Loughborough University

This item was submitted to Loughborough's Research Repository by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

Hyperbolic systems with non-diagonalisable principal part and variable multiplicities, II: microlocal analysis

PLEASE CITE THE PUBLISHED VERSION

https://doi.org/10.1016/j.jde.2020.05.038

PUBLISHER

Elsevier

VERSION

VoR (Version of Record)

PUBLISHER STATEMENT

This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

LICENCE

CC BY 4.0

REPOSITORY RECORD

Garetto, Claudia, Christian Jäh, and Michael Ruzhansky. 2020. “Hyperbolic Systems with Non-diagonalisable Principal Part and Variable Multiplicities, II: Microlocal Analysis”. Loughborough University. https://hdl.handle.net/2134/12472778.v1.

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 269 (2020) 7881–7905

Journal of Differential Equations

www.elsevier.com/locate/jde

Hyperbolic systems with non-diagonalisable principal part and variable multiplicities, II: Microlocal analysis ☆

Claudia Garetto [a,*], Christian Jäh [b], Michael Ruzhansky [c,d]

[a] *Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, United Kingdom*

[b] *Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstraße 3-5, D-37037 Göttingen, Germany*

[c] *Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium*

[d] *School of Mathematical Sciences, Queen Mary University London, Mile End, London, E1 4NS, United Kingdom*

Received 2 April 2020; accepted 30 May 2020

Abstract

In this paper we continue the study of non-diagonalisable hyperbolic systems with variable multiplicity started by the authors in [1]. In the case of space dependent coefficients, we prove a representation formula for solutions that allows us to derive results of regularity and propagation of singularities.
© 2020 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

MSC: primary 35L45 ; secondary 46E35

Keywords: Hyperbolic systems; Fourier integral operators; Microlocal analysis; Propagation of singularities

Contents

1. Introduction 7882
   1.1. Notations and preliminary notions 7883
   1.2. Assumptions on the matrices $A(x, D_x)$ and $B(t, x, D_x)$ 7884

☆ The authors were supported in parts by the FWO Odysseus Project G.0H94.18N: Analysis and Partial Differential Equations, the Leverhulme Trust Grant RPG-2017-151 and by EPSRC Grant EP/R003025/1.

* Corresponding author.

E-mail addresses: c.garetto@lboro.ac.uk (C. Garetto), christian.jaeh@mathematik.uni-goettingen.de (C. Jäh), m.ruzhansky@qmul.ac.uk, michael.ruzhansky@ugent.be (M. Ruzhansky).

https://doi.org/10.1016/j.jde.2020.05.038
0022-0396/© 2020 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

2. Auxiliary results . . . 7885
2.1. Composition of FIOs and regularising effect . . . 7886
3. Solution representations . . . 7888
3.1. The 2×2 case . . . 7888
3.2. Inversion of the operator L_1 . . . 7889
3.3. Representation formulas . . . 7890
3.4. The $m \times m$ case . . . 7891
3.5. Regularity results . . . 7894
4. Propagation of singularities . . . 7895
5. Application: higher order hyperbolic equations . . . 7897
5.1. Well-posedness in Sobolev spaces . . . 7898
5.2. Representation formula for the solution . . . 7902
References . . . 7905

1. Introduction

In this work, we continue the study of non-diagonalisable systems that begun in [1] by proving a result on solution representations and propagation of singularities for a hyperbolic system with x-dependent principal part. Let us consider

$$\begin{cases} D_t u = A(x, D_x)u + B(t, x, D_x)u + f(t, x), & (t, x) \in [0, T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

with the usual notation $D_t = -\mathrm{i}\partial_t$ and $D_x = -\mathrm{i}\partial_x$. We assume that $A(x, D_x) = \big(a_{ij}(x, D_x)\big)_{i,j=1}^m$ is an $m \times m$ matrix of pseudo-differential operators of order 1, i.e., $a_{ij} \in \Psi^1_{1,0}(\mathbb{R}^n))$ and that $B(t, x, D_x) = \big(b_{ij}(t, x, D_x)\big)_{i,j=1}^m$ is an $m \times m$ matrix of pseudo-differential operators of order 0, i.e., $b_{ij} \in C([0, T], \Psi^0_{1,0}(\mathbb{R}^n))$. We also assume that the matrix A is upper triangular and hyperbolic, i.e.,

$$A(x, D_x) = \Lambda(x, D_x) + N(x, D_x) = \mathrm{diag}(\lambda_1(x, D_x), \lambda_2(x, D_x), \dots, \lambda_m(x, D_x)) + N(x, D_x)$$

with real eigenvalues $\lambda_1(x, \xi), \lambda_2(x, \xi), \dots, \lambda_m(x, \xi)$ and

$$N(x, D_x) = \begin{pmatrix} 0 & a_{12}(x, D_x) & a_{13}(x, D_x) & \cdots & a_{1m}(x, D_x) \\ 0 & 0 & a_{23}(x, D_x) & \cdots & a_{2m}(x, D_x) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & a_{m-1m}(x, D_x) \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We recall that the well-posedness of this kind of systems has been proven in anisotropic Sobolev spaces in [1] under specific assumptions on the lower order terms.

Propagation of singularities for systems with vanishing iterated Poisson brackets has been studied by several authors as Iwasaki and Morimoto [6] who studied 3×3 systems where the twice iterated Poisson bracket vanishes and Ichinose [5] studied 2×2 systems under the

same condition. In [8], Rozenblum considered smoothly diagonalisable systems with transversally intersecting characteristics, and derived a formula for the propagation of its singularities. Consequently, the transversality condition was removed in [7], replaced by a weaker condition of intersection of finite order at points of multiplicity, with propagation of singularities result as well. Here we extend the results of [7] to non-diagonalisable hyperbolic systems with variable multiplicity. The paper is organised as follows. Section 2 collects some important notions of Fourier integral operators and related integral operators relevant to our problem. The main well-posed result and corresponding representation formula is proven in Section 3. Section 4 is devoted to propagation of singularities. The paper ends in Section 5 with an application of our results to higher order hyperbolic equations with multiplicities.

1.1. Notations and preliminary notions

For the convenience of the reader we recall here some notations and preliminary notions that we will use throughout the paper.

Let $\mu \in \mathbb{R}$. We recall that $S^{\mu}_{1,0}(\mathbb{R}^n)$ is the space of symbols of order μ and type $(1,0)$, i.e., $a = a(x,\xi) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to $S^{\mu}_{1,0}(\mathbb{R}^n)$ if there exist constants $C_{\alpha,\beta} > 0$ such that

$$\forall \alpha, \beta \in \mathbb{N}_0^n \ : \ |\partial_x^{\alpha}\partial_{\xi}^{\beta} a(x,\xi)| \leq C_{\alpha,\beta}\, \langle \xi \rangle^{m-|\beta|} \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

with $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$.

The set of pseudo-differential operators associated to the symbols in $S^{\mu}_{1,0}(\mathbb{R}^n)$ is denoted by $\Psi^{\mu}_{1,0}(\mathbb{R}^n)$. If the symbol has an extra (continuous) dependence on $t \in [0,T]$ we will use the notations $C([0,T], S^{\mu})$ and $C([0,T], \Psi^{\mu}_{1,0})$ for symbols and operators, respectively. For the sake of simplicity we will adopt the abbreviated notations S^{μ} and Ψ^{μ} for $S^{\mu}_{1,0}(\mathbb{R}^n)$ and $\Psi^{\mu}_{1,0}(\mathbb{R}^n)$, respectively, and CS^{μ} and $C\Psi^{m}$ for $C([0,T], S^{\mu})$ and $C([0,T], \Psi^{\mu}_{1,0})$, respectively.

With I^{μ} we denote the class of Fourier Integral Operators with amplitude in S^{μ}, i.e., of operators of the type

$$I_{\varphi}(a)(f)(t,x) = \int_{\mathbb{R}^n} e^{\mathrm{i}\varphi(t,x,\xi)} a(t,x,\xi)\, \widehat{f}(\xi) d\xi,$$

where φ is a phase function and $\hat{f}$ is the Fourier transform of f. This notation is standard and further details can be found in [1] and the references therein, for instance [3]. In this paper we will use the short expression *integrated Fourier integral operator* to denote an operator of the type

$$\int_0^t \int_{\mathbb{R}^n} e^{\mathrm{i}\varphi(t,s,x,\xi)} a(t,s,x,\xi)\, \widehat{f}(\xi,s) d\xi\, ds,$$

where the Fourier transform of $f = f(y,s)$ is meant with respect to the variable y.

By $L^p_{\alpha}(\mathbb{R}^n)$, we denote the Sobolev space $(I-\Delta)^{-\frac{\alpha}{2}} L^p(\mathbb{R}^n)$. As usual, we set $H^s = L^2_s$. By $\|\cdot\|_{L^p_{loc}(\mathbb{R}^n)}$ we denote any localisation of the $L^p(\mathbb{R}^n)$-norm, i.e. the estimate $\|f\|_{L^p_{loc}(\mathbb{R}^n)} \leq C$ means that for all $\chi \in C_0^{\infty}(\mathbb{R}^n)$, we have the estimate $\|\chi f\|_{L^p(\mathbb{R}^n)} \leq C$, where the constant C

may depend on χ. Since we only work in $\mathbb{R}^n$ and no confusion can arise, we drop the indication of $\mathbb{R}^n$ from here on.

Finally, we recall that the Poisson bracket of two differentiable functions $f = f(x,\xi)$ and $g = g(x,\xi)$ is defined as

$$\{f\,,\,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i}.$$

1.2. Assumptions on the matrices $A(x,D_x)$ and $B(t,x,D_x)$

In this paper, we make the following assumptions on lower order terms and multiplicities:

(H1) (**Lower order terms**) The entries of the matrix $B(t,x,D_x) = [b_{ij}(t,x,D))]_{i,j=1}^{m}$ belong to $C([0,T],\Psi^0)$ and are of decreasing order below the diagonal, i.e.,

$$b_{ij} \in C([0,T],\Psi^{j-i}) \quad \text{for } i > j. \tag{1.2}$$

(H2) (**Multiplicities**) There exists $M \in \mathbb{N}$ such that if $\lambda_j(x,\xi) = \lambda_k(x,\xi)$ for some $j,k \in \{1,\dots,m\}$ and $\lambda_j(x,\xi)$ and $\lambda_k(x,\xi)$ are not identically equal near (x,ξ) then there exists some $N \le M$ such that

$$\lambda_j(x,\xi) = \lambda_k(x,\xi) \;\Rightarrow\; H^N_{\lambda_j}(\lambda_k) := \{\lambda_j,\{\lambda_j,\dots,\{\lambda_j,\lambda_k\}\}\dots\} \neq 0,$$

where the Poisson bracket $\{\cdot,\cdot\}$ in $H^N_{\lambda_j}$ is iterated N times.

Remark 1.1. In [1], for A depending on t as well and under the condition (H1) on B, we proved that for any $s \in \mathbb{R}$, $u_k^0 \in H^{s+k-1}$, $k = 1,\dots,m$, and $f_k \in C([0,T],H^{s+k-1})$, $k = 1,\dots,m$, the Cauchy problem (1.1) has a unique anisotropic Sobolev solution u with components $u_k \in C\left([0,T],H^{s+k-1}\right)$, $k = 1,\dots,m$. In this paper we do the microlocal analysis of solutions in the case of A depending only on x.

Remark 1.2. The condition (H2) was introduced in [7, p.3]. For $1 < p < \infty$ and $\alpha = (n-1)|1/p - 1/2|$, it was proved in [7] that when the matrix $A(x,D_x)$ is smoothly microlocally diagonalisable, with smooth eigenspaces and real eigenvalues λ_j, $j = 1,\dots,m$, fulfilling the condition (H2) then for every compactly supported initial data $u_0 \in L^p_\alpha \cap L^2_{comp}$ the Cauchy problem (1.1) has a unique solution u such that $u(t,\cdot) \in L^p_{loc}$ for every $t \in [0,T]$. Previously, a similar result had been proved in L^2 by Rozenblum in [8], in the special case of (H2) with $N = 1$.

We are now ready to state the main result of our paper. This is a representation formula for the solution u which shows how this depends on initial data and right-hand side. This dependence is given in terms of integral operators (of Fourier type) modulo regularising operators of order N, i.e. mapping H^s into H^{s+N}, for any $s \in \mathbb{R}$.

Theorem 1.3. *Let $n \ge 1$, $m \ge 2$, and let*

$$\begin{cases} D_t u = A(x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T]\times\mathbb{R}^n, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $A(x, D_x)$ is an upper-triangular matrix of pseudo-differential operators of order 1 and $B(t, x, D_x)$ is a matrix of pseudo-differential operators of order 0, continuous with respect to t. Let u_0 and f have components u_j^0 and f_j, respectively, with $u_j^0 \in H^{s+j-1}(\mathbb{R}^n)$ and $f_j \in C([0,T], H^{s+j-1})$ for $j = 1, \dots, m$. Then, under condition (H1) and (H2), we have the following:

(i) *the Cauchy problem above has a unique anisotropic Sobolev solution u, i.e., $u_j \in C([0,T], H^{s+j-1})$ for $j = 1, \dots, m$;*
(ii) *for any $N \in \mathbb{N}$, the components u_j, $j = 1, \dots, m$, of the solution u are given by*

$$u_j(t,x) = \sum_{l=1}^{m} \left(\mathcal{H}_{j,l}^{l-j}(t) + R_{j,l}(t)\right) u_l^0 + \left(\mathcal{K}_{j,l}^{l-j}(t) + S_{j,l}(t)\right) f_l, \tag{1.3}$$

where $R_{j,l}$, $S_{j,l} \in \mathcal{L}(H^s, C([0,T], H^{s+N-l+j}))$ and the operators $\mathcal{H}_{j,l}^{l-j}$, $\mathcal{K}_{j,l}^{l-j} \in \mathcal{L}(C([0,T], H^s), C([0,T], H^{s-l+j}))$ are integrated Fourier Integral Operators of order $l-j$.

For the convenience of the reader we recall here Theorem 1 in [1] which proves already assertion (i) in Theorem 1.3.

Theorem 1.4 ***(Theorem 1 in [1]).*** *Let $n \geq 1$, $m \geq 2$, and let*

$$\begin{cases} D_t u = A(t,x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $A(t,x,D_x) = (a_{ij}(t,x,D_x))_{i,j=1}^m$ is an upper-triangular matrix of pseudo-differential operators of order 1 and $B(t,x,D_x) = (b_{ij}(t,x,D_x))_{i,j=1}^m$ is a matrix of pseudo-differential operators of order 0, continuous with respect to t. Assume that (H1) holds. Then, given $u_j^0 \in H^{s+j-1}(\mathbb{R}^n)$ and $f_j \in C([0,T], H^{s+j-1})$ for $j = 1, \dots, m$, the Cauchy problem (1.4) has a unique anisotropic Sobolev solution u, i.e., $u_j \in C([0,T], H^{s+j-1})$ for $j = 1, \dots, m$.

2. Auxiliary results

This section contains some auxiliary results on Fourier integral operators and related integral operators that we will use throughout the paper. For the convenience of the reader, we begin by recalling some notations introduced in [1].

For each eigenvalue $\lambda_j(x,\xi)$ of $A(x,\xi)$, we will be denoting by $G_j^0\theta$ the solution to

$$\begin{cases} D_t w = \lambda_j(x,D_x)w + b_{jj}(t,x,D_x)w, \\ w(0,x) = \theta(x), \end{cases}$$

and by $G_j g$ the solution to

$$\begin{cases} D_t w = \lambda_j(x,D_x)w + b_{jj}(t,x,D_x)w + g(t,x), \\ w(0,x) = 0. \end{cases}$$

The b_{jj} are the diagonal elements of the lower order term $B(t,x,D_x)$ in (1.1). The operators G_j^0 and G_j can be locally represented by a Fourier integral operator and an integrated Fourier integral operator, respectively, i.e.,

$$G_j^0\theta(t,x)=\int_{\mathbb{R}^n} e^{\mathrm{i}\varphi_j(t,x,\xi)}c_j(t,x,\xi)\widehat{\theta}(\xi)d\xi \tag{2.1}$$

and

$$G_j g(t,x)=\int_0^t\int_{\mathbb{R}^n} e^{\mathrm{i}\varphi_j(t,s,x,\xi)}C_j(t,s,x,\xi)\widehat{g}(s,\xi)d\xi ds, \tag{2.2}$$

with $\varphi_j(t,s,x,\xi)$ solving the eikonal equation

$$\begin{cases}\partial_t\varphi_j=\lambda_j(x,\nabla_x\varphi_j(t,s,x,\xi)),\\ \varphi_j(s,s,x,\xi)=x\cdot\xi,\end{cases}$$

and $\varphi_j(t,x,\xi)=\varphi_j(t,0,x,\xi)$. Note that the amplitudes C_j in (2.2) have asymptotic expansions $\sum_{k=0}^{+\infty}C_{j,-k}$ where the element $C_{j,-k}(s,x,\xi)$ is of order $-k$, $k\in\mathbb{N}$, and satisfies transport equations with initial data at $t=s$. By construction, $c_j(t,x,\xi)=C_j(t,0,x,\xi)$. In the above construction of propagators for hyperbolic equations, we have $c_j\in S^0$, so that $G_j^0\in I^0$.

Further, to simplify the analysis of the regularising part in (1.3) we introduce the notation

$$\mathcal{E}_j(t,s)g(s,x)=\int_{\mathbb{R}^n} e^{\mathrm{i}\varphi_j(t,s,x,\xi)}C_j(s,x,\xi)\widehat{g}(s,\xi)d\xi, \tag{2.3}$$

i.e., the integrated Fourier integral operator G_j can now be written as

$$G_j g(t,x)=\int_0^t \mathcal{E}_j(t,s)g(s,x)ds. \tag{2.4}$$

2.1. Composition of FIOs and regularising effect

In this section, we state and prove auxiliary results that are crucial to the proof of the solution representation formula stated in Theorem 1.3. In particular, we investigate the mapping properties of compositions and powers of Fourier integral operators and integrated Fourier integral operators as in (2.1) and (2.2). This will be useful when analysing the regularising part of our representation formula (1.3).

2.1.1. Integrated Fourier integral operators

The composition of integrated Fourier integral operators like in (2.2) was studied in [7]. Their result, that we recall in the sequel, is crucial for our proof and is a generalisation of a previous result by Rozenblum in [8].

Let $t_1, t_2, \dots, t_l \in [0, T]$, $\bar{t} = (t_1, t_2, \dots, t_l)$ and let $H(\bar{t})$ be the operator

$$H(\bar{t}) = e^{i\lambda_{j_1} t_1} e^{i\lambda_{j_2}(t_2 - t_1)} \cdot \dots \cdot e^{i\lambda_{j_l}(t_l - t_{l-1})} e^{-i\lambda_{j_l} t_l}, \tag{2.5}$$

where λ_i are pseudo-differential operators of order 1.

By [3], $H(\bar{t})$ is a (parameter dependent) Fourier integral operator and its canonical relation $\Lambda^{\bar{t}} \subseteq T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is given by

$$\Lambda^{\bar{t}} = \left\{ (x, p, y, \xi) \,:\, (x, p) = \Psi^{\bar{t}}(y, \xi) \right\},$$

where

$$\Psi^{\bar{t}} = \Phi_{j_1}^{t_1} \circ \dots \circ \Phi_{j_l}^{t_l - t_{l-1}} \circ \Phi_{j_{l+1}}^{-t_l}$$

and the Φ_j^t are the transformations corresponding to a shift by t along the trajectories of the Hamiltonian flow defined by the λ_j.

Theorem 2.1 *(Thm 2.1 in [7]). With the above notation, assume that not all λ_is are identical to each other and let (H2) be satisfied for the λ_j in (2.5). Further, suppose that $D(\bar{t}) \in \Psi^0$. Then, the operator*

$$Q_l = \int_0^t \int_0^{t_1} \dots \int_0^{t_{l-1}} D(\bar{t}) H(\bar{t})\, dt_l \dots dt_1$$

belongs to $\mathcal{L}(H^s, H^{s+N(l)})$, where $N(l) \to +\infty$ as $l \to +\infty$.

Remark 2.2. If the global estimate in the definition of the symbols classes $S^m_{1,0}$ in [1] is replaced with an estimate that holds locally on every compact set, then the conclusions of Theorem 2.1 hold true if $\mathcal{L}(H^s, H^{s+N(l)})$ is replaced by $\mathcal{L}(H^s_{comp}, H^{s+N(l)}_{loc})$.

Theorem 2.1 allows us to investigate the composition $G_i G_j$.

2.1.2. The composition $G_i G_j$

Let G_i and G_j be two operators as in (2.2). Then, we can write

$$G_i G_j u = \int_0^t \int_0^{t_1} \mathcal{E}_i(t, t_1) \mathcal{E}_j(t_1, t_2) u\, dt_2 dt_1,$$

with $\mathcal{E}_i, \mathcal{E}_j$ as in (2.4). If we now iterate this k times, we obtain

$$(G_iG_j)^k = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{2k-1}} \mathcal{E}_i(t,t_1)\mathcal{E}_j(t_1,t_2)\cdot\ldots\cdot\mathcal{E}_i(t_{2k-3},t_{2k-2})\mathcal{E}_j(t_{2k-2},t_{2k-1})\,d\bar{t},$$

where $\bar{t} = dt_1 \ldots dt_{2k-1}$. This is an operator of the same type as Q_l above so we can apply Theorem 2.1 and obtain the following corollary.

Corollary 2.3. *Let conditions (H1) and (H2) be satisfied. Let $j \in \{1,\ldots,m\}$, $s \in \mathbb{R}$. Then, for all $N \in \mathbb{N}$, $k \in \{1,\ldots,m\}$, there exists $M \in \mathbb{N}$ such that*

$$\left(\prod_{i=1}^{k} G_{\sigma(i)}\right)^M \in \mathcal{L}\left(C([0,T],H^s), C([0,T],H^{s+N})\right),$$

where σ is an element of the symmetric group over $\{1,\ldots,m\}$.

Remark 2.4. The same conclusion as in Corollary 2.3 holds true if we have a product that contains a collection G_j^0's as long as there is at least one integrated version G_j present.

3. Solution representations

This section is devoted to the proof of Theorem 1.3. For the sake of simplicity and for the advantage of the reader we give first a detailed explanatory proof for 2×2 systems and we then pass to consider the $m\times m$ case. We adopt the notations introduced in Section 2.

3.1. The 2×2 case

Let us consider the system

$$\begin{cases} D_t u = A(x,D_x)u + B(t,x,D_x)u + f(t,x), & (t,x)\in[0,T]\times\mathbb{R}^n,\\ u|_{t=0} = u_0, & x\in\mathbb{R}^n, \end{cases} \tag{3.1}$$

where $u_0(x) = \left(u_1^0(x), u_2^0(x)\right)^T$, $f(t,x) = \left(f_1(t,x), f_2(t,x)\right)^T$, and with the operators $A(x,D_x)$ and $B(t,x,D_x)$ given by

$$A(x,D_x) = \begin{pmatrix} \lambda_1(x,D_x) & a_{12}(x,D_x) \\ 0 & \lambda_2(x,D_x) \end{pmatrix} \tag{3.2}$$

and

$$B(t,x,D_x) = \begin{pmatrix} b_{11}(t,x,D_x) & b_{12}(t,x,D_x) \\ b_{21}(t,x,D_x) & b_{22}(t,x,D_x) \end{pmatrix}.$$

We suppose that all entries of $A(x,D_x)$ belong to $\Psi^1_{1,0}$ and all entries of $B(t,x,D_x)$ belong to $C\Psi^0_{1,0}$.

As detailed in Subsection 2.2 in [1], we obtain the equations

$$\begin{aligned} u_1 &= G_1^0(u_1^0) + G_1(f_1) + G_1(a_{12}u_2) + G_1(b_{12}u_2) \\ &= G_1^0(u_1^0) + G_1(f_1) + G_1((a_{12}+b_{12})u_2), \\ u_2 &= G_2^0(u_2^0) + G_2(f_2) + G_2(b_{21}u_1), \end{aligned}$$

and with that

$$\begin{aligned} u_1 &= G_1^0(u_1^0) + G_1((a_{12}+b_{12})G_2^0(u_2^0)) + G_1(f_1) + G_1((a_{12}+b_{12})G_2(f_2)) \\ &\quad + G_1((a_{12}+b_{12})G_2(b_{21}u_1)), \\ u_2 &= G_2^0(u_2^0) + G_2(b_{21}G_1^0(u_0^1)) + G_2(b_{21}G_1(f_1)) + G_2(f_2) \\ &\quad + G_2(b_{21}G_1((a_{12}+b_{12})u_2)). \end{aligned} \tag{3.3}$$

We note that the operators $G_1 \circ (a_{12}+b_{12}) \circ G_2 \circ b_{21}$ and $G_2 \circ b_{21} \circ G_1 \circ (a_{12}+b_{12})$ are of order 0 under the assumption (H1) made on the lower order terms; here in particular $b_{21} \in \Psi^{-1}$. From (3.3), we have

$$\begin{aligned} &u_1 - G_1((a_{12}+b_{12})G_2(b_{21}u_1)) \\ &\qquad = G_1^0(u_1^0) + G_1((a_{12}+b_{12})G_2^0(u_2^0)) + G_1(f_1) + G_1((a_{12}+b_{12})G_2(f_2)) \end{aligned}$$

and

$$\begin{aligned} &u_2 - G_2(b_{21}G_1((a_{12}+b_{12})u_2)) \\ &\qquad = G_2^0(u_2^0) + G_2(b_{21}G_1^0(u_0^1)) + G_2(b_{21}G_1(f_1)) + G_2(f_2). \end{aligned} \tag{3.4}$$

3.2. Inversion of the operator L_1

Adopting the notations introduced in [1] we introduce the operator

$$L_1 := \mathbf{I} - G_1 \circ (a_{12}+b_{12}) \circ G_2 \circ b_{21} = \mathbf{I} - \mathcal{G}_1^0$$

Note that $\mathcal{G}_1^0 = G_1 \circ (a_{12}+b_{12}) \circ G_2 \circ b_{21}$ is of order 0 and from the Sobolev mapping properties of Fourier Integral Operators (see Lemma 1 in [1]) the norm of this operator can be estimated by a constant times the length of the time interval $[0,T]$. So it can be made as small as wanted by a suitable choice of T. It follows that L_1 is invertible for T small enough and its inverse can be written as sum of a Neumann series. More precisely, under the assumptions (H1) and (H2) from Corollary 2.3 we get that for every $N \in \mathbb{N}$ the operator L^{-1} can be written as a finite sum of powers of the operator $\mathcal{G}_1^0$ modulo some regularising operator mapping $C([0,T], H^s)$ into $C([0,T], H^{s+N}))$, i.e., for every $N \in \mathbb{N}$, there exists $M \in N$ such that

$$L_1^{-1} = \sum_{k=0}^{+\infty} \left(\mathcal{G}_1^0\right)^k$$

$$= \sum_{k=0}^{M} \left(\mathcal{G}_1^0\right)^k \mod \mathcal{L}(C([0,T], H^s), C([0,T], H^{s+N})).$$

It is important to remark here that the estimates needed to ensure the small norm of the operator $\mathcal{G}_1^0$, do not depend on the initial data and therefore one can repeat the same argument covering the original interval $[0, T]$.

3.3. Representation formulas

We now apply the operator L_1^{-1} as written above to both sides of the equality

$$L_1 u_1 = G_1^0(u_1^0) + G_1((a_{12} + b_{12})G_2^0(u_2^0)) + G_1(f_1) + G_1((a_{12} + b_{12})G_2(f_2)).$$

We obtain the following representation for u_1, where

$$R_1 = L_1^{-1} - \sum_{k=0}^{M} \left(\mathcal{G}_1^0\right)^k$$

is a regularising operator, i.e., $R_1 \in \mathcal{L}(C([0,T], H^s), C([0,T], H^{s+N}))$:

$$\begin{aligned} u_1 = &\underbrace{\sum_{k=0}^{M} \left(\mathcal{G}_1^0\right)^k G_1^0(u_1^0)}_{=\mathcal{H}_{1,1}^{1-1}u_1^0} + \underbrace{\sum_{k=0}^{M} \left(\mathcal{G}_1^0\right)^k G_1((a_{12}+b_{12})G_2^0(u_2^0))}_{=\mathcal{H}_{1,2}^{2-1}u_2^0} \\ &+ \underbrace{\sum_{k=0}^{M} \left(\mathcal{G}_1^0\right)^k G_1(f_1)}_{=\mathcal{K}_{1,1}^{1-1}f_1} + \underbrace{\sum_{k=0}^{M} \left(\mathcal{G}_1^0\right)^k G_1((a_{12}+b_{12})G_2(f_2))}_{=\mathcal{K}_{1,2}^{2-1}f_2} \\ &+ R_1 G_1^0(u_1^0) + R_1 G_1((a_{12}+b_{12})G_2^0(u_2^0)) \\ &+ R_1 G_1(f_1) + R_1 G_1((a_{12}+b_{12})G_2(f_2)). \end{aligned}$$

Denoting $R_1 G_1^0$ and $R_1 G_1((a_{12} + b_{12})G_2^0$ by $R_{1,1}$ and $R_{1,2}$ respectively, and $R_1 G_1$ and $R_1 G_1((a_{12} + b_{12})G_2$ by $S_{1,1}$ and $S_{1,2}$, respectively, we have that

$$u_1 = \sum_{l=1}^{2} \left(\mathcal{H}_{1,l}^{l-j}(t) + R_{1,l}(t)\right) u_l^0 + \left(\mathcal{K}_{1,l}^{l-j}(t) + S_{1,l}(t)\right) f_l,$$

where

- the operators $H_{1,l}^{l-1}$ and $K_{1,l}^{l-1}$ are of order $l-1$ and therefore map $C([0,T], H^s)$ into $C([0,T], H^{s-l+1})$,
- $R_{1,1}$ and $S_{1,1}$ map H^s into $C([0,T], H^{s+N})$,
- $R_{1,2}$ and $S_{1,2}$ map H^s into $C([0,T], H^{s+N-1})$.

This means that $\mathcal{H}^{1-l}_{1,l}$, $\mathcal{K}^{1-l}_{1,l} \in \mathcal{L}(C([0,T],H^s), C([0,T],H^{s-l+1}))$ and $R_{1,l}$, $S_{1,l} \in \mathcal{L}(H^s, C([0,T],H^{s+N-l+1}))$. The same argument is true for u_2. We have in this way obtained the representation formula stated in Theorem 1.3.

3.4. The $m \times m$ case

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Throughout this proof we refer to the proof of Theorem 1 in [1], i.e. Theorem 1.4 in this paper. Theorem 1.4 proves the well-posedness of this Cauchy problem in anisotropic Sobolev spaces. It remains to prove the representation formula for the components of the solution u. We begin by observing that under our hypotheses we can write

$$u_i = U_i^0 + \sum_{j<i} G_{i,j}^{j-i}(u_j) + \sum_{i<j\le m} G_{i,j}^{1}(u_j),$$

for $i = 1,\dots,m$, where $G_{i,j}^{j-i}$ and $G_{i,j}^{1}$ are operators of order $j-i$ and 1, respectively and

$$U_i^0 = G_i^0 u_j^0 + G_i(f_i).$$

We begin by substituting

$$u_m = U_m^0 + \sum_{j<m} G_{m,j}^{j-m}(u_j),$$

into

$$u_{m-1} = U_{m-1}^0 + \sum_{j<m-1} G_{m-1,j}^{j-m+1}(u_j) + G_{m-1,m}^{1}(u_m).$$

We get

$$\begin{aligned} u_{m-1} &= U_{m-1}^0 + \sum_{j<m-1} G_{m-1,j}^{j-m+1}(u_j) + G_{m-1,m}^{1} U_m^0 + \sum_{j<m} G_{m-1,m}^{1} G_{m,j}^{j-m}(u_j) \\ &= (U_{m-1}^0 + G_{m-1,m}^{1} U_m^0) + \sum_{j<m-1} (G_{m-1,j}^{j-m+1}(u_j) + G_{m-1,m}^{1} G_{m,j}^{j-m}(u_j)) \\ &\quad + G_{m-1,m}^{1} G_{m,m-1}^{-1} u_{m-1}. \end{aligned}$$

Since all the operators above are of order ≤ 0 we conclude that the operator

$$L_{m-1} = I - G_{m-1,m}^{1} G_{m,m-1}^{-1} := I - \mathcal{G}_{m-1}^{0}$$

is invertible on a sufficiently small interval $[0,T]$ and, therefore,

$$u_{m-1} - G^1_{m-1,m}G^{-1}_{m,m-1}u_{m-1} = (U^0_{m-1} + G^1_{m-1,m}U^0_m) + \sum_{j<m-1}(G^{j-m+1}_{m-1,j}(u_j) + G^1_{m-1,m}G^{j-m}_{m,j}(u_j)), \tag{3.5}$$

yields

$$u_{m-1} = L^{-1}_{m-1}\widetilde{U}^0_{m-1} + L^{-1}_{m-1}\sum_{j<m-1}\widetilde{G}^{j-m+1}_{m-1}u_j, \tag{3.6}$$

with $\widetilde{U}^0_{m-1}$ and $\widetilde{G}^{j-m+1}_{m-1}$ defined by the right-hand side of (3.5). In particular,

$$\widetilde{U}^0_{m-1} = U^0_{m-1} + D^1_{m-1,m}U^0_m, \tag{3.7}$$

where $G^1_{m-1,m} = D^1_{m-1,m}$ is an integrated Fourier integral operator with symbol of order 1. Note that we choose the notation $D^1_{m-1,m}$ in order to have simpler notations for the compositions of operators in the computations below. Substituting u_m and u_{m-1} into u_{m-2} and making use of (3.6) we find a similar formula to (3.6) for u_{m-2} (see (24) in [1]) with $\widetilde{U}^0_{m-2}$ defined as follows:

$$\widetilde{U}^0_{m-2} = U^0_{m-2} + G^1_{m-2,m-1}L^{-1}_{m-1}\widetilde{U}^0_{m-1} + G^1_{m-2,m}U^0_m + G^1_{m-2,m}G^{-1}_{m,m-1}L^{-1}_{m-1}\widetilde{U}^0_{m-1}. \tag{3.8}$$

Hence, by implementing (3.7) in (3.8) we have

$$\widetilde{U}^0_{m-2} = U^0_{m-2} + G^1_{m-2,m-1}L^{-1}_{m-1}(U^0_{m-1} + D^1_{m-1,m}U^0_m) + G^1_{m-2,m}U^0_m + G^1_{m-2,m}G^{-1}_{m,m-1}L^{-1}_{m-1}(U^0_{m-1} + D^1_{m-1,m}U^0_m).$$

By collecting the terms U^0_{m-1} and U^0_m we conclude that $\widetilde{U}^0_{m-2}$ can be written as

$$\widetilde{U}^0_{m-2} = U^0_{m-2} + D^1_{m-2,m-1}U^0_{m-1} + D^2_{m-2,m}U^0_m,$$

where the operators $D^1_{m-2,m-1}$ and $D^2_{m-2,m}$ are of order 1 and 2, respectively. By iterating the same argument we prove that for every $j = 1, \dots, m-1$,

$$\widetilde{U}^0_j = U^0_j + \sum_{k>j} D^{k-j}_{j,k}U^0_k,$$

where $k-j$ is the order of the operator $D^{k-j}_{j,k}$. For a precise construction of the operators $D^{k-j}_{j,k}$ we refer the reader to the proof of Theorem 1 in [1]. Since

$$u_1 = \widetilde{U}^0_1 + \mathcal{G}^0_1 u_1,$$

where the operator $I - \mathcal{G}^0_1$ is invertible with inverse L^{-1}_1 (see [1]) we conclude that

$$u_1 = L_1^{-1}\left(U_1^0 + \sum_{k>1} D_{1,k}^{k-1} U_k^0\right) = L_1^{-1}\left(G_1^0 u_1^0 + G_1(f_1) + \sum_{k>1} D_{1,k}^{k-1}(G_k^0 u_k^0 + G_k(f_k))\right).$$

We now argue as in the case 2×2 and we apply Corollary 2.3 which holds thanks to the hypotheses (H1) and (H2). We obtain that for every N there exists M such that $L_1^{-1} = \sum_{j=0}^{M} (\mathcal{G}_1^0)^j$ modulo some regularising operator R_1 mapping $C([0,T], H^s)$ into $C([0,T], H^{s+N})$ and therefore

$$u_1 = \sum_{j=0}^{M} (\mathcal{G}_1^0)^j \left(G_1^0 u_1^0 + G_1(f_1) + \sum_{k>1} D_{1,k}^{k-1}(G_k^0 u_k^0 + G_k(f_k))\right) + R_1\left(G_1^0 u_1^0 + G_1(f_1) + \sum_{k>1} D_{1,k}^{k-1}(G_k^0 u_k^0 + G_k(f_k))\right).$$

By collecting all the terms with u_k^0 and all the terms with f_k, $k = 1, \dots, m$ we see that

$$u_1 = \sum_{l=1}^{m} \left(\mathcal{H}_{1,l}^{l-1}(t) + R_{1,l}(t)\right) u_l^0 + \left(\mathcal{K}_{1,l}^{l-1}(t) + S_{1,l}(t)\right) f_l,$$

where $\mathcal{H}_{1,l}^{l-1}$ and $\mathcal{K}_{1,l}^{l-1}$ have order $l-1$ and $R_{1,l}$ and $S_{1,l}$ are regularising. This is due to the fact $(\mathcal{G}_1^0)^j$ is an operator of order 0 as well as G_k^0 and G_k, and $D_{1,k}^{k-1}$ is an operator of order $k-1$. The regularising operator R_1 generates $R_{1,l}$ and $S_{1,l}$. These last two operators map $C([0,T], H^s)$ into $C([0,T], H^{s+N-l+1})$. We have therefore proven the second assertion of this theorem for $j = 1$. Following the proof of Theorem 1 in [1] we now have that

$$u_2 = \widetilde{U}_2^0 + \mathcal{G}_2^0 u_2 + \widetilde{G}_2^{-1} u_1,$$

where the operator $\mathcal{G}_2^0$ is of order zero and its definition involves invertible operators L_{m-1}, $L_{m-2}, \dots, L_2$ and $\widetilde{G}_2^{-1}$ is of order -1. Hence, by inverting the operator $L_2 = I - \mathcal{G}_2^0$ on a sufficiently small interval $[0,T]$ we have

$$u_2 = L_2^{-1}\widetilde{U}_2^0 + L_2^{-1}\widetilde{G}_2^{-1} u_1.$$

By the definition of $\widetilde{U}_2^0$ and by the representation formula for u_1 obtained above we can write

$$u_2 = L_2^{-1}\left(G_2^0 u_2^0 + G_2(f_2) + \sum_{k>2} D_{2,k}^{k-2}(G_k^0 u_k^0 + G_k(f_k))\right) + L_2^{-1}\widetilde{G}_2^{-1}\left(\sum_{l=1}^{m} \left(\mathcal{H}_{1,l}^{l-1}(t) + R_{1,l}(t)\right) u_l^0 + \left(\mathcal{K}_{1,l}^{l-1}(t) + S_{1,l}(t)\right) f_l\right).$$

Note that the operators above are of order $l-2$. By arguing as for u_1 and by writing L_2^{-1} as a finite number of powers of $\widetilde{G}_2^{-1}$ plus a regularising operator we arrive at the formula

$$u_2 = \sum_{l=1}^{m} \left(\mathcal{H}_{2,l}^{l-2}(t) + R_{2,l}(t)\right) u_l^0 + \left(\mathcal{K}_{2,l}^{l-2}(t) + S_{2,l}(t)\right) f_l,$$

with the desired order and regularising properties. We conclude the proof by iterating the same scheme. From formula (28) in [1] we obtain for $j > 2$ the following expression for u_j:

$$\begin{aligned} u_j &= L_j^{-1}\left(U_j^0 + \sum_{k>j} D_{j,k}^{k-j} U_k^0\right) + L_j^{-1}\left(\sum_{k<j} \widetilde{G}_j^{k-j} u_k\right) \\ &= L_j^{-1}\left(G_j^0 u_j^0 + G_j(f_j) + \sum_{k>j} D_{j,k}^{k-j}(G_k^0 u_k^0 + G_k(f_k))\right) \\ &\quad + L_j^{-1}\left(\sum_{k<j} \widetilde{G}_j^{k-j}\left(\sum_{l=1}^{m}\left(\mathcal{H}_{k,l}^{l-k}(t) + R_{k,l}(t)\right) u_l^0 + \left(\mathcal{K}_{k,l}^{l-k}(t) + S_{k,l}(t)\right) f_l\right)\right), \end{aligned}$$

where the operator involved are of order $l - j$. Writing L_j^{-1} by Neumann series we can conclude that

$$u_j = \sum_{l=1}^{m} \left(\mathcal{H}_{j,l}^{l-j}(t) + R_{j,l}(t)\right) u_l^0 + \left(\mathcal{K}_{j,l}^{l-j}(t) + S_{j,l}(t)\right) f_l,$$

where $\mathcal{H}_{j,l}^{l-j}$ and $\mathcal{K}_{j,l}^{l-j}$ have order $l - j$ and $R_{j,l}$ and $S_{j,l}$ map $C([0,T], H^s)$ into $C([0,T], H^{s+N-l+j})$. □

Remark 3.1. Note that in [1], we defined S^m by global estimates on $\mathbb{R}^n \times \mathbb{R}^n$. If one replaces that definition with a locally over every compact sets version then Theorem 1.3 still holds true with the spaces $\mathcal{L}(H^s, C([0,T], H^{s+N}))$ and $\mathcal{L}(C([0,T], H^s), C([0,T], H^{s+N-l+j}))$ replaced by the spaces $\mathcal{L}(H^s_{comp}, C([0,T], H^{s+N}_{loc}))$ and $\mathcal{L}(C([0,T], H^s_{comp}, C([0,T], H^{s+N-l+j}_{loc}))$, respectively.

3.5. Regularity results

We conclude this section with some regularity results in L^p and Hölder spaces. These are obtained by arguing as in [7] Theorem 2.2 and Theorem 3.1.

Theorem 3.2. *Let $1 < p < \infty$ and $\alpha = (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. Let $A(x, D_x)$ be an $m \times m$ upper-triangular matrix of pseudo-differential operators of order 1 and suppose that the eigenvalues $\lambda_i(x,\xi) \in S^1$ of $A(x,\xi)$ are real and satisfy (H2). Assume further, that $B(t, x, D_x)$ is an $m \times m$ matrix of pseudo-differential operators of order 0 satisfying (H1). Then, for any compactly supported $u_0 \in L^p_\alpha \cap L^2_{comp}$, the solution $u = u(t,x)$ of the Cauchy problem (1.1) satisfies $u(t,\cdot) \in L^p_{loc}$, for all $t \in [0,T]$. Moreover, there is a positive constant C_T such that*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^p_{loc}} \leq C_T \|u_0\|_{L^p_\alpha}.$$

Local estimates can be obtained in other spaces as well, for $s \in \mathbb{R}$ and α as above. In detail, assuming u_0 below is compactly supported, we have

- $u_0 \in L^p_{s+\alpha}$ implies $u(t,\cdot) \in L^p_s$;
- $u_0 \in C^{s+\frac{n-1}{2}}$ implies $u(t,\cdot) \in C^s$;
- for $1 < p \le q \le 2$, $u_0 \in L^p_{s-\frac{1}{q}+\frac{n}{p}-\frac{n-1}{2}}$ implies $u(t,\cdot) \in L^q_s$.

4. Propagation of singularities

We now want to analyse the solution u under a microlocal point of view. In particular we want to see how its wavefront set is related to the wavefront set of the initial data. Thanks to the assumptions (H1) and (H2) and the representation formula in Theorem 1.3 we are able to extend the result of propagation of singularities in [7] to systems with non-diagonalisable principal part (in upper-triangular form). For the sake of the reader we recall below some basic notions of microlocal analysis which can be found in [4] and [3].

Definition 4.1 *(Def. 8.1.2 in [4], Def. 2.5.2 in [3]).* Let $v \in \mathcal{D}'(\mathbb{R}^n)$. The wave front set $\mathrm{WF}(v) \subseteq T^*(\mathbb{R}^n) \setminus \{0\} := \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ is defined via its complement as follows: (x_0, ξ_0) belongs to $(\mathrm{WF}(u))^c$ if and only if there exists a $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and a conic neighbourhood Γ of ξ_0 such that for every $N \in \mathbb{N}$ there exists a positive constant $C_{N,\chi}$ such that

$$|\mathcal{F}(\chi u)(\xi)| \le C_{n,\chi} \langle \xi \rangle^{-N},$$

for all $\xi \in \Gamma$.

Let us now discuss the propagation of singularities for operators Q_l from Theorem 2.1, given by

$$Q_l = \int_0^t \int_0^{t_1} \dots \int_0^{t_{l-1}} D(\bar{t}) H(\bar{t})\, dt_l \dots dt_1. \tag{4.1}$$

Similar analysis was done in [7]. It is clear that singularities propagate along broken Hamiltonian flows. Let

$$J = \{j_1, \dots, j_{l+1}\},\ 1 \le j_k \le m,\ j_k \neq j_{k+1}.$$

We recall from the definition of $H(\bar{t})$ that its canonical relation $\Lambda^{\bar{t}} \subseteq T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is given by

$$\Lambda^{\bar{t}} = \left\{ (x, p, y, \xi) :\ (x, p) = \Psi^{\bar{t}}(y, \xi) \right\},$$

where

$$\Psi^{\bar{t}} = \Phi^{t_1}_{j_1} \circ \dots \circ \Phi^{t_l - t_{l-1}}_{j_l} \circ \Phi^{-t_l}_{j_{l+1}}$$

and the Φ_j^t are the transformations corresponding to a shift by t along the trajectories of the Hamiltonian flow defined by the λ_j. Also, recall that $t_1, t_2, \dots, t_l \in [0, T]$, $\bar{t} = (t_1, t_2, \dots, t_l)$, and $H(\bar{t})$ is the operator

$$H(\bar{t}) = e^{i\lambda_{j_1} t_1} e^{i\lambda_{j_2}(t_2 - t_1)} \cdot \dots \cdot e^{i\lambda_{j_l}(t_l - t_{l-1})} e^{-i\lambda_{j_l} t_l}, \tag{4.2}$$

where λ_i are pseudo-differential operators of order 1.

Let $\Phi_J(t, x, \xi)$ be the corresponding broken Hamiltonian flow. It means that points follow bicharacteristics of λ_{j_1} until meeting the characteristic of λ_{j_2}, and then continue along the bicharacteristic of λ_{j_2}, etc. In this procedure the singularities may accumulate if wave front sets for different broken trajectories project to the same point of X. We can rewrite (4.1) as

$$Q_l = \int_\Delta I(\bar{t}) d\bar{t},$$

where $\bar{t} = (t_1, \dots, t_l)$ ranges over the simplex

$$\Delta = \{0 \le t_l \le t_{l-1} \le \dots \le t_1 \le t\}$$

in $\mathbb{R}^l$ and

$$I(\bar{t}) = Z(t_1) \circ \dots \circ Z(t_l),$$

with $Z(t_j)$ found from (4.2). It is then possible to treat it as a standard Fourier integral operator with the change of variables $\bar{t} = \zeta |\xi|^{-1}$. Let K be a cone in $\mathbb{R}^N = \mathbb{R}^{n+l}$ and let

$$Iu(x) = \int_K \int_Y e^{i\varphi(x,y,\theta)} a(x, y, \theta) u(y) dy d\theta$$

be a Fourier integral operator with integration over the cone K with respect to θ. Let K_j be K or a face of K. Let $\varphi_j(x, y, \theta_j) = \varphi|_{K_j}, \theta_j \in K_j$. Let $\Lambda_j \subset T^*X \times T^*X$ be a Lagrangian manifold with boundary:

$$\Lambda_j = \left\{ \left(x, \frac{\partial \varphi_j}{\partial x}, y, -\frac{\partial \varphi_j}{\partial y} \right) : \frac{\partial \varphi_j}{\partial \theta_j} = 0 \right\}.$$

For $G \subset T^*Y$, let

$$\Lambda_j(G) = \{z \in T^*X : \exists \zeta \in G : (z, \zeta) \in \Lambda_j\}.$$

Then we have the following statement on the propagation of singularities, see [7].

Theorem 4.2. *Let $u \in \mathcal{D}'(Y)$. Then $WF(Iu) \subset \bigcup_j \Lambda_j(WF(u))$.*

Consequently, combining these observations with Theorem 1.3 we obtain the following property.

Corollary 4.3. *Let $n \geq 1$, $m \geq 2$, and let*

$$\begin{cases} D_t u = A(x, D_x)u + B(t, x, D_x)u + f(t, x), & (t, x) \in [0, T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $A(x, D_x)$ is an upper-triangular matrix of pseudo-differential operators of order 1 and $B(t, x, D_x)$ is a matrix of pseudo-differential operators of order 0, continuous with respect to t. Recall that under condition (H1) and (H2), for any $N \in \mathbb{N}$, the components u_j, $j = 1, \dots, m$, of the solution u are given by

$$u_j(t, x) = \sum_{l=1}^{m} \left(\mathcal{H}_{j,l}^{l-j}(t) + R_{j,l}(t) \right) u_l^0 + \left(\mathcal{K}_{j,l}^{l-j}(t) + S_{j,l}(t) \right) f_l, \tag{4.3}$$

where $R_{j,l}$, $S_{j,l} \in \mathcal{L}(H^s, C([0, T], H^{s+N-l+j}))$ and the operators

$$\mathcal{H}_{j,l}^{l-j}, \mathcal{K}_{j,l}^{l-j} \in \mathcal{L}(C([0, T], H^s), C([0, T], H^{s-l+j}))$$

are integrated Fourier Integral Operators of order $l - j$.

Consequently, up to any Sobolev order (depending on N), the wave front set of u_j is given by

$$WF(u_j(t, \cdot)) \subset \left(\bigcup_{l=1}^{m} WF(\mathcal{H}_{j,l}^{l-j}(t) u_l^0) \right) \bigcup \left(\bigcup_{l=1}^{m} WF(\mathcal{K}_{j,l}^{l-j}(t) f_l) \right), \tag{4.4}$$

with each of the wave front sets for terms in the right hand side of (4.4) given by the propagation along the broken Hamiltonian flow as in Theorem 4.2.

We conclude the paper by presenting some applications of Theorem 1.3 and Theorem 1.4.

5. Application: higher order hyperbolic equations

In this section we want to study the well-posedness of the Cauchy problem

$$\begin{cases} D_t^m u = \sum_{j=0}^{m-1} A_{m-j}(t, x, D_x) D_t^j u + f(t, x), & (t, x) \in [0, T] \times \mathbb{R}^n, \\ D_t^{k-1} u(0, x) = g_k(x), & k = 1, ..., m, \end{cases} \tag{5.1}$$

where each $A_{m-j}(t, x, D_x)$ is a scalar differential operator of order $m - j$ with continuous and bounded coefficients depending on t and x. As usual, $D_t = \frac{1}{\mathrm{i}}\partial_t$ and $D_x = \frac{1}{\mathrm{i}}\partial_x$. Let $A_{(m-j)}$ denote the principal part of the operator A_{m-j}. We assume that the problem above is hyperbolic, i.e., the characteristic equation

$$\tau^m = \sum_{j=0}^{m-1} A_{(m-j)}(t, x, \xi)\tau^j \equiv \sum_{j=0}^{m-1} \sum_{|\gamma|=m-j} a_{m-j,\gamma}(t, x)\xi^\gamma \tau^j$$

has m real valued roots $\lambda_1, \lambda_2, \cdots, \lambda_m$. In addition we work under the hypothesis that the roots λ_i, $i = 1, \dots, m$ are symbols of order 1, i.e.,

$$\lambda_i \in C([0,T], S^1),$$

for all $i = 1, \dots, m$. For this reason we assume that

(**H0**) the coefficients of the equation above are continuous in t and smooth in x, with bounded derivatives of any order $\alpha \in \mathbb{N}_0^n$ with respect to x.

We will make use first of Theorem 1.4 and then of Theorem 1.3.

5.1. Well-posedness in Sobolev spaces

We begin by reducing the m-order partial differential equation in (5.1) into a first order system of pseudo-differential equations. Let $\langle D_x \rangle$ be the pseudo-differential operator with symbol $\langle \xi \rangle$. The transformation

$$u_k = D_t^{k-1} \langle D_x \rangle^{m-k} u,$$

with $k = 1, ..., m$, makes the Cauchy problem (5.1) equivalent to the following system

$$D_t \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_m \end{pmatrix} = \begin{pmatrix} 0 & \langle D_x \rangle & 0 & \dots & 0 \\ 0 & 0 & \langle D_x \rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle D_x \rangle \\ b_1 & b_2 & \dots & \dots & b_m \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdot \\ f \end{pmatrix}, \tag{5.2}$$

where

$$b_j = A_{m-j+1}(t, x, D_x) \langle D_x \rangle^{j-m},$$

with initial condition

$$u_k|_{t=0} = \langle D_x \rangle^{m-k} g_k, \qquad k = 1, ..., m. \tag{5.3}$$

The matrix in (5.2) can be written as $A + B$ with

$$A = \begin{pmatrix} 0 & \langle D_x \rangle & 0 & \dots & 0 \\ 0 & 0 & \langle D_x \rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle D_x \rangle \\ b_{(1)} & b_{(2)} & \dots & \dots & b_{(m)} \end{pmatrix},$$

where $b_{(j)} = A_{(m-j+1)}(t, x, D_x) \langle D_x \rangle^{j-m}$ and

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ b_1 - b_{(1)} & b_2 - b_{(2)} & \dots & \dots & b_m - b_{(m)} \end{pmatrix}.$$

It is clear that the eigenvalues of the symbol matrix $A(t,x,\xi)$ are the roots $\lambda_j(t,\xi)$, $j=1,...,m$.

We want to apply Theorem 1.4 to our Cauchy problem. This means to find under which hypotheses the equation in (5.1) can be reduced into a first order system with upper-triangular principal part and lower order terms of suitable order as in (H1).

Theorem 5.1. *Let*

$$\begin{cases} D_t^m u = \sum_{j=0}^{m-1} A_{m-j}(t,x,D_x) D_t^j u + f(t,x), & (t,x)\in[0,T]\times\mathbb{R}^n, \\ D_t^{k-1}u(0,x) = g_k(x), & k=1,...,m, \end{cases}$$

where each $A_{m-j}(t,x,D_x)$ is a differential operator of order $m-j$ with continuous and bounded coefficients depending on t and x as in (H0). Let $A_{(m-j)}$ denote the principal part of the operator A_{m-j}. Assume that the roots of the corresponding characteristic polynomial are real valued symbols

$$\lambda_i \in C([0,T],S^1), \quad i=1,\dots,m,$$

and that

$$A_{m-j+1}(t,x,\xi)\in C([0,T],S^0)$$

for all $j=1,\dots,m-1$. If $f\in C([0,T],H^{s+m-1})$ and $g_k\in H^{s+m-1}(\mathbb{R}^n)$ for all $k=1,\dots,m$ then the Cauchy problem (5.1) has a unique solution $u\in C^{m-1}([0,T],H^{s+m-1})$.

Proof. We consider the associated reduced system with principal part given by the matrix

$$A=\begin{pmatrix} 0 & \langle D_x\rangle & 0 & \dots & 0 \\ 0 & 0 & \langle D_x\rangle & \dots & 0 \\ \dots & \dots & \dots & \dots & \langle D_x\rangle \\ b_{(1)} & b_{(2)} & \dots & \dots & b_{(m)} \end{pmatrix},$$

where $b_{(j)}=A_{(m-j+1)}(t,x,D_x)\langle D_x\rangle^{j-m}$. The operators $b_{(j)}$ are of order 1 so, since we assume that $A_{m-j+1}(t,x,\xi)\in C([0,T],S^0)$ for $j=1,\dots,m-1$, it follows that $b_{(j)}\equiv 0$ for $j=1,\dots,m-1$. This means that the Sylvester matrix A is actually upper-triangular and that the matrix B of the lower order terms is of the following type:

$$B=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ b_1 & b_2 & \dots & \dots & b_m-b_{(m)} \end{pmatrix},$$

with $b_j=A_{m-j+1}(t,x,D_x)\langle D_x\rangle^{j-m}$, for $j=1,\dots,m-1$ and $b_m-b_{(m)}=A_1(t,x,D_x)-A_{(1)}(t,x,D_x)$. Since $A_{m-j+1}(t,x,\xi)\in C([0,T],S^0)$ for $j=1,\dots,m-1$ we have that b_j is a pseudo-differential operator of order $j-m$ for $j=1,\dots,m$. We are therefore under the assumptions of Theorem 1.4 for the matrices A and B. Since $g_k\in H^{s+m-1}(\mathbb{R}^n)$ the initial data

$$\langle D_x\rangle^{m-k} g_k$$

of the reduced Cauchy problem belong to the space $H^{s+k-1}(\mathbb{R}^n)$ for all $k = 1, \dots, m$. Thus, by Theorem 1.4 there exists a unique solution $u(t,x)$ to the Cauchy problem under consideration such that

$$D_t^{k-1}\langle D_x\rangle^{m-k} u \in C([0,T], H^{s+k-1}),$$

for $k = 1, \dots, m$. By Sobolev mapping properties of pseudo-differential operators it follows that $u \in C^{m-1}([0,T], H^{s+m-1})$. □

5.1.1. Second order examples

(i) The equation

$$D_t^2 u = a_2(t,x)u + (a_1(t,x)D_x + b(t,x))D_t u + f(t,x),$$

where $x \in \mathbb{R}$ and a_1 is real valued falls into the class of equations considered in the previous theorem. Indeed, the characteristic polynomial

$$\tau^2 - a_1(t,x)\tau\xi$$

has two real roots and $A_2 = a_2(t,x)$ is an operator of order 0.

(ii) Let us now consider the second order Cauchy problem

$$\begin{cases} D_t^2 u = a^2(t)D_x^2 u + b_1(t)D_x u + b_2(t)D_t u + b_3(t)u + f(t,x), \\ u(0,x) = g_0(x), \\ D_t u(0,x) = g_1(x), \end{cases}$$

where $(t,x) \in [0,T] \times \mathbb{R}$, the equation coefficients are continuous and $a \in C^1$ with $a(t) \geq 0$. Making use of the standard reduction into first order system of pseudo-differential equations we have that the Cauchy problem above is equivalent to

$$\begin{aligned} D_t U = &\begin{pmatrix} 0 & \langle D_x\rangle \\ a^2(t)D_x^2\langle D_x\rangle^{-1} & 0 \end{pmatrix} U \\ &+ \begin{pmatrix} 0 & 0 \\ b_1(t)D_x\langle D_x\rangle^{-1} + b_3(t)\langle D_x\rangle^{-1} & b_2(t) \end{pmatrix} U + \begin{pmatrix} 0 \\ f(t,x) \end{pmatrix}, \end{aligned} \tag{5.4}$$

where $U = (u_1, u_2)^T = (\langle D_x\rangle u, D_t u)^T$ and $U(0,x) = U_0 = (\langle D_x\rangle g_0, g_1)^T$. In the sequel we will denote the right-hand side of the system above with

$$A(t, D_x)U + B(t, D_x) + F,$$

where A and B are defined by operators of order 1 and 0, respectively. The principal part matrix

$$A(x,\xi) = \begin{pmatrix} 0 & \langle\xi\rangle \\ a^2(t)\xi^2\langle\xi\rangle^{-1} & 0 \end{pmatrix}$$

is not upper triangular and it is not diagonalisable because of the zeros of the coefficient a. However it can be reduced into upper-triangular form. We refer here the reader to [2] and to the appendix in [1]. The matrix A has $\lambda_1(t,\xi) = -a(t)\xi$ and $\lambda_2(t,\xi) = a(t)\xi$ as eigenvalues. It follows that

$$h^{(1)} = \begin{pmatrix} 1 \\ a(t)\xi\langle\xi\rangle^{-1} \end{pmatrix}$$

and

$$h^{(2)} = \begin{pmatrix} 1 \\ -a(t)\xi\langle\xi\rangle^{-1} \end{pmatrix}$$

are eigenvectors corresponding to λ_1 and λ_2, respectively. Choose now $h^{(1)}$ (the argument is analogous with $h^{(2)}$). The matrix

$$T_1 = (h^{(1)}, e_2) = \begin{pmatrix} 1 & 0 \\ a(t)\xi\langle\xi\rangle^{-1} & 1 \end{pmatrix}$$

is invertible. Its inverse is

$$T_1^{-1} = \begin{pmatrix} 1 & 0 \\ -a(t)\xi\langle\xi\rangle^{-1} & 1 \end{pmatrix}$$

and

$$T_1^{-1}AT_1 = \begin{pmatrix} -a(t)\xi & \langle\xi\rangle \\ 0 & a(t)\xi \end{pmatrix} = \begin{pmatrix} \lambda_1(x,\xi) & \langle\xi\rangle \\ 0 & \lambda_2(x,\xi) \end{pmatrix}.$$

Note that the operator $T_1^{-1}(t,D_x)A(t,D_x)T_1(t,D_x)$ can be therefore written as

$$\begin{pmatrix} \lambda_1(t,D_x) & \langle D_x\rangle \\ 0 & \lambda_2(t,D_x) \end{pmatrix}.$$

We can now use this transformation to reduce the system $D_tU = AU$ into upper-triangular form. More precisely, for $U = T_1V$, we have that the system

$$D_tU = A(t,D_x)U + B(t,D_x)U + F$$

is equivalent to

$$\begin{aligned} D_tV &= (T_1^{-1}AT_1)(t,D_x)V + T_1^{-1}(BT_1 + D_tT_1)V + T_1^{-1}F, \\ &= \begin{pmatrix} \lambda_1(t,D_x) & \langle D_x\rangle \\ 0 & \lambda_2(t,D_x) \end{pmatrix} V + \begin{pmatrix} 0 & 0 \\ b(t,\xi) & b_2(t) \end{pmatrix} V + F, \end{aligned}$$

where

$$b(t,\xi) = (b_1(t) + b_2(t)a(t) + D_ta(t))\xi\langle\xi\rangle^{-1} + b_3(t)\langle\xi\rangle^{-1}.$$

Under the assumption that $a \in C^1([0,T])$ we easily see that the eigenvalues λ_1 and λ_2 belong to $C([0,T], S^1)$. In addition, condition (H1) is fulfilled if the symbol above is of order -1. This is the case when

$$b_1(t) + b_2(t)a(t) + D_t a(t) = 0,$$

for all $t \in [0,T]$ (for instance when $b_2 \equiv 0$ and $b_1 = -D_t a$. Since T^{-1} is a matrix of pseudo-differential operators of order 0 we have that $V(0,x) = V_0 = T_1^{-1}(0, D_x)U_0$ has the same regularity properties of U_0. We can therefore apply Theorem 1.3 to this system and obtain the following result.

Theorem 5.2. *Let*

$$\begin{cases} D_t^2 u = a^2(t) D_x^2 u + b_1(t) D_x u + b_2(t) D_t u + b_3(t) u + f(t,x), \\ u(0,x) = g_0(x), \\ D_t u(0,x) = g_1(x), \end{cases}$$

where $a(t) \geq 0$ is of class C^1, the lower order coefficients $b_i(t)$, $i = 1,2,3$, are continuous and

$$b_1(t) + b_2(t)a(t) + D_t a(t) = 0,$$

for all $t \in [0,T]$. Let $s \in \mathbb{R}$. If $f \in C([0,T], H^{s+1})$ and $g_0, g_1 \in H^{s+1}(\mathbb{R}^n)$ then there exists a unique solution $u \in C^1([0,T], H^{s+1})$ of the Cauchy problem above.

In particular it follows that if $f \in C^\infty([0,T] \times \mathbb{R})$ and compactly supported with respect to x and $g_0, g_1 \in C_c^\infty(\mathbb{R})$ then this Cauchy problem is C^∞ well-posed.

5.2. Representation formula for the solution

We now assume that the principal part of the equation

$$D_t^m u = \sum_{j=0}^{m-1} A_{m-j}(t,x,D_x) D_t^j u + f(t,x)$$

depends only on x, i.e.,

$$D_t^m u = \sum_{j=0}^{m-1} A_{(m-j)}(x,D_x) D_t^j u + \sum_{j=0}^{m-1} (A_{m-j} - A_{(m-j)})(t,x,D_x) D_t^j u + f(t,x).$$

If in addition to the hypotheses of Theorem 5.1 we assume that the roots of this equation fulfil condition (H2) we obtain the following representation formula by straightforward application of Theorem 1.3.

Theorem 5.3. *Let*

$$\begin{cases} D_t^m u = \sum_{j=0}^{m-1} A_{m-j}(t,x,D_x) D_t^j u + f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ D_t^{k-1} u(0,x) = g_k(x), & k = 1, ..., m, \end{cases}$$

where each $A_{m-j}(t,x,D_x)$ is a differential operator of order $m-j$ with continuous and bounded coefficients depending on t and x as in (H0). Let the principal part $A_{(m-j)}$ of the operator A_{m-j} be independent of t and let the roots λ_i, $i = 1, \dots, m$, of the corresponding characteristic polynomial be real valued symbols of order 1 fulfilling condition (H2). Assume that

$$A_{m-j+1}(t,x,\xi) \in C([0,T], S^0)$$

for all $j = 1, \dots, m-1$. If $f \in C([0,T], H^{s+m-1})$ and $g_k \in H^{s+m-1}(\mathbb{R}^n)$ for all $k = 1, \dots, m$, then for any $N \in \mathbb{N}$ and $j = 1, \dots, m$ the solution $u \in C^{m-1}([0,T], H^{s+m-1})$ to the Cauchy problem can be written as

$$D_t^{j-1} u(t,x) = \sum_{l=1}^{m} \left(\mathcal{H}_{j,l}^{l-m}(t) + R_{j,l}(t) \right) \langle D_x \rangle^{m-l} g_l + \left(\mathcal{K}_{j,m}^{0}(t) + S_{j,m}(t) \right) f, \tag{5.5}$$

where

(i) $\mathcal{H}_{j,l}^{l-m}$ *is an integrated Fourier Integral Operator of order $l-m$,*
(ii) $\mathcal{K}_{j,m}^{0}$ *is an integrated Fourier Integral Operator of order 0,*
(iii) $R_{j,l} \in \mathcal{L}(H^s, C([0,T], H^{s+N-l+m}))$,
(iv) $S_{j,m} \in \mathcal{L}(H^s, C([0,T], H^{s+N}))$.

Proof. Since we work under the assumptions (H1) and (H2) we can apply Theorem 1.3 to the system in (5.2) where the matrix A of the principal part is only depending on x and is upper-triangular. Note that the right-hand side is of the type

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}$$

and the initial condition is given by

$$u_k|_{t=0} = \langle D_x \rangle^{m-k} g_k, \qquad k = 1, ..., m.$$

A straightforward application of Theorem 1.3 allows us to write

$$D_j^{j-1} \langle D_x \rangle^{m-j} u = \sum_{l=1}^{m} \left(\mathcal{H}_{j,l}^{l-j}(t) + R_{j,l}(t) \right) \langle D_x \rangle^{m-l} g_l + \left(\mathcal{K}_{j,m}^{m-j}(t) + S_{j,m}(t) \right) f,$$

where $R_{j,l} \in \mathcal{L}(H^s, C([0,T], H^{s+N-l+j}))$, $S_{j,m} \in \mathcal{L}(H^s, C([0,T], H^{s+N-m+j}))$ and the operators $\mathcal{H}_{j,l}^{l-j}$ and $\mathcal{K}_{j,m}^{m-j}$ are of order $l-j$ and $m-j$, respectively. It follows that

$$D_j^{j-1}u = \sum_{l=1}^{m} \langle D_x\rangle^{-m+j}\left(\mathcal{H}_{j,l}^{l-j}(t) + R_{j,l}(t)\right)\langle D_x\rangle^{m-l} g_l + \langle D_x\rangle^{-m+j}\left(\mathcal{K}_{j,m}^{m-j}(t) + S_{j,m}(t)\right) f,$$

for all $j = 1, \dots, m$. By composition of pseudo- and Fourier integral operators we easily see that

- $\langle D_x\rangle^{-m+j}\mathcal{H}_{j,l}^{l-j}(t)$ is of order $l-m$,
- $\langle D_x\rangle^{-m+j} R_{j,l} \in \mathcal{L}(H^s, C([0,T], H^{s+N-l+m}))$,
- $\langle D_x\rangle^{-m+j}\mathcal{K}_{j,m}^{m-j}$ is of order 0,
- $\langle D_x\rangle^{-m+j} S_{j,m} \in \mathcal{L}(H^s, C([0,T], H^{s+N}))$.

This shows that u can be written as in (5.5) and completes the proof. □

5.2.1. Example

Let us consider an m order homogeneous differential operator

$$A(x, D_t, D_x), \qquad t \in [0,T],\ x \in \mathbb{R},$$

such that its symbol (which coincides with its principal symbol) is

$$A(x,\tau,\xi) = \Pi_{i=1}^{m}(\tau - a_i(x)\xi).$$

Assume that all the coefficients a_i are real valued, smooth and bounded. Assume that the derivatives of the coefficients a_i, $i = 1, \dots, n$, are bounded as well. It follows that the roots of the characteristic polynomials above are symbols of order 1 and the operator A is in general a hyperbolic operator with multiplicities. We work under the assumption that when the equation $A(x, D_t, D_x)u = f$ is transformed into a first order system of pseudo-differential equations the matrix of the system is upper-triangular (for a second order example see Subsection 5.1.1(i)). Since there are no lower order terms then condition (H1) is trivially fulfilled. We can now investigate the well-posedness of the Cauchy problem

$$\begin{aligned} A(x, D_t, D_x)u &= f(t,x), \\ D_t^{k-1}u(0,x) &= g_k, \end{aligned} \tag{5.6}$$

$k = 1, \dots, m-1$. We have that if $f \in C^\infty([0,T], \mathbb{R})$ and $g_k \in C_c^\infty(\mathbb{R})$ for all $k = 1, \dots, m-1$ the Cauchy problem (5.6) has a unique solution $u \in C^\infty([0,T] \times \mathbb{R})$. In addition if the roots

$$\lambda_i(x,\xi) = a_i(x)\xi, \quad i = 1, \dots, m$$

fulfil the condition (H2) then the representation formula (5.5) holds. This for instance happens when the coefficients a_i have distinct first derivatives in the points of multiplicities, i.e. $a_i(x) = a_j(x)$ implies $a_i'(x) \neq a_j'(x)$.

References

[1] C. Garetto, C. Jäh, M. Ruzhansky, Hyperbolic systems with non-diagonalisable principal part and variable multiplicities, I: well-posedness, Math. Ann. (2018) 1432–1807.
[2] T. Gramchev, M. Ruzhansky, Cauchy problem for some 2×2 hyperbolic systems of pseudo-differential equations with nondiagonalisable principal part, in: Studies in Phase Space Analysis with Applications to PDEs, in: Progr. Nonlinear Differential Equations Appl., vol. 84, Birkhäuser/Springer, New York, 2013, pp. 129–145.
[3] L. Hörmander, Fourier integral operators. I, Acta Math. 127 (1971) 79–183.
[4] L. Hörmander, The Analysis of Linear Partial Differential Operators I, 2nd edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, Heidelberg, New York, 1990.
[5] W. Ichinose, Propagation of singularities for a hyperbolic system with double characteristics, Osaka J. Math. 19 (1) (1982) 171–187.
[6] C. Iwasaki, Y. Morimoto, Propagation of singularities of solutions for a hyperbolic system with nilpotent characteristics. II, Commun. Partial Differ. Equ. 9 (15) (1984) 1407–1436.
[7] I. Kamotski, M. Ruzhansky, Regularity properties, representation of solutions, and spectral asymptotics of systems with multiplicities, Commun. Partial Differ. Equ. 32 (1) (2007) 1–35.
[8] G.V. Rozenblyum, Spectral asymptotic behavior of elliptic systems (engl. translation), J. Sov. Math. 21 (5) (1983) 837–850.