneering paper [2] Ambrosetti and Prodi studied the operator $L = \Delta + f(\cdot)$ as a differentiable map between $C^{2,\alpha}(\bar{D})$ and $\mathcal{C}^{0,\alpha}(\bar{D})$, and discovered the following phenomenon. Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ denote the Dirichlet eigenvalues of the Laplacian for the domain D. The authors have shown that provided f is strictly convex, with $f(0) = 0$, and

$$0 < \lim_{z \to -\infty} \frac{f(z)}{z} < \lambda_1 < \lim_{z \to \infty} \frac{f(z)}{z} < \lambda_2, \tag{1.2}$$

then

(1) there is a closed connected manifold $\mathcal{M}_1 \subset \mathcal{C}^{0,\alpha}(\bar{D})$ of codimension 1, with the property that there exist $\mathcal{M}_0, \mathcal{M}_2$ such that $\mathcal{C}^{0,\alpha}(\bar{D}) \backslash \mathcal{M}_1 = \mathcal{M}_0 \sqcup \mathcal{M}_2$,
(2) the Dirichlet problem (1.1) has no solution if $g \in \mathcal{M}_0$, has a unique solution if $g \in \mathcal{M}_1$, and has exactly two solutions if $g \in \mathcal{M}_2$.

The problem formulates in the wider context of invertibility of differentiable maps between Banach spaces, in fact, $\mathcal{M}_1$ is the set of elements u on which the Fréchet derivative of L is not locally invertible. Also, as it is seen from condition (1.2), this split behaviour shows that the existence and multiplicity of solutions is conditioned by the crossing of the nonlinear term with the principal eigenvalue of the linear part.

Following this fundamental observation, much work has been done in the direction of relaxing the conditions or generalizing to further non-linear partial differential equations or systems. A first contribution has been made by Berger and Podolak proposing a useful reformulation of the problem. Write

$$L_1 u = \Delta u + \lambda_1 u, \quad f_1(u) = f(u) - \lambda_1 u, \quad g = \rho\varphi_1 + h,$$

where φ_1 is the principal eigenfunction of the Dirichlet Laplacian, h is in the orthogonal complement of φ_1, L^2-normalized to 1, and $\rho \in \mathbb{R}$, so that (1.1) becomes

$$L_1 u + f_1 u = \rho\varphi_1 + h \text{ in } \mathrm{D}, \quad u = 0 \text{ in } \partial\mathrm{D}. \tag{1.3}$$

In [6] it is then shown that there exists $\rho^*(h) \in \mathbb{R}$, continuously dependent on h, such that for $\rho > \rho^*(h)$ the equivalent Dirichlet problem has no solution, for $\rho = \rho^*(h)$ it has a unique solution, and for $\rho < \rho^*(h)$ it has exactly two solutions. For further early developments we refer to the works of Kazdan and Warner [24] relaxing the assumptions, Dancer [16] and Amann and Hess [1] identifying a suitable growth condition on f_1, and Ruf and Srikanth [32]

turning to the super-linear case. More recent papers exploring different perspectives include [3,15,19–21,27,35,36], and for useful surveys we refer to de Figueiredo [18] and Mahwin [26]. For non-local Hamilton–Jacobi equations see [17], and for systems of non-local equations [28].

Let $\mathrm{D} \subset \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^2$ boundary, $s \in (0,1)$, and consider the fractional Laplacian-like operator

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \nabla u(x) \cdot y \mathbb{1}_{\{|y|\le 1\}}) \frac{k(y/|y|)}{|y|^{d+2s}}\, dy,$$

where $k : \mathbb{S}^{d-1} \to (0,\infty)$ is a symmetric (i.e., $k(z) = k(-z)$), Borel-measurable function satisfying the non-degeneracy condition

$$0 < \Lambda_1 \le k(z) \le \Lambda_2, \quad \text{for all } z \in \mathbb{S}^{d-1}.$$

As it can be seen from Lemma 2.9 below, our proof techniques use fine boundary behaviour of the solutions of the Dirichlet problem. It is known from [30] that such a behaviour may not hold for a general non-degenerate kernel k defined on $\mathbb{R}^d$. This is the reason why we restrict ourselves to the kernel functions of above type.

Motivated by the problem (1.3), in this paper we are interested in the existence and multiplicity of solutions of

$$\begin{cases} Lu + f(x,u) + \rho\Phi_1 + h(x) = 0 & \text{in } \mathrm{D}, \\ u = 0 & \text{in } \mathrm{D}^c, \end{cases} \tag{1.4}$$

where Φ_1 is the Dirichlet principal eigenfunction of L in D, $\rho \in \mathbb{R}$, and $h \in \mathcal{C}^\alpha(\bar{D})$ for some $\alpha > 0$. We also assume that $\|\Phi_1\|_\infty = 1$. Below we will consider viscosity solutions, however, we will also discuss a sufficient condition on f so that every viscosity solution becomes a classical solution.

Let $V \in \mathcal{C}(\bar{D})$, which will be referred to as a potential. We use the notation $\mathcal{C}_{\mathrm{b},+}(\mathbb{R}^d)$ for the space of non-negative bounded continuous functions on $\mathbb{R}^d$. Also, we denote by $\mathcal{C}^{2s+}(\mathrm{D})$ the space of continuous functions on D with the property that if $\psi \in \mathcal{C}^{2s+}(\mathrm{D})$, then for every compact subset $\mathrm{K} \subset \mathrm{D}$ there exists $\gamma > 0$ with $f \in \mathcal{C}^{2s+\gamma}(\mathrm{K})$. Define

$$\mathfrak{F}(\lambda, \mathrm{D}) = \left\{ \psi \in \mathcal{C}_{\mathrm{b},+}(\mathbb{R}^d) \cap \mathcal{C}^{2s+}(\mathrm{D}) \ : \ \psi > 0 \text{ in D, and } L\psi - V\psi + \lambda\psi \le 0 \right\}.$$

The principal eigenvalue of $-L + V$ is defined as

$$\lambda^*(-L+V) = \sup \left\{ \lambda \ : \ \mathfrak{F}(\lambda, \mathrm{D}) \neq \emptyset \right\}. \tag{1.5}$$

For easing the notation, we will simply write λ^* for the above. This widely used characterization of the principal eigenvalue originates from the seminal work of Berestycki et al. [4]. Descriptions in a similar spirit for a different class of non-local Schrödinger operators have been obtained in [5], while in [11,17,33] non-local Pucci operators have been considered. Recently, we proposed in [13] a probabilistic approach using a Feynman–Kac representation to establish characterizations of the principal eigenvalue and the corresponding semigroup solutions.

Our first result concerns the existence of the principal eigenfunction and of a solution of the Dirichlet problem.

Theorem 1.1. *Suppose that $V, g \in \mathcal{C}^\alpha(\bar{D})$ for some $\alpha > 0$. The following hold:*

(a) There exists a unique $\Psi_1 \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_{\mathrm{b},+}(\mathbb{R}^d)$, $\|\Psi_1\|_\infty = 1$, *satisfying*

$$-L\Psi_1 + V\Psi_1 = \lambda^*\Psi_1 \ \text{ in } \mathrm{D}, \quad \Psi_1 > 0 \ \text{ in } \mathrm{D}, \quad \Psi_1 = 0 \ \text{ in } \mathrm{D}^c. \tag{1.6}$$

(b) Suppose $\lambda^* > 0$. *Then there exists a unique* $u \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}(\mathbb{R}^d)$ *satisfying*

$$-Lu + Vu = g \ \text{ in } \mathrm{D}, \quad u = 0 \ \text{ in } \mathrm{D}^c. \tag{1.7}$$

We will also need the following refined weak maximum principle for viscosity solutions.

Theorem 1.2. *Suppose that* $V \in \mathcal{C}^\alpha(\bar{D})$ *and* $\lambda^* > 0$. *Let* $u \in \mathcal{C}_\mathrm{b}(\mathbb{R}^d)$ *be a viscosity subsolution of* $-Lu + Vu \le 0$ *and* $v \in \mathcal{C}_\mathrm{b}(\mathbb{R}^d)$ *be a viscosity supersolution of* $-Lv + Vv \ge 0$ *in* D. *Furthermore, assume that* $u \le v$ *in* D^c. *Then* $u \le v$ *in* $\mathbb{R}^d$.

Remark 1.1. On completion of this paper we have learnt that [17] obtained results similar to Theorems 1.1 and 1.2, using a different technique than ours. In our understanding, these and related methods in the literature, applied also for other purposes, depend on the comparability of the used non-local operators with the Riesz kernel and the fractional Laplacian. We emphasize that in this paper we develop a path integration-based approach which, like in the framework first set in [11–13], is applicable for a large class of non-local operators (Markov generators of Lévy processes) without a similar restriction, also covering qualitatively different jump kernels. Since this will need further probabilistic machinery, it will be presented elsewhere, and we limit ourselves to the fractional Laplacian here. A recent paper [9] deals with a class of non-local operators and extends some of results of this paper using a probabilistic framework.

Next we impose the following Ambrosetti–Prodi type condition on f.

Assumption [AP]. Let $f : \bar{D} \times \mathbb{R} \to \mathbb{R}$ be such that

(1) f is Hölder continuous in x, locally with respect to u, and locally Lipschitz continuous in u, uniformly in $x \in \bar{D}$,

(2) there exist $V_1, V_2 \in \mathcal{C}^\alpha(\bar{D})$, for some $\alpha > 0$, such that

$$\lambda^*(-L - V_1) > 0 \quad \text{and} \quad \lambda^*(-L - V_2) < 0, \tag{1.8}$$

$$f(x,q) \ge V_1(x)q - C \quad \text{for all } q \le 0,\ x \in \bar{D}, \tag{1.9}$$

$$f(x,q) \ge V_2(x)q - C \quad \text{for all } q \ge 0,\ x \in \bar{D}, \tag{1.10}$$

(3) f has at most linear growth, i.e., there exists a constant $C > 0$ such that

$$|f(x,q)| \le C(1 + |q|),$$

for all $(x,q) \in \bar{D} \times \mathbb{R}$, or

(3') $L = -(-\Delta)^s$ (i.e. k is constant), $d > 1 + 2s$ and there exists a positive continuous function a_0 such that

$$\lim_{q\to\infty} \frac{f(x,q)}{q^p} = a_0(x), \quad \text{for some} p \in \left(1, \frac{d+2s}{d-2s}\right),$$

where the above limit holds uniformly in $x \in \bar{D}$.

When referring to Assumption [AP] below, we will understand that conditions (1), (2) and one of (3) or (3′) hold. In what follows, we assume with no loss of generality that $f(x,0) = 0$, otherwise h can be replaced by $h - f(\cdot,0)$.

Now we are ready to state our main result on the fractional Ambrosetti–Prodi problem.

Theorem 1.3. *Let Assumption [AP] hold. Then there exists $\rho^* = \rho^*(h) \in \mathbb{R}$ such that for $\rho < \rho^*$ the Dirichlet problem (1.4) has at least two solutions, at least one solution for $\rho = \rho^*$, and no solution for $\rho > \rho^*$.*

To prove our main Theorem 1.3, like in classic proofs such as in [18,19], in our context too the viscosity characterization of the principal eigenfunction plays a key role. In Theorem 1.1 therefore first we obtain such a characterization. The refined maximum principle shown in Theorem 1.2 will also be a key object towards the proof of the fractional Ambrosetti–Prodi phenomenon. We will rely on our recent work [13], in which we proposed a method based on Feynman–Kac representations to establish Aleksandrov–Bakelman–Pucci (ABP) estimates for semigroup solutions of non-local Dirichlet problems for a large class of operators, including but going well beyond the fractional Laplacian. We will also show that every classical solution in our context here is also a semigroup solution and thus a generalized ABP estimate can be established for these solutions, which will be essential for obtaining the a priori estimates.

2. Proofs

2.1. Preliminaries

We begin by recalling some notations and results from [12,13], which will be used below. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $(X_t)_{t\geq 0}$ be an isotropic Lévy process on this space with infinitesimal generator L. Given a function $V \in \mathcal{C}(\bar{D})$, called potential, the corresponding Feynman–Kac semigroup is given by

$$T_t^{\mathrm{D},V} f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\,\mathrm{d}s} f(X_t) \mathbb{1}_{\{t<\tau_{\mathrm{D}}\}} \right], \quad t > 0,\ x \in \mathrm{D},\ f \in L^2(\mathrm{D}),$$

where

$$\tau_{\mathrm{D}} = \inf\{t > 0 :\ X_t \notin \mathrm{D}\}$$

is the first exit time of the process $(X_t)_{t\geq 0}$ from the domain D. When $L = -(-\Delta)^s$, $0 < s < 1$, it is shown in [12, Lem 3.1] that $T_t^{\mathrm{D},V}$, $t > 0$, is a Hilbert–Schmidt operator on $L^2(\mathrm{D})$ with continuous integral kernel in $(0,\infty) \times \mathrm{D} \times \mathrm{D}$. Moreover, every operator T_t has the same purely discrete spectrum, independent of t, whose lowest eigenvalue is the principal eigenvalue λ^* having multiplicity one, and the corresponding principal eigenfunction $\Psi \in L^2(\mathrm{D})$ is strictly positive. We also have from [12, Lem. 3.1] that $\Psi \in \mathcal{C}_0(\mathrm{D})$, where $\mathcal{C}_0(\mathrm{D})$ denotes the class of continuous functions on $\mathbb{R}^d$ vanishing in D^c. Since Ψ is an eigenfunction in semigroup sense, we have for all

$t > 0$ that

$$e^{-\lambda^* t}\Psi(x) = T_t\Psi(x) = \mathbb{E}^x\left[e^{-\int_0^t V(X_s)\,\mathrm{d}s}\Psi(X_t)\mathbb{1}_{\{t<\tau_{\mathrm{D}}\}}\right], \quad x \in \mathrm{D}. \tag{2.1}$$

Let $(\mathrm{D}_n)_{n\in\mathbb{N}}$ be a collection of strictly decreasing domains with the property that $\cap_{n\geq 1}\mathrm{D}_n = \mathrm{D}$, and each D_n having its boundary satisfying the exterior cone condition. Denote by λ_n^* the principal eigenvalue in sense of (1.5). The following result will be useful below (see also, [13, Lem. 4.2]).

Proposition 2.1. *The following hold:*

(1) For every $n \in \mathbb{N}$ we have $\lambda^* > \lambda_n^*$ and $\lim_{n\to\infty}\lambda_n^* = \lambda^*$.

(2) Let $\tilde{V} \geq V$ and suppose that for an open set $\mathrm{U} \subset \mathrm{D}$ we have $\tilde{V} > V$ in U. Then $\lambda_{\tilde{V}}^* > \lambda_V^*$, where λ_V^* and $\lambda_{\tilde{V}}^*$ denote the principal eigenvalues corresponding to the potentials V and $\tilde{V}$, respectively.

Proof. Existence of a unique principal eigenfunction follows from [33, Th. 1.1] (see also [17]). Note that [33] considers the case $s > \frac{1}{2}$ due to the presence of the drift term and the same proof would go through in our setting. Using [34, Th. 1.3] we can show that the eigenfunction belongs to $\mathcal{C}^{2s+}(\mathrm{D})$. Then the strict monotonicity of the eigenvalue with respect to domains follows from [33, Theorem 5.1]. Using the arguments of [11, Th. 1.6] it can be shown that $\lim_{n\to\infty}\lambda_n^* = \lambda^*$. Part (2) again follows from [33, Th. 5.1]. □

Since the theory developed in [12] is probabilistic while here we are concentrating on viscosity solutions, we point out the relationship between these notions of solution (compare also with [13, Rem. 3.2], [10, Sec. 3.1]). We say that $u \in \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ is a semigroup sub-solution of

$$-Lu + Vu \leq g \quad \text{in } \mathrm{D},$$

if we have for all $x \in \mathrm{D}$ and all $t \geq 0$ that

$$u(x) \leq \mathbb{E}^x\left[e^{-\int_0^{t\wedge\tau_{\mathrm{D}}} V(X_s)\mathrm{d}s}u(X_{t\wedge\tau_{\mathrm{D}}})\right] + \mathbb{E}^x\left[\int_0^{t\wedge\tau_{\mathrm{D}}} e^{-\int_0^s V(X_p)\mathrm{d}p}g(X_s)\,\mathrm{d}s\right],$$

Semigroup super-solutions are defined in an analogous way.

Lemma 2.1. *Suppose that $V, g \in \mathcal{C}(\bar{D})$, and let u satisfy*

$$-Lu + Vu \leq g \quad \text{in } \mathrm{D}. \tag{2.2}$$

We have the following:

(1) If $u \in \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ is a semigroup sub-solution (resp., super-solution) of (2.2), then it is also a viscosity sub-solution (resp., super-solution).

(2) If $u \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ is a classical sub-solution (super-solution) of (2.2), then it is also a semigroup sub-solution (resp., super-solution).

Proof. Consider part (1). Choose a point $x \in \mathrm{D}$, and let $\varphi \in \mathcal{C}^2(\mathrm{D})$ be a test function that (strictly) touches u at x from above, i.e., for a ball $\mathcal{B}_r(x) \subset \mathrm{D}$ we have $\varphi(x) = u(x)$, and $\varphi(y) > u(y)$ for $y \in \mathcal{B}_r(x)\backslash\{x\}$. Define

$$\varphi_r(y) = \begin{cases} \varphi(y) & y \in \mathcal{B}_r(x), \\ u(y) & y \in \mathcal{B}_r^c(x). \end{cases}$$

To show that u is viscosity solution, we need to show that $-L\varphi_r(x) + V(x)u(x) \leq g(x)$. Consider a sequence of functions $(\varphi_{r,n})_{n\in\mathbb{N}} \subset \mathcal{C}^2(\mathcal{B}_r(x)) \cap \mathcal{C}(\mathbb{R}^d)$ with the property that $\varphi_{2,n} = \varphi_r$ outside $\mathcal{B}_{r+\frac{1}{n}}(x)\backslash\mathcal{B}_r(x)$, $\varphi_{r,n} \geq u$, and $\varphi_{r,n} \to \varphi_r$ almost surely, as $n \to \infty$. Since u is a semigroup subsolution, we have that for all $t \geq 0$

$$u(x) \leq \mathbb{E}^x\left[e^{-\int_0^{t\wedge\tau_{\mathrm{D}}} V(X_s)\mathrm{d}s} u(X_{t\wedge\tau_{\mathrm{D}}})\right] + \mathbb{E}^x\left[\int_0^{t\wedge\tau_{\mathrm{D}}} e^{-\int_0^s V(X_p)\mathrm{d}p} g(X_s)\,\mathrm{d}s\right].$$

It is direct to show that the process $(Y_t)_{t\geq 0}$, $Y_t = e^{-\int_0^{t\wedge\tau_{\mathrm{D}}} V(X_s)\mathrm{d}s} u(X_{t\wedge\tau_{\mathrm{D}}}) + \int_0^{t\wedge\tau_{\mathrm{D}}} e^{-\int_0^s V(X_p)\mathrm{d}p} g(X_s)\,\mathrm{d}s$, is a submartingale with respect to the natural filtration of $(X_{t\wedge\tau_{\mathrm{D}}})_{t\geq 0}$, see also [13], hence by optional sampling we obtain for all $t \geq 0$ that

$$u(x) \leq \mathbb{E}^x\left[e^{-\int_0^{t\wedge\tau_r} V(X_s)\mathrm{d}s} u(X_{t\wedge\tau_r})\right] + \mathbb{E}^x\left[\int_0^{t\wedge\tau_r} e^{-\int_0^s V(X_p)\mathrm{d}p} g(X_s)\,\mathrm{d}s\right], \tag{2.3}$$

where τ_r denotes the first exit time from the ball $\mathcal{B}_r(x)$. On the other hand, by applying Itô's formula on $\varphi_{r,n}$ we obtain

$$\begin{aligned}&\mathbb{E}^x\left[e^{-\int_0^{t\wedge\tau_r} V(X_s)\mathrm{d}s}\varphi_{r,n}(X_{t\wedge\tau_{\mathrm{D}}})\right] - \varphi_{r,n}(x)\\ &\quad= \mathbb{E}^x\left[\int_0^{t\wedge\tau_r} e^{-\int_0^s V(X_p)\mathrm{d}p}(L\varphi_{r,n} - V\varphi_{r,n})(X_s)\,\mathrm{d}s\right],\end{aligned}$$

for all $t \geq 0$. Combining this with (2.3) gives

$$\begin{aligned}&\mathbb{E}^x\left[\int_0^{t\wedge\tau_r} e^{-\int_0^s V(X_p)\mathrm{d}p}(L\varphi_{r,n} - V\varphi_{r,n})(X_s)\,\mathrm{d}s\right]\\ &\qquad + \mathbb{E}^x\left[\int_0^{t\wedge\tau_r} e^{-\int_0^s V(X_p)\mathrm{d}p} g(X_s)\,\mathrm{d}s\right] \geq 0.\end{aligned}$$

On dividing both sides by t and letting $t \to 0$, it follows that

$$L\varphi_{r,n}(x) - V(x)\varphi_{r,n}(x) + g(x) \geq 0.$$

Thus by letting $n \to \infty$, we obtain

$$-L\varphi_r(x) + V(x)\varphi_r(x) \leq g(x),$$

which proves the first part of the claim.

Next consider part (2). By the property of u we note that Lu is continuous in D. Consider a sequence of open sets $\mathrm{K}_n \Subset K_{n+1} \Subset \mathrm{D}$ and $\cup_n K_n = \mathrm{D}$. For fixed n, let $(\psi_m)_{m\in\mathbb{N}} \subset \mathcal{C}^2(\mathrm{D}) \cap \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ be a sequence of functions satisfying

$$\sup_{x\in\bar{\mathrm{K}}_n} |Lu(x) - L\psi_m(x)| + \sup_{x\in\mathbb{R}^d} |u(x) - \psi_m(x)| \to 0, \quad \text{as} \quad m \to \infty.$$

Applying Itô's formula to ψ_m, we get that

$$\begin{aligned}&\mathbb{E}^x\left[e^{-\int_0^{t\wedge\tau_n} V(X_s)\mathrm{d}s}\psi_m(X_{t\wedge\tau_n})\right] - \psi_m(x)\\ &\quad= \mathbb{E}^x\left[\int_0^{t\wedge\tau_n} e^{-\int_0^s V(X_p)\mathrm{d}p}(L\varphi_m - V\varphi_m)(X_s)\,\mathrm{d}s\right],\end{aligned}$$

where τ_n denotes the first exit time from the set K_n. First letting $m \to \infty$ and then $n \to \infty$ above, and using the fact that $\tau_n \uparrow \tau_{\mathrm{D}}$ almost surely, we obtain for every $t \geq 0$,

$$\mathbb{E}^x \left[e^{-\int_0^{t\wedge\tau_{\mathrm{D}}} V(X_s)\mathrm{d}s} u(X_{t\wedge\tau_n}) \right] - u(x) \geq -\mathbb{E}^x \left[\int_0^{t\wedge\tau_{\mathrm{D}}} e^{-\int_0^s V(X_p)\mathrm{d}p} g(X_s)\,\mathrm{d}s \right].$$

This shows that u is a semigroup subsolution. □

2.2. Proof of Theorem 1.1

Now we are ready to prove our first theorem.

Proof. First consider (a). As discussed in Proposition 2.1, there exists an eigenpair $(\lambda^*, \Psi) \in \mathbb{R} \times \mathcal{C}_0(\mathrm{D})$ with $\Psi > 0$ in D, satisfying

$$-L\Psi = (\lambda^* - V)\Psi \quad \text{in D}, \quad \text{and} \quad \Psi = 0 \quad \text{in D}^c, \tag{2.4}$$

in viscosity sense. By [11, Th. 2.6] we have $\Psi \in \mathcal{C}^\alpha(\mathbb{R}^d)$ for some $\alpha > 0$, independent of Ψ. Since V is Hölder continuous, it follows that $(\lambda^* - V)\Psi$ is Hölder continuous in $\bar{D}$. A combination of (2.4) and [34, Th. 1.3] gives that $\Psi \in \mathcal{C}^{2s+}(\mathrm{D})$, implying existence for (1.6). Using Lemma 2.1 we also note that (λ^*, Ψ) satisfies (2.1). Simplicity of λ^* again follows from [33, Th. 1.2].

Next we consider (b). The main idea in proving (1.7) is to use Schauder's fixed point theorem. Consider a map $\mathcal{T} : \mathcal{C}_0(\mathrm{D}) \to \mathcal{C}_0(\mathrm{D})$ defined such that for every $\psi \in \mathcal{C}_0(\mathrm{D})$, $\mathcal{T}\psi = \varphi$ is the unique viscosity solution of

$$-L\varphi = g - V\psi \quad \text{in D}, \quad \text{and} \quad \varphi = 0 \quad \text{in D}^c. \tag{2.5}$$

Using [11, Th. 2.6] we know that

$$\|\mathcal{T}\psi\|_{\mathcal{C}^s(\mathbb{R}^d)} \leq c_1(\|g\|_\infty + \|V\psi\|_\infty),$$

for a constant $c_1 = c_1(\mathrm{D}, d, s)$. This implies that $\mathcal{T}$ is a compact linear operator. It is also easy to see that $\mathcal{T}$ is continuous.

In a next step we show that the set

$$\mathcal{B} = \left\{ \varphi \in \mathcal{C}_0(\mathrm{D}) \ : \ \varphi = \mu\mathcal{T}\varphi \ \text{ for some } \ \mu \in [0,1] \right\}$$

is bounded in $\mathcal{C}_0(\mathrm{D})$. For every $\varphi \in \mathcal{B}$ we have

$$-L\varphi = \mu g - \mu V\varphi \quad \text{in D}, \quad \text{and} \quad \varphi = 0 \quad \text{in D}^c, \tag{2.6}$$

for some $\mu \in [0,1]$. As argued above, we note that $\varphi \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$. Thus by Lemma 2.1 we see that φ is a semigroup solution of (2.6). To show boundedness of $\mathcal{B}$ it suffices to show that for a constant c_2, independent of μ, we have

$$\sup_{x\in\bar{D}} |\varphi(x)| \leq c_2 \sup_{x\in\bar{D}} |g(x)|. \tag{2.7}$$

Once (2.7) is established, the existence of a fixed point of $\mathcal{T}$ follows by Schauder's fixed point theorem. Since every solution of (1.7) is a semigroup solution and $\lambda^* > 0$, the uniqueness of the solution follows from [13, Th. 4.5]. To obtain (2.7) recall from [13, Cor. 4.3] (which basically uses Proposition 2.1 above) that

$$\lambda^*_{\mu V} = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^x \left[e^{-\int_0^t \mu V(X_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{\mathrm{D}} > t\}} \right], \quad x \in \mathrm{D}. \tag{2.8}$$

Let $\lambda_0^* > 0$ be the principal eigenvalue corresponding to the potential $V = 0$. Then from the concavity of the map $\mu \mapsto \lambda^*_{\mu V}$, which results from (2.8) by applying Young's inequality, it follows that

$$\lambda^*_{\mu V} \geq \lambda^*_V \wedge \lambda^*_0 = 2\delta > 0.$$

Hence by using (2.8) and the continuity of $\mu \mapsto \lambda^*_{\mu V}$, we find constants $c_3 > 0, \mu_0 > 1$, such that for every $\mu \in [0, \mu_0]$ we have

$$\mathbb{E}^x \left[e^{-\int_0^t \mu V(X_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{\mathrm{D}} > t\}} \right] \leq c_3 e^{-\delta t}, \quad t \geq 0,\ x \in \mathrm{D}. \tag{2.9}$$

Since φ is a semigroup solution, we have that

$$\varphi(x) = \mathbb{E}^x \left[e^{-\int_0^t \mu V(X_s)\,\mathrm{d}s} \varphi(X_t) \mathbb{1}_{\{\tau_{\mathrm{D}} > t\}} \right] + \int_0^t T_s^{\mathrm{D},\mu V} g(x)\mathrm{d}s.$$

Letting $t \to \infty$, using (2.9) and Hölder inequality, it is easily seen that the first term at the right hand side of the above vanishes. Again by (2.9), we have for $x \in \mathrm{D}$

$$\left| T_t^{\mathrm{D},\mu V} g(x) \right| \leq c_3 \sup_{x \in \bar{D}} |g|\, e^{-\delta t}, \quad t \geq 0.$$

Thus finally we obtain

$$\sup_{x \in \bar{D}} |\varphi(x)| \leq \frac{c_3}{\delta} \sup_{x \in \bar{D}} |g(x)|,$$

yielding (2.7). □

2.3. Proof of Theorem 1.2

First we show that the $\mathcal{C}^2$-class of test functions can be replaced by functions of class $\mathcal{C}^{2s+}$ in the definition of the viscosity solution.

Lemma 2.2. *Let $u \in \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ be a viscosity subsolution of $(-\Delta)^s u + Vu \leq g$ in* D. *Consider $x \in \mathrm{D}$. Suppose that there exists an open set $\mathrm{N} \Subset \mathrm{D}$, containing x, and a function $\varphi \in \mathcal{C}^{2s+}(\bar{N})$ satisfying $\varphi(x) = u(x)$ and $\varphi > u$ in $\mathrm{N}\backslash\{x\}$. Define*

$$\varphi_{\mathrm{N}}(y) = \begin{cases} \varphi(y) \text{ for } y \in \mathrm{N}, \\ u(y) \text{ for } y \in \mathbb{R}^d \backslash \mathrm{N}. \end{cases}$$

Then we have $-L\varphi_{\mathrm{N}}(x) + V(x)u(x) \leq g(x)$.

Proof. Consider a sequence of functions in $(\varphi_m)_{m\in\mathbb{N}}$, $\mathcal{C}^2$ in a neighbourhood of x, and such that $\|\varphi_m - \varphi\|_{\mathcal{C}^{2s+\alpha}(\bar{N})} \to 0$, for some $\alpha > 0$, as $m \to \infty$. This is possible since $\varphi \in \mathcal{C}^{2s+}(\bar{N})$. Let

$$\delta_m = \min_{\bar{N}} (\varphi_m - u).$$

Then $\hat{\varphi}_m = \varphi_m - \delta_m$ touches u from above in $\bar{N}$. Since $\sup_{\bar{N}} |\hat{\varphi}_m - u| \to 0$, it follows that there exists a sequence $(x_m)_{m\in\mathbb{N}} \in N$ such that $x_m \to x$, $\delta_m \to 0$, as $m \to \infty$, and $\hat{\varphi}_m(x_m) = u(x_m)$. Set

$$\varphi_{N,m}(y) = \begin{cases} \hat{\varphi}_m(y) \text{ for } y \in N, \\ u(y) \quad \text{for } y \in \mathbb{R}^d \backslash N. \end{cases}$$

By the definition of the viscosity subsolution we find

$$-\frac{C(d,s)}{2}\int_{B_r(x)}\frac{\varphi_{N,m}(x_m+y)+\varphi_{N,m}(x_m-y)-2\varphi_{N,m}(x_m)}{|y|^{d+2s}}\,\mathrm{d}y$$
$$-\frac{C(d,s)}{2}\int_{B_r^c(x)}\frac{\varphi_{N,m}(x_m+y)+\varphi_{N,m}(x_m-y)-2\varphi_{N,m}(x_m)}{|y|^{d+2s}}\,\mathrm{d}y$$
$$+V(x_m)u(x_m)\leq g(x_m),$$

where $C(d,s)$ is the normalizing constant for fractional Laplacian and $r>0$ is chosen to satisfy $B_{2r}(x)\Subset N$. It is easily seen that we can let $m\to\infty$ above and use the continuity of V,g,u to obtain

$$-L\varphi_N(x)+V(x)u(x)\leq g(x),$$

which shows the claim. □

Next we prove our second theorem stated in the previous section.

Proof of Theorem 1.2. Let $w=u-v$. By [14, Th. 5.9] it then follows that

$$-Lw+Vw\leq 0\quad\text{in D},\tag{2.10}$$

in viscosity sense. Note that $w\leq 0$ in D^c, while we need to show that $w\leq 0$ in $\mathbb{R}^d$. Suppose, to the contrary, that $w^+>0$ in D. Using Proposition 2.1, we find a domain $\mathrm{D}_1\Supset\mathrm{D}$ with a $\mathcal{C}^1$-boundary and $\lambda_1^*>0$, where λ_1^* is the principal eigenvalue for D_1 and potential V. In fact, we may take V as a $\mathcal{C}^\alpha$-extension from D to D_1. Let $\Psi_1\in\mathcal{C}^{2s+}(\mathrm{D}_1)\cap\mathcal{C}_0(\mathrm{D}_1)$ be the corresponding positive principal eigenfunction. Thus we have

$$-L\Psi_1+V\Psi_1=\lambda_1^*\Psi_1\ \text{ in }\ \mathrm{D}_1\quad\text{and}\quad\Psi_1=0\ \text{ in }\ \mathrm{D}_1^c.\tag{2.11}$$

Define

$$c_0=\inf\big\{c\in(0,\infty)\ :\ c\Psi_1-w>0\ \text{ in }\ \mathrm{D}\big\}.$$

Since $\min_{\mathrm{D}}\Psi_1>0$, it follows that c_0 is finite, and $w^+>0$ implies that $c_0>0$. Then $\Phi=c_0\Psi_1-w$ necessarily vanishes at some point, say $x_0\in\mathrm{D}$. This follows from the fact that $w^+=0$ on $\partial\mathrm{D}$. Thus $c_0\Psi_1$ lies above u on all of $\mathbb{R}^d$ and touches w at x_0. Hence by (2.10) and Lemma 2.2 it follows that

$$-L(c_0\Psi_1)(x_0)+V(x_0)(c_0\Psi_1(x_0))\leq 0.$$

This leads to a contradiction as the left hand side of the above expression equals $\lambda_1^*(c_0\Psi_1(x_0))>0$ by (2.11). □

2.4. Proof of Theorem 1.3

Now we turn to proving our main result on the fractional Ambrosetti–Prodi phenomenon. The strategy of proof will be divided in the following steps.

(1) First we find ρ_1 such that for every $\rho\leq\rho_1$ there exists a minimal solution of (1.4). This will be done in Lemmas 2.3 and 2.4 below.
(2) Next we find $\rho_2>\rho_1$ such that no solution of (1.4) above ρ_2 exists. This is the content of Lemmas 2.7 and 2.8.
(3) Finally, we follow the arguments in [18] to find the bifurcation point ρ^*.

We begin by showing the existence of a sub/super-solution, which will be used for constructing a minimal solution.

Lemma 2.3. *Let Assumption [AP] hold. The following hold:*

(1) For every $\rho \in \mathbb{R}$ there exists $\underline{u} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ satisfying $\underline{u} \leq 0$ in D *and*

$$-L\underline{u} \leq f(x, \underline{u}) + \rho\Phi_1 + h(x) \quad \text{in D.}$$

(2) There exists $\bar{\rho}_1 < 0$ such that for every $\rho \leq \bar{\rho}_1$ there exists $\bar{u} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ satisfying $\bar{u} \geq 0$ in D *and*

$$-L\bar{u} \geq f(x, \bar{u}) + \rho\Phi_1 + h(x) \quad \text{in D.}$$

(3) We can construct $\underline{u}$ to satisfy $\underline{u} \leq \hat{u}$, for every super-solution $\hat{u}$ of

$$-L\hat{u} \geq f(x, \hat{u}) + \rho\Phi_1 + h(x) \quad \text{in D,}$$

with $\hat{u} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$.

Proof. Consider $\rho \in \mathbb{R}$. Let $C_2 = 2\sup_{\bar{D}}|h| + 2|\rho| + C$, where C is the same constant as in (1.9) and (1.10). Since $\lambda^*(-L - V_1) > 0$ by (1.8), it follows from Theorem 1.1(2) that there exists a unique $\underline{u} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ satisfying

$$-L\underline{u} - V_1\underline{u} = -C_2 + h(x) + \rho\Phi_1 \quad \text{in D.} \tag{2.12}$$

Recalling that $\Phi_1 \in \mathcal{C}^{\alpha}(\mathbb{R}^d) \cap \mathcal{C}_0(\mathrm{D})$ by [11, Th. 2.6], the right hand side of the (2.12) is Hölder continuous in D. By our choice of C_2 we see that

$$-L\underline{u} - V_1\underline{u} \leq 0,$$

and hence, by Theorem 1.2 we have $\underline{u} \leq 0$ in $\mathbb{R}^d$. Therefore, by making use of (1.9) we get that

$$-L\underline{u} \leq f(x, \underline{u}) + h(x) + \rho\Phi_1 \quad \text{in D,} \quad \text{and} \quad \underline{u} = 0 \quad \text{in } \mathrm{D}^c.$$

This proves part (1).

Now we proceed to establish (2). Due to Assumption [AP] there exists a constant C_1 satisfying $f(x, q) \leq C_1(1 + q^p)$, for all $(x, q) \in \bar{D} \times [0, \infty)$. We consider the unique function $\bar{u} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ satisfying

$$-L\bar{u} - h^+ - C_1 = 0 \quad \text{in D.} \tag{2.13}$$

Using [11, Th. 2.6, Eqn (2.3)] we find $c_1 = c_1(d, s, \mathrm{D}) > 0$, such that

$$\sup_{x \in \mathrm{D}} \frac{|\bar{u}(x)|}{d^s(x)} \leq c_1, \tag{2.14}$$

where $d(\cdot)$ is the distance function from the boundary of D. Since $-L\bar{u} \geq 0$, it also follows from Hopf's lemma [11, Th. 2.4] that $\bar{u} > 0$ in D. Since $-L\Phi_1 = \lambda_0^*\Phi_1 \geq 0$ in D, another application of Hopf's lemma gives a constant $c_2 > 0$ satisfying

$$\frac{\Phi_1(x)}{d^s(x)} \geq c_2, \quad x \in \mathrm{D}.$$

Combining the above with (2.14) and choosing $-\bar{\rho}_1 > 0$ large, we find for every $\rho \leq \bar{\rho}_1$ that

$$-\rho\Phi_1 \geq C_1 c_1^p d^{sp} \geq C_1 \bar{u}^p, \quad \text{for } x \in \mathrm{D}.$$

Hence by (2.13) we have for $\rho \leq \rho_0$

$$-L\bar{u} \geq f(x, \bar{u}) + \rho\Phi + h \quad \text{in D.}$$

This proves (2).

Now we come to (3). Note that

$$-L\hat{u} \geq f(x,\hat{u}) - |\rho| - \|h\|_\infty \quad \text{in D}.$$

Since the minimum of two viscosity super-solutions is again a viscosity super-solution, we note that $w = \hat{u} \wedge 0$ is a viscosity super-solution of

$$-Lw \geq f(x,w) - |\rho| - \|h\|_\infty \geq V_1 w - C - |\rho| - \|h\|_\infty \quad \text{in D}, \tag{2.15}$$

by (1.9). On the other hand, by our choice of C_2 in (2.12) we have

$$-L\underline{u} - V_1\underline{u} \leq -C - |\rho| - \|h\|_\infty \quad \text{in D}. \tag{2.16}$$

Combining (2.15), (2.16) and [14, Th. 5.9], we obtain

$$-L(w-\underline{u}) - V_1(w-\underline{u}) \geq 0 \quad \text{in D},$$

in viscosity sense, and $w-\underline{u} = 0$ in D^c. Hence by Theorem 1.2 we have $w \geq \underline{u}$ in $\mathbb{R}^d$, implying $\hat{u} \geq w \geq \underline{u}$ in $\mathbb{R}^d$. This yields part (3). □

Using Lemma 2.3 we can now prove the existence of a minimal solution.

Lemma 2.4. *For $\rho \leq \bar{\rho}_1$, where $\bar{\rho}_1$ is the same value as in Lemma 2.3, there exists $u \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ satisfying*

$$(-\Delta)^s u = f(x,u) + \rho\Phi_1 + h(x) \quad \textit{in D}. \tag{2.17}$$

Moreover, the above u can be chosen to be minimal in the sense that if $\tilde{u} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ is another solution of (2.17), then $\tilde{u} \geq u$ in $\mathbb{R}^d$.

Proof. The proof is based on the standard monotone iteration method. Denote $m = \min_{\bar{D}} \underline{u}$ and $M = \max_{\bar{D}} \bar{u}$. Let $\theta > 0$ be a Lipschitz constant for $f(x,\cdot)$ on the interval $[m,M]$, i.e.,

$$|f(x,q_1) - f(x,q_2)| \leq \theta|q_1 - q_2| \quad \text{for } q_1, q_2 \in [m,M],\ x \in \bar{D}.$$

Denote $F(x,u) = f(x,u) + \rho\Phi(x) + h(x)$. Consider the solutions of the following family of problems:

$$\begin{aligned} &-Lu^{(n+1)} + \theta u^{(n+1)} = F(x,u^{(n)}) + \theta u^{(n)} \quad \text{in D},\\ &u^{(n+1)} = 0 \quad \text{in } \mathrm{D}^c. \end{aligned} \tag{2.18}$$

By Theorem 1.1(2) Eq. (2.18) has a unique solution, provided $u^{(n)}$ is Hölder continuous in $\bar{D}$. We set $u^{(0)} = \underline{u}$. Since $u^{(0)} \in \mathcal{C}^\alpha(\mathbb{R}^d)$ by [11, Th. 2.6], it follows from [34] that $u^{(1)} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}^\alpha(\mathbb{R}^d)$. Thus by successive iteration it follows that $u^{(n)} \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}^\alpha(\mathbb{R}^d)$, for all $n \geq 0$. Hence all solutions of (2.18) are classical solutions. Again, it is routine to check from (2.18) and Theorem 1.2 that $u^{(0)} \leq u^{(n)} \leq u^{(n+1)} \leq \bar{u}$ in D. This implies $\sup_{\mathbb{R}^d} |u^{(n)}| \leq M - m$, for all n. Thus applying [11, Th. 2.6] we obtain

$$\sup_{n\in\mathbb{N}} \|u^{(n)}\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} \leq \kappa_1,$$

for some constants α, κ_1. Hence there exists $u \in \mathcal{C}^\alpha(\mathbb{R}^d) \cap \mathcal{C}_0(\mathrm{D})$ such that $u^{(n)} \to u$ in $\mathcal{C}_0(\mathrm{D})$ as $n \to \infty$. Using the stability of viscosity solutions, it then follows that u is a viscosity solution to

$$-Lu = F(x,u) \quad \text{in D},$$
$$u = 0 \quad \text{in D}^c.$$

We can now apply the regularity estimates from [34] to show that $u \in \mathcal{C}^{2s+}(\mathrm{D})$.

To establish minimality we consider a solution $\tilde{u}$ of (2.17) in $\mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$. From Lemma 2.3(3) we have $\underline{u} \le \tilde{u}$ in $\mathbb{R}^d$. Thus $\bar{u}$ can be replaced by $\tilde{u}$, and the above argument shows that $u \le \tilde{u}$. □

Now we derive a priori bounds on the solutions of (1.4). Our first result bounds the negative part of solutions u of (1.4). We recall that under the standing assumptions on f, any viscosity solution of (1.4) is an element of $\mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$, and thus also a classical solution.

Lemma 2.5. *Let Assumption [AP](2) hold. There exists a constant $\kappa = \kappa(d, s, \mathrm{D}, V_1)$, such that for every solution u of (1.4) with $\rho \ge -\hat{\rho}$, $\hat{\rho} > 0$, we have*

$$\sup_{\mathrm{D}} |u^-| \ \le \ \kappa(C + \hat{\rho} + \|h\|_\infty),$$

where C is the same constant as in (1.9).

Proof. First observe that if u is a solution to (1.4) for some $\rho \ge -\hat{\rho}$, then

$$Lu + f(x,u) \le \hat{\rho} + \|h\|_\infty \quad \text{in D}.$$

Defining $w = u \wedge 0$ we see that w is a viscosity super-solution of the above equation, i.e.,

$$Lw + f(x,w) \le \hat{\rho} + \|h\|_\infty \quad \text{in D}, \quad \text{and} \quad w = 0 \quad \text{in D}^c.$$

From (1.9) it then follows that

$$Lw + V_1 w \le C + \hat{\rho} + \|h\|_\infty \quad \text{in D}, \quad \text{and} \quad w = 0 \quad \text{in D}^c,$$

in viscosity sense. Let $v \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$ be the unique solution of

$$Lv + V_1 v = C + \hat{\rho} + \|h\|_\infty \quad \text{in D}, \quad \text{and} \quad v = 0 \quad \text{in D}^c.$$

Existence follows from Theorem 1.1(2). Applying Theorem 1.2, we get $-w \le -v$ in $\mathbb{R}^d$. Since v is also a semigroup solution by Lemma 2.1, we obtain from [13, Th. 4.12 and Rem. 3.9] that with a constant $\kappa = \kappa(s, d, \mathrm{D}, V_1)$

$$\sup_{x \in \bar{D}} |v| \ \le \ \kappa(C + \hat{\rho} + \|h\|_\infty)$$

holds. Thus $u^- = -w \le \kappa(C + \hat{\rho} + \|h\|_\infty)$, for $x \in \mathrm{D}$, and the result follows. □

Our next result provides a lower bound on the growth of the solution for large ρ.

Lemma 2.6. *Let Assumption [AP](1)–(2) hold. For every $\hat{\rho} > 0$ there exists $C_3 > 0$ such that for every solution u of (1.4) with $\rho \ge -\hat{\rho}$ we have*

$$\rho^+ \le C_3(1 + \|u^+\|_\infty) \le C_3(1 + \|u\|_\infty).$$

Proof. Let $\varphi = u - \frac{\rho}{\lambda_0^*}\Phi_1$. Then we have $\varphi \in \mathcal{C}^{2s+}(\mathrm{D}) \cap \mathcal{C}_0(\mathrm{D})$. Also,

$$\begin{aligned} -L\varphi(x) &= f(x,u) + \rho\Phi_1 + h - \rho\Phi_1 \\ &= f(x,u) - h \geq f(x,u^+) + f(x,-u^-) - \|h\|_\infty \geq -C_4(1+u^+(x)), \end{aligned}$$

with a constant $C_4 = C_4(\|h\|_\infty, \|V_2\|_\infty, C, \hat{\rho})$, where in the last estimate we used Lemma 2.5 and (1.10). Thus

$$L\varphi \leq C_4(1+u^+) \quad \text{in D}.$$

By an application of [13, Th. 4.12 and Rem 3.9] it then follows that with a constant C_5,

$$\sup_{\mathrm{D}}(-\varphi)^+ \leq C_5 C_4(1+\|u^+\|_\infty)$$

holds. Pick $x \in \mathrm{D}$ such that $\Phi_1(x) = 1$; this is possible since $\|\Phi_1\|_\infty = 1$ by assumption. It gives

$$\frac{\rho}{\lambda_0^*} - u(x) \leq (-\varphi(x))^+ \leq C_5 C_4(1+\|u^+\|_\infty),$$

which, in turn, implies

$$\rho \leq \lambda_0^* \left(C_4 C_5 + (1+C_4 C_5)\|u^+\|_\infty\right),$$

proving the claim. □

One may notice that we have not used the second condition in (1.8) so far. The next result makes use of this condition to establish an upper bound on the growth of u.

Lemma 2.7. *Let Assumption [AP](3) hold. For every $\hat{\rho} > 0$ there exists C_0 such that for every solution u of (1.4), for $\rho \geq -\hat{\rho}$ we have*

$$\|u\|_\infty \leq C_0. \tag{2.19}$$

In particular, there exists $\rho_2 > 0$ such that (1.4) does not have any solution for $\rho \geq \rho_2$.

Proof. Suppose, to the contrary, that there exists a sequence $(\rho_n, u_n)_{n\in\mathbb{N}}$ satisfying (1.4) with $\rho_n \geq -\hat{\rho}$ and $\|u_n\|_\infty \to \infty$. From Lemma 2.5 it follows that $\|u_n^+\|_\infty = \|u_n\|_\infty$. Define $v_n = \frac{u_n}{\|u_n\|_\infty}$. Then

$$-Lv_n = H_n(x) = \frac{1}{\|u_n\|_\infty}\left(f(x,u_n) + \rho_n\Phi_1 + h\right) \quad \text{in D}. \tag{2.20}$$

Since $\|H_n\|_\infty$ is uniformly bounded by Lemmas 2.5 and 2.6, it follows by [11, Th. 2.6] that

$$\sup_{n\geq 1}\|v_n\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} < \infty.$$

for some $\alpha > 0$. Hence we can extract a subsequence of $(v_n)_{n\in\mathbb{N}}$, denoted by the original sequence, such that it converges to a continuous function $v \in \mathcal{C}_0(\mathrm{D})$ in $\mathcal{C}(\mathbb{R}^d)$. Denote

$$G_n(x) = \frac{1}{\|u_n\|_\infty}(f(x,-u_n^-(x)) + h(x) - C + V_2(x)u_n^-(x) - \rho_n^-\Phi_1(x)).$$

Then using (1.10) and (2.20), we get

$$-Lv_n - V_2 v_n \geq G_n \quad \text{in D}.$$

Using Lemma 2.1 we have that v_n is a semigroup super-solution, i.e., for every $t > 0$

$$v_n(x) \geq \mathbb{E}^x \left[\int_0^{t\wedge\tau_{\mathrm{D}}} e^{\int_0^s V_2(X_p)\mathrm{d}p} G_n(X_s)\,\mathrm{d}s \right] + \mathbb{E}^x \left[e^{\int_0^t V_2(X_s)\,\mathrm{d}s} v_n(X_t) \mathbb{1}_{\{t<\tau_{\mathrm{D}}\}} \right]. \tag{2.21}$$

Letting $n \to \infty$ in (2.21) and using the uniform convergence of G_n and v_n, we obtain

$$v(x) \geq \mathbb{E}^x \left[e^{\int_0^t V_2(X_s)\,\mathrm{d}s} v(X_t) \mathbb{1}_{\{t<\tau_{\mathrm{D}}\}} \right] \quad \text{for all } x \in \mathrm{D},\ t \geq 0. \tag{2.22}$$

Since $\|v\|_\infty = 1$ and $v \geq 0$ in $\mathbb{R}^d$, it is easily seen from (2.22) that $v > 0$ in D. Again, by Lemma 2.1, we see that

$$-Lv - V_2 v \geq 0 \quad \text{in D},$$

in viscosity sense. Hence it follows that $\lambda^*((-\Delta)^s - V_2) \geq 0$, contradicting (1.8). This proves the first part of the result. The second part follows by Lemma 2.6 and (2.19). □

The following result will be useful for tackling the super-linear case. We note that the general idea of the a priori bound below has its origins in the work of Gidas and Spruck [22], see also the more recent [7] for a non-local version.

Lemma 2.8. *Let Assumption [AP](3′) hold. Then for every $\hat{\rho} > 0$ there exists C_0 such that for every solution u of (1.4), with $\rho \geq -\hat{\rho}$ we have*

$$\|u\|_\infty \leq C_0 \max\left\{1, |\rho|^{\frac{1}{p}}\right\}. \tag{2.23}$$

In particular, there exists $\rho_2 > 0$ such that (1.4) does not have any solution for $\rho \geq \rho_2$.

Proof. First we establish (2.23) for all $\rho \geq 1$. Suppose, to the contrary, that there exists $(u_n, \rho_n)_{n\in\mathbb{N}}$, $\rho_n \geq 1$, satisfying (1.4) with the property that

$$\|u_n\|_\infty \geq n\rho_n^{\frac{1}{p}}, \quad n \geq 1. \tag{2.24}$$

This then implies that $\|u_n\|_\infty \to \infty$ as $n \to \infty$, and

$$\|u_n\|_\infty^{-p} \rho_n \to 0 \quad \text{as} \quad n \to \infty. \tag{2.25}$$

Let $x_n \in \mathrm{D}$ be such that $u(x_n) = u^+(x_n) = \|u_n\|_\infty$. Such a choice is possible due to Lemma 2.5. Write

$$\gamma_n = \|u_n\|_\infty^{-\frac{p-1}{2s}} \quad \text{and} \quad \theta_n = \mathrm{dist}(x_n, \partial\mathrm{D}).$$

Using compactness, we may also assume that $x_n \to x_0 \in \bar{D}$ as $n \to \infty$. We split the proof into two cases.

Case 1 Assume $\limsup_{n\to\infty} \frac{\theta_n}{\gamma_n} = \infty$. Defining $w_n(x) = \frac{1}{\|u_n\|_\infty} u_n(\gamma_n x + x_n)$, we have in $\frac{1}{\gamma_n}(\mathrm{D} - x_n)$ that

$$(-\Delta)^s w_n = \frac{1}{\|u_n\|_\infty^p}\big(f(\gamma_n x + x_n, u_n(x)) + \rho_n \Phi(\gamma_n x + x_n) + h(\gamma_n x + x_n)\big). \tag{2.26}$$

We choose a subsequence, denoted is the same way, such that $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} = \infty$. Then for any given $k \in \mathbb{N}$ there is a large enough n_0 satisfying $\mathcal{B}_k(0) \subset \frac{1}{\gamma_n}(\mathrm{D} - x_n)$ for all $n \geq n_0$. Therefore, the right hand side of (2.26) is uniformly bounded in $\mathcal{B}_k(0)$. Since $\|w_n\| = w_n(0) = 1$, it follows that for some $\alpha > 0$, $\|w_n\|_{\mathcal{C}^\alpha(\mathcal{B}_{k/2}(0))}$ is bounded uniformly in n (see [14]). Thus we can extract a subsequence $(w_n)_{n\in\mathbb{N}}$ such that $w_n \to w \in \mathcal{C}_{\mathrm{b},+}(\mathbb{R}^d)$ locally uniformly. Hence, by the stability of viscosity solutions

$$(-\Delta)^s w = a_0(x_0) w^p \quad \text{in } \mathbb{R}^d, \quad w(0) = 1.$$

By the strong maximum principle we also have $w > 0$. However, no such solution can exist due to the Liouville theorem [29, Th. 1.2], and hence we have a contradiction in this case.

Case 2 Suppose that $\limsup_{n\to\infty} \frac{\theta_n}{\gamma_n} < \infty$. First we show that for a positive constant κ

$$\liminf_{n\to\infty} \frac{\theta_n}{\gamma_n} \geq \kappa. \tag{2.27}$$

Note that using Lemma 2.5 and Assumption [AP](3′) we can find a constant κ_1 satisfying

$$\kappa_1(1 + \|u_n\|_\infty^{p-1}) \operatorname{sgn}(u_n) u_n \geq f(x, u_n) \quad \text{for } x \in \mathrm{D},\ n \geq 1.$$

Indeed, using Assumption [AP](3′) it follows that for $u_n(x) \geq \ell$, for some $\ell > 0$, we have

$$f(x, u_n(x)) \leq 2\|a_0\|_\infty u_n^p(x) \leq 2\|a_0\|_\infty \|u_n\|_\infty^{p-1} u_n(x).$$

Then the estimate follows from the local Lipschitz property of f and Lemma 2.5. Hence, using (1.4) we obtain

$$-(-\Delta)^s u_n + \kappa_1(1 + \|u_n\|_\infty^{p-1})\operatorname{sgn}(u_n) u_n \geq -\rho_n - \|h\|_\infty \quad \text{in D}.$$

Denote by $C_n = \rho_n + \|h\|_\infty$. Applying Lemma 2.1 we get that for $t \geq 0$,

$$\|u_n\|_\infty = u_n(x_n) \leq e^{\kappa_1(1+\|u_n\|_\infty^{p-1})t} \|u_n\|_\infty \, \mathbb{P}^{x_n}(\tau_{\mathrm{D}} > t) + e^{\kappa_1(1+\|u_n\|_\infty^{p-1})t} t C_n.$$

It follows from the proof of [8, Th. 1.1] that there exist constants κ_2 and $\eta \in (0,1)$, not depending on x_n, such that for $t = \kappa_2 \theta_n^{2s}$ we have

$$\mathbb{P}^{x_n}(\tau_{\mathrm{D}} > t) \leq \eta.$$

Inserting this choice of t in the above expression we obtain

$$1 \leq e^{\kappa_1(1+\|u_n\|_\infty^{p-1})t}\left[\eta + \kappa_2 \theta_n^{2s} \frac{C_n}{\|u_n\|_\infty}\right] = e^{\kappa_1(1+\|u_n\|_\infty^{p-1})t}\left[\eta + \kappa_2 \frac{\theta_n^{2s}}{\gamma_n^{2s}} \frac{C_n}{\|u_n\|_\infty^p}\right].$$

Thus by the assertion and (2.25) it follows that for all large n we have

$$\kappa_1 \kappa_2 \theta_n^{2s}(1 + \|u_n\|_\infty^{p-1}) \geq \log \frac{2}{\eta}.$$

This gives (2.27), since $\theta_n \to 0$.

Hence we may assume that, up to a subsequence,

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} = b \in (0,\infty)$$

holds. Then using again an argument similar to above, we obtain a positive bounded solution

$$(-\Delta)^s w = a_0(x_0) w^p \quad \text{in } \mathbb{R}^d_+,$$

see, for instance, the arguments in [25, Lem. 5.3]. This again contradicts [29, Th. 1.1].

Thus (2.24) can not hold and this proves our result when $\rho \geq 1$. For the remaining case $\rho \in [-\hat{\rho}, 1]$, note that we can rewrite

$$\rho\Phi_1 + h = \Phi_1 + \tilde{h} \quad \text{where} \quad \tilde{h} = h - \Phi_1 + \rho\Phi_1.$$

Note that $\|\tilde{h}\|_\infty$ is uniformly bounded for $\rho \in [-\hat{\rho}, 1]$. Then (2.24) follows from the previous argument. The other claim follows by (2.23) and Lemma 2.6. □

With the above results in hand, we can now proceed to prove Theorem 1.3. Define

$$\mathcal{A} = \big\{\rho \in \mathbb{R} : \quad (1.4) \quad \text{has a viscosity solution}\big\}.$$

By Lemma 2.4 we have that $\mathcal{A} \neq \emptyset$, and Lemmas 2.7 and 2.8 imply that $\mathcal{A}$ is bounded from above. Define

$$\rho^* = \sup \mathcal{A}.$$

Note that if $\rho' < \rho^*$, then $\rho' \in \mathcal{A}$. Indeed, there is $\tilde{\rho} \in (\rho', \rho^*) \cap \mathcal{A}$ and the corresponding solution $u^{(\tilde{\rho})}$ of (1.4) with $\rho = \tilde{\rho}$ is a super-solution at level ρ', i.e.,

$$-Lu^{(\tilde{\rho})} \geq f(x, u^{(\tilde{\rho})}) + \rho'\Phi_1 + h(x) \quad \text{in D}, \quad \text{and} \quad u = 0 \quad \text{in D}^c.$$

Using Lemma 2.3(3) and from the proof of Lemma 2.4 we have a minimal solution of (1.4) with $\rho = \rho'$. Next we show that there are at least two solutions for $\rho < \rho^*$.

Recall that $d : \bar{D} \to [0,\infty)$ is the distance function from the boundary of D. We can assume that d is a positive $\mathcal{C}^1$-function in D. For a sufficiently small $\varepsilon > 0$, to be chosen later, consider the Banach space

$$\mathfrak{X} = \left\{ \psi \in \mathcal{C}_0(\mathrm{D}) \ : \ \left\| \frac{\psi}{d^s} \right\|_{\mathcal{C}^\varepsilon(\mathrm{D})} < \infty \right\}.$$

In fact, it is sufficient to consider any ε strictly smaller than the parameter $\alpha < s \wedge (1-s)$ in [31, Th. 1.2]. Since d^s is s-Hölder continuous in $\bar{D}$, it is routine to check that $\mathfrak{X} \subset \mathcal{C}^\varepsilon(\bar{D})$.

For $\rho \in \mathbb{R}$ and $m \geq 0$ we define a map $K_\rho : \mathfrak{X} \to \mathfrak{X}$ as follows. For $v \in \mathfrak{X}$, $K_\rho v = u$ is the unique viscosity solution (see Theorem 1.1(b)) to the Dirichlet problem

$$-Lu + mu = f(x,v) + \rho\Phi_1 + h(x) + mv \quad \text{in D}, \quad \text{and} \quad u = 0 \quad \text{in D}^c.$$

It follows from [31, Th. 1.2] that $u \in \mathfrak{X}$.

Lemma 2.9. *Let $\rho < \rho^*$. Then there exist $m \geq 0$ and an open $\mathcal{O} \subset \mathfrak{X}$, containing the minimal solution, satisfying* $\deg(I - K_\rho, \mathcal{O}, 0) = 1$.

Proof. We borrow some of the arguments of [18] with a suitable modification. Pick $\bar\rho \in (\rho, \rho^*)$ and let $\bar u$ be a solution of (1.4) with $\rho = \bar\rho$. It then follows that

$$-L\bar u > f(x,\bar u) + \rho\Phi_1 + h(x) \quad \text{in D} \quad \text{and} \quad u = 0 \quad \text{in D}^c.$$

and by Lemma 2.3(i) we have a classical subsolution

$$-L\underline{u} < f(x,\underline{u}) + \rho\Phi_1 + h(x) \quad \text{in D} \quad \text{and} \quad u = 0 \quad \text{in D}^c.$$

Then Lemma 2.3(3) supplies $\underline{u} \leq \bar u$ in $\mathbb{R}^d$, hence the minimal solution u of (1.4) satisfies $\underline{u} \leq u \leq \bar u$ in $\mathbb{R}^d$. Note that for every $\psi \in \mathfrak{X}$, the ratio $\frac{\psi}{d^s}$ is continuous up to the boundary. Define

$$\mathcal{O} = \left\{ \psi \in \mathfrak{X} \,:\, \underline{u} < \psi < \bar u \text{ in D}, \;\; \frac{\underline{u}}{d^s} < \frac{\psi}{d^s} < \frac{\bar u}{d^s} \text{ on } \partial\text{D}, \;\; \|\psi\|_{\mathfrak{X}} < r \right\},$$

where the value of r will be chosen conveniently below. It is evident that $\mathcal{O}$ is bounded, open and convex. Also, if we choose r large enough, then the minimal solution u belongs to $\mathcal{O}$. Indeed, note that

$$-L(u-\underline{u}) + \left(\frac{f(x,u) - f(x,\underline{u})}{u - \underline{u}} \right)^+ (u - \underline{u}) \geq 0 \quad \text{in D}.$$

By strong maximum principle it follows that $\underline{u} < u$ in D. Hence by [23, Lem. 1.2] we have

$$\min_{\partial\text{D}} \left(\frac{u}{d^s} - \frac{\underline{u}}{d^s} \right) > 0.$$

The results in [23] are proved for $(-\Delta)^s$ (i.e. k constant), but the similar argument works with the barrier function constructed in [30, Lem. 3.4] giving us Hopf's lemma in our setting. Similarly, we can compare also u and $\bar u$.

Define m to be a Lipschitz constant of $f(x,\cdot)$ in the interval $[\min \underline{u}, \max \bar u]$. Also, define

$$\tilde f(x,q) = f\left(x, (\underline{u}(x) \vee q) \wedge \bar u(x)\right) + m(\underline{u}(x) \vee q) \wedge \bar u(x).$$

Note that f is bounded and Lipschitz continuous in q, and also non-decreasing in q. We define another map $\tilde K_\rho : \mathfrak{X} \to \mathfrak{X}$ as follows: for $v \in \mathfrak{X}$, $\tilde K_\rho v = u$ is the unique viscosity solution of

$$-Lu + mu = \tilde f(x,v) + \rho\Phi + h \quad \text{in D}, \quad \text{and} \quad u = 0 \quad \text{in D}^c. \tag{2.28}$$

It is easy to check that K_ρ is a compact mapping. Using again [30, Th. 1.2], we find r satisfying

$$\sup \left\{ \|\tilde K_\rho v\|_{\mathfrak{X}} \,:\, v \in \mathfrak{X} \right\} < r.$$

We fix this choice of r. Using the regularity estimate of [34], we see that the solution u in (2.28) is in $\mathcal{C}^{2s+}(\mathrm{D})$. Therefore,

$$\begin{aligned} &- L(u-\underline{u}) + m(u-\underline{u}) \\ &\quad > \tilde{f}(x,v) - m\underline{u} - f(x,\underline{u}) \geq \tilde{f}(x,\underline{u}) - m\underline{u} - f(x,\underline{u}) = 0. \end{aligned}$$

Hence by [23, Th. 2.1, Lem 1.2] we have $\underline{u} < u$ in D and

$$\min_{\partial \mathrm{D}} \left(\frac{u}{d^s} - \frac{\underline{u}}{d^s} \right) > 0.$$

The other estimates can be obtained similarly. Finally, this implies that $\tilde{K}_\rho v \in \mathcal{O}$, for all $v \in \mathfrak{X}$. Moreover, $0 \notin (I - \tilde{K}_\rho)(\partial \mathrm{D})$. Then by the homotopy invariance property of degree we find that $\deg(I - \tilde{K}_\rho, \mathcal{O}, 0) = 1$. Since $\tilde{K}_\rho$ coincides with K_ρ in $\mathcal{O}$, we obtain $\deg(I - K_\rho, \mathcal{O}, 0) = 1$. □

Similarly as before, define $\mathcal{S}_\rho : \mathfrak{X} \to \mathfrak{X}$ such that for $v \in \mathfrak{X}$, $u = \mathcal{S}_\rho v$ is given by the unique solution of

$$-Lu = f(x,v) + \rho\Phi_1 + h(x) \quad \text{in D}, \quad \text{and} \quad u = 0 \quad \text{in D}^c.$$

Then the standard homotopy invariance of degree gives that $\deg(I - \mathcal{S}_\rho, \mathcal{O}, 0) = 1$. This observation will be helpful in concluding the proof below.

Proof of Theorem 1.3. Using Lemma 2.9 we can now complete the proof by using [18,19]. Recall the map $\mathcal{S}_\rho$ defined above, and fix $\rho < \rho^*$. Denote by $\mathcal{O}_R$ a ball of radius R in $\mathfrak{X}$. From Lemmas 2.7 and 2.8 we find that

$$\deg(I - \mathcal{S}_{\tilde{\rho}}, \mathcal{O}_R, 0) = 0 \quad \text{for all } R > 0,\ \tilde{\rho} \geq \rho_2.$$

Using again Lemmas 2.7, 2.8 and [31, Th. 1.2], we obtain that for every $\hat{\rho}$ there exists a constant R such that

$$\|u\|_{\mathfrak{X}} < R$$

for each solution u of (1.4) with $\tilde{\rho} \geq -\hat{\rho}$. Fixing $\hat{\rho} > |\rho|$ and the corresponding choice of R, by homotopy invariance $\deg(I - \mathcal{S}_\rho, \mathcal{O}_R, 0) = 0$. We can choose R large enough so that $\mathcal{O} \subset \mathcal{O}_R$. Since $\deg(I - \mathcal{S}_\rho, \mathcal{O}, 0) = 1$, as seen above, using the excision property we conclude that there exists a solution of (1.4) in $\mathcal{O}_R \backslash \mathcal{O}$. Hence for every $\rho < \rho^*$ there exist at least two solutions of (1.4). The existence of a solution at $\rho = \rho^*$ follows from the a priori estimates in Lemmas 2.7 and 2.8, the estimate in [11, Th. 2.6], and the stability property of the viscosity solutions. This completes the proof of Theorem 1.3. □

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Anup Biswas
Department of Mathematics
Indian Institute of Science Education and Research
Dr. Homi Bhabha Road
Pune 411008
India
e-mail: anup@iiserpune.ac.in

József Lőrinczi (✉)
Department of Mathematical Sciences
Loughborough University
Loughborough LE11 3TU
UK
e-mail: J.Lorinczi@lboro.ac.uk

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