The role of differential and integral calculus in schools: a review

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THE ROLE OF DIFFERENTIAL AND INTEGRAL CALCULUS IN SCHOOLS: A REVIEW.

by

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ABSTRACT

One of the most conceptually difficult topics to be studied at school level is calculus. Its notation and subtle implications are but two of its main ares of difficulty for students. The views of exponents of the calculus on methods of approach to the subject vary as much as the different notations of the calculus. The degree of difficulty with which calculus is perceived also varies from writer to writer. Experience shows, however, that, whatever the approach, the concepts of calculus are difficult to impart. Nevertheless, it is an important subject not only for those students who are mathematically able, but also for the large numbers who need to use it in other curricular areas.

In this dissertation we examine the role of calculus in schools and look at the traditional methods of approach. The emphasis is on methodology and applications without recourse to rigour except where this is relevant to the theme. New approaches to the calculus are considered with particular reference to the influence of the microcomputer. Its role as a teaching aid in the process of trying to impart the concepts of the calculus is a recurring feature throughout this work.

In the analysis of the methods of approach to the calculus in schools, use is made of personal experience in teaching calculus and, based on that experience, some attempt is made to shed light on the areas of the literature which have proved difficult to impart.

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1.1 A brief historical perspective.

Mathematics, with a history extending beyond 4000 years, is a constantly growing subject and the rapidity with which it has developed over the last century is clearly reflected in changes in mathematical curricula, although this reflection is often delayed. These changes were necessary for social reasons: pupils trained on aspects of traditional mathematics syllabuses in schools floundered at university level as they encountered a new style of university mathematics which bore no resemblance to that which they encountered in schools. The high influx of students into universities was accompanied by a high failure rate which became politically important [1] (p 101). It was necessary, therefore, for changes in curricula and teaching styles to accommodate the transition between secondary and tertiary level.

The use of mathematics as a tool in many ares of industry and commerce has also given rise to the need for changes in curricula. A number of students who do not aspire to university mathematics need to be acquainted with the many applications of mathematics relevant to their chosen careers in science, technology and commerce. They need, therefore, to be introduced to the new applications of mathematics, particularly with reference to the use of computers, before embarking on the world of work. Here, we look at the changes in mathematics curricula with specific reference to the influences of the calculus on these changes.

Newtonian calculus was fairly common knowledge amongst engineers, scientists and mathematicians throughout the nineteenth century. It was widely used in mathematical physics, theoretical physics and analytical mechanics. Further applications of mathematical analysis to the theories of electricity and magnetism, and a desire to achieve a greater understanding of the 'real world', led to applications of mathematics to what is now called mathematical modelling - an important feature of which is the identification, construction and solution of differential and integral equations.

Calculus is one of the areas of mathematics which contributed to the increasing gulf between university and school mathematics [1]. Hardy's book on Pure Mathematics attempted to bridge the gap between school and university mathematics. It failed because it introduced students to a style of mathematics which was not an extension of school mathematics and which they found difficult [1]. The traditional approach to calculus and the implications of its generality caused more problems than it solved; even for university students. It was out of concern for this situation that mathematics projects such as SMP [2] and SMSG [3] were set up. Other groups seeking to bridge the gap between school and university mathematics have been formed, for example The Mathematical Association (MA) and The Association of Teachers of Mathematics (ATM). The main aim of all such groups is to seek to present conceptually difficult mathematical ideas in an elementary and understandable way. The ideas of the calculus are widely used in careers such as engineering and science [22]. Many students of average ability, and indeed a wider range of students, have benefited from the many varied approaches to the concepts of elementary calculus in schools. For example, the computer approach to concepts of calculus: integration, differentiation and limiting processes [11] have served to clarify some of the fundamental ideas of calculus.

1.2 The role of the calculus.

Men were starting to study the material world intensively around 1600 A.D. The motion of all kinds of moving objects -

from planets to pendulums - were of great concern. From that study came the knowledge of the stars, atoms and machines that we enjoy today. Calculus is an example of the kind of development which grew out of the desire to understand the 'real world' more fully, and, as a consequence, its use has been indispensable in every development in science and mathematics from 1600 to 1900 A.D. [4]. It is to be found in the theory of heat, light, gravitation, sound, electricity and magnetism and the flow of water - hydrodynamics. Calculus enabled Maxwell to predict radio twenty years before physicists could demonstrate radio experimentally and it played a vital role in Einstein's theory of 1916 and the new atomic theories of the nineteen twenties. Apart from these and the many other applications in science and technology, calculus stimulated the appearance of new branches of mathematics : modelling and numerical methods. In this century, therefore, even those ares of mathematics which do not use calculus directly are connected in some way with subjects related to calculus [4].

Calculus, then, plays a vital role in the development of mathematics and the enhancement of mathematical education, both for the pure and applied mathematician. (A pure mathematician is one who studies mathematics for its own sake, whilst an applied mathematician is one who studies mathematics in order to deal with some aspects of the 'real world': science, economics, engineering and medicine).

In the past, the view of calculus was that it is a difficult subject. Gradually, particularly in England, teachers began to realise its use in the solution of problems by a much easier root than an appeal to the purely algebraic methods. In English schools students had two or three years of calculus. Some mathematicians argued that this was inadequate; that calculus was too difficult for all but a few, and the conceptual difficulty of the calculus should preclude its study by average students and its exposition must be left to well qualified mathematicians [4]. The substance of this view prevails today, but new directions and methods in mathematical teaching - the use of computers, for example,- have shown that it is possible to enable a wide range of students to gain some appreciation of the elementary concepts of a conceptually difficult subject. A question which remains is one which seeks to determine the way in which calculus should be taught or introduced in schools. The answer, hopefully, lies in much of the discussion which follows in this work.

1.3 Rigour versus understanding.

Another area of concern with regard to calculus in schools has been the extent to which the traditionally rigorous approach to calculus should be compromised to facilitate easier understanding by a wider range of students. Early this century, Thompson [6] attempted to address this problem and, as a consequence, he was ridiculed by many purists of the day. He attempted to show that it was possible to simplify the ideas of the calculus in order that it was more appealing to students. The opposite view is held by Schwarzenberger [5] who argues that it is not possible to simplify calculus. Coupled with this concern is the argument as to when calculus should be introduced in schools and the possible consequences of students moving to university courses in science and technology without adequate grounding in the subject.

The purists continue to argue that to rob calculus of its rigour is to rob it of its beauty; to simplify it to the extent that it is not treated seriously. The above is the substance of Feit's article [7]. He advocates that an introductory course on calculus should be presented in a manner that is understandable to students without sacrificing rigour. The influx of students whose mathematical backgrounds vary considerably but who have need of mathematics - calculus - in their fields of study, has forced many professors to realise that not every student of engineering or science is a mathematician, or, indeed, in love with the subject. This has brought about changes both at school and university level in methods of approach to mathematics and subjects dependant upon mathematical applications. To some extent, some text books [23, 24] have made strenuous efforts to reduce their dependence upon symbolic manipulations and seek to explain mathematical ideas without stifling students with complexities beyond their understanding. As a result, many areas of mathematics which were previously discussed at graduate level have found their way into sixth form courses. For example, hyperbolic functions and integration and differentiation of these are part of some A Level Further Mathematics syllabuses.

Most text books on calculus, particularly those that have 'cornered' the market, are written for students with a degree of maturity and mathematical sophistication. Calculus is not a course taught exclusively to prospective students of pure mathematics, yet it is one area of mathematics which has successfully resisted the winds of change [7]. Students are often bombarded with catalogues of mathematical definitions, theorems, rules and exceptions to rules. It should be clear to mathematical educators that it is not necessary to understand every possible meaning and application of a fundamental idea in calculus before using it. Feit argues that, in any topic of mathematics, certain facts must be learned, but these should be no more than is required at each stage. His ideal book on calculus would be one which works like the brain. He writes:

> "Our thinking process involves grasping a few details, attempting to apply them to the whole, grasping some more details and applying these, and so the main idea

> becomes clearer. The process continues, as at each step we embrace more details and let them percolate through the system, thus extracting a better picture of the whole. A book should imitate this process by presenting details as and when they are needed."

No attempt is being made here to suggest that rigour should be sacrificed for a more humorous or conversational approach to problem solving. What should not be sacrificed, however, is students' understanding of the subject matter for the desire on the part of authors to impress instructors. Feit's recommendation is that students should be taken 'by the hand' and shown how a problem can be changed, in varying degrees, to give better understanding and different results. This is in keeping with the approaches and recommended practices advocated by Cockcroft [8] in paragraphs 240 to 243.

1.4. The Computer Influence.

Many of the applications of mathematics in science and technology would not be possible without the use of computers. The whole study of numerical methods, for example, and its use in solving differential equations would be a tedious, if not impossible exercise. The computer is an influence within mathematics which is changing the attitudes of mathematicians to mathematics and, as a consequence, the ways in which mathematics is taught today. Throughout this dissertation reference would be made to the use of computers in mathematics teaching, with specific reference its use in approaches to some aspects of calculus.

The arrival of this new technology brings with it the beginnings of a new phase in the development of mathematical education - the second major revolution in thirty years. The introduction of the 'New Mathematics' in the 1960's was largely due to internal forces : teachers and professional mathematicians dissatisfied with the mathematics curriculum. The new technology - computers - will change our cultural environment in a way which will force mathematics education to respond [9]. Currently, some teachers are bemused by the pace of this change. The development of new software and hardware is seen as a threat to teachers who must ensure that the far-reaching implications, not simply the present practical applications, provide a vision of the possibilities of their (computers) use in mathematical education in the future.

Winkelmann [10] argues that in addition to the technical considerations, there are social factors relating to teachers and their work in the classroom. He suggests that we should distinguish three different levels of computer use :

Its use in principle - a consideration of how information technology might be used for teaching specific topics in a particular discipline;

Its use in practice - actual use in the classroom, using available equipment and software;

Its use in reality - a measure of the relevance of computers in certain disciplines and of their use by teachers.

Both employers and agents of government have acknowledged the use of computers in schools. Indeed, the availability of computers in most schools is a direct response to political initiatives and governmental directives, that education authorities should be aware of the changing technological era and the need to make necessary changes in curriculum planning to reflect that change.

Mathematics, in particular, has benefited from the new technology and associated software. There is much imaginative software developed for use in the teaching of mathematics. This covers areas of the subject from counting to calculus [11]. However, research shows that 39% of teachers used the computer very rarely in their mathematics teaching, whilst 33% never use it at all [12]. This is a situation which needs to be remedied if all teachers and students are to share in what Tall [9] refers to as an exciting new phase of development. The Council for Educational Technology highlighted in their report [13], the need for continuing curriculum development to produce materials and software, and an increase in the amount of research into the impact of the new technology on the mathematics curriculum. A measure of the responsibility for the direction of changes in mathematics teaching must be shared by teachers who would otherwise be forced to accept the new technology as an imposition rather than a welcomed aid to their work in the classroom.

1.5 Calculus and GCSE.

Many radical changes have taken place in the education system of England and Wales over a period of time. One that is seen as the most radical is the advent of GCSE [14]. This new examination may be seen to combine two previous certificates : CSE and O level. Its introduction has brought with it implications for all subject areas including mathematics. The point of concern here is its implication for calculus.

GCE O level included elementary calculus. The option existed for the elementary treatment of ideas such as velocity and acceleration, maxima and minima, and areas under curves. These provided a prelude to a more detailed treatment in the sixth form. For example:

(i) The equation of a curve is $y = 12 + 4x - x^2$. Show that the curve has local maximum and find the maximum value.

(ii) Calculate the area bounded by the arc of the curve $y = 12 + 4x - x^2$ between $x = -2$ and $x = +2$, the ordinate at $x = 2$, and the x-axis. (Oxford GCE, Autumn 1984) [25].

Calculus is absent from GCSE mathematics. The consequences of this exclusion can only be assessed with time. One possible consequence, however, could be the 'handicap' this may cause at sixth form level. The A level course is a difficult course to complete in the time allocated. Previously, the fifth year provided an opportunity to discuss, in a less formal way, some ideas - including calculus - to be considered in greater detail in the sixth form. Calculus plays a large role in A level mathematics and the physical sciences. It is present, too, in economics and statistics at A level. Students who pursue these subjects at A level in a combination of subjects which exclude mathematics have thus been deprived of an introduction to relevant ideas in calculus in their fifth year. The Higginson Report [30] cites the introduction of AS levels as a means of meeting the needs of such students.

Research in the United States by Spresser [16] shows that students who do well on university courses in mathematics, engineering and science had three years of an introduction to ideas of the calculus in secondary school. The A level course is essentially a two-year sixth form course. Applying Spresser's findings to the situation which exists in UK schools would lead to highlight the importance of an introduction to calculus - even as an option - in the fifth year. Indeed, his research [15] shows calculus in year five would be advantageous. It must be acknowledged that the US system is very different from that in the UK. Nevertheless, the point being made here is that there is an omission from

GCSE - calculus - which is of fundamental importance in later A level work in mathematics.

Some teachers argue that within the framework of GCSE and with the emphasis on investigations by Cockcroft [8] and others [33], it is possible to embrace a numerical approach to calculus using investigations. Experience shows, however, that there is some unwillingness on the part of teachers to cover any topic that is not specifically mentioned in a syllabus related to a particular course. GCSE coursework is not compulsory for mathematics until 1991,and those who do not welcome its inclusion and all the assessment objectives it entails, may well argue that a numerical approach to calculus, at this stage, is neither relevant nor desired. It is necessary for teachers to carefully observe the consequences for A level mathematics. Only then would GCSE be forced to remedy the exclusion of calculus from its framework or, indeed, justify its exclusion.

Tall [44] argues that this new qualification - GCSE - will impinge on the teaching of calculus at A level and that, when it does so, two opposing factors will come into play that are peculiar to the British system. Firstly, the broader span of GCSE will contain less of the material which is currently regarded as essential prerequisites for the calculus (for example, the reduced emphasis on algebraic manipulation). This would add to the dilemma for teachers. It would make it difficult to reach the same level of performance at A level, producing pressure on teachers to reduce content and an increasing dependence upon investigational elements through coursework. Secondly, the agreed 'common core' shared by A level examination boards will, in practice, present strong opposition to major changes. The way ahead may be unclear, but it is sensible for both teachers and pupils to familiarise themselves with the computer methods of approach. The deep structure and subtle ideas of the calculus are well served by an exploration of the graphical approach at A level.

1.6 GCE A level and the calculus.

The first examinations for GCE A level were held in 1951 and, since then, have been administered by eight GCE Boards in England and Wales. Their syllabuses and procedures are scrutinized by the Secondary Examination Council, but they are, in the main, independent bodies. A levels have been widely accepted as setting recognised standards of academic excellence by educationists and industrialists. The variety of syllabuses provide an opportunity for candidates and teachers to choose the one which best suits their needs.

Since 1951, however, there has been some concern about the nature and structure of A levels and the failure of the GCE Boards to identify, with sufficient clarity, their aims, objectives and criteria for assessment. The Higginson Report [31] records the common perception that, over the years, syllabuses have become too voluminous and candidates are overburdened with having to memorise a large amount of information to the exclusion of other important demands [paragraph 1.3]. Another criticism hinges on the narrowness of programmes of study and a system which seems to encourage premature specialisation. Mathematics has, to some extent, addressed the first concern by making examination papers less rigorous and with the inclusion of a more comprehensive formulae sheet with its A level papers. There is greater scope, therefore, to examine a knowledge and application of principles.

The recent introduction of Advanced Supplementary (AS) level syllabuses has been a step towards broadening students' experience and catering for a wider range of non-specialist students.Any A level reform should recognise: the need to appeal to less formalised mathematics and embrace aspects of decision mathematics and problem solving mathematics [31,32]; the role of the mathematical model and modelling processes [33] in the 16-19 curriculum; and the greater crosscurricular role mathematics plays in the new technological age.

The Higginson Report [31] stresses the importance of rigour [paragraph 2.7 - 2.11], a concept with which mathematics in general, and calculus in particular, is quite familiar. Rigour, it argues, has several components: assessing and rewarding higher level skills; modes of assessment which have claim to exactitude; setting out plainly and precisely, though not necessarily in detail, what is to be studied and learned, what is to be assessed and how that assessment is to be made; identifying aims and objectives and establishing syllabuses; setting and marking of questions and awarding grades. The implications for mathematical education and teachers are enormous. Rigour should no longer be confined to methods of exposition of subject matter, but to other areas which combine with teaching approaches to achieve attainment and certification which sustain public confidence and the currency of its certificates with employers. The future of mathematics depends, to a large extent, on the ability of curriculum developers and teachers to respond positively to these challenges.

The wide choice of syllabuses and the independence of the syllabus writers contribute to the difficulty teachers experience in choosing a particular Examination Board. The report [31] calls for all syllabuses to be governed by common general principles and subject specific principles. Often schools have to justify their choice of Examination Board to employers who sometimes entertain the view that some A level examinations are more difficult than others and, as such, the higher grade categories are easier to obtain on some A level papers. A common set of principles, aims and objectives, applicable to all GCE Boards, and the new identified compulsory common core would serve to facilitate greater comparability between Boards. Employers would then be able to accept with confidence what it means when applicants state that they have completed a course in A level mathematics and achieved a particular grade.

The following looks at the calculus content of four 'major' Examining Boards in an attempt to ascertain the degree of commonality between these Boards, as far as calculus is concerned. It is acknowledged that there are eight Examining Boards and that there is no significance in the choice of this particular group of four. The aim here is simply to show the types of calculus content at A level for the 1990's and to see whether or not there is some adherence to the agreed national common core. Furthermore, it is acknowledged that there are other syllabuses at this level: Further mathematics, Mathematics with applications, AS level, Pure mathematics and Applied mathematics as single subjects. The following analysis of the calculus content of these four Examining Boards at A level suffices to facilitate and substantiate a conclusion to the above objective.

Notes and abbreviations.

Pm : Pure mathematics with Mechanics.

Ps : Pure mathematics with Statistics.

The above combinations consist of 3 papers:

Paper 1 - Pure mathematics
(common to papers 2 and 3).

Paper 2 - Mechanics.

Paper 3 - Statistics.

Examining Boards and Syllabuses.

LOND : University of London School Examination Board.
Pure mathematics with Mechanics - 371.
Pure mathematics with Statistics - 374.

AEB : The Associated Examining Board.
Pure mathematics with Mechanics - 636.
Pure mathematics with Statistics - 646.

JMB : Joint Matriculation Board.
Pure mathematics with Mechanics - P1 + Me1.
Pure mathematics with Statistics - P1 + S.

OXF : University of Oxford Delegacy of Local Examinations.
Pure mathematics with Mechanics - 9850/1/2.
Pure mathematics with Statistics - 9850/1/3.

An entry of Pm or Ps separately implies that the particular topic is only offered in the mechanics or statistics paper respectively. For example, differentiation of a vector with respect to a scalar variable is offered by London and AEB in the mechanics part of Pure mathematics with mechanics.An entry of Pm and Ps together implies essentially that the particular topic occurs in both combinations.

Examining Boards

Syllabus Content	LOND	AEB	OXF	JMB
The derivative as a limit.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
The gradient of a tangent as the limit of the gradient of a chord	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Differentiation of standard functions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Differentiation of sums, products, quotients.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Differentiation of composite and inverse functions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Implicit functions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Parametric functions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Rates of change.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Tangents and normals.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Maxima, minima, points of inflexion.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Curve sketching.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Connected rates of change.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Small increments and approximations.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Areas under curves.	Pm Ps	Pm Ps		Pm Ps

Syllabus Content	LOND	AEB	OXF	JMB
	Examining Boards			
Integration as a reverse of differentiation.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Indefinite and definite integration.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Integration of standard functions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Techniques of integration - decomposition.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Linear and non-linear substitutions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Integration by parts and partial fractions.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Volumes of revolution.	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Centres of mass and centroids.	Pm Ps	Pm		Pm
Trapezium rule.	Pm Ps	Pm Ps	Pm Ps	
First order differential equations with variables separable. (*)	Pm Ps	Pm Ps	Pm Ps	Pm Ps
Differentiation of a vector w.r.t. a scalar variable.	Pm	Pm		

Examining Boards

Syllabus Content	LOND	AEB	OXF	JMB
Kinematics of a particle dv/dt = f(v) or f(t), vdr/dx = f(x) or f(v) and their solutions.	Pm	Pm	Pm	Pm
Velocity and acceleration as vectors.	Pm	Pm		Pm
Velocity and acceleration when position is a function of time.	Pm	Pm	Pm	Pm Ps
Probability density function.	Ps	Ps	Ps	Ps
Normal distribution.	Ps	Ps	Ps	Ps
Poisson distribution.	Ps	Ps	Ps	Ps
Newton - Raphson process.	Pm Ps		Pm Ps	
Simpson's rule.		Pm Ps	Pm Ps	
Integration of a vector w.r.t. scalar variable.		Pm		
Simple harmonic motion.		Pm		Pm
Mean values of functions. (*)				Pm Ps
Velocity and acceleration of a point moving in a straight line. (*)				Pm Ps
Continuous distribution.	Ps	Ps	Ps	Ps

(*) Implies that this topic is not part of the agreed national common core.

The above analysis shows that of the four Examining Boards chosen there is adherence to the agreed national common core. There is evidence, however, of the independence of Examining Boards to include what they wish. For example, first order differential equations with variables separable is not part of the agreed national common core; nevertheless, all four boards include this topic but only JMB acknowledges it in their syllabus description.

The combinations analysed above: pure mathematics with mechanics and pure mathematics with statistics, are two of the most popular at A level. Students choose these for different reasons: they may have an engineering bias and so opt for Pm whilst those with a business orientation choose Ps; on the other hand, only one option may be available at a particular school. The choice in the latter case is, more often than not, the choice of the teacher or school.

Calculus, however, is part of a wider syllabus and it is the content of that syllabus, with its aims and objectives, which form the basis of the earlier discussion. Any change in content of a particular syllabus is likely to have some effect upon calculus, the specific nature of which is conjecture at this stage. The possibility, however, in the changing situation, is a reduction in the amount of calculus at A level and balancing that reduction with an introduction of the more applicable elements of calculus at AS level. AEB, for example, is offering three syllabuses at AS level in 1990 with reduced calculus content. These are:

Mathematics: (Pure with Applications) - 994,
Mathematics: Calculus and Particle Dynamics - 995,
Mathematics with Applications (Contrasting) - 384.

The first two are intended to serve as a complement to other mathematical and scientific studies, whilst the third hopes to cater for candidates studying Arts and Humanity subjects at A level [53].

1.7 The needs of industry and commerce.

Much attention has been drawn to the mathematical needs of the school leaver in an industrial society [63]. Cockcroft [8] made specific reference to these needs. Harris [62] argues that much of the mathematics needs of industry and commerce , as advocated by Cockcroft, are not actually used in these domains. The prevalence of computers and calculators in industry and commerce have eradicated, to some extent, the need for school leavers to be able to physically perform many skills: roots of numbers; calculations with trigonometry; multiply and divide fractions; solve quadratic equations, to name a few [62] (page 9).

Fitzgerald [22] also argues that the mathematical demands on employees are not static for too long. He attributes this to the continual development of new machines, industrial processes and commercial practices. All these lead to a shift of emphasis on the mathematical needs of industry and commerce. The following is a list of tasks where, he observed, the need for mathematical skills has been reduced:

> Modern computer systems allow immediate access to any point in a stored file, so that visual/manual searching, involving recognition of numerical (and, incidentally, alphabetical) order is less in evidence.
>
> Less substitution into algebraic formulae is required, because the functions are absorbed into computer programs.

A craftsman working with a computerised machine programmed by a technician may not need to use any trigonometry, nor may the technician if he is using a computer-aided system.

Computer-aided design facilities can absorb many complex applications of geometry and trigonometry and make conventional drawing-board skills redundant.

Graph-plotters, linked to computers, can produce third-angle projections, isometric and perspective drawings, line-graphs, bar-charts, pie-charts and many other displays depending on the programs being used, so that there is less need to produce these by hand.

There are many specialised tasks which are being programmed into computers such as deciding how to cut various sizes of glass from large standard sheets in order to minimise waste - previously done by experience and estimation [22] (page 14 - 15).

New and increasing mathematical demands, however, embrace elements of: problem solving - modelling and investigations; transformation geometry; keyboard skills; number bases - binary, octal and hexadecimal are in common use; analysis of numerical data; graphical displays - bar-chart, pie-chart and line graphs; coordinates and algebraic thinking [22] (page 15 - 22). Some of these are contained in a pre-A level course of study. The challenge for mathematical education, then, is great. Not only is it required to meet these needs but it is also required to meet the needs of students who wish to pursue degree courses which prepare them for work in more technologically advanced branches of industry: aircraft design, fluid dynamics, electrical and mechanical engineering.

New syllabuses in mathematics (section 1.6) are attempting to cater for this need. They serve to facilitate a much greater variety of choice for students and signal a marked improvement to the way in which A level mathematics is responding to the increasing and changing demands put upon it by the new technological age.

No improvement, however, in a subject area can be achieved without implications. If A level mathematics, indeed mathematics in general, is to survive in a rapidly changing technological age and continue to serve in its many crosscurricular roles, the debate and discussion between syllabus writers, curriculum planners and teachers on the issues raised by the Higginson report (despite its rejection) must continue. The fundamental aim should be to provide courses which are relevant, stimulating and interesting.

Chapter 2: Some problems of teaching calculus.

2.1 Teaching calculus.

This dissertation seeks to discuss the role and applications of calculus in the school curriculum. It examines the way in which calculus is introduced at school level, and looks at some of the methods of approach though not in detail.

There is an argument against the presentation of any calculus in schools. This argument has its basis in the belief that many teachers only show the student how to perform mechanical processes without the necessary emphasis on principles. Furthermore, there is the claim that the study of small amounts of calculus in secondary schools induces false confidence in students and that, as a result, they often flounder in university work [20]. Presser's research [15,16] shows the opposite and supports the introduction to calculus at this stage.

It is fair, however, to say that the possibility always exists for some degree of misunderstanding in university subjects which began at school level. Insufficient time and attention to detail contributes to this. The claim, however, merely serves to emphasize the need to ensure that mechanical processes and the understanding of the underlying concepts, at each stage, complement each other.

Early this century, Thompson [6] startled the mathematical world with a treatment of the calculus which was seen as a deviation from the norm. He used an approach to calculus which he argued was simple and lacked mathematical rigour. Methods similar to Thompson's have always been used by mathematicians and scientists as a way of finding results which they subsequently prove by other means. Schwarzenberger [5] discusses Thompson's approach and argues that it evades some of the important concepts in the fundamental development of the calculus. The report of the Mathematical Association [17] asserts that the concepts of the calculus are too important to be treated in a trivial way. It concludes that there is no part of mathematics for which the methods of approach and development are more important than the calculus - a view with which most teachers will concur. This degree of importance, it argues, is partly because of the novelty of the notation, but mainly because of intrinsic difficulties - the ideas of a limit, for example. These occur very early in the development of the calculus and, as such, any approach to calculus should be gradual [7] without apology for the frequency of appeal to the principles rather than the processes. Much of the chaos which ensues after a hurried introduction to calculus has its basis in the confusion that arises when students find themselves unable to deal with any matter which is slightly outside the usual routine.

Any approach to elementary calculus must seek to address three main themes [18]. These are:

a) The study of gradients which leads to the idea of the derivative;
b) The process of finding simple, plane areas which leads to the finding of the definite integral;
c) The relationship between rates of change of a quantity and the rates of change of a function.

It is important that the principles underlying each of these themes should be given their own importance whilst due regard is paid to the inter-relationships between them. Shuard and Neill [18] assert that often they combine harmoniously in many applications of the calculus to describe mathematically some situations in the 'real world'.

One of the main problems students face is that, more often than not, the work is based on a great deal of algebraic manipulations at a stage when they are still trying to gain confidence in dealing with algebra. Indeed, the formal definition of the derivative of a function is often more off-putting than useful. The predominance of electronic calculators and computers in schools makes it easier for students to see that the derivative of x^3 at $x=a$ is $3a^2$, than it is for them to have seen, and not understood, the formal algebraic manipulations which lead to the same answer.

It is possible, therefore, with these facilities, to make the ideas of the calculus accessible to a wide range of students seeking to make themselves familiar with it. The pleasure of calculus could be available even to those students who would not have previously studied it because of their inability to manipulate algebraic expressions.

Teachers must never fail to understand that whilst the basic ideas of calculus hinge on the idea of a limit and subtle properties of the number system [19], complete understanding is not possible at an early stage. Numerical and graphical approaches [9], facilitated by the availability of computers, should not be seen as excessive time wasting but as a better way of nurturing sound intuitive understanding.

2.2 A teacher's dilemma.

There was a time when calculus was looked upon by many as being abstruse and lying beyond the boundaries of elementary mathematics. It occupied the mathematics and science syllabuses of universities and colleges. Earlier we discussed the developments in engineering and science which have made greater demands on mathematics, to the extent that, the desirability of bringing such a powerful mathematical tool within the reach of a wider circle of students, led to the gradual simplification of its presentation. Today calculus forms part of all A-Level syllabuses. It used to be an option in some O-Level syllabuses but is absent from GCSE.

Most teachers of mathematics have completed courses in analysis and advanced calculus. They are aware and acknowledge the overwhelming beauty of the logical structure of the calculus when assembled in an imposing form. Calculus is wide and deep in its ramifications and applications [21] and the temptation is continually present to include much more than the limitations imposed by time would allow. Here lies the dilemma for teachers. Faced with a lengthy examination syllabus, of which calculus is a part, it is often difficult to give calculus the treatment it deserves. The result is an exposition on the calculus which falls short of what is deemed necessary for such an important exercise. The absence of calculus from GCSE is likely to make the demand on time much tighter.

An elementary knowledge of algebra, trigonometry and the fundamental principles of geometry are necessary ingredients for the basis on which the ideas of the calculus would be built. Many students of calculus are weak in one or more of these areas. It is not surprising, therefore, that the principles of the calculus take second place in a treatment which has to ensure that mechanical processes are familiar for examination purposes. A teacher's role in ensuring that the right balance exists between principles and processes is often influenced by two factors: the realization of the need for a detailed and thoughtful approach and the punitive consequences of failure to complete the requirements of a syllabus as a result of insufficient time. The success or failure of an introduction to calculus depends upon the right balance between these two factors, whatever the method of approach. The absence of calculus from the framework of GCSE makes striking that balance a more delicate exercise. Both teachers and pupils have been denied the opportunity to explore ideas of the calculus before a fuller treatment at sixth form level.

2.3 Problems of notation.

All areas of mathematics have notations which are associated with them. None, however, can claim to have had as many different notations which have persisted in such regular and popular use as in the treatment of calculus [18]. Each notation has its meaning, advantages and disadvantages. Their survival is attributable to the fact that they emphasize different aspects of the development, and so are useful at different times and in different circumstances.

Thurston [50] argues that although mathematics is, by its very nature, precise and logical - for the most part - there is one topic which falls short of the ideal: the definition and meaning of dy/dx. In a first course on calculus this meaning is often obscure. The consequence is that students, although they use the given definition of dy/dx, do not really understand the meaning of it. It is important that some discussion precedes any definition since crucial to this definition is what, to students, seems to be dividing by zero. What follows attempts to look at this point before considering an approach to the definition of dy/dx.

The idea of dividing by zero is as controversial as a definition or meaning of infinity. Perhaps the easiest solution here is to get students to perform this operation on their calculators. They would undoubtedly get an error message. It is usually sufficient to conclude from this that division by zero is not possible.

Students often argue - falsely of course - that $0 \div 0 = 1$ because $5 \div 5 = 1$, $10 \div 10 = 1$ and so on. It is useful to trade on their experiences in elementary algebra here, stressing that the correct argument stems from the fact that division is the inverse of multiplication. As a consequence

$$x = \frac{p}{q}$$

is the (unique) solution of the equation :

$$qx = p$$

The equation :

$$0x = k, \text{ k constant, } (k \neq 0).$$

has no solution, but the equation :

$$0x = 0$$

has every value of x as a solution. So neither $k \div 0$ nor $0 \div 0$ is possible.

This clarification should not be seen as a trivial exercise as students who have not grasped these rather fundamental facts would fail to appreciate the meaning of:

$$\underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\delta y}{\delta x}\right),$$

thinking that it is of the form $0 \div 0$, and conclude that the answer is 1. Alternative, $0 \div 0 = 0$, because $0 \div 5 = 0$, $0 \div 10 = 0$ and so on. Hence, $\underset{\delta x \to 0}{\mathrm{Lt}} \left[\frac{\delta y}{\delta x}\right] = 0$ is another common, erroneous conclusion.

Traditionally, an introduction to differentiation uses the notations:

$$\delta y,\ \delta x \text{ and } \frac{dy}{dx}$$

A typical starting point is to find the gradient of the curve $y=x^2$ at point P(x,y)

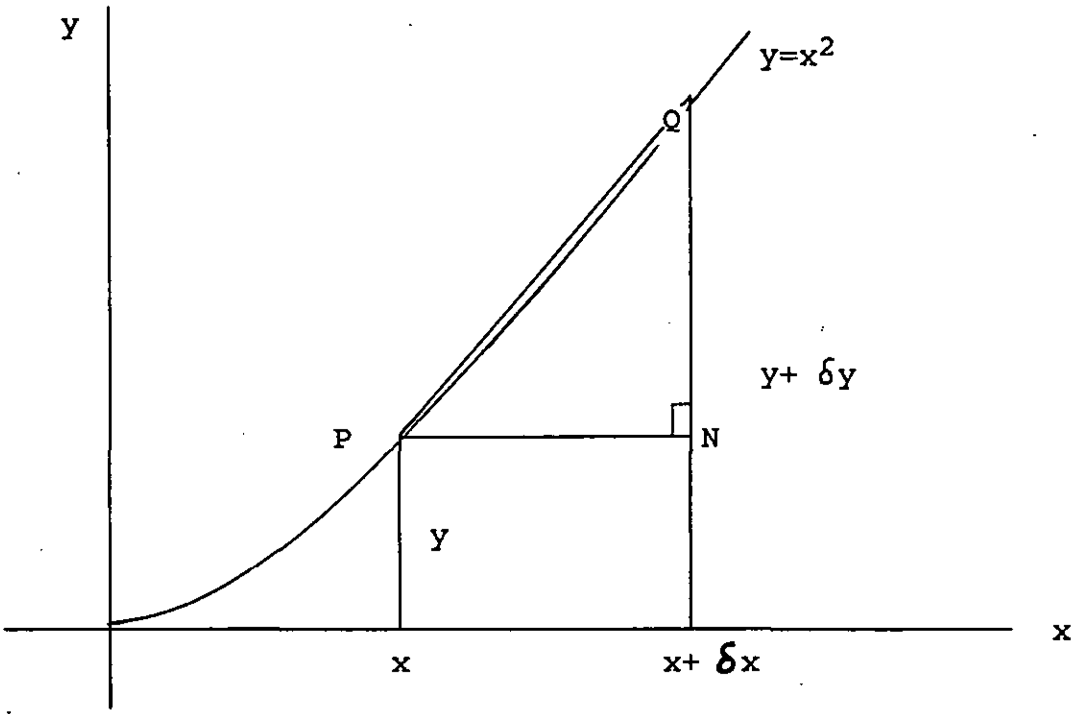


Fig.2.3(a).

Let Q be a neighbouring point.(The idea of 'neighbourhood' is not considered here!) Q has co-ordinates $(x+\delta x,\ y+\delta y)$.
The gradient of PQ $= \frac{QN}{PN}$

$$= \frac{\delta y}{\delta x} \quad \text{where } \delta x \neq 0$$

Q lies on $y=x^2$ and as such satisfies its equation.

$$y + \delta y = (x + \delta x)^2$$
$$= x^2 + 2x.\delta x + \delta x^2$$

But $y = x^2$

$$\therefore \delta y = 2x.\delta x + \delta x^2$$

and

$$\frac{\delta y}{\delta x} = 2x + \delta x \quad \text{\_\_\_\_\_\_\_} \quad (1)$$

But

$$f(x + \delta x) = (x + \delta x)^2 \quad \text{(see section 3.2)}$$

and

$$\frac{f(x+h) - f(x)}{h}$$

reduces to the same as the right hand side of equation (1) when $h = \delta x$.

We now consider what happens as Q moves to P. Obviously $\delta x \longrightarrow 0$. In otherwords we can make $(2x + \delta x)$ as close as we please to $2x$ by taking δx small enough.

So $$\underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta y}{\delta x}\right) = \underset{\delta x \to 0}{\text{Lt}} (2x + \delta x) = 2x, \text{ from (1).}$$

We write, $\frac{dy}{dx}$ for $\underset{\delta x \longrightarrow 0}{\text{Lt}} \left(\frac{\delta y}{\delta x}\right)$, so that the derivative, or the derived function, of $y = x^2$ is:

$$\frac{dy}{dx} = 2x$$

The teaching problems of $\delta x \to 0$ has already been discussed. $\frac{\delta y}{\delta x}$ is a fraction and often $\frac{dy}{dx}$ is taken to be a fraction. It is important that this crucial difference is emphasized. The relationship between them is that $\frac{dy}{dx}$ is the notation used to represent $\underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta y}{\delta x}\right)$. It is the limit of a ratio but it is not itself a ratio.

There are two further concerns here: the quantity δx is sometimes mistaken to be $\delta \cdot x$ and the student now has to accustom himself to a notation in which δx is a single symbol and not a product; the ambiguous use of x. In the definition of the function $y = x^2$, x can be any real number, but P is a point (x, y) where x is a fixed value. The strengths of δx and δy lie and emerge in the applications of the ideas of differentiation, in particular small changes and rates of change.

The function notation is used to find the gradient of the <u>function</u> at a point with known x - co-ordinate.

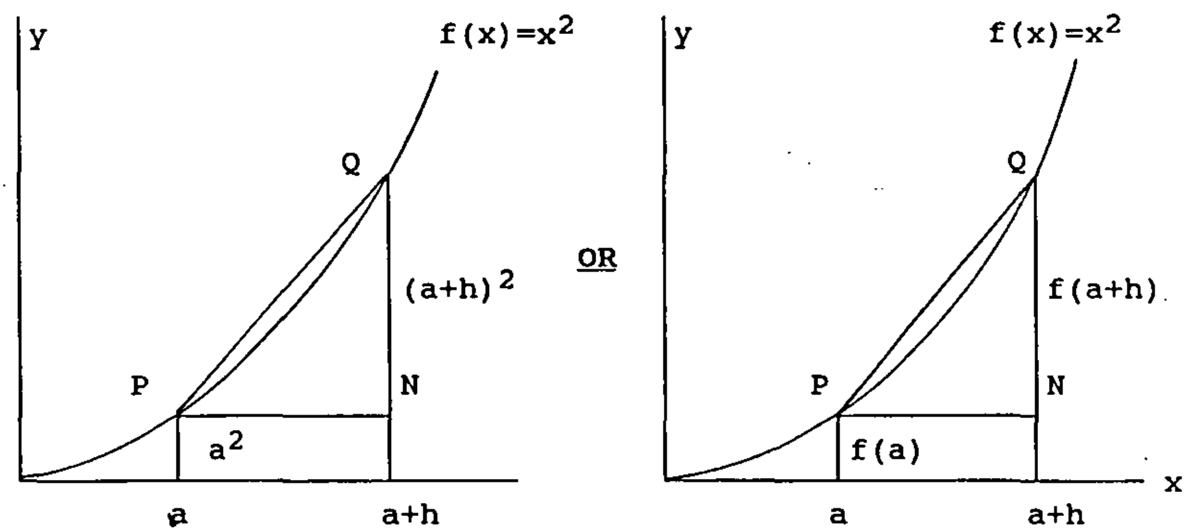


Fig.2.3(b).

This approach considers another point Q, on the graph, with x - coordinate (a+h), where $h \neq o$. The gradient of the function is given by the gradient of PQ, which is $\frac{QN}{PN}$.

Gradient of PQ = $\frac{(a+h)^2 - a^2}{h}$ or $\frac{f(a+h)-f(a)}{h}$ where $h \neq 0$

$$= \frac{2ah + h^2}{h} \quad , h \neq 0$$

$$= \underline{2a + h} \quad , h \neq 0$$

The discussion now hinges on what happens as $Q \longrightarrow P$, so that $h \longrightarrow 0$. Here, too, we can make $(2a+h)$ as close as we please to $2a$ by taking h as small as we please. Hence, the derivative of the function

$$f(x) = x^2$$

at the point $x=a$ is $2a$. We symbolise this derivative with the notation $f'(x)$ - f dash x. So for:

$$f(x) = x^2$$
$$f'(a) = 2a.$$

The derived function of $f(x)=x^2$ is $x \longmapsto 2x$.

Despite its length this notation is clearer for beginners of calculus. There is no mystery about h, whereas δx is viewed with some doubt. The point a, is clear and unambiguous and this method serves to facilitate the recognition of the distinction between the <u>derivative</u> and the <u>derived function</u>. However, if this is used as the initial approach in an introduction to calculus, the notation $\frac{dy}{dx}$ cannot be delayed for too long because $\frac{dy}{dx}$ is more widely used in the very many applications of calculus.

Whatever the notation used, it is important that it is concise. It should be easy to manipulate. It is important that teacher and student work in both notations as they are advantageous in different ways, and neither is a sufficient substitute for the other. A suggestion here is that teachers should use, at each stage of the development of the calculus, the notation which seems most appropriate for the communication of the ideas they wish to impart. Much, too, will depend upon the level of ability of the students concerned.

2.4 Students' view and understanding of the calculus.

Many significant changes have occurred recently which present an opportunity for teachers to look seriously at the way in which calculus is taught. The most important and obvious is the arrival of the microcomputer in schools in general and the mathematics classroom in particular. It brings with it the possibility for new investigative methods of approach to mathematics and Tall [43] argues that pupils following the experimental approach to calculus have a significantly better geometric insight into the notion of differentiation.

Students' first encounter with calculus is usually through considering two points A and B on a graph and considering how the 'chord' AB tends to the tangent at A as B approaches A (section 4.2). The informal language at this early stage introduces unforeseen difficulties since the interpretation of phrases such as 'tending to a limit' or 'as close as we please' is not easy at this stage. The graphical approach to calculus, however, using computers [11], enables students to see this limiting process and closeness of approach to a particular limit or, indeed, the tangent at A referred to above. Another difficulty, however, is the interpretation of the word 'chord'. In geometry the 'chord' is a finite line segment between two points on a curve, whereas the tangent to the curve is seen by students as an infinite line.

Schwarzenberger [5] argues that the mathematical difficulties of classical analysis do not lend themselves to simple explanation and that any attempt to formalize the ideas of calculus will implicitly contain underlying difficulties which haunt students. Tall [43] suggests that the above argument demonstrates the mathematical difficulties of analysis, not the cognitive difficulties of the calculus. Any attempt to simplify 'high-powered' mathematics, he argues, will contain inherent difficulties for the learner. What is absent is the inability of any such attempt to address itself to the alternative possibility of seeing the basic concepts of the subject matter from the pupils' point of view. They are unable, therefore, to progressively build up to the ideas from their current position.

The task of teachers, therefore, is to somehow try to enable students to get an intimation of the whole concept first, then they would be in a better position to organize their own thoughts and thinking processes to cope with these ideas. The microcomputer is a resource in the mathematics classroom which facilitates this method of approach. It enables a cognitive approach to the calculus without the prerequisites of limiting processes and chords approaching tangents, based on the fact that the derivative is not just the gradient of the tangent, but the gradient of the graph itself. The role of the computer in mathematical education is a study in its own right. Its invaluable contribution to the study and presentation of calculus in schools cannot be emphasized too strongly.

Chapter 3: Important basic ideas in learning calculus.

3.1 Real numbers and continuity.

3.2 Functions.

3.3 Graphs of real functions.

3.4 Gradient.

3.5 Rate of change and gradient.

3.1 Real numbers and continuity

Mathematical ideas concerning numbers and limits underpin much of the work on calculus. Frequently we make mathematical assumptions at various points in teaching calculus to beginners in order to, one hopes, nurture an intuitive understanding of important ideas. For example, the limit idea occurs in several situations, two of the most important being: the limit of a sequence and the limit of a function.

The definition that a sequence (s_n) of real numbers tends to a limit s iff: "Given any positive real number $\varepsilon > 0$, there exists $N \in Z$ (which might depend upon ε) such that

$$|s_n - s| < \varepsilon \text{ for all } n > N."$$

This definition, like the definition of the limit of a function, embraces - in addition to the idea of smallness or as close as we please - the idea of a real number tending to a limit and the process of arriving at that limit. The idea of a real number needs to be explained to students. A (positive) real number can be represented by a length of a line. Freudenthal [48] argues that real numbers should be identified as points on a line. The problem here is one of limited accuracy as it is difficult to distinguish between a line segment of $\sqrt{2}$ and one of length 1.414, since not only are they different, but one is irrational and the other rational, a vital distinction in pure mathematics [49]. Shuard [18] argues that the real number line gives a particularly good picture of the ordering of real numbers. It serves also to facilitate the idea that every point on the line corresponds to a real number. This, in turn, leads to the idea of the property of real numbers called completeness. It is usual for the idea of completeness to be taken for granted in a first course in calculus [18] (p 262).

A first course in calculus also assumes that students intuitively realize that most of the functions they encounter are continuous, that is to say: everywhere in their domains. Their graph can be drawn without 'lifting pencil from paper'. However, the ideas of continuity, real number and a rigorous discussion of the algebra of limits is beyond the scope of an A level course [18] (p 254-279). Intuitive ideas of these, however, are relevant to any discussion in an introductory course on integration and differentiation [49], but care is necessary in attempting to strike the right balance between rigorous definitions (such as that of a limit of a sequence given above) and sound intuitive understanding. Mathematics is a difficult enough subject to understand without the additional hazards which are introduced by misguided attempts to provide the wrong sort of motivation or help.

3.2 Functions.

It is desirable that we begin the study of calculus by clarifying ideas about the meaning of a function, since it is essential to the understanding of the subject. Students will have some intuitive ideas about functions from their study of elementary algebra. Indeed, they will appreciate that the growth of a plant is an example of a functional relation. If variations in temperature, moisture and sunlight remain constant then the progress of a plant is a function of time. The fact that we are unable to express this relationship in a mathematical form is not relevant to the theme.

Elementary physics asserts that the formula for a falling body is

$$s = \frac{1}{2} g t^2$$

Where s is the distance fallen,
t is the time taken
g is the acceleration due to gravity - assumed constant.

In the above equation s and t are variables and the distance s fallen depends on the time t. Distance s, therefore, is a "function" of time.

Most students now have a background in mathematics which uses modern ideas about functions. Familiarity with functions and their associated notations provides students with a greater opportunity for understanding the differentiation process. Many texts in calculus do not take advantage of this. Here, we discuss the function notation and how it can be used to facilitate an introduction to the differentiation process.

The literature on functions used ideas of domains and co-domains and a rule which establishes the relationship between them. This rule tells how the domain is mapped into the co-domain. A typical description of a function is:

$$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{given by} \quad x \longrightarrow x^2$$

or

$$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{given by} \quad f(x) = x^2$$

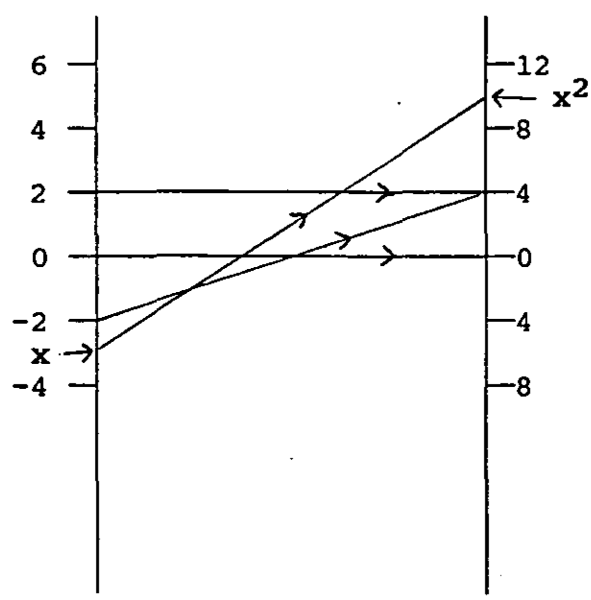
Here the domain is the set of real numbers which is mapped into the codomain - also $\mathbb{R}$ - by a rule which takes x in $\mathbb{R}$ into its square. There are several representations of functions. The following are three different representations of the above function:

($\longrightarrow$ stands for 'maps to' in this context)

a)

$\mathbb{R}$		$\mathbb{R}$
1	$\longrightarrow$	1
2	$\longrightarrow$	4
x	$\longrightarrow$	x^2
Domain		Codomain

b)



Domain Codomain

c)

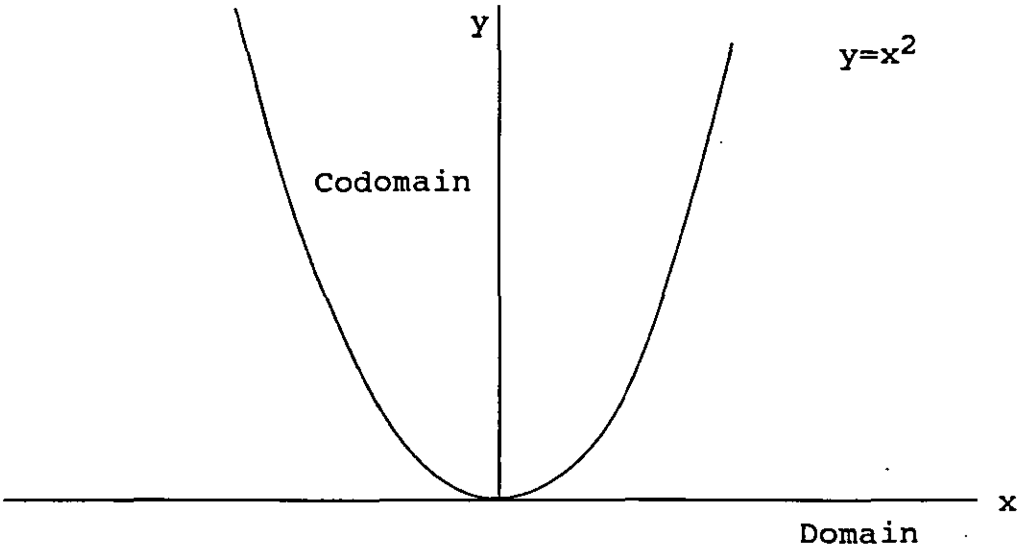


Fig. 3.2(a).

Each of the above diagrams is an adequate representation of the function but (c) - the cartesian graph - is more useful. It contains far more information than an arrow diagram. It represents a function in a manner that is useful in calculus and shows that

$$y = x^2$$

is the equation of the graph of the function

$$f:x \longrightarrow x^2$$

Mathematicians now appreciate the importance of stating the domain and codomain of functions. There is the tendency to speak of 'the function f' rather than 'the function f(x)'. The symbol f(x) is usually reserved for the image of an element x in the domain, so that it is acceptable to write:

$$x \longrightarrow f(x)$$

It follows, therefore, that

$$f: x \longrightarrow x^2 = f(x)$$
$$f: (x+h) \longrightarrow (x+h)^2 = f(x+h)$$

Here x and x+h are in the domain on which the function f is defined. f(x+h) and f(x) are the images of (x+h) and x respectively. In particular f(a) and f(a+h) are the images of a and (a+h) in the domain of f. So the expression

$$\frac{f(a+h) - f(a)}{h}$$

would have some meaning to students. It is easier to relate this expression to the idea of finding the derivative of a function at x=a. Essentially, the expression

$$\frac{f(a+h) - f(a)}{h}$$

is the gradient of the 'tangent' AB in Fig. 3.2(b).

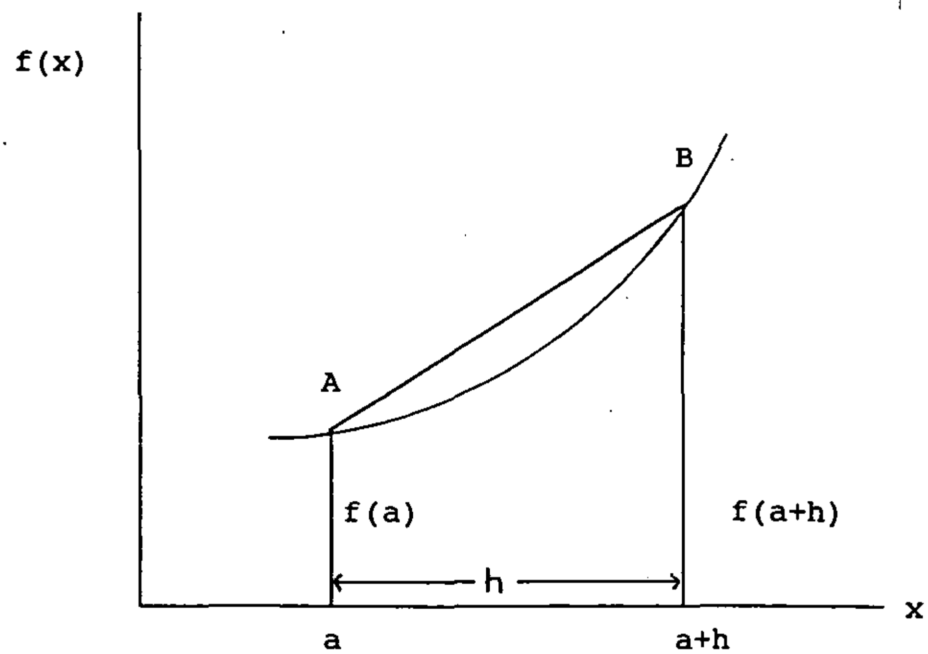


Fig. 3.2(b).

The limit as h tends to zero is the derivative of f(x) at x=a

$$\text{i.e.} \quad \underset{h \to 0}{\mathrm{Lt}} \left[\frac{f(a+h) - f(a)}{h} \right]$$

is the definition of the derivative of f(x) at x=a.
We can now differentiate and integrate functions from the knowledge and appreciation of their nature and composition. New directions in mathematical education have caused most examining bodies to have due regard to present day, modern treatment of mathematical concepts. The new approach to calculus is clearly reflected in most examination syllabuses.

In concluding the above discussion it is necessary to draw the attention of students to a more general representation of a function. Essentially it is a rule which defines a particular type of correspondence or relation between domain and co-domain.(Fig.3.2(c))). The correspondence must be one to one or many to one and each and every member of the domain must have an image in the co-domain.

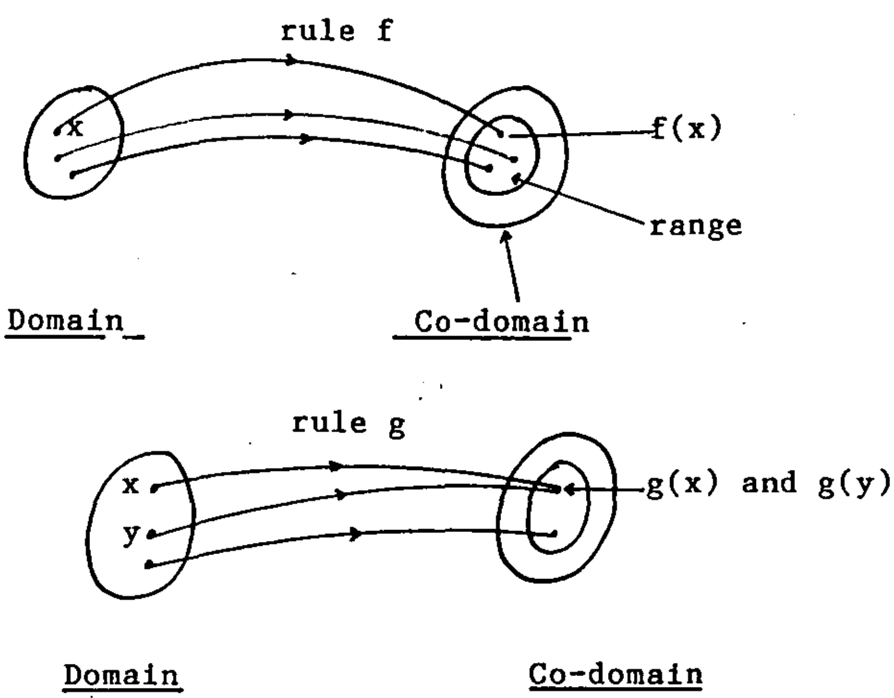


Fig.3.2(c)

3.3 Graphs of real functions.

It is also important to look at graphs of real functions without reference to a particular rule. This serves to show the general representation of the terms 'domain', 'co-domain' and 'range' discussed in 3.2.

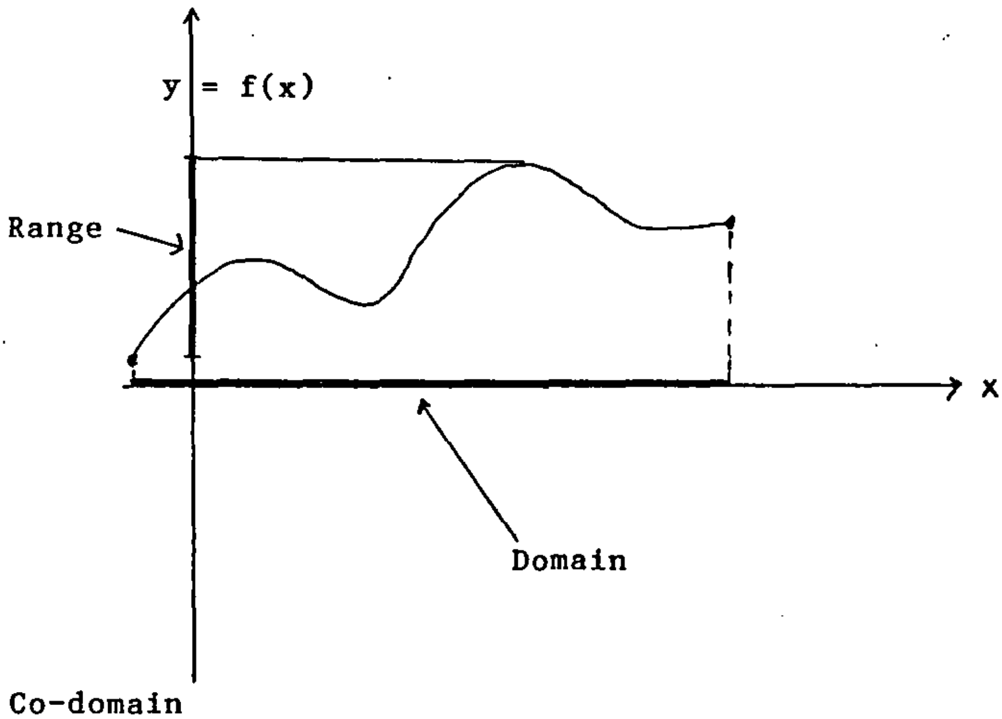


Fig.3.3(a).

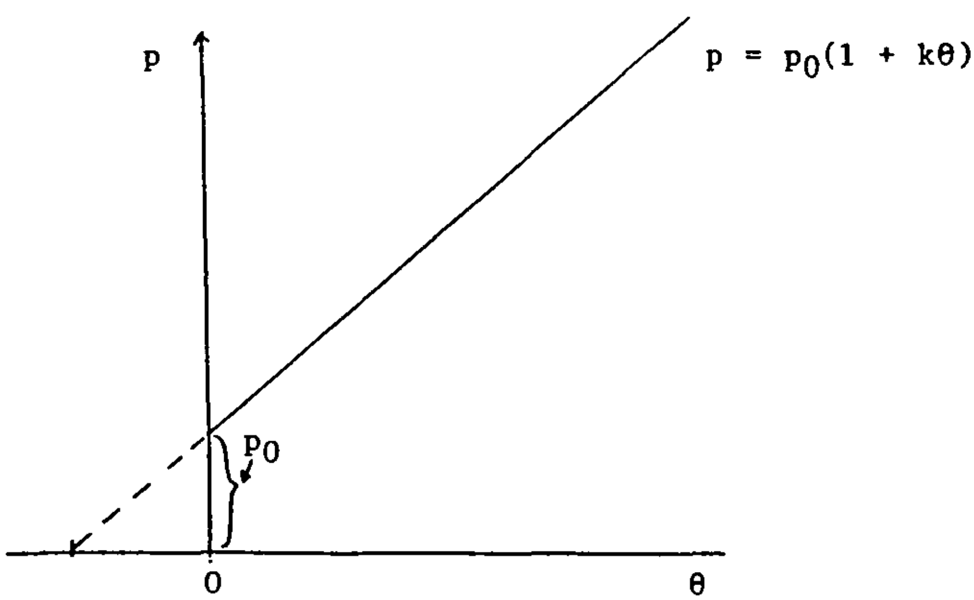
3.4 Gradient

There are many basic ideas which require systematic discussion at the beginning of any course in calculus.These ideas need not be introduced in a formal manner [11] but in such a way that use is made of students' prior experiences in algebra and geometry. So far we have looked at some of those ideas: real numbers, functions as a rule, domain, co-domain, range and limits. Here we look at another idea: gradient, which must be assimilated before a student can, with any degree of understanding, pursue a course in calculus. Students should be familiar with the straight line and its general equation

$$y = mx + c.$$

They should know the meaning of m, the gradient, and the meaning of c. Most students are familiar with finding gradients of line segments between two points. A good approach seeks to invite students to consider problems like this and, on the basis of that experience, arrive at the relationship between the gradient of the line segment and the line of which it is part [20]. The value of c can be estimated or calculated.

The value of c is more useful, however, in practical work such as the physical sciences. In A level physics, students often need to plot the relationship between two variables. An example of this is the relationship between pressure and temperature of a constant volume of gas (Charles Law) [51,52].(See Fig.3.4(a)).



Pressure vs Temperature (constant volume).

Fig.3.4(a).

In the above diagram the value of c corresponds to the pressure when temperature is zero. This value is p_0, the intercept on the 'pressure axis'.

In addition to the many simple numerical exercises which can be used to consolidate the meaning of m, there are several available software which can be used to give a graphic representation of a concept. Tall's Graphic Calculus [9] looks at the whole idea of gradient in relation to several functions including the straight line. He uses the function notation - which we discussed earlier - as a means of defining m, the gradient of a straight line. Whatever the approach to an interpretation or meaning of m, it is important that students undertake one or both exercises. Certainly the computer approach would serve to confirm quite easily that any line segment of the same line would give the same gradient m. The above exercise does prove invaluable in facilitating a smooth transition to, and understanding of, the general form

$$y = mx + c.$$

From this point of understanding we can consider the algebraic representation of the gradient of any straight line. (Fig.3.4(b)).

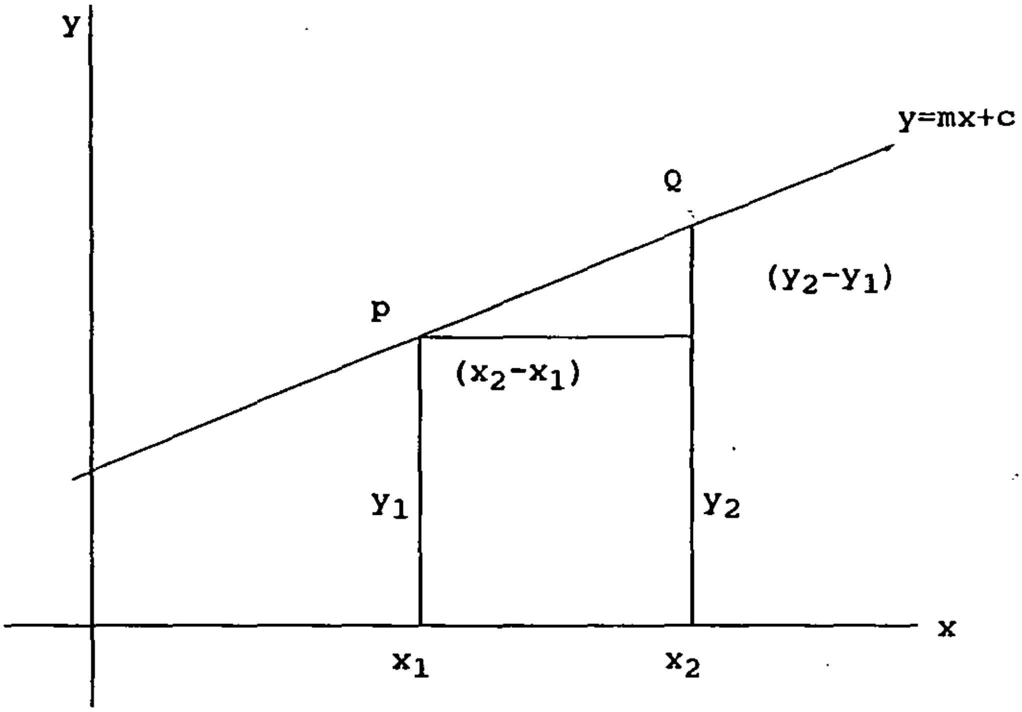


Fig. 3.4(b).

If P and Q are two points on that line, then the gradient of the line is the gradient of the line segment PQ.

$$\text{Gradient of PQ} = \frac{y_2-y_1}{x_2-x_1}$$

The abundance of calculators in schools enables calculations of gradients of straightlines when x and y values are in decimal form; thus ending any restriction of using whole numbers for ease of calculations. The concept of ratio is subtly concealed in the calculation of gradients. Many students need to be convinced that any size of right-angled triangle would yield the same gradient. Furthermore, geometry tells us that all such right-angled triangles are similar and the fundamental property of similar triangles is that their corresponding sides are in the same ratio. A verification of these facts through practical exercises with some tabulation would serve as a useful exercise.

3.5 Rate of change and gradient.

There are many practical experiments in the discrete real world which can be used to introduce the idea of rate of change and its relationship with gradient. Here we look at one such experiment which can be carried out in a classroom or laboratory. It requires much student participation and discussion. The experiment uses a pipette containing water at a height h. Water is allowed to drip into a container and the change of height with time is recorded. A question and answer method of discussion is used.

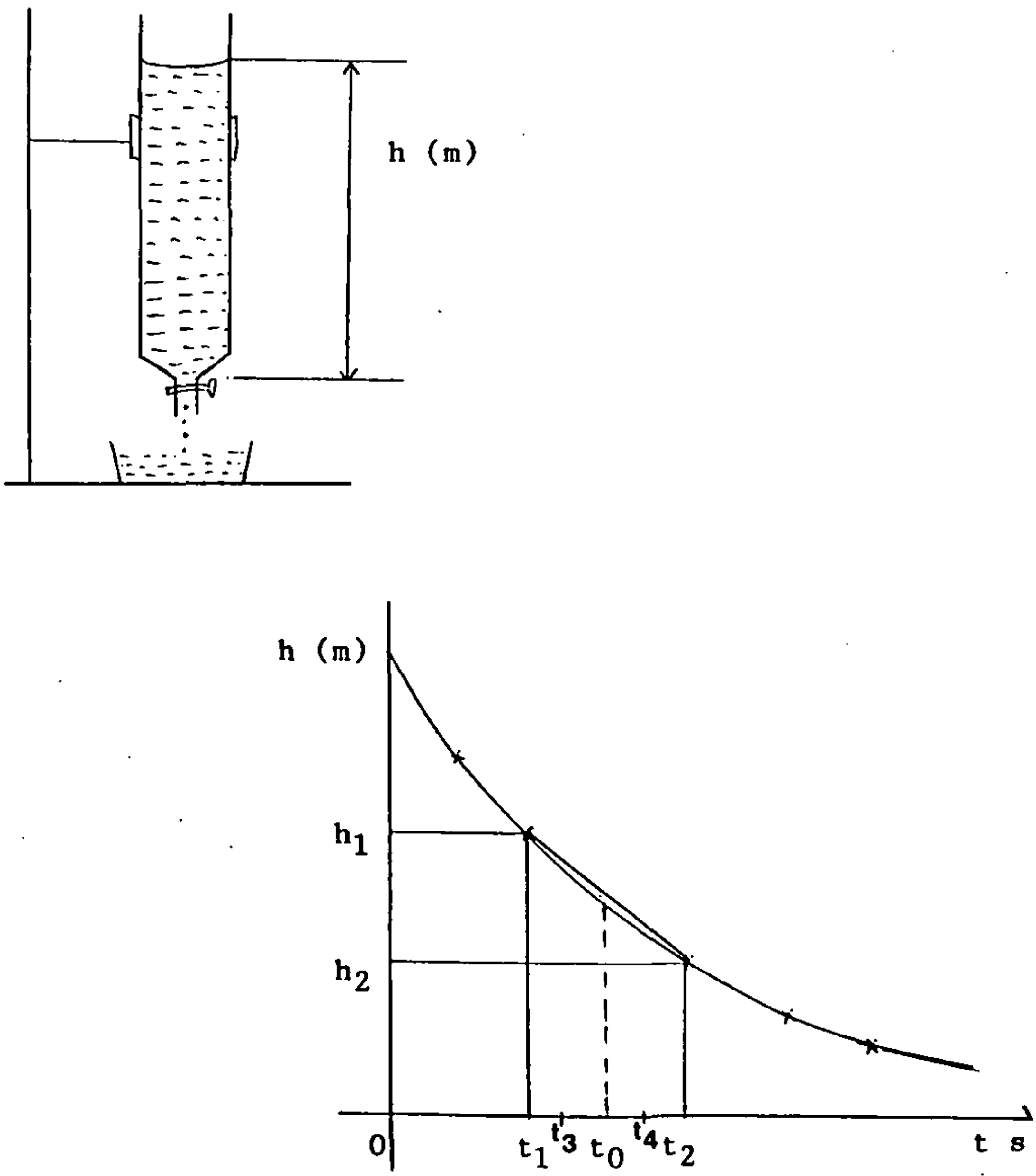


Fig. 3.5(a).

Q. What is the rate of change of height at time t_0 s?

A. We make a prediction.

Average rate of change of height = $\dfrac{h_2 - h_1}{t_2 - t_1}$

Q. Is this a good prediction of the rate of change at t_0?

A. No, this is only an approximation.

Q. How could this prediction be improved?

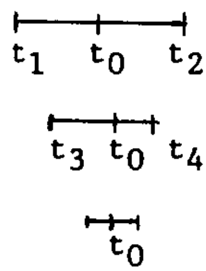
A. We could take measurements at t_3 and t_4.

Predict average rate of change of height = $\dfrac{h_4 - h_3}{t_4 - t_3}$

Q. Can we improve on this?

A. Yes, we can chose smaller intervals which include t_0.

For example:



Q. Can we do this indefinitely?

A. Not in the real world because of inadequate measuring instruments and because of measurement errors.

We need, therefore, to adopt another approach to arrive at the average rate of change of height at t_0.

Q. What can we do?

A. Make a mathematical model.

For example:

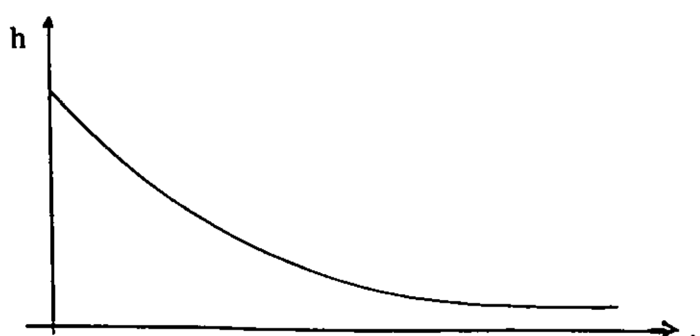


Fig. 3.5(b).

Q. What do you notice about the curve?

A. It is continuous.

Here we can pause to re-affirm the ideas of the continuous real number line as opposed to a discrete real world.

Q. What process can we use (with the curve) to arrive at our goal?

A. The process of finding the limit (gradient of the curve) at t_0.

For example:

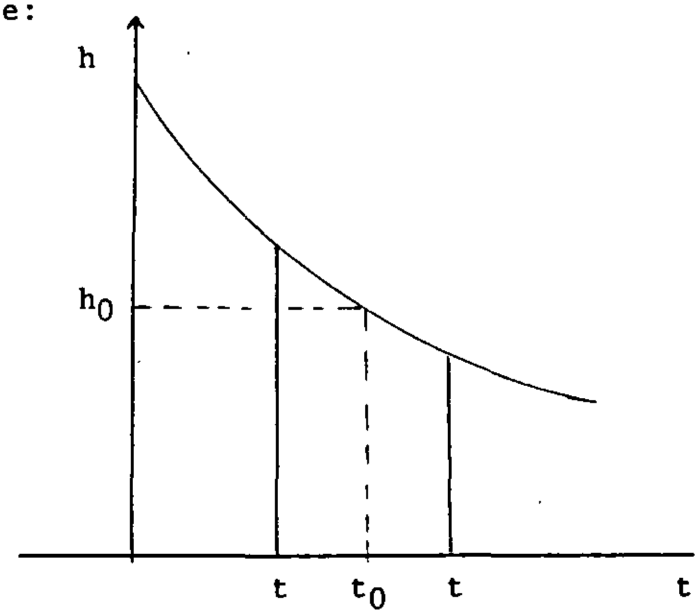


Fig. 3.5(c).

Here we need to consider two limits: one as t approaches t_0 from the right and the other as t approaches t_0 from the left. We need

$$\lim_{t \to t_0} \left[\frac{h - h_0}{t - t_0}\right] = L^+, \text{ the limit from the right.}$$

and,

$$\lim_{t \to t_0} \left[\frac{h_0 - h}{t_0 - t}\right] = L^-, \text{ the limit from the left.}$$

We can form a derivative, if and only if, $L^+ = L^-$. Denoting this limit by L, we can say that

$$L = dh/dt$$

$$L = h'(t)$$

and the function is differentiable at t_0.

Prediction of instantaneous rate of change of height at t_0 is given by dh/dt.

The concept of differentiability should not be 'laboured' too much at this stage, since it requires more rigorous analysis than that which is appropriate to the likely levels of ability of A level students.

Chapter 4: Traditional approaches to teaching differentiation.

4.1 Introduction.

It is usual for the introduction to differentiation in schools to be concerned initially with the finding of gradients of simple curves at known points: the gradient of the function $f(x)=x^2$ - much used, but simple enough to manipulate - at the point (3,6), say. The equations of straight lines and their interpretations have been dealt with in Chapter 3. It is necessary, therefore, only to refer to their importance in any attempt to teach the concepts of differentiation, at this point.

Traditionally, lessons begin with the definition of the gradient of the graph of a function at a point as the gradient of the tangent to the graph at that point. This is followed by geometrical representations of chords and their approaches to the tangent - the limiting process. Three methods of introducing differentiation are considered here:

a) The chord approach
b) The scale factor approach
c) The tangent approach.

a) The Chord Approach

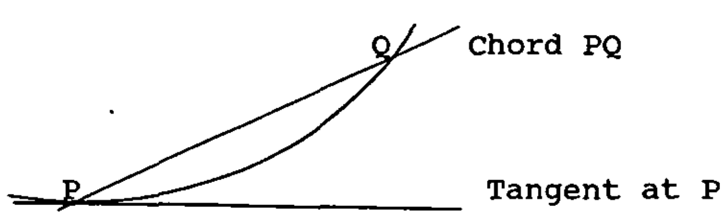


Fig.4.1(a).

This method was attributed to Newton and formed the basis of elementary calculus for schools in almost all early text books on calculus and is still widely used in current text books.

b) The Scale Factor Approach

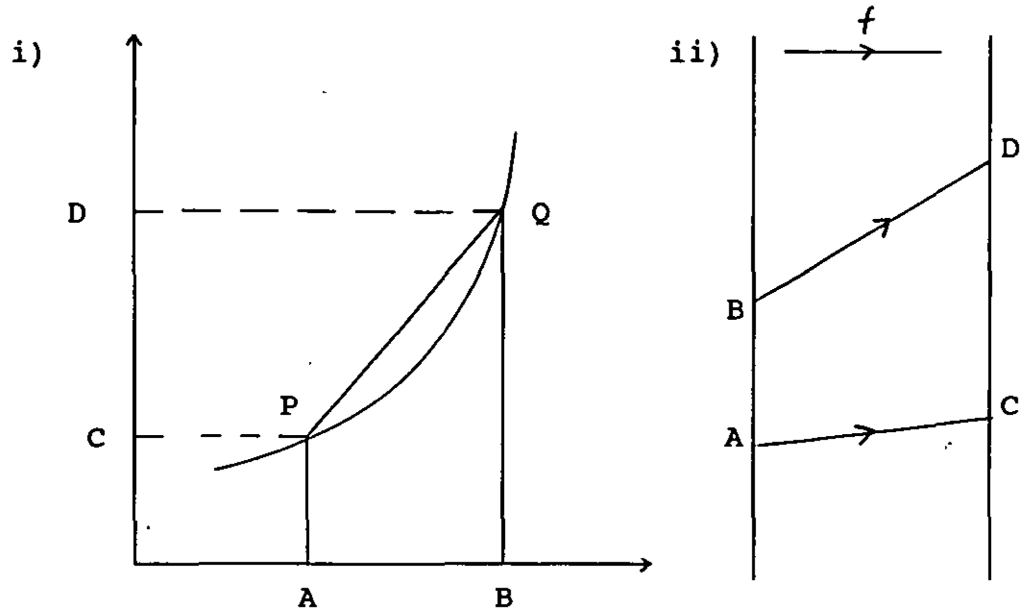
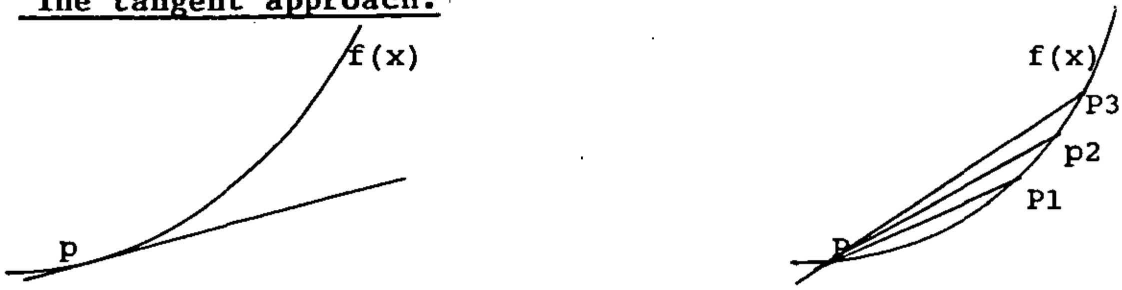


Fig.4.1(b).

Here the cartesian graph in Fig.4.1(b). - (i) and the arrow diagram (ii) are used. In (i) the ratio $\frac{CD}{AB}$ is a measure of the gradient of the chord PQ, but in (ii) the same ratio is a measure of the scale factor of the enlargement from AB to CD. This is essentially the same as the chord approach in terms of calculations, but different in terms of the geometrical representation. This method of approach to differentiation was used and pioneered by S.M.P. - Schools Mathematics Project - in their 'A' level course.

c) The tangent approach.



Best Linear Approximation

Bad Linear Approximation

Fig.4.1(c).

This approach to differentiation appears in calculus and Elementary Functions, (Montgomery and Jones), [57]. Initially this treatment was intended for 16 year olds, but elementary ideas of calculus are introduced, in some cases, at a much earlier or later age. The fundamental concepts, however, are relatively easy to impart and understand. The two simple diagrams show clearly that of all possible straight lines PP_3, PP_2, PP_1, drawn through the point P, none approximate more closely to the gradient of the function at P than the tangent at P.

4.2 The chord approach.

The three approaches to differentiation referred to above are discussed in more detail here. The chord approach is considered first.

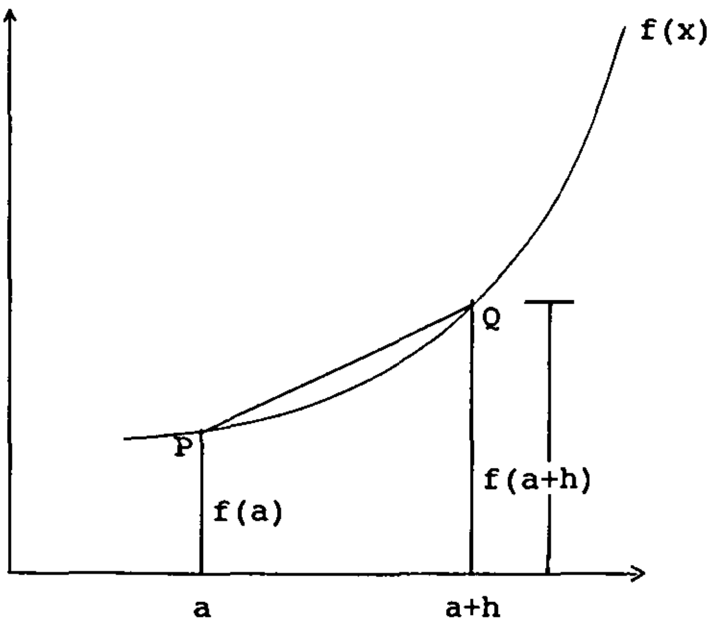


Fig.4.2(a).

The gradient of PQ for the function f(x) is given by:

$$\text{Gradient PQ} = \frac{f(a+h) - f(a)}{h} \quad \text{see Fig.4.2(a).}$$

For simple functions it is possible to tabulate the values of the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

for a given or different values of a, and a corresponding sequence of values of h. Hitherto, such calculations would have been tedious, but the easy access to calculators and computers in schools would facilitate this rather important exercise. It is good practice to choose different values of a and sequences of h for each student to investigate so that collectively, through discussion, the many different calculations would serve to give a broader understanding of the limiting process.

For example, finding the gradient of the tangent to the graph of $f(x)=x^2$ at the point (2,4), say.

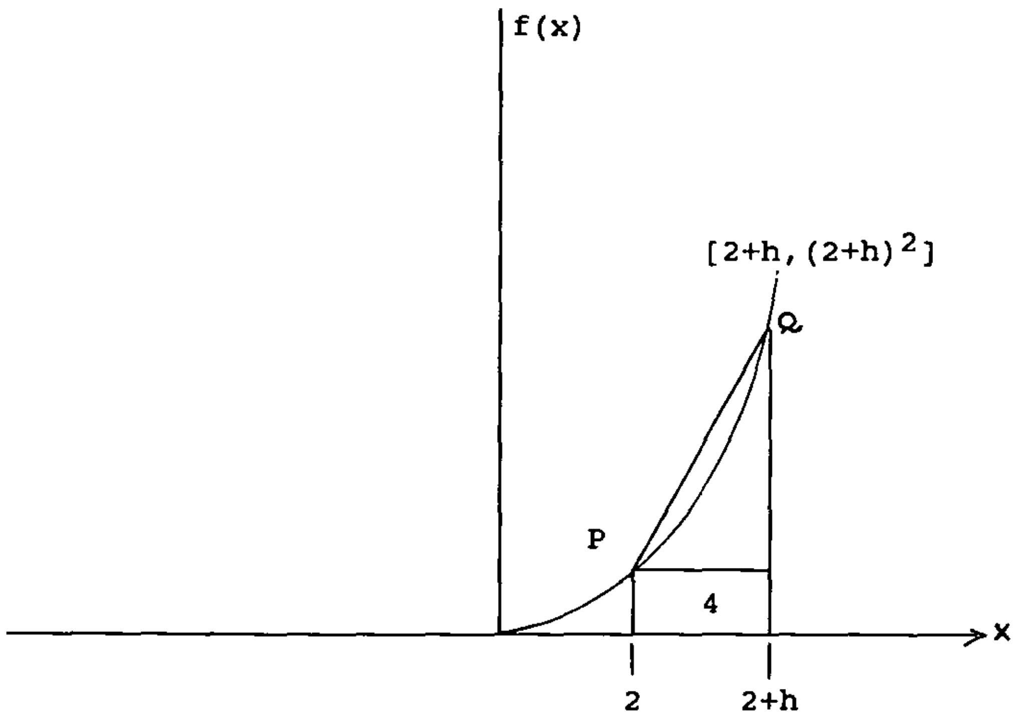


Fig.4.2(b).

P is the point (2,4)

Here: a=2

$f(a)=4$, and choose h = 4.

Let the sequence for h be that shown in the tabulation.

h	f(a+h)	$\frac{f(a+h)-f(a)}{h}$
4	36	8
3	25	7
1	16	6
2	9	5
0.75	7.5625	4.75
0.5	6.25	4.5
0.25	5.0625	4.25
0.05	4.2025	4.05
0.03	4.1209	4.03
0.02	4.0804	4.02
0.01	4.0401	4.01
.	.	.
.	.	.
.	.	.
.	.	.
0.0001	4.0004	4.0001

Table 4.2(b).

Clearly, as Q tends to P the gradient of the tangent at P tends to 4. Since h cannot be zero, it is necessary to emphasize that the closeness of this approximation to 4 is achieved by taking h as close as we please to zero.

It is only after repeated calculations of this sort that any attempt should be made to generalise. The general form of the gradient of PQ:

$$\text{gradient of PQ} = \frac{f(a+h) - f(a)}{h}$$

$$= \frac{(2+h)^2 - f(2)}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

$$= \frac{4h + h^2}{h}$$

$$= 4 + h,$$

now has some meaning. As P and Q are essentially different points and h cannot be zero, the gradient of PQ can be made as close as we please to 4 by choosing h small enough.

4.3 The scale factor approach.

Secondly, the scale factor approach appeals to concepts of scale factor and local scale factor which may prove to be an additional and unnecessary burden to most students. A reasonably sound knowledge of sets and intervals is required here. The closed interval AB[a, a+h] is mapped onto the interval CD[f(a), f(a+h)] in such a way that

A is mapped onto C
B is mapped onto D [58],

See Fig.4.3(a).

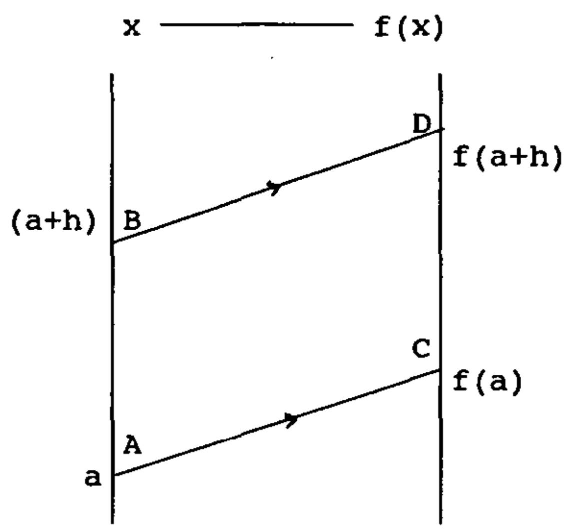


Fig.4.3(a).

The ratio $\frac{CD}{AB}$ is equivalent to the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

with CD and AB being equal to [f(a+h) - f(a)] and h respectively. The difficulties here are two fold: the problem of signs and the fact that as Q get closer to P, the quotient.

$$\frac{f(a+h) - f(a)}{h}$$

becomes increasingly difficult to visualize. Some interpretation of

$$\underset{h \to 0}{\mathrm{Lt}} \left[\frac{f(a+h) - f(a)}{h}\right]$$

is required. This is referred to as the local scale factor at P, another concept which is geometrically obscure [58].

Finally, for some functions, the case may arise in which a particular element in the domain - the closed interval [a, (a+h)] - is mapped onto an element outside the closed interval [f(a), f(a+h)]. For example, the closed interval [-1,2] for the function $f(x)=x^2$

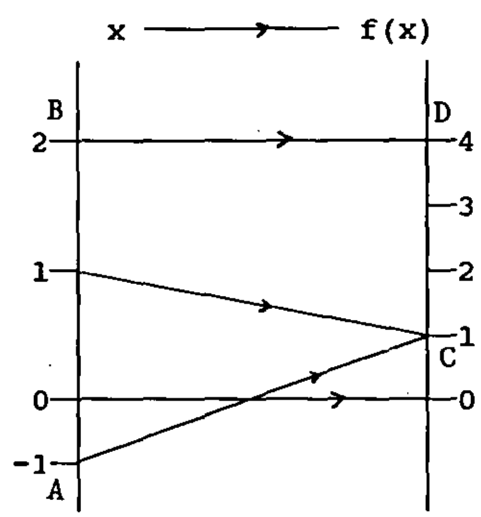


Fig.4.3(b).

Here $f(1) = f(-1) = 1$

$f(2) = 4$

and

$f(0) = 0$

So whilst A is mapped onto C and B onto D, there is a point on AB which is mapped onto a point outside the interval [1,4]. The additional conceptual difficulties here, renders this method unsuitable as an introductory course in calculus - particularly for average students. It is perhaps for this reason that such an approach is absent from most texts.

4.4 The tangent approach.

The third method of approach - the tangent approach - seeks to find an equation of a straight line which is tangential to the graph of a function at a particular point. This line, if it exists, is unique. Essentially the idea hinges on showing that the best linear approximation near a particular point is achieved when a non linear function is replaced by a linear function. That linear function is the tangent to the curve at the point in question.

A discussion of smallness and error terms is necessary here. For whilst the function

$$y = x^2 + 2x + 1$$

gives a curve, the function

$$y = 2x + 1$$

gives a straight line. The function $y = 2x + 1$ is a linear approximation of

$$y = x^2 + 2x + 1$$

when x is small, since x^2 - the error term - is negligible compared to x.

So $y = 2x + 1$ is a tangent to the function $y = x^2 + 2x + 1$ at some point. That point is easily found by solving

$$x^2 + 2x + 1 = 2x + 1$$

giving, x=0. When x=0, y=1 and thus y=2x+1 is a tangent to $y= x^2 + 2x + 1$ at the point (0,1)

The following diagram shows the geometrical representation of the above.

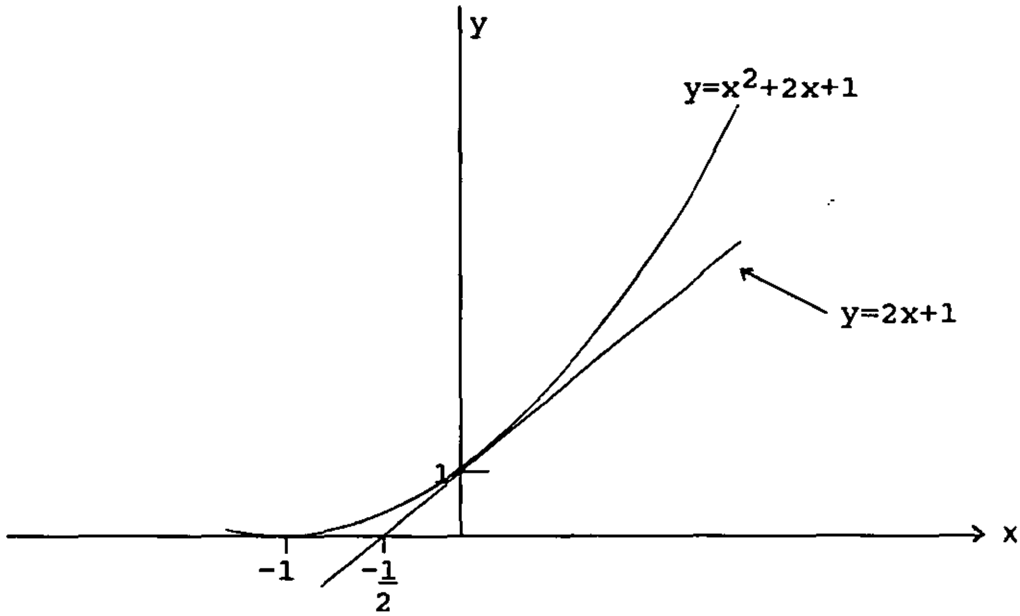


Fig.4.4(a).

The generalization of this method of approach to that of finding the equation of the tangent to a function at a general point x=a, is difficult. The level of algebraic manipulation and geometrical representation and appreciation may be beyond the scope of many students. The well-used function $f(x)=x^2$ is sufficient to demonstrate the salient points in the general case. Using equations of higher degree may serve only to confuse rather than enlighten. The following simple analysis proves an adequate explanation of the basic concepts.

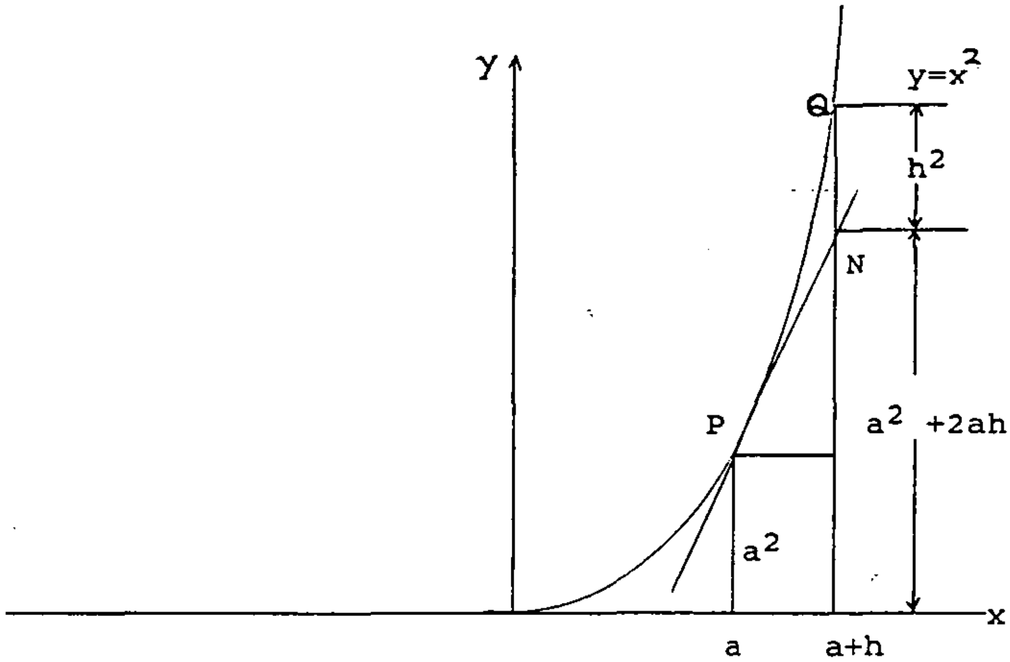


Fig.4.4(b).

For any point x=a, consider the point x=a+h - Fig.4.4(b).

Here $x^2 = (a+h)^2$

$= a^2 + 2ah + h^2$

The linear part of this expression in h - $a^2 + 2ah$- corresponds to the point N on the tangent at P. So as h becomes closer to zero, h^2 is negligible compared to h and PQ becomes PN. PN is thus the best linear approximation of the tangent at P.

4.5 Summary.

In concluding the analysis of these three major methods of approach to differentiation, it is necessary to make the following remarks:

(i) whilst all three methods are individually adequate, a choice must be made so that matching is achieved (for example, the level of ability of the students concerned might determine which method is used and when);

(ii) able and above average students should almost certainly be exposed to all three as each serves to elucidate the concept of the derivative.

Chapter 5: Traditional approaches to teaching integration.

5.1 Introduction.

5.2 The definite integral.

5.3 Areas under curves - methods of summation.

5.1 INTRODUCTION - The Reverse of Differentiation

In Chapter 2, Section 2.2, reference was made to the restrictions the lack of time put on teachers when they attempt to teach calculus and give it the full treatment it deserves. The whole concept of integration is associated with summation but time does not allow the rigours of the summative process to be thoroughly examined.

One of its aspects is the converse of differentiation and it is this aspect that is most commonly used in schools as an introduction to integration.

From this viewpoint, the problem to be solved is that of finding a function which when differentiated produces a given function. The process of finding the integral, however, is seldom as simple as it seems. For example, the function whose derivative is 2x is required. Intuitively, a reasonable guess is x^2, and indeed

$$\frac{d}{dx}(x^2) = 2x$$

but also,

$$\frac{d}{dx}(x^2 + 2) = 2x$$

and

$$\frac{d}{dx}(x^2 + \tfrac{1}{2}) = 2x$$

In short, it seems that there are a family of functions for which the derived function is identical. The function approach to calculus has been used throughout this work and the analysis of this concept would use that approach.

In general, if $y = f(x)$, the derived function

$$\frac{dy}{dx} = f'(x)$$

similarly, if $y = f(x) + c$, where c is any constant,

$$\frac{dy}{dx} = f'(x)$$

The reverse process asserts that, if $\frac{dy}{dx} = f'(x)$, and $f(x)$ is any one of those functions whose derivative is $f'(x)$, then there are a family of functions of the form,

$$y = f(x) + c,$$

each one obtained by giving c all possible values. The general function $f(x) + c$, where c is arbitrary is called the complete primitive of the derived function $f'(x)$.

The completeness of the family needs to be shown and explained. It is complete because no further function can be found such that its derivative is $f'(x)$. This is easy to demonstrate.

For if, $y_1 = f_1(x)$ and $y_2 = f_2(x)$ are two functions

for which $\frac{dy}{dx} = f'(x)$, then

$$\frac{d}{dx}(y_1 - y2) = \frac{dy_1}{dx} - \frac{dy_2}{dx} = f'(x) - f'(x) = 0$$

From our knowledge of differentiation, any function whose derivative is zero for all values of x must be a constant. It follows, therefore, that any two solutions of the differential equation,

$$\frac{dy}{dx} = f'(x)$$

can only differ by a constant. So if $y = f(x)$ is one solution, every other solution takes the form

$$y = f(x) + c$$

It is useful to show at this stage that graphs of the complete primitive are a family of curves which may be obtained by drawing any one of them and moving it at right angles to the x axis. (See fig 5.1a). For example if $\frac{dy}{dx} = 2x$ then $y = x^2 + c$.

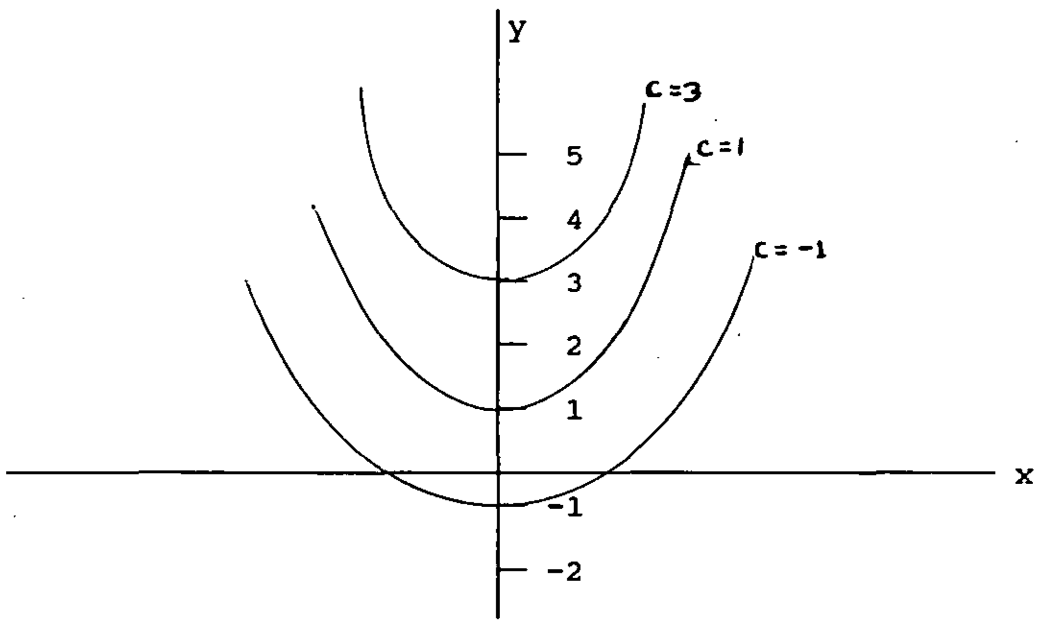


FIG 5.1(a)

Experience shows the above discussion to be invaluable especially in that it eradicates the suspicion with which the constant of integration, c, is usually viewed. It is therefore unhelpful at the initial stage to confront students with the formula

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{\_\_\_\_\_\_} \quad (i)$$

and then try to explain the relevance of c. Furthermore, the emphasis is on the reverse process of differentiation and students by this stage are usually familiar with the formula,

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{\_\_\_\_\_\_\_\_\_\_} \quad \text{(ii)}$$

Both formulas (i) and (ii) use the function $f(x) = x^n$ on which to operate. Students are often confused by this as it appears that the function and the derived function are identical. The generalization

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

should be shown to come from the considerations of the complete primitives referred to earlier. There are many simple functions for which the derived functions are known:

a) $f'(x) = 2x$ ------- $y = x^2 + c$

b) $f'(x) = 3x^2$ ----- $y = x^3 + c$

c) $f'(x) = 4x^3$ ------ $y = x^4 + c$

Students can see that if 2x is the result of the differentiation - as in $\frac{d}{dx}(x^2) = 2x$ - then the integral must contain a constant factor of x such that it cancels with the 2 in 2x

Therefore,

$$\int 2x \, dx = \frac{2x^2}{2} + c$$

$$\int 2x \, dx = \underline{x^2 + c}$$

In other words, the power of n increases by one and the reciprocal of the new power is the constant factor required. Having established this fact, it is less difficult to show and justify:

a) $\int x \, dx = \frac{x^2}{2} + c$

b) $\int x^2 \, dx = \frac{x^3}{3} + c$

c) $\int x^3 \, dx = \frac{x^4}{4} + c$

which are more direct applications of the formula,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

Finally, the integral in the above form is indefinite. This term is not easy to explain if the method of approach is to present students with the formula. However, from the previous discussion, each integral is one of the family of primitives and its precise form depends on the choice of the value c. Its "indefinite" nature is thus clearly established.

It is not easy to formulate a set of rules by which any function may be integrated nor indeed always possible to recognise the function from which the derivative was obtained. Many methods of integration exist and, in general, these consist of transposing and manipulating functions in such a way that they assume the form of standard functions whose integrals are known. The great advantage of integration is that the result can always be checked by differentiation. Students should be encouraged not to miss out this check.

Using algebraic and trigonometric methods in conjunction with the definition of the derivative students can verify that the derived functions of

$$f(x) = \sin x \qquad \text{and} \qquad f(x) = e^x$$

are

$$f'(x) = \cos x \qquad \text{and} \qquad f'(x) = e^x$$

respectively. This implies that,

$$\int \cos x \, dx = \sin x + c$$

and

$$\int e^x dx = e^x + c$$

Alternatively, the above results can be deduced from a knowledge of the power series expansion for sinx and e^x. It is assumed, here, that students are aware of such expansions.

From $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{— —}$$

we have,

$$\frac{d}{dx}(e^x) = 0+1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{— — —}$$

ie $$\frac{d}{dx}(e^x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{— — —}$$

$$\frac{d}{dx}(e^x) = e^x$$

Similarly, $$\int e^x\,dx = \int (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \_\_\ \_\_\ \_\_)\,dx$$

$$= (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \_\_\ \_\_\ \_\_) + c$$

$$\int e^x dx = (e^x - 1) + c$$

$$= e^x + (c-1)$$

$$\int e^x dx = e^x + c_1$$ where $c_1 = (c-1)$ is a constant.

The power series for sinx and cosx are given by:

$$\text{Sinx} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \_\_\ \_\_\ \_\_$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{-- -- --}$$

Students can use simple differentiation to show that:

$$\frac{d}{dx}(\sin x) = \cos x \longrightarrow \int \cos x dx = \sin x + c$$

and

$$\frac{d}{dx}(\cos x) = -\sin x \longrightarrow \int \sin x dx = -\cos x + c$$

The method of approach to the above conclusions is left to the teacher. However, it is good teaching practice to allow students to verify the results for themselves by one of the alternative methods.

5.2 The Definite integral.

Usually the definite integral is introduced in relation to areas under curves. This association stems from the work of the mathematicians of classical Greece, like Archimedes, who pioneered work in this area. Students therefore often feel that the only appreciation of the definite integral is one which relates specifically to the finding of an area under a curve. It should be emphasized that it is a method of summation.

From the discussion in section (5.1), the integral was indefinite because of the arbitrary choice of c. It is not always possible to draw the graph of the curve to which the integral refers and as such any association with a specific area is difficult to see.

The characteristic of the definite integral is its independence of the constant c, but from a teaching point of view it is better to demonstrate this, in general terms, before formulating the conclusion. A popular example is to consider the motion of a particle moving with a velocity $v ms^{-1}$.

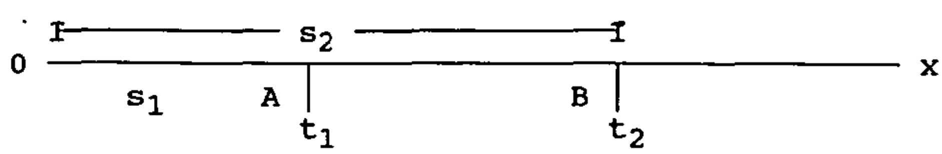


FIG 5.2 (a)

If the particle moves along Ox and is at points A and B at times t_1 and t_2 respectively with corresponding distances s_1 and s_2 from 0 - FIG 5.2 (a) - then AB represent the finite distance travelled between times t_1 and t_2. This distance is $s_1 - s_2$. Clearly both s_2 and s_1 depend on time, so we may write

$$S_2 = f(t_2) \text{ and, } S_1 = f(t_1)$$

In general $s = f(t)$ ———————— (a)

So $s_2 - s_1 = f(t_2) - f(t_1)$ is the distance travelled

by the particle in the interval t_1 to t_2.

But velocity $v = \frac{ds}{dt}$

$$v(t_2) = \frac{ds_2}{dt_2} = f'(t_2) \text{ --------- (b)}$$

and $$v(t_1) = \frac{ds_1}{dt_1} = f'(t_1) \text{ ——— (c)}$$

From (a),(b) and (c) $$s_2 = f(t_2) + c$$

$$s_1 = f(t_1) + c$$

and $$\underline{s_2 - s_1 = f(t_2) - f(t_1)}$$

So the distance travelled (s_2-s_1) is independent of the origin from which s is measured, that is, it does not depend on the value of c.

The following argument reinforces the arguments above. For if

$$v = \frac{ds}{dt} = 3t^2 - 2t$$

$$s = \int vdt = t^3 - t^2 + c$$

$$s_2 - s_1 = (t_2^3 - t_2^2 + c) - (t_1^3 - t_1^2 + c)$$

$$= (t_2^3 - t_2^2) - (t_1^3 - t_1^2),$$ which is independent of c.

Therefore, we write $s = \int_{t_1}^{t_2} vdt$ is the definite integral on the interval t_1 to t_2.

The treatment of the definite integral and its formal definition - the limit of a sum - in most texts on calculus is beyond the scope of the majority of students. It is necessary to use concrete examples, as above, to consolidate the concept. The use of the area beneath a curve should not be neglected as it is the most popular treatment of the definite integral in school texts. The usual rigorous analysis associated with it should, however, be avoided. Students can often see that the shaded area - (see FIG 5.2(b)) - corresponds to

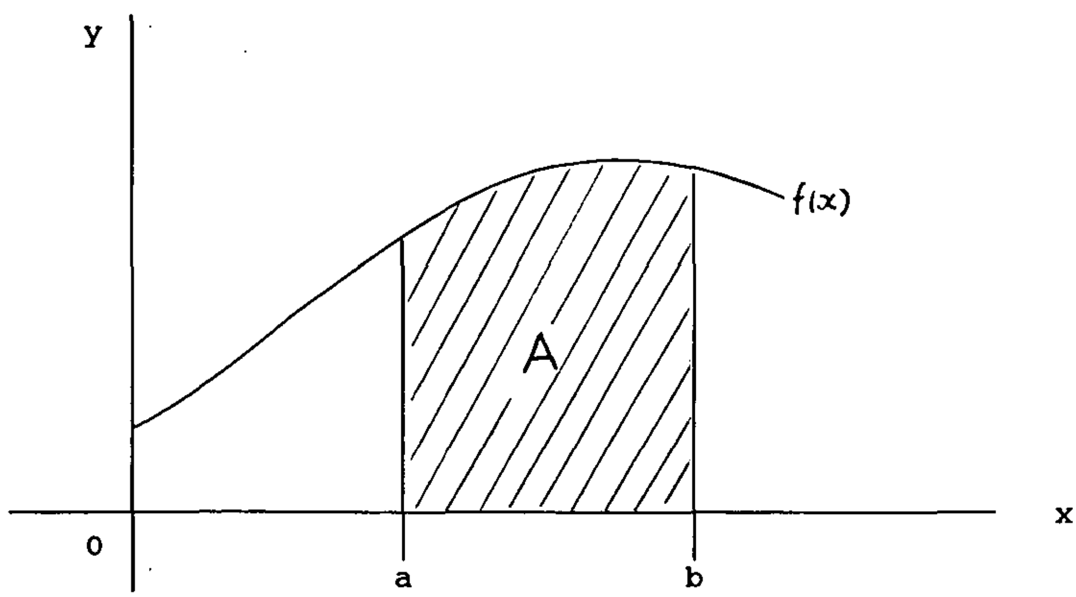


FIG 5.2(b)

the area "up" to x=b minus the area "up" to x=a. The formal definition

$$A = \int_a^b f(x)\,dx, \text{ (discussed in section 5.3)}$$

is not so easy to impart. A trivial example such as f(x)=x is helpful here.

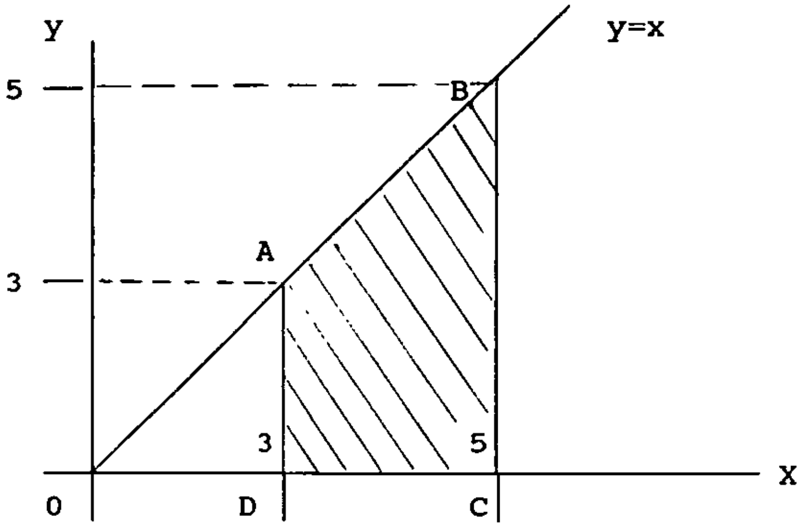


FIG 5.2 (c)

In FIG 5.2 (c) Area of ABCD = Area of ΔOBC - Area of ΔODA

$$= \frac{1}{2}(5x5) - \frac{1}{2}(3x3)$$

$$= \underline{8 \text{ units}}$$

Applying the formula $A = \int_3^5 x\,dx$, we get

$$A = \left[\frac{x^2}{2}\right]_3^5$$

$$= \frac{5^2}{2} - \frac{3^2}{2}$$

$$= \underline{8 \text{ units}}, \quad \text{as before}$$

One important observation is that there is no consistency on the treatment of definite and indefinite integral in texts on calculus. Some writers introduce the definite integral first, failing in the process to make the important link between integration and differentiation. Integration is a summation, and evaluation is through anti-differentiation using the fundamental theorem. Such an approach shows the relationship between and the importance of the processes of finding areas and of summation.

5.3 Areas under curves - methods of summation

In earlier work on areas of geometric shapes students will have found areas of rectangles and used this knowledge to discover the areas of triangles and trapezia. As a start to areas 'under' curves it is possible to draw on this past experience. In finding areas of rectangles the process is usually one of counting squares; dividing the rectangle into squares and counting them and relating the number of squares to a relationship between the length and width of the square.

A rectangle can be divided into two triangles, the area of each being half the area of the rectangle. So any triangle with base equal to the length of a rectangle and height equal to its width, has an area equal to $\frac{1}{2}$ LxB or $\frac{1}{2}$ base x height. A trapezium on the other hand can be divided into two triangles.

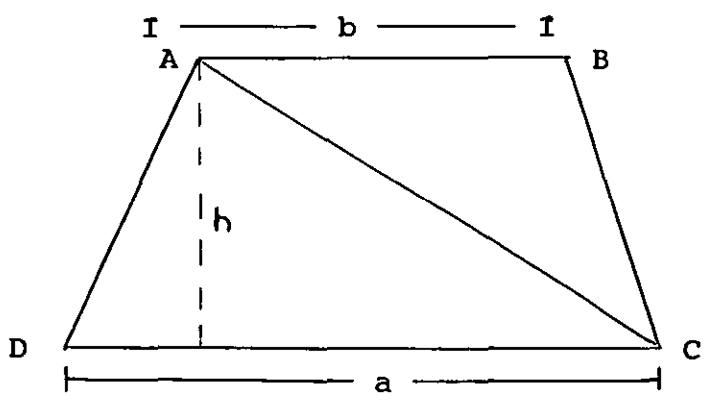


FIG 5.3 (b)

Area of: $\Delta ABC = \frac{1}{2} bh$

$\Delta ADC = \frac{1}{2} ah$

Area of trapezium = Area of Δ ABC + Area of Δ ADC

$$= \frac{1}{2} bh + \frac{1}{2} ah$$

$$= \frac{1}{2} (a+b)h$$

In FIG 5.2 (c) we saw how an area under a line could be found without the use of calculus, but calculus gave the answer quickly. A similar treatment can be applied to a rectangle provided the equations of the straight lines which make up the rectangle are known.

The most popular approach to areas 'under' curves is the one which uses a knowledge of the areas of rectangles, triangle and trapezia even though the graph or curve does not assume a straightline form. The method often employed is that of approximating rectangles or trapezia.

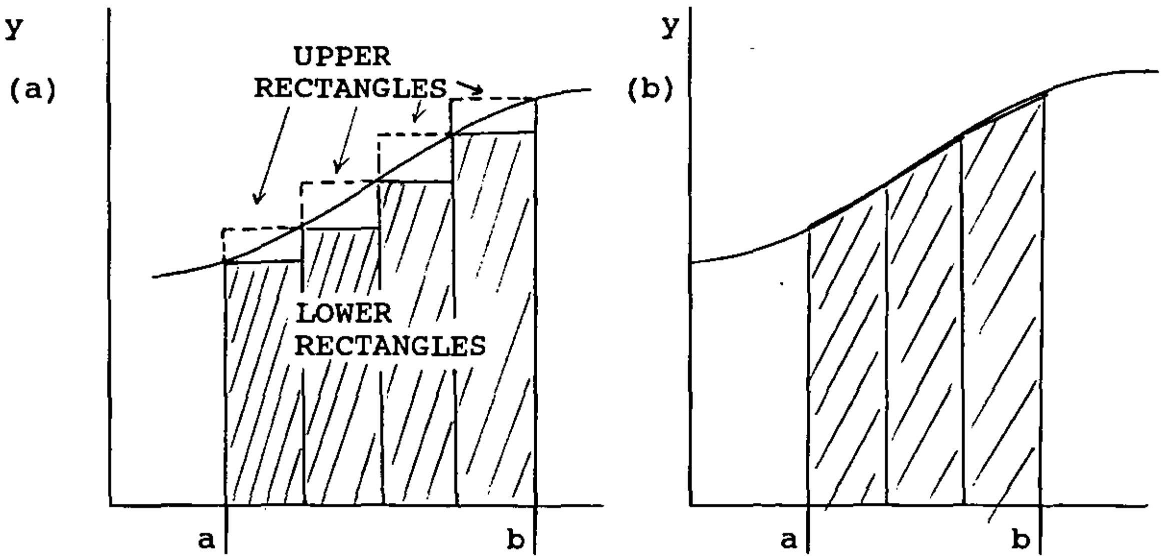


FIG 5.3 (c)

In FIG 5.3 (c) part (a), the shaded rectangles and the 'dotted' "upper rectangles" are both approximations to the area under the graph. So students can see and appreciate the inequality;

Area of lower rectangles $\leq$ Area under graph $\leq$ Area of upper rectangles.

In b) the area under the graph approximates to the area of the shaded trapezia. It is difficult here, though to establish an inequality with any degree of accuracy. It is not clear whether the area of the shaded trapezia is greater or less than the area under the graph.

The "sandwich" inequality, however, enables us to approximate as close as we please to the area under the graph by making the areas of the upper and lower rectangles as close together as we please. A simple demonstration using the function $f(x)=x^2$ would prove the point.

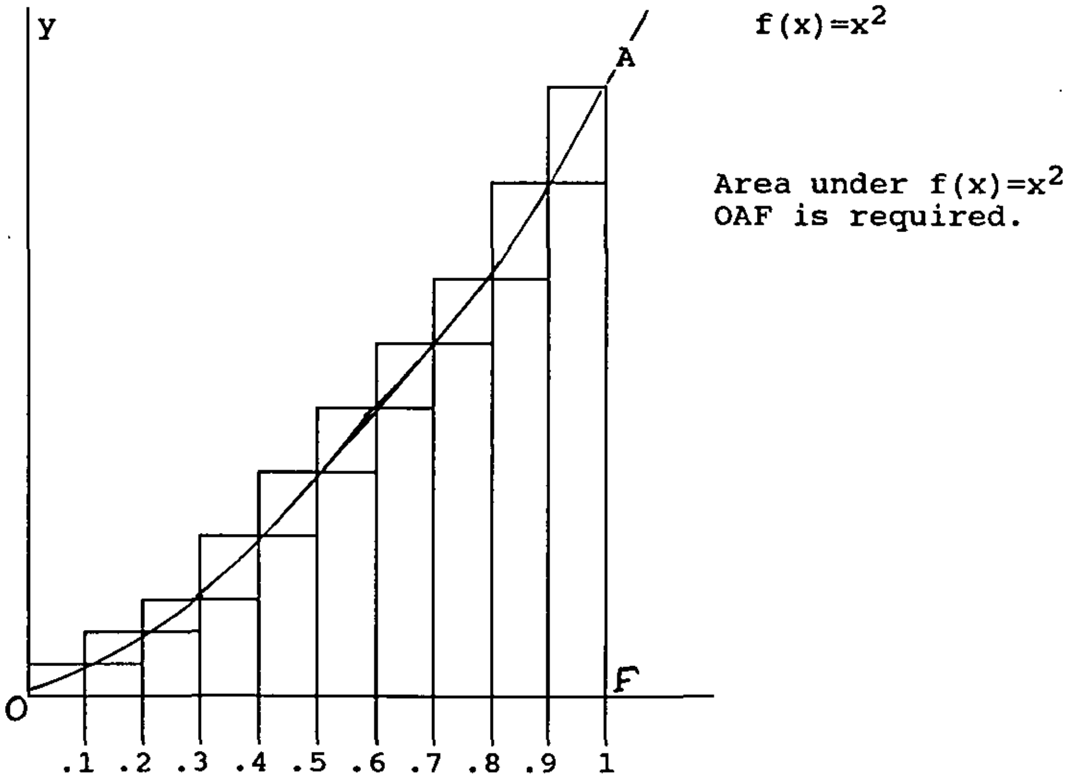


FIG 5.3 (d)

Area under lower rectangles $\leq$ Area under $f(x)=x^2$ $\leq$ Area under upper Rectangles

Using ten rectangles of width 0.1 we have:

$$0.1(0^2 + 0.1^2 +- - +0.9^2) \leq \text{Area under } f(x)=x^2 \leq 0.1(.1^2+0.2^2+--1.0^2)$$

$$0.285 \leq \text{Area under } f(x)=x^2 \leq 0.385 - - - (1)$$

Using twenty rectangles of width 0.05 we have

$$0309 \leq \text{Area under } f(x)=x^2 \leq 0.359 - - - (2)$$

From the inequalities (1) and (2) we see that the area under the curve can be made as close as we please to the true area, by adjusting the width of the rectangles. The calculations involved are made easier by the use of calculators. Nevertheless, a formal definition of area under a graph is requires if students are to be able to use a formula in other less amenable cases.

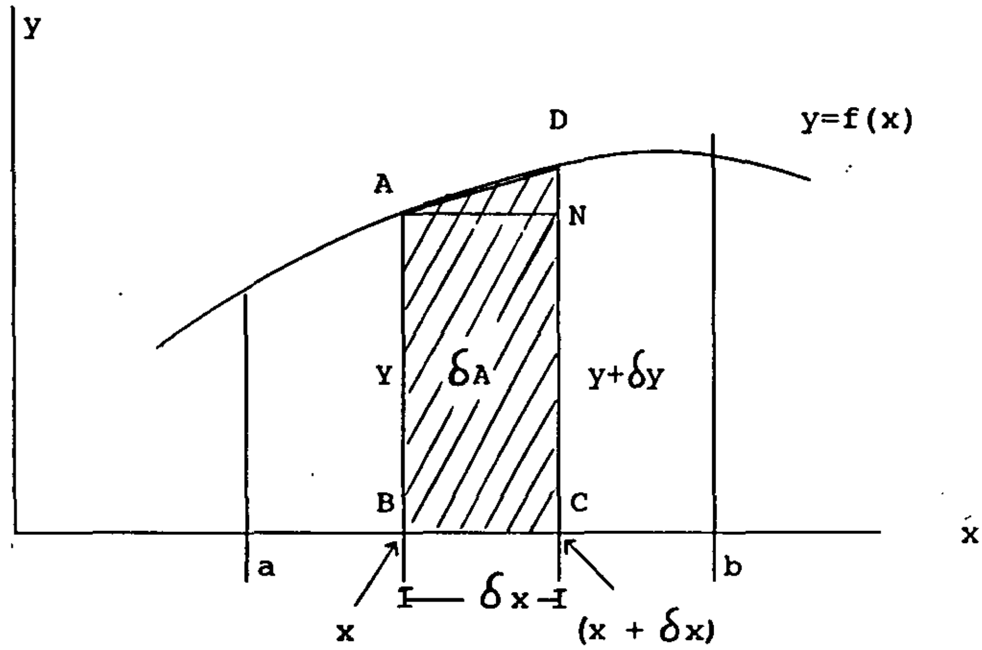


FIG 5.3 (e)

Most of the literature on areas under curves consider an element of the required area A. In FIG 5.3(e),

B is the point [x,o]

C is the point [x+δx,0]

A is the point [x,y]

D is the point [x+δx,y+δy]

When δx and δy are corresponding increases in x and y. Using previous argument y.δx is an approximation to the area of δA, so we may write,

$$\delta A \simeq y.\delta x$$

The limiting process was discussed in Chapter 3, so it is reasonable here to consider what happens when δx becomes as close as we please to zero. As $\delta x \longrightarrow 0$ the area of $\triangle$ AND decreases to the extent that δA becomes very close to y.δx, Considering the required area as the sum of all elementary areas $y.\delta x$,

$$\text{Area } A \simeq \sum_{a}^{b} y.\delta x.$$

As $\delta x \to 0$,

$$\lim_{\delta x \to 0} \sum_{a}^{b} y.\delta x = \int_{a}^{b} y \, dx$$

or,

$$A = \int_{a}^{b} y \, dx$$

This approach has the advantage of introducing students to the idea of integration as a summation. Practical work with calculators as in the previous example (FIG 5.3 (d)) also reinforces the idea of summation. The first method uses the ideas of the reverse process and the indefinite integral. Both methods embrace ideas about integration which are complementary and students should be made aware of these.

Chapter 6: New approaches to integration and deferentiation.

6.1 Introduction.

Some of the traditional approaches to calculus have been discussed. To some extent the method of approach to calculus in schools is dependent upon the attitude of the teacher. He can either introduce the subject with an understanding that he, the teacher, knows all about it and the students must learn it, or he can attempt to justify the first steps of the subject in the light of the students' previous experiences.

Shuard and Neill [18] (p 63), discuss a method of approach to integration without a knowledge of differentiation. They argue that the availability of calculators in schools make it possible for students to have first-hand numerical experience of finding areas under graphs. They acknowledge, however, that the unwieldiness of the resulting algebra is beyond the scope of most students and, as such, a more suitable method is required. They also discuss a method of finding the derivative of $f(x) = x^n$ (p 37). This hinges on the definition of the derivative

$$f'(a) = \underset{h \to 0}{\mathrm{Lt}} \frac{(a+h)^n - f(a)}{h}$$

Here, too, the algebra is tedious and may well be beyond the capability of most students. The conclusion depends on prior knowledge of the binomial theorem or Pascal's triangle of binomial coefficients. Teacher-led discussion could show, however, that the above definition reduces to

$$f'(a) = na^{n-1}$$

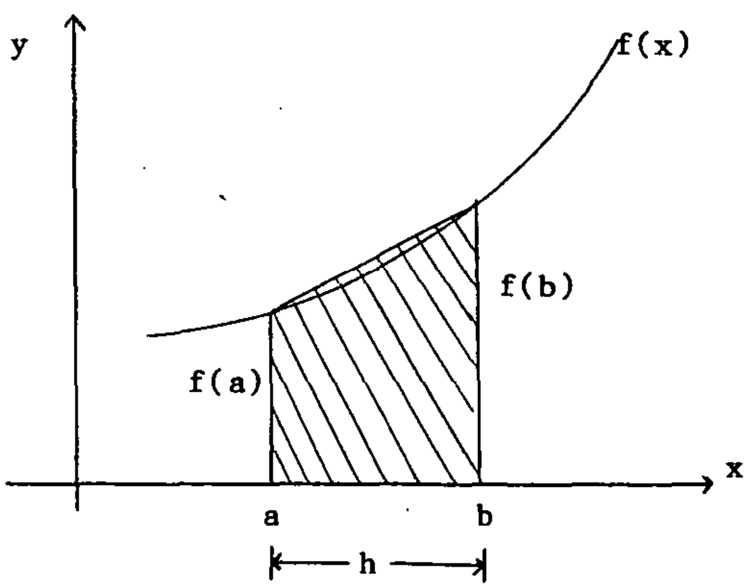
provided h is small enough. Both of the above methods involve a numerical element but whilst suitable for some students, may be inappropriate for others because of the algebraic content.

6.2 Numerical approach.

Here we look at how, in the light of the new technology, a numerical approach provides a suitable introduction to a first course in calculus. In most cases it is simple, easy to understand and appeals to concepts with which most students are familiar. We have already discussed a numerical approach to the derivative (p 57). There we showed how simple calculation by calculator or computer could lead to the derivative of $f(x) = x^2$, at $x = 2$, using the idea of gradient which is familiar to most students. This numerical work, then, has two aims: it introduces students to the ideas of the derivative and the derived function in very concrete situations; it facilitates, by their own intuitive generalisation, that for

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$
$$\text{and } f'(a) = na^{n-1}$$

We have also discussed one method of approximating the area under a curve (p 77). There we used the "sandwich" inequality and the process of approximating rectangles. Many elementary accounts of numerical integration, however, use the trapezium rule [65]. We illustrate this rule using one interval only.



$$\int_a^b f(x)\ dx \doteqdot \frac{1}{2}\ h(f(a) + f(b))$$

Fig.6.2(a).

The trapezium rule uses a linear approximation to the function (i.e the chord over the interval [a,b]). In a practical application, however, we may specify any number of subdivision of the interval [a,b]. The growing availability of programmable calculators and microcomputers in schools makes this sort of analysis totally feasible, rightly shifting the emphasis from the production of answers by standard integration to interpretation of results. This rule can be used to obtain integrals which give a lead to theory not yet established. For example, we can approach the logarithmic function by evaluation of

$$\int_1^a \frac{1}{x}\,dx \text{ for } a=1,2,3,\ldots..$$

or investigate π by evaluating

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

We examine the latter as an example of a typical A level question.

Example.

Use the trapezium rule with eight strips to find an approximation for

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

Compare your answer with the exact value of the integral to find an estimate for π.

In this case we use strips of width 0.125 (i.e h=0.125) and appeal to the generalisation of the rule:

$$\text{Area } A = \frac{h}{2}[y_0+y_n+2(y_1+y_2+........+y_{n-1})]$$

$$\left[\text{Area } A \simeq \int_a^{b=a+nh} f(x)\ dx\right]$$

With the help of calculators students can verify that for

$$y = \frac{1}{1+x^2}$$

$y_0 = 1$	$y_1 = 0.9846$
$y_2 = 0.9412$	$y_3 = 0.8767$
$y_4 = 0.8$	$y_5 = 0.71791$
$y_6 = 0.64$	$y_7 = 0.5664$
$y_8 = 0.5$	

Therefore area
$=0.5(0.125)[1+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)$
$+0.5]$
$=0.7847$
The exact value of the integral is $\pi/4$, hence an estimate of is given by

$$\pi \simeq 4(0.7847)$$
$$\underline{\simeq 3.1388}$$

The challenge now is for students to try and find closer estimates by taking more strips.This exercise is useful although it would not give the best estimate of π to three decimal places. It provides, however, opportunity for discussion of results and an appreciation of integration as a summation.

6.3 Applications of differentiation.

In discussing the applications of differentiation, the rules of differentiation such as the product, function of a function and quotient rules will be assumed where necessary. Emphasis is placed on those areas which appear in the introductory courses on calculus and often form part of first examinations. We will be mainly concerned with the applications of differentiation in finding rates of change, maximum and minimum values and elementary curve sketching.

Classifying points of zero gradients.

In Chapter 3 the equation of straight lines of the form $y=mx+c$ was discussed. The gradient of any such line is zero if $m=0$.

The equation then takes the form

$$y=c, \text{ a constant.}$$

In other words, whatever the change in x, the value of y remains unchanged. Elementary co ordinate geometery tells us that any line of the form

$$y=k, \text{ a constant}$$

is parallel to the x-axis. It is necessary, therefore, for zero gradients to be associated with horizontal lines.

Furthermore, if $y=mx+c$ is the equation of the tangent to $f(x)$ at $x=a$, then

$$f'(a) = m$$

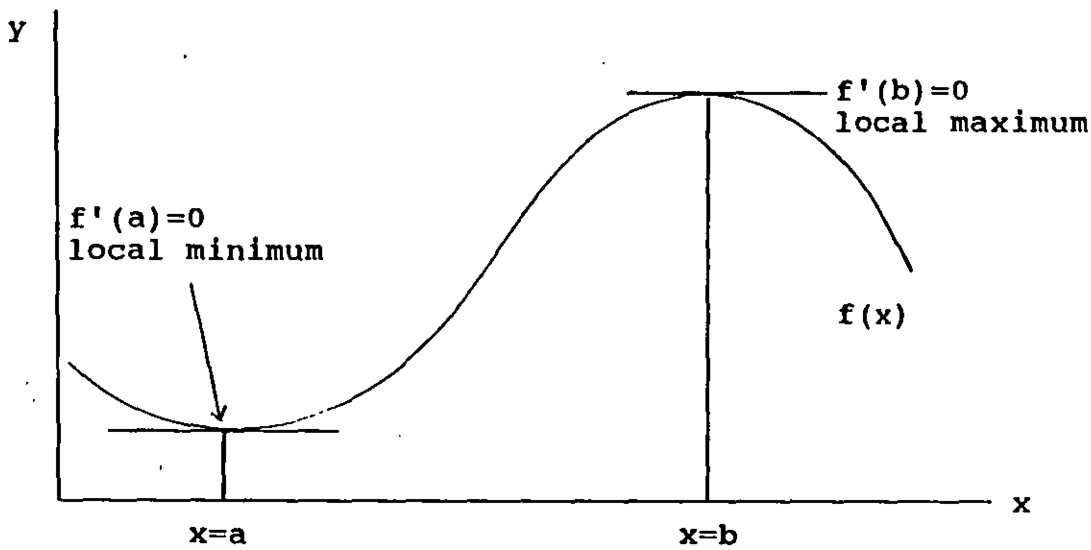
The condition, therefore, for the function $f(x)$ to have zero gradient at a point $x=a$, is that

$$f'(x)=0$$

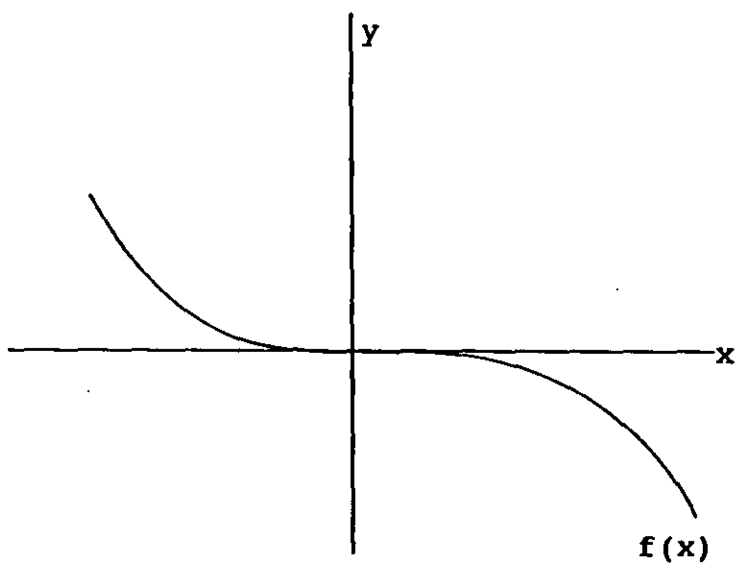
at $x=a$, or $\qquad f'(a)=0$

The whole discussion of zero gradients of tangents relates to the investigation of points where a function turns and a consideration as to whether these points correspond to a maximum or minimum value. We define a turning point to be one about which the gradient of the tangent changes sign. We also look at horizontal points of inflexion at this stage.

i)



ii)



points of inflexion.

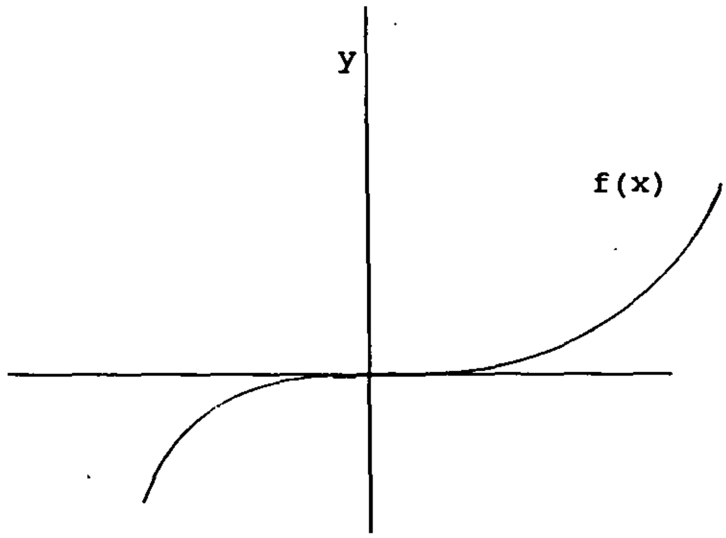


Fig.6.3(a).

As a prelude to the discussion it is helpful to briefly recapitulate on the gradients of tangents with specific reference to positive and negative gradients. It is insufficient to simply say that lines which pass between the first and third quadrants have positive gradients, whilst those which pass through the second and fourth quadrant have negative gradients. It is also unsatisfactory to say that the gradients can be got from the equations of the straightline. It is evident that

$$y = 2x + 3$$

and

$$y = 5 - 4x$$

have a positive gradient of 2 and a negative gradient of -4 respectively, but in the whole discussion on the nature of turning points, the equations of the tangents in the neighbourhood of stationary points are not known. The conclusion is based entirely on positive and negative slopes or gradients. A simple cartesian graph would clarify the point.

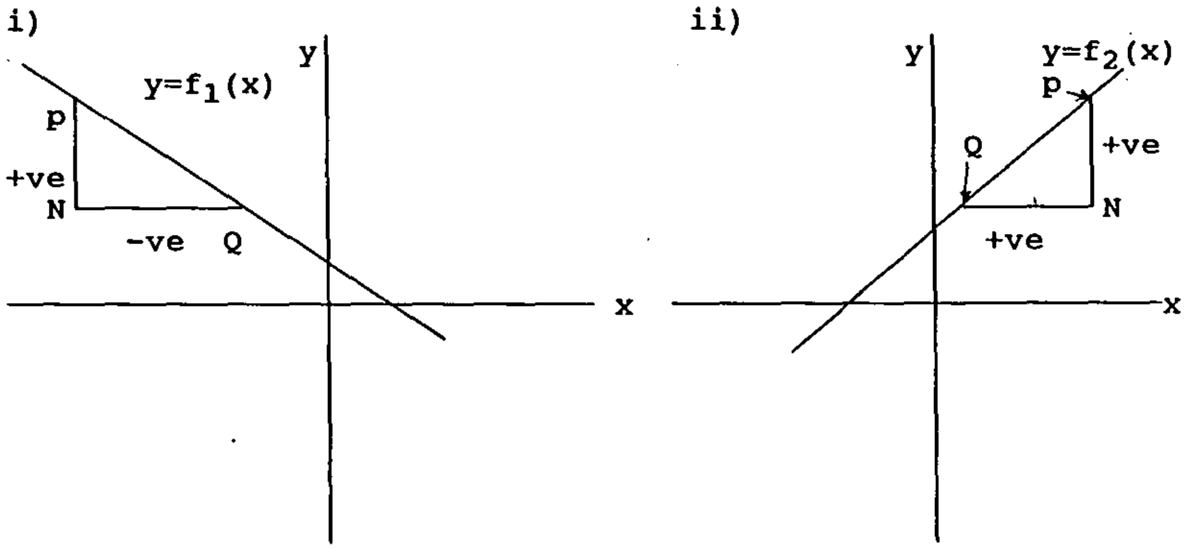


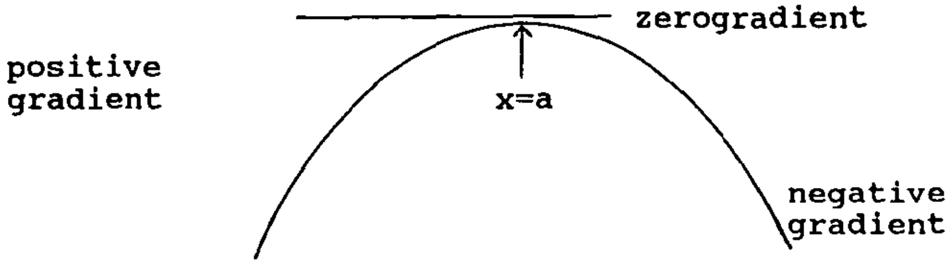
Fig.6.3(b).

In the above diagram members of the family of straightlines drawn from the second to the fourth quadrant - Fig.6.3(b), (i) - and the first and third quadrants are drawn - Fig. 6.3(b), (ii). In each case the gradient of each line represented by the ratio $\frac{PN}{QN}$. Using the normal sign convention, the ratio $\frac{PN}{QN}$ is negative in (i) and positive in (ii).

Some active participation by students in the calculation of similar ratios $\frac{PN}{QN}$ for different lines would serve to consolidate the idea of positive and negative slopes or gradients. Finally, the relationship between the gradient of the tangent and f'(x) at the point x=a needs to be repeated here. It is helpful to remind students that f'(a) is the gradient of the tangent to the curve f(x) at x=a. The discussion of the nature of turning points of a function can now begin in earnest.

The four different points of zero gradients were shown in Fig.6.3(a). The contours of these are used here:

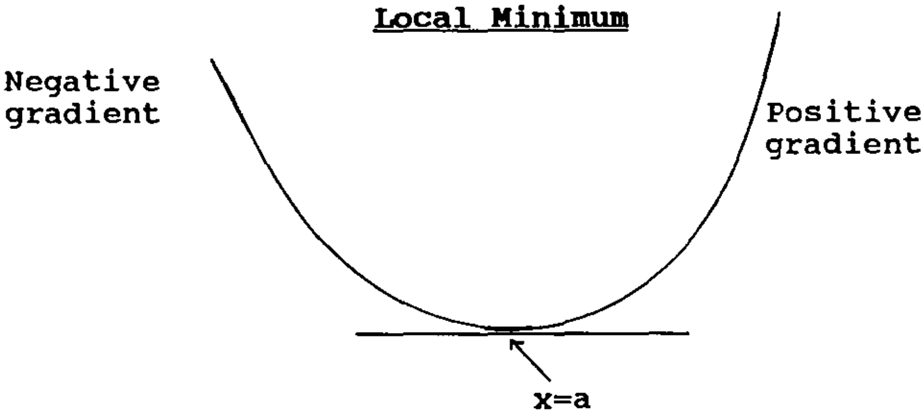
i)



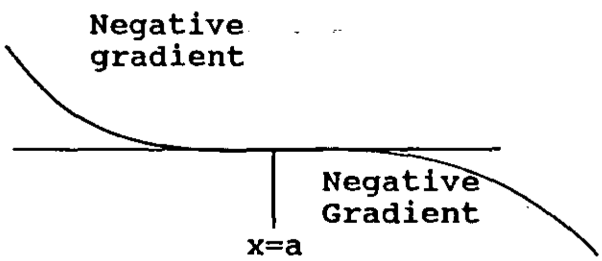
Local Maximum

ii)

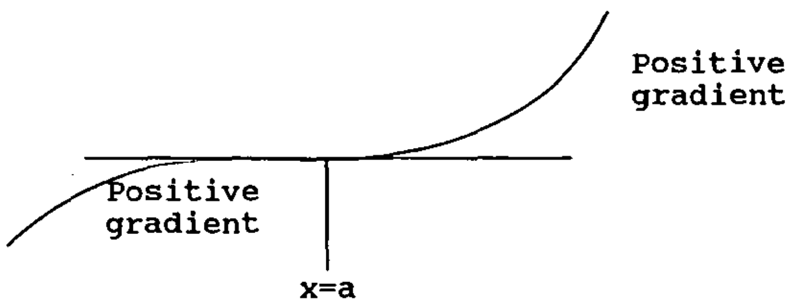
Local Minimum



iii)



iv)

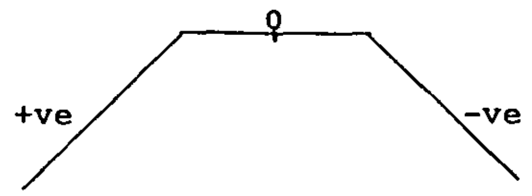


POINTS OF INFLEXION

Fig.6.3(c).

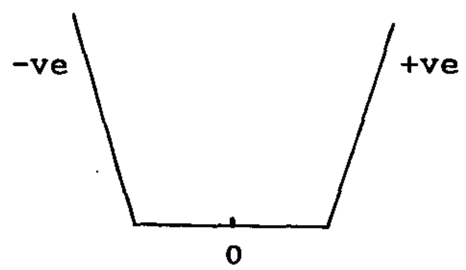
Some modern texts on calculus summarise diagram 6.3(c) in the following way:

i)



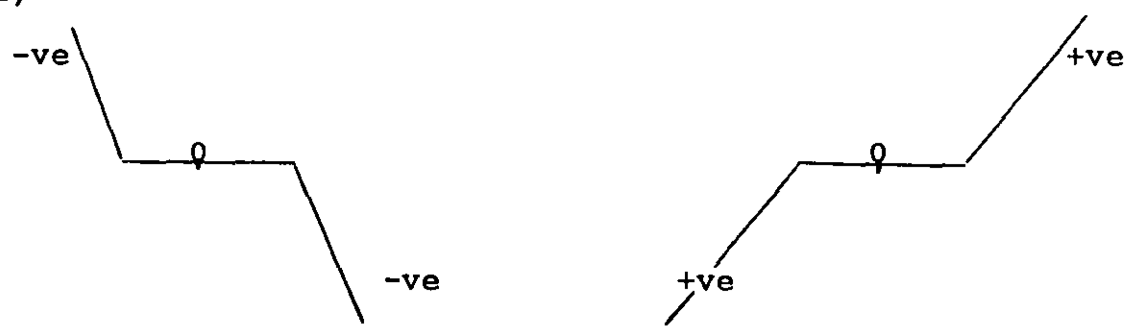
LOCAL MAXIMUM

ii)



LOCAL MINIMUM

iii)



POINTS OF INFLEXION

Fig.6.3(d).

The above diagrams refer to the sign of the gradient of the tangent in the neighbourhood of the point in question. Experience shows that when used in isolation it causes confusion, but used in conjunction with Fig.6.3(c), the relationship between the contour of the turning points and the behaviour of the tangents - in terms of the signs of their gradients about these points - is very clearly made and understood.

In concluding this discussion of zero gradients, the use of the terms local maximum and local minimum must be justified. More often than not the terms "maximum point" and "minimum point" are used to refer to local maximum and local minimum. This is quite misleading because often the <u>local</u> maximum is not the maximum point of the function. For instance the function

$$f(x) = x(4-x)^2 \qquad : x \in \text{Real Numbers.}$$

has a local maximum at $x=\frac{4}{3}$ and a local maximum at $x=4$. However, $x = \frac{4}{3}$ does not correspond to the maximum value of the function, nor does $x=4$ with the minimum value. See sketch in Fig.6.3(e).

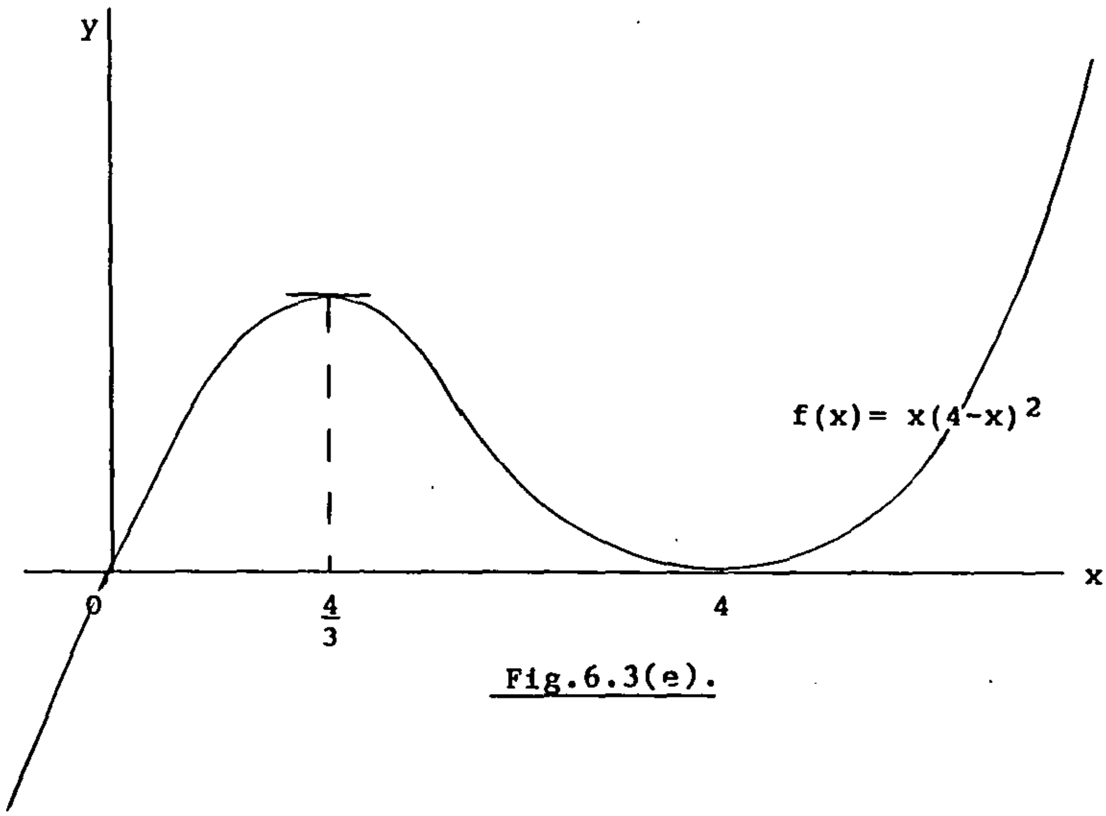


Fig.6.3(e).

Students do find it difficult, however, to apply the technique outlined so far in all but simple cases. For example

$$f(x) = x^2$$
$$f(x) = x(x-1)$$
$$f(x) = x(3-x)$$
$$f(x) = x^3 - 6x^2 + 9x - 2$$

The types of functions which cause difficulty are those of the form f(x) = Q(x) where Q(x) is a rational function. One such function is considered here.

Consider $f(x) = \dfrac{x+1}{(x+2)^2}$, $x \neq -2$

$$f'(x) = \frac{-x}{(x+2)^3}$$

But $f'(x)=0$ for only one value of x: $x=0$;
when $x > 0$, $f'(x)$ is always negative;
when $-2 < x < 0$, $f'(x)$ is always positive.

So the point of zero gradient, $x=0$, corresponds to a local maximum.

The reason for the difficulty here is their apparent inability to discuss the range of values of x for which algebraic expressions of the type

$\dfrac{x}{(k-x)^2}$ are valid.

One other type of application of zero gradients is worthy of some mention here: the type which involves an element of geometry as a requirement for the solution to be possible. For example, the finding of the maximum volume of a cone inscribed in a sphere of radius R.

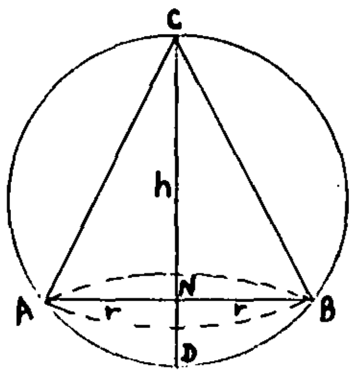


Fig.6.3(f).

with reference to Fig.6.3(f) let the height of the cone be h and radius of its base be r. The property of intersecting chords of a circle implies that

$AN \times NB = CN \times ND$

i.e. $r^2 = (2R-h) \times h$

Volume of cone ABC $= V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi h^2(2R-h)$$

$$V = \frac{1}{3}\pi(2Rh^2-h^3)$$

$$\frac{dV}{dh} = \frac{1}{3}\pi(4Rh - 3h^2)$$

$\frac{dV}{dh} = 0$ when $h=0$, or $h= \frac{4}{3}R$

The volume V assumes the value zero when h=0 or h=2R. The point of zero gradient occurs in this range. Furthermore, V is positive for all values in the range $0 < h < 2R$. Hence the point of zero gradient $h=\frac{4}{3}R$ corresponds to a maximum value and

$$V_{max} = \frac{32\pi R^3}{81}$$

The reasons for the difficulties in problems or applications of this type are the following

i) failure to recognise the geometry of the figure;

ii) the volume equation cannot be differentiated as it stands because r and h are functions of each other;

iii) the concept of local maximum or minimum does not apply here; the points of zero gradient correspond to maximum or minimum values (discrete values).

Teachers must ensure that applications of the theory and use of zero gradients include problems of this kind.

<u>The Second derivative - its meaning and use.</u>

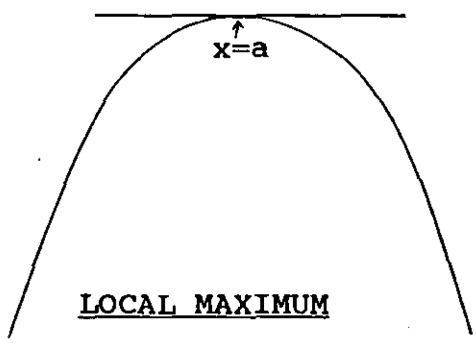
The main thrust of the arguments here is the use of the second derivative as a means of classifying points of zero gradients. In terms of a definition, the second derivative, $f''(a)$, is given by

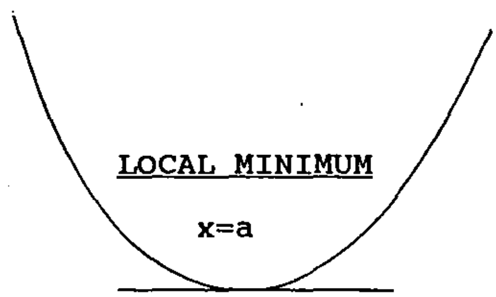
$$f''(a) = \underset{h \to 0}{\mathrm{Lt}} \frac{f'(a+h) - f'(a)}{h}$$

This definition, however, is somewhat formidable for teaching purposes. It is better to introduce the concept of the rate of change of the first derivative. In other words it should be made quite clear that just as $f'(x)$ gives useful information about $f(x)$, so also $f''(x)$ - the second derived function - gives useful information about $f'(x)$ and hence $f(x)$.

In Fig.6.3(b) and Fig.6.3(c) the behaviour of function and tangents about the point of zero gradient was considered. There it can be seen that if $f'(a)$ is positive then the function is increasing. So if $f''(a)$ is positive then the function $f'(x)$ is increasing; that is the gradient of $f(x)$ is increasing. An increase in the gradient of $f(x)$ is associated with the graph of f(x) bending to the left; that is, in the direction of x increasing the path of f(x) inclines leftwards. Similarly, if f"(a) is negative then f'(x) is decreasing and the gradient of f(x) is decreasing. In this case the graph of the function bends to the right. A summary of the analysis so far is shown diagrammatically Fig.6.3(g)

In the direction of x increasing we have:





i) f(x) bends to the right
   f'(x) decreasing
   f"(x) < 0 about x=a

ii) f(x) bends to the left
   f'(x) increasing
   f"(x) > 0, about x=a

Fig.6.3(g).

Part (i) corresponds to a local maximum whilst part (ii) corresponds to a local minimum.

The point of inflexion is much more difficult to classify using this analysis. The characteristic of a point of inflexion is that about the stationary point the function changes the direction in which it is bending. The crucial factor here is that a point of inflexion does not necessarily occur at a point of zero gradient. (see Fig.6.3(h)).

It is not necessary to consider horizontal inflexions separately. We need only stress that the criteria for a point of inflexion is that $f''(x)$ changes sign - in all cases - about that point. We can show, therefore, that if $f'(x) = 0$ and $f'(x)$ does not change sign then $f''(x)$ must. For example:

$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$f''(x) = 6x$$
$$\text{and} \quad f'(0) = 0 = f''(0).$$

Here, $f'(x)$ is positive about $x = 0$, but $f''(x)$ changes sign. Therefore $x = 0$ corresponds to a point of inflexion.

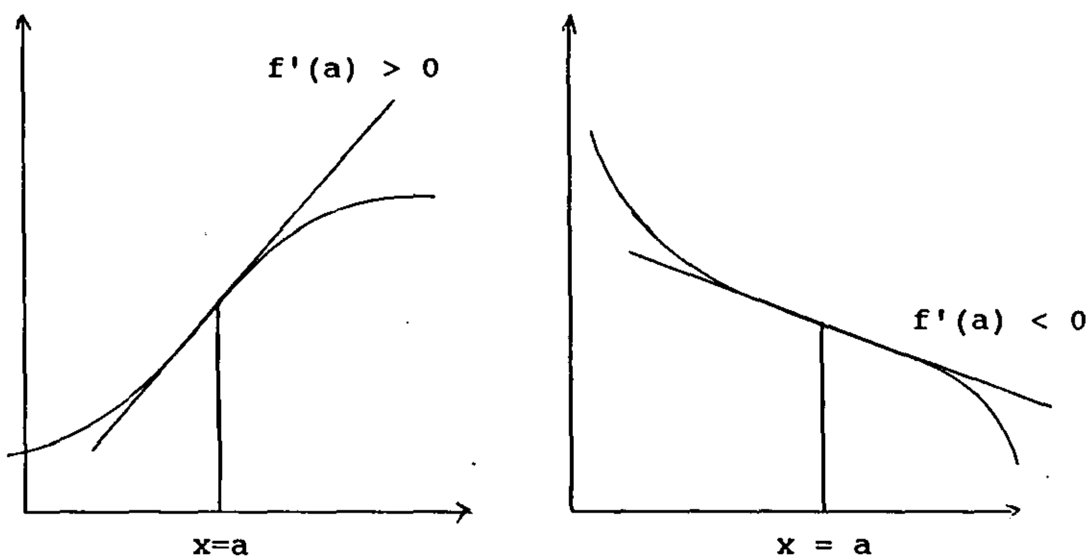


Fig.6.3(h).

Many students, however, falsely conclude that a point of inflexion necessarily occurs at $x=a$ if $f''(a)=0$. This very common error of argument can easily be corrected as it takes very little time to show that for

$$f(x) = x^4$$
$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

and

$$f''(0) = 0 = f'(0)$$

Furthermore, $f''(x)$ does not change sign about $x=0$; indeed it is positive for all values of x other than $x=0$.

A good teaching strategy is to leave the consideration of the second derivative as a means of determining the nature of turn points for a later stage of the development. In short, it should not be over-emphasized at the early stage. Students can safely and competently resolve each case by considering the sign of $f'(x)$ about $x=a$. It is necessary to remark, however, that 'non-horizontal' points of inflexion can only occur when $f''(a)=0$, since $f''(x)$ only changes sign when it goes through $f''(a)=0$. A later and more advanced treatment can appeal to Taylor's Theorem.

<u>An aid to curve sketching.</u>

This section uses two examples to demonstrate the uses of zero gradients in curve sketching. In the process it would be necessary to recall some of the relevant guidelines which assist the sketching of some functions. For example,

i) a possible domain of the function;
ii) any obvious points: what happens when $x=0$ or $f(x)=0$;
iii) behaviour of the function when x is large or small (positive or negative);
iv) points of discontinuity and the behaviour of the function near these;
v) turning points.

The above are by no means the only considerations in any systematic approach to curve sketching, nor are they all necessary for each function. Although this point is stressed, experience shows that students are often happier with the final shape of the graph of the function when all the above guidelines have been tested.

The following two examples are chosen in such away as to use all of the above considerations between them:

Sketch a) $f(x) = (x-1)(x-2)(x-3)$

b) $f(x) = \frac{2}{x-1} + \frac{2}{4-x}$

Sketch a) $f(x) = (x-1)(x-2)(x-3)$

when $x=1$, 2 or 3, $f(x)=0$

when $x=o$, $f(x) = f(0) = -6$

Expanding $f(x)$ we get:

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$f(x)$ turns where $f'(x) = 0$

i.e. $3x^2 - 12x + 11 = 0$

$x = 1.42$ and $x = 2.58$ (both 2dp)

For $1 < x < 2$, $f(x)$ is positive,

and $2 < x < 3$, $f(x)$ is negative.

When x is large and positive, f(x) is large and positive.

When x is large and negative, f(x) is large and negative.

Finally, f(x) is continous everywhere.

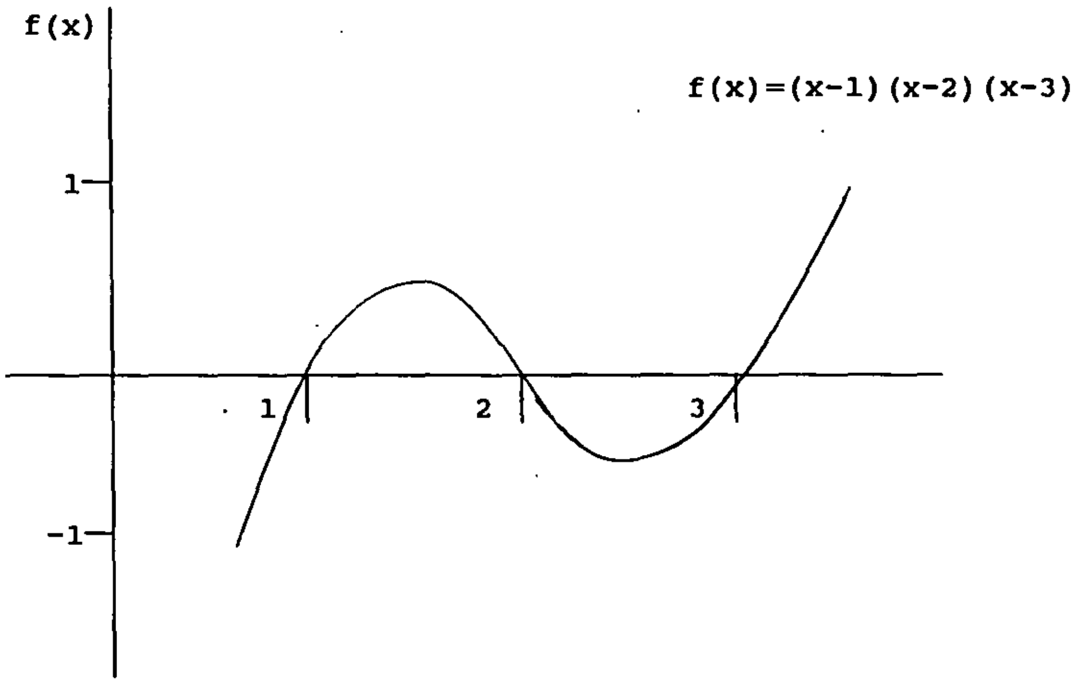


Fig.6.3(i).

The local maximum of $f(x) \simeq 0.385$, when $x = 1.42$

The local minimum of $f(x) \simeq -0.385$, when $x = 2.58$

Sketch b) $f(x) = \frac{2}{x-1} + \frac{2}{4-x}$ $\left(= \frac{6}{(4-x)(x-1)}\right)$

Discontinuities occur when $x=1$ and $x=4$

As: $x \longrightarrow \infty^+$, $f(x) \longrightarrow 0$

: $x \longrightarrow \infty^-$, $f(x) \longrightarrow 0$

For: $1 < x < 4$, $f(x)$ is positive.

: $x < 1$, $f(x)$ is negative.

: $x < 4$, $f(x)$ is negative.

When $x=0$, $f(x) = -1.5$

$$\frac{dy}{dx} = f'(x) = \frac{12x - 30}{[(4-x)(x-1)]^2}$$

Turning points occur when $f'(x)=0$

$$f'(x)=0 \quad \text{when } x=2.5$$

For $2 < a < 2.5$, $f'(a) < 0$

$2.5 < a < 3$, $f'(a) > 0$

$x = 2.5$, $f(x) \cong 2.67$ corresponds to a local minimum.

Sketch

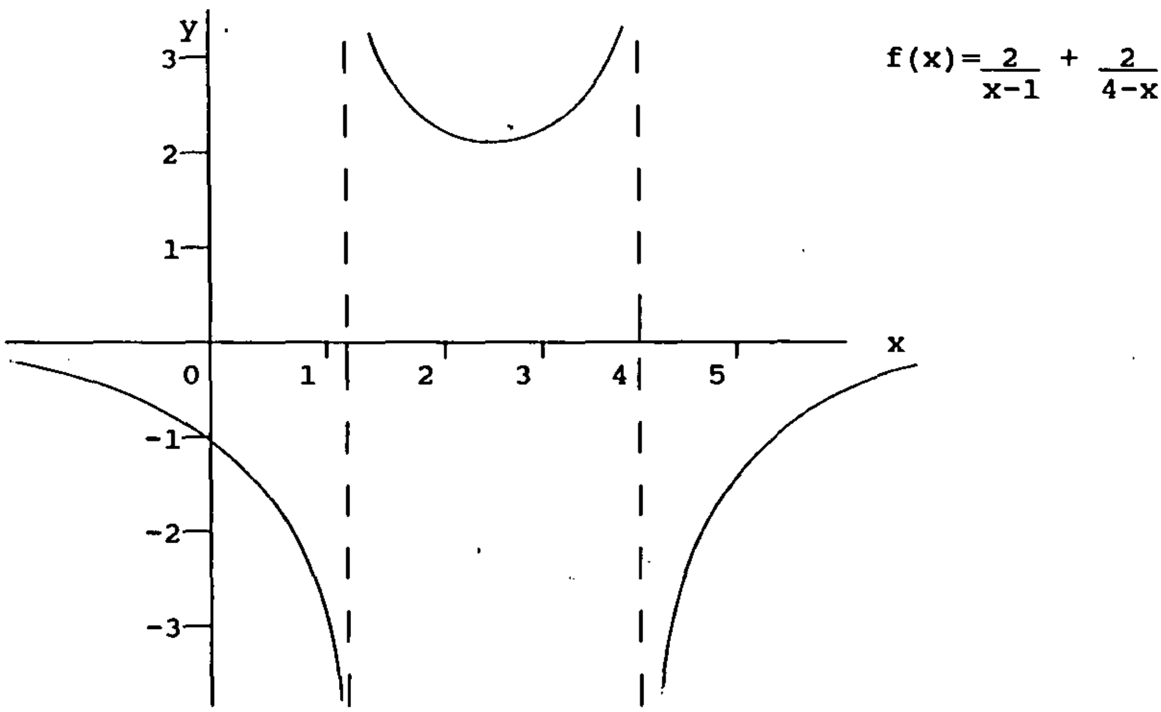
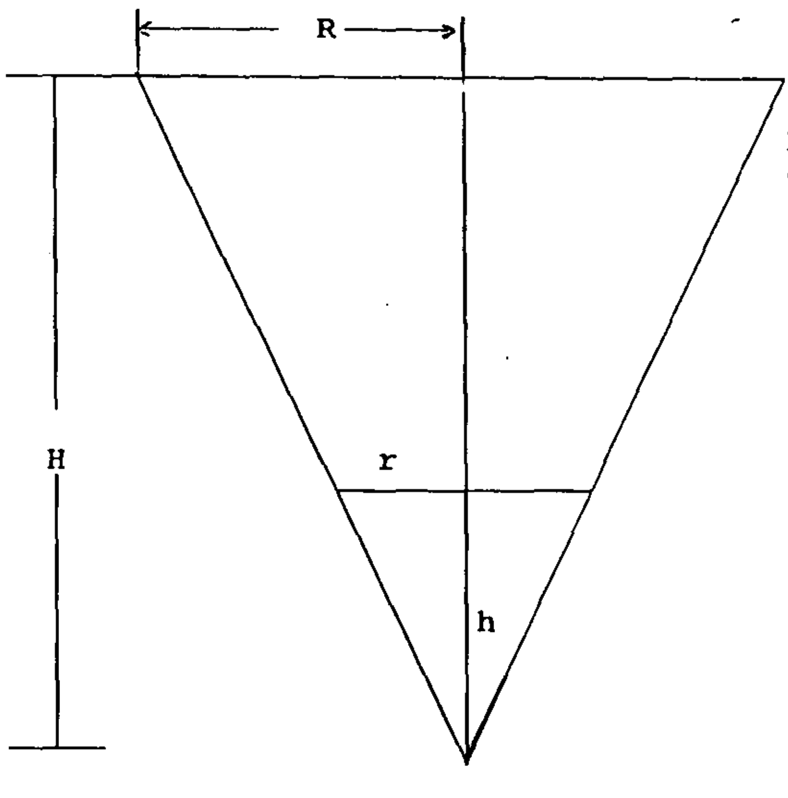


Fig.6.3(j).

Rates of change and small changes.

This is one of the most widely used applications of differentiation. It is ever present in the physical sciences and examples of such applications will be discussed in Chapter 7 - crosscurricular applications. The concept of rates of change was discussed at length in chapter 3. Here a very popular examination type application is given.

This application is chosen because it embraces the concepts of elementary geometry and rates of change and small changes. The physical situation considered is leaking tap dripping water into right circular cone at a certain rate. We wish to calculate the rate at which the surface area of the water is increasing at a certain height. The discussion would be non-specific, with any necessary assignments to quantities made, to prove that the rate of change of surface area is inversely proportional to the height of water at the time considered. See Fig.6.3(k).



H is the height of the cone. h is the height of water at time t, and r the radius of the surface of water.

Fig.6.3(k).

From the geometry of the Figure,

$$\frac{r}{R} = \frac{h}{H}$$

$$r = \frac{hR}{H}$$

The surface area of water at height h is given by

$$A = \pi r^2$$

i.e. $$A = \frac{\pi h^2R^2}{H^2}$$

$$\frac{dA}{dt} = \frac{2\pi hR^2}{H^2} x \frac{dh}{dt} \quad \text{----- (a)} \quad \text{from} \quad \frac{dA}{dt} = \frac{dA}{dh} x \frac{dh}{dt}$$

Now the volume of the water at height h is given by

$$V = \frac{1}{3}\pi r^2h$$

i.e. $$V = \frac{1}{3} \frac{\pi h^3R^2}{H^2}$$

$$\frac{dV}{dt} = \frac{\pi h^2R^2}{H^2} x \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{dV}{dt} x \frac{H^2}{h^2R^2\pi}$$

The rate at which the tap is leaking is given by, $\frac{dV}{dt}$,

suppose $\frac{dV}{dt} = 0.2 \ cm^3s^{-1}$, say.

Then $$\frac{dh}{dt} = \frac{0.2 \ x \ H^2}{\pi h^2R^2} \quad \text{------------} \quad \text{(b)}$$

From (a) and (b) we get

$$\frac{dA}{dt} = \frac{2\pi hR^2}{H^2} \times \frac{0.2\ H^2}{\pi h^2R^2}$$

$$\frac{dA}{dt} = \frac{0.4}{h}$$

In other words, the rate of increase in the surface area of water is inversely proportional to the height h.

This problem is similar to that discussed in relation to Fig.6.3(f). Evidently it involves more rigorous analysis but the underlying teaching points must not be overlooked. This type of problem embraces concepts of elementary geometry and the concepts of implicit differentiation which students usually find difficult to manipulate. Teachers need to stress the importance of the geometry of a figure to the solution of a mathematical problem to which the figure relates. In applications of differentiation students need to be made aware of the dependance of the variables involved on each other. This is crucial to the solution of the problem as can be seen in the previous example. Another difficulty- for students - arises when attempts are made to differentiate the expression

$$A = \frac{\pi h^2R^2}{H^2}$$

Both R and H are constant and the surface area, A, varies with h. It is easy for students to find $\frac{dA}{dh}$, but in trying to find $\frac{dA}{dt}$, they often fail to recognise the function of a function rule

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$$

In short, A and h are 'functions' of each other but they are both functions of time, hence the application of the above rule. Experience in the classroom shows that students need to be reminded of this fact quite often.

6.4 Applications of integration.

The discussion on applications of integration will be concerned with those areas which occur most frequently in school mathematical examinations. Other applications specific to particular curricular areas will be dealt with in the Chapter on Cross Curricular Applications. The examples chosen and the relevant discussion serve to highlight particular concerns. Any relevant formulae will be assumed.

Finding Areas Under Curves

In section (5.3) we discussed the general approach to finding areas under curves. Most of the problems using this application are concerned with finding a defined area beneath a graph of some given function. Often it is necessary to sketch the function first in order to see precisely which area is required. This is an exercise in itself and will not be considered here. Instead we study specific problems and analyse some of the difficulties that students experience.

Consider the case of finding

a) The area under the graph of $y=x^2$

b) The area bounded by the graph $y=x^2$, the y-axis and the line $y=4$

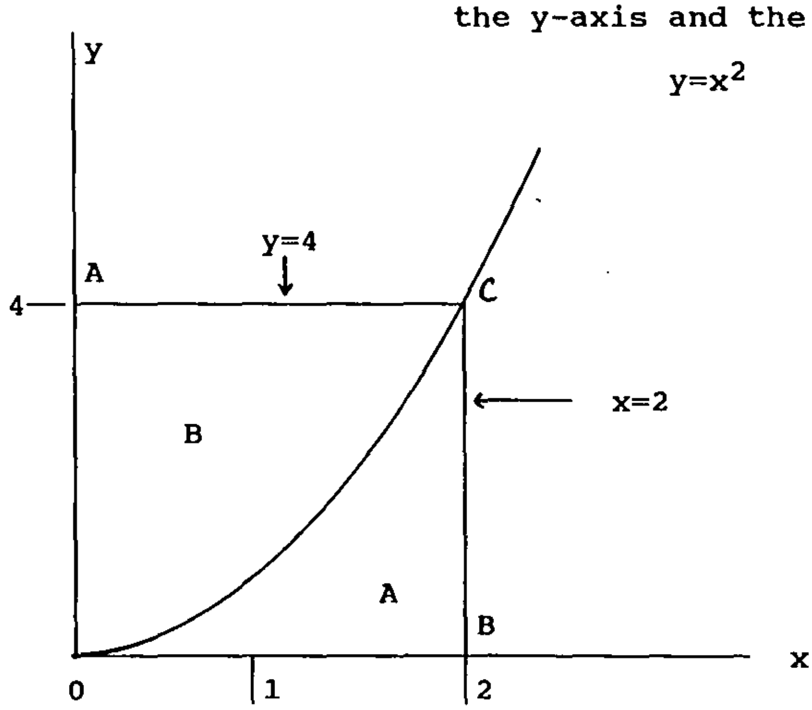


Fig.6.4(a).

The required areas for parts (a) and (b) are labelled A and B respectively - Fig.6.4(a). Most students can cope with part (a), but it is included here because it will form part of the discussion of part (b).
The required area A is given by

$$A = \int_0^2 y\,dx$$

$$= \int_0^2 x^2\,dx$$

$$= \frac{8}{3} \text{ units}^2$$

To find the area B, however, experience shows that a number of students perform the calculation for A and use the relation that

Area B = Area of rectangle AOBC - Area A

giving

$$\text{Area } B = 8 - \frac{8}{3}$$

$$\text{i.e. } B = \frac{16}{3} \text{ units}^2$$

Now this is quite correct, but students will often do this even if part (a) of the question was not given. The reason for this lies in the fact that most text books discuss the finding of areas under curves with reference to the x axis and almost without exception conclude that

$$\text{Area under curve } A = \int_a^b y\,dx$$

This is a clear case of mechanical processes versus principles. Section 2.2 refers to the need for teachers to ensure that the right balance exists between principles and mechanical processes.

The difficulty lies in students' inability to fully grasp the process by which the area is found. The whole process of finding areas revolves around considering an element of the required area and summing all such areas. Teachers need to reinforce and demonstrate this. See Fig.6.4(b).

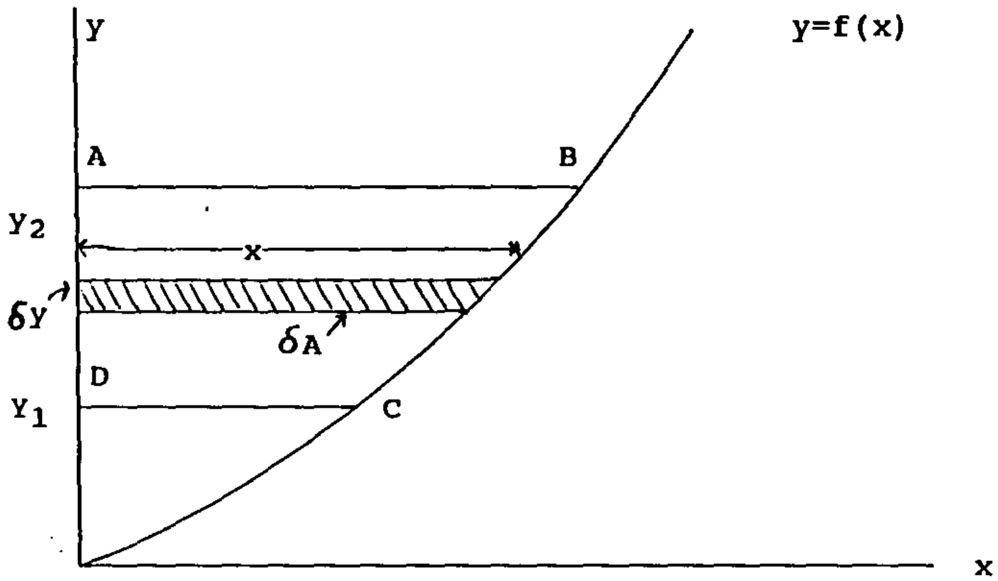


Fig.6.4(b).

The area ABCD is required and the element of area δA is given by

$$\delta A \simeq x.\delta y$$

The required limiting process is one in which $\delta y \longrightarrow 0$ and not $\delta x \longrightarrow 0$

$$\therefore \underset{\delta y \longrightarrow 0}{Lt} \left(\frac{\delta A}{\delta y}\right) = x$$

Yielding

$$A = \int_{Y_1}^{Y_2} x \, dy$$

(reinforcing summation: $A \propto \Sigma \delta A$)

This result is absolutely vital and must be stressed. The question of finding the area B is then one of using,

$$B = \int_0^4 x \, dy$$

i.e. $$B = \int_0^4 y^{1/2}\,dy$$

$$B = \left[\frac{2y^{3/2}}{3}\right]_0^4$$

$$\underline{B = \frac{16}{3}\ \text{units}^2} \quad \text{(as before).}$$

It is acknowledged that in specific questions the first method of solution may well be easier, but the point being made here is that students often demonstrate a lack of understanding of the fundamental principles in finding areas under curves.

Another example which highlights this difficulty is one in which two curves overlap. For example the area between the intersection of the curves $y=x^2-2x$ and $y=4-x^2$. A sketch of these curves is shown below.

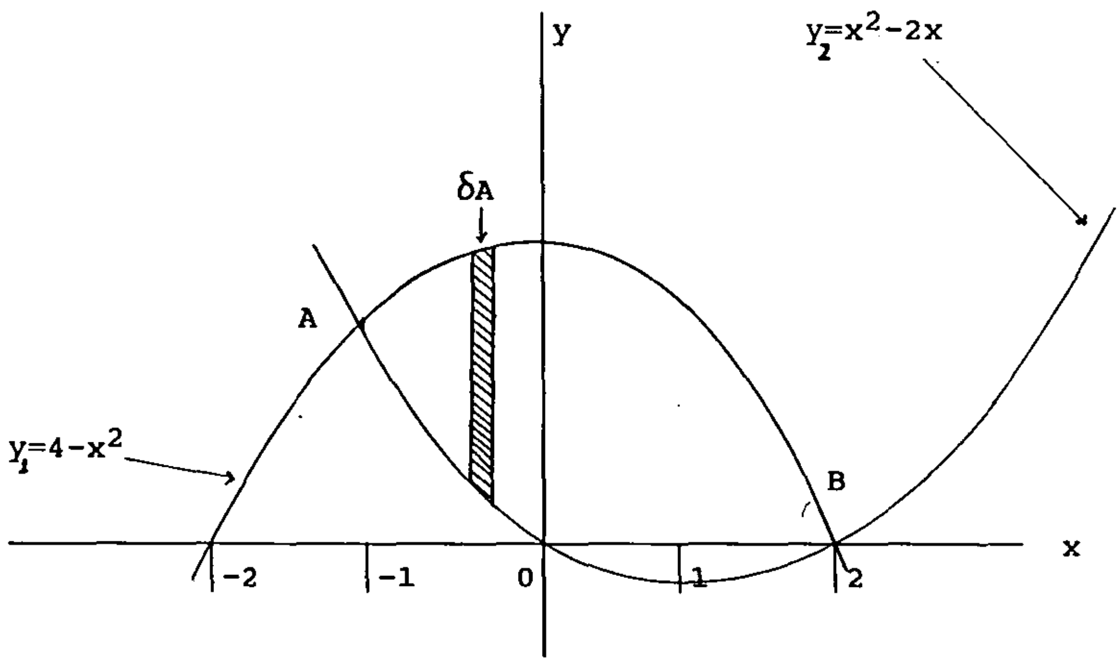


Fig.6.4(c)

The advice here should be to label each curve as Y_1 and Y_2

$$y_1 = 4-x^2$$

$$y_2 = x^2-2x$$

In addition to the difficulty under discussion, there is also the further problem of finding the limits of integration. These are obviously where the curves meet. They meet where

$$y_1 = y_2$$

$$\text{i.e. } 2x^2 - 2x - 4 = 0$$

$$\underline{x=-1} \text{ or } \underline{x=+2}$$

The element of area is shaded and $(y_1-y_2)\cdot\delta x$ is the area of this element. The required area is given by

$$A = \int_{-1}^{2} (y_1-y_2)\,dx$$

$$= \int_{-1}^{2} (4+2x-2x^2)\,dx$$

$$= 9 \text{ units}^2$$

The detail of the solutions is not a priority here. It is the difficulties which students encounter which are of some concern. These types of question occur frequently at examination level and more often than not the solutions submitted by students reveal most clearly that the underlying principles and techniques referred to here have been misunderstood. It is very important that adequate discussion and demonstration accompany the teaching of areas under curves as this application is typical of many arising in other branches of science.

Velocity and Acceleration

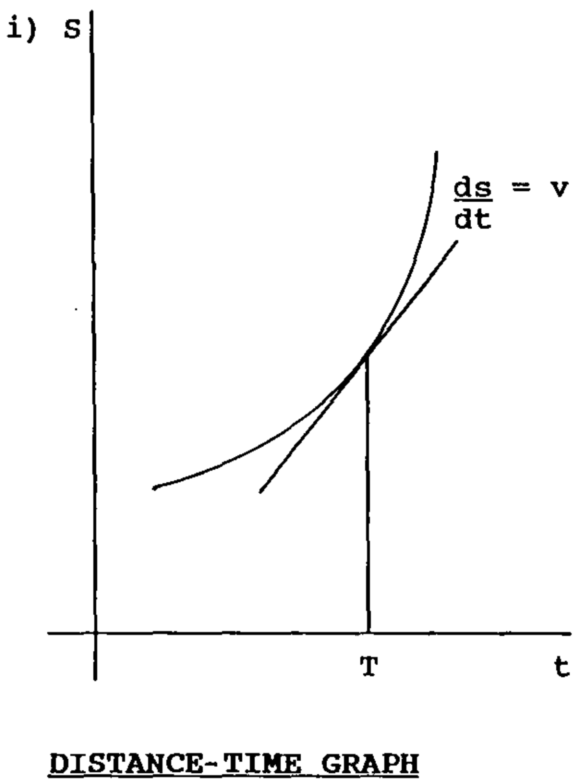
An important example of the use of integration and differentiation occurs in mechanics and familiarity with this subject is assumed in this section. Students are familiar with the discussions of falling bodies and particles moving through space with various velocities. Here, and in a teaching context, we can tie up some of the points made earlier on areas under graphs and indeed areas of geometric shapes like triangles, rectangles and trapezia.

We bring the following definitions from our previous discussions on differentiation and the definite and indefinite integral. We assert that if a particle moves with velocity vms $^{-1}$ and is at a distance Sm from a point of reference, then,

$$v = \frac{ds}{dt} \quad \text{or} \quad v(t) = s'(t)$$

and using the indefinite integral

$$s(t) = \int v(t)dt \quad \text{or} \quad s = \int vdt$$



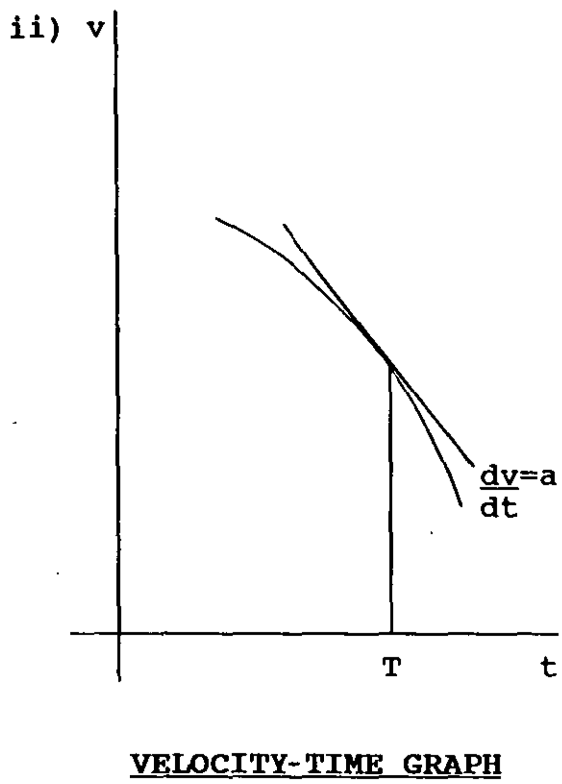


Fig.6.4(d).

Similarly, we have discussed the second derivative s"(t) and here

$$s''(t) = v'(t)$$

But $v'(t) = \frac{dv}{dt}$ gives the acceleration a(t). So

using the indefinite integral

$$v(t) = \int a(t)dt$$

or

$$v = \int a\ dt$$

In each case $\frac{ds}{dt}$ and $\frac{dv}{dt}$ are gradients of the tangent to each graph Fig.6.4(d).

Consider i)a particle moving with constant velocity v over a time interval t_1 to t_2.

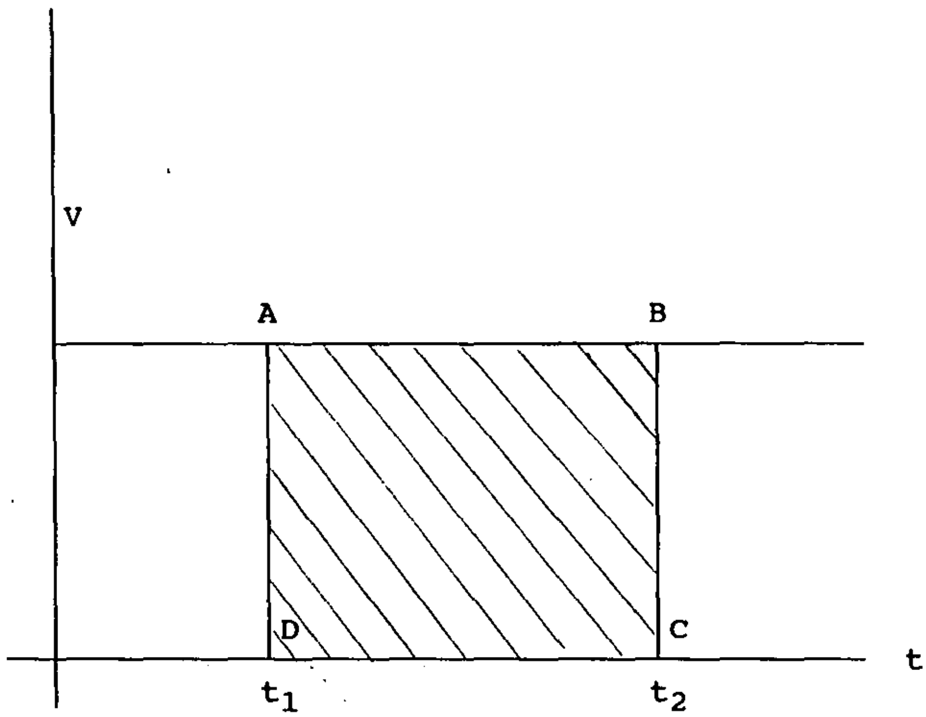


Fig.6.4(e).

The distance or displacement in the time interval t_1 to t_2 is given by the area of the rectangle.

i.e. $s = v(t_2-t_1)$

We can appeal to the definite integral for if $v_o = k\,m\,s^{-1}$ where k is constant,

$$s = \int v\,dt$$

gives

$$s = \int_{t_1}^{t_2} k\,dt$$

$$= \Big[kt \Big]_{t_1}^{t_2}$$

i.e. $s = k(t_2-t_1)$

or $s = v_o(t_2-t_1)$

ii) a body falling vertically with constant acceleration (here, the acceleration due to gravity)

The equation of the velocity-time graph is given by

$$V = 10t$$

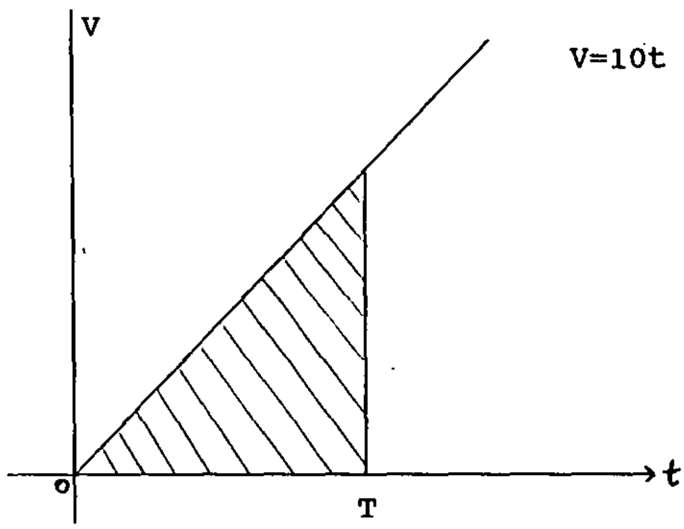


Fig.6.4(f).

The distance travelled in time T is the area of the shaded triangle given by

$$S = \frac{1}{2}\,10T^2$$

$$S = 5T^2$$

Here, too, the definite integral gives the answer,

$$S = \int_{0}^{T} v\,dt$$

$$= \int_{0}^{T} 10t\,dt$$

$$= 5T^2$$

The point being made here is that in a number of cases students fail to represent the motion of the particle in graphical form. The consequence is that they become involved in tedious applications of equations of motion when the formula for the area of triangle, rectangle or trapezium gives the answer. The following example is quite common in examinations:

Example: A body moves from rest with acceleration $8ms^{-2}$ for 2 seconds then moves with constant velocity for 4 seconds before coming to rest in a further 3 seconds. Find the distance travelled.

The easiest method of solution is to draw a diagram which shows graphically the motion of the body.

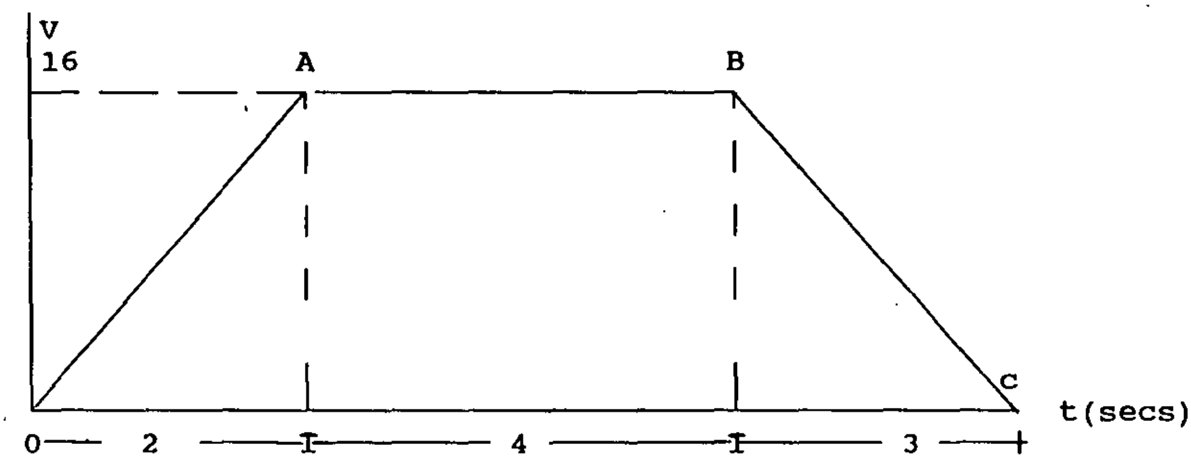


Fig.6.4(g)

The only calculation needed for the solution is the velocity at A - This is given by

$$V = 8t$$

$$V = 16\ ms^{-1}$$

The distance travelled is the area of the trapezium OABC

$$S = \frac{1}{2}\ (AB + OC)\ \times 16$$

$$S = 104m$$

Evidently a sketch or diagrammatic representation of the motion of moving bodies is not always possible, and the method of solution depends upon evaluating integrals. A vector example is chosen here to demonstrate this. Also it serves to draw attention to an important fact not often mentioned in texts.

Consider the motion of a particle with velocity vector given by

$$V = 3t^2\hat{i} + 2t\hat{j} + \hat{k}$$

We wish to find the distance of the particle from the origin given that when t=o, the position of the particle is given by $2\hat{i}-3\hat{j}+\hat{k}$.

$$\text{Now} \quad \bar{s} = \int \bar{v}\,dt \text{ give}$$
$$\bar{s} = \int (3t^2\hat{i}+2t\hat{j}+\hat{k})\,dt$$
$$\bar{s} = t^3\hat{i} + t^2\hat{j} + t\hat{k} + \bar{c}$$

The vector $\bar{c}$ is point of concern here. In dealing with the indefinite integral mention is made to the constant of integration. No reference, however, is made to the fact that the constant of integration is relative to the domain of application. Here the domain of application is vectors and $\bar{c}$ is a constant vector. However, scalar examples of c should be discussed first.

Using the initial condition we get

$$\bar{c} = 2\hat{i} - 3\hat{j} + \hat{k}$$

and

$$\bar{s} = (t^3+2)\hat{i} + (t^2-3)\hat{j} + (t+1)\hat{k}$$

The above concern may appear trivial but students often feel that $\bar{c}$ is not constant because it is a vector, even though they may well have done work on vectors.

In general, however, the velocity time graph is more useful than it might first appear. It gives information on the change of velocity with time and the gradient at any point gives the acceleration at that instant. Furthermore, the area under the graph - as we have just seen - gives the distance travelled.

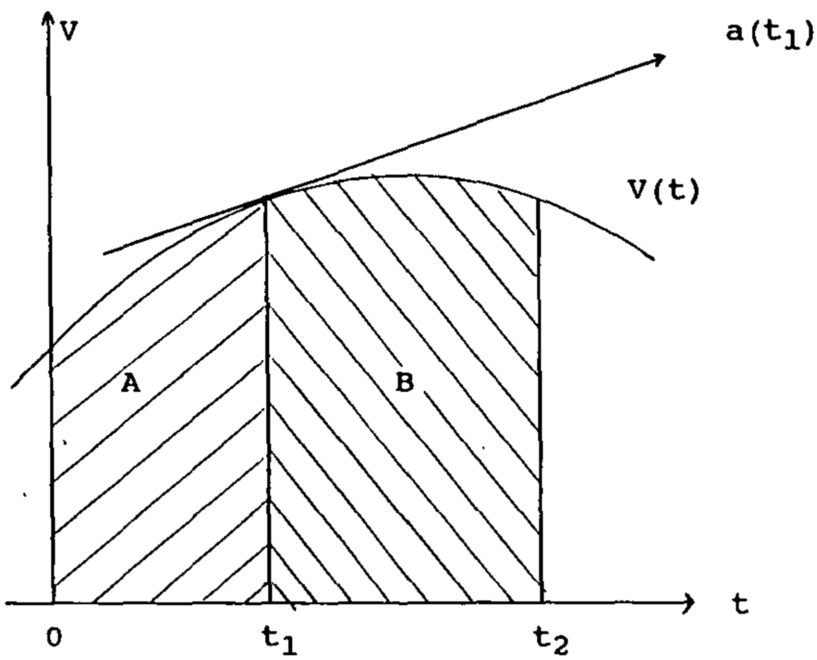


Fig.6.4(h).

In Fig.6.4(h), area A gives the distance travelled in the interval 0 to t_1 and area B the distance travelled in the interval t_1 to t_2. The acceleration $a(t_1)$ is the gradient of the tangent to the curve at t_1. In short, the velocity-time graph contains all the information students need on the motion of the particle. Students need to be able to deduce these 'bits' of information when they are required.

Volumes of Revolution

This is an equally popular application which makes frequent appearances at examination level. At times volumes of revolution appears as an application in its own right or tied up in some way with areas under curves. The methods are similar and as such the concerns expressed previously on areas are equally valid here.

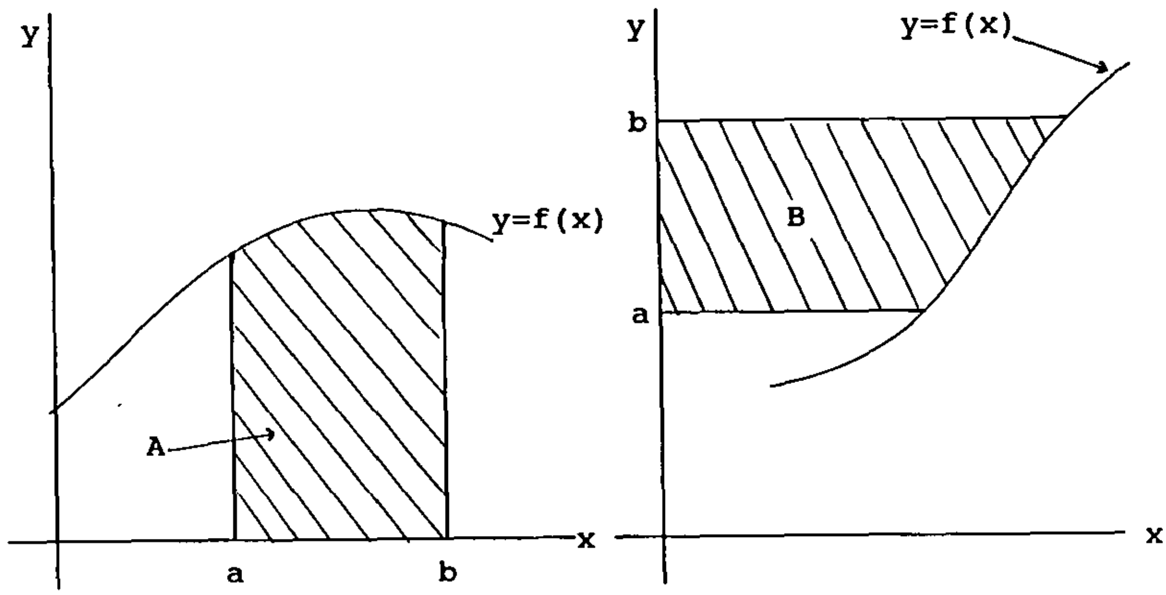


Fig.6.4(i)

The volume generated by rotating area A about the x-axis is given by

$$V = \pi \int_a^b y^2 \, dx$$

(π because elementary volume is a cylinder of volume $\pi y^2 \delta x$)
Whilst rotating B about the y-axis gives

$$V = \pi \int_a^b x^2 dy$$

(See Fig.6.4(i))

One of the applications of volumes of revolution which occur in a number of texts is to find the volume of a cone. Students are required to prove or verify using the definite integral that the volume of a cone is given by

$$V = \frac{1}{3} \pi r^2 h$$

where h is the vertical height and r the radius of the base. Frequently they are told to use a line of gradient $\frac{r}{h}$ which passes through the origin and rotate it about the x-axis. (FIG 6.4(j))

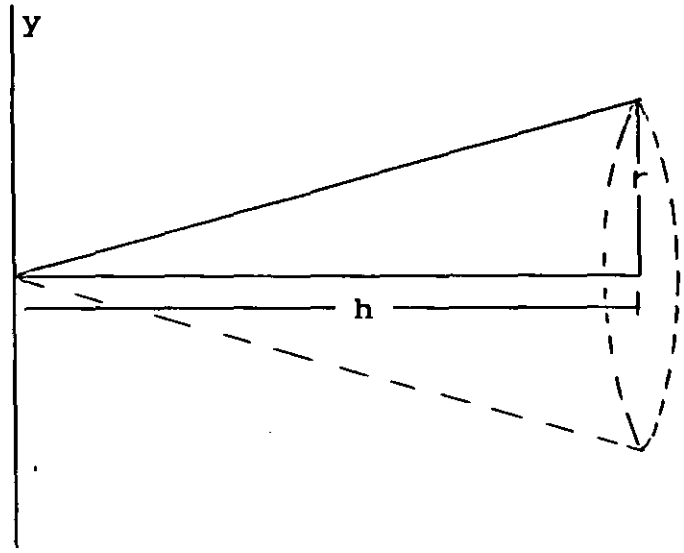


Fig.6.4(j)

The volume then is given by

$$V = \int_0^h \pi y^2 \, dx$$

$$= \int_0^h \pi \left[\frac{r}{h} x \right]^2 dx$$

$$= \pi \left[\frac{r^2}{h^2} \cdot \frac{x^3}{3} \right]_0^h$$

$$= \frac{1}{3} \pi r^2 h$$

Whilst this is a good example to use to test the formula for volumes of revolution, the restriction of choosing a line of gradient $\frac{r}{h}$ is an unnecessary one to make. Firstly, students often forget the stipulation on the function to use, and would find it difficult to do this question in an examination if it were not given. Secondly, they often ask what function should be integrated to get the required answer. The failure, here, is to recognise that if the straight line has gradient $\frac{r}{h}$ it must be of the form

$$y = \frac{r}{h} . x$$

since it passes through the origin. Many students try to find the required volume using

$$V = \int_0^h \pi\left(\frac{r}{h}\right)^2 dx \ !$$

Finally, any line of the form $y = mx$, $m\ (\neq 0)$, rotated about the x-axis within the limits 0 to h is sufficient to deduce the answer required.

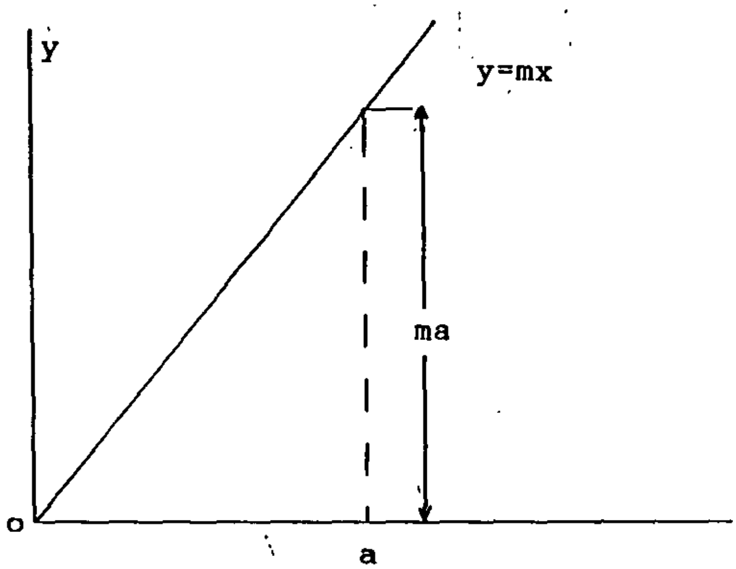


Fig.6.4(k).

Volume generated is given by

$$V = \int_0^a \pi\, y^2 dx$$

$$= \int_0^a \pi\, (mx)^2\, dx$$

$$= \pi . \frac{m^2}{3} a^3$$

The solid generated is clearly a cone of height a and base radius ma.

$$V = \pi . \frac{m^2}{3} a^3$$

can be written

$$V = \frac{1}{3} . \pi . (ma)^2 . a$$

Since $h=a$ and $r=ma$

$$V = \frac{1}{3}\pi r^2 h$$

Whilst a line of gradient $\frac{r}{h}$ passing through the origin does the 'trick' nicely, it is really a source of confusion to most students who cannot see why that choice of line was made.

In concluding, students often do not recognise the relationship between volume of revolution generated and area. The volume generated by rotating $y=f(x)$ about the x-axis between $x=a$ and $x=b$ is given by

$$V = \int_a^b \pi y^2 dx$$

$$= \int_a^b \pi [f(x)]^2 dx$$

There is, therefore, a function $F(x) = \pi[f(x)]^2$ for which

$$V = \int_a^b F(x)dx$$

The right hand side of this equation is essentially the area under the graph of $F(x)$.

This serves to emphasise that integration is, essentially, a summation.

Chapter 7: Crosscurricular applications.

7.1 Why teach applications?

In Chapter 1 we referred to the many areas other than mathematics in which calculus is used. Mention was made to the use of mathematics as a tool rather than a subject whose traditional exposition is for the purist mathematician. The field of applications, in the context of the school curriculum, is one in which mathematics is used as a means of enabling students to take their places in society. There are many students who will not use mathematics in their later careers but will need, as citizens, to estimate and make judgements or decisions during their working lives.

The following are four mathematical and pedagogical reasons for teaching applications [1] :

a) For motivation: an application of a piece of mathematics to a practical problem can motivate students to learn mathematics thus consolidating the concepts.

b) For cultural reasons: many English mathematicians contributed in some way to the development of calculus. Newton's contribution to calculus is an example. It is insufficient to learn calculus without some knowledge or appreciation of what led to its development and how it was used by Newton and others.

c) For fear of something 'worse': the view here is that teaching mathematics is the responsibility of mathematics teachers. They should also be responsible for teaching applications as only then can they reduce the possibility of crudity, unsuitability and inaccuracy which is likely to accompany a treatment by non-mathematicians.

d) To teach recognition of structure out of 'noise': this is similar in a sense to making order out of chaos as a reason for teaching mathematics. The essence of this reason for teaching applications is that teachers should try to nurture an investigative approach to problem solving (Cockcroft) which will enable them to identify mathematical structures within a variety of situations.

More often than not, however, the social reasons for teaching applications are more important. In every walk of life and perhaps every day the ordinary citizen needs to make judgements and decisions. These need to be rational if they are not to give rise to greater inconveniences and punitive penalties. Decisions and judgements can be more rational if they are approached in an analytical and quantitative way.

7.2 Calculus in applications.

The importance of gradients of lines was discussed in section 3.4. Gradients of lines lead to the notion of differentiation and the derivative [9]. With an idea of gradients, simple problems of maxima and minima can be attempted. There are an abundance of situations in the 'real world' where the above ideas of the calculus can be employed.

Often we speak of exponential growth and decay or population growth of species. These require the use of differential equations, the solutions of which employ methods of calculus; integration, for example. Embedded in the definition of the integral is the concept of area and often we are required to find areas under or enclosed by curves.

The motivating examples mentioned so far may appear to be applications which are prevalent in classical physics or a purely mathematical domain. They are, however, equally applicable in the biological sciences: blood flow and muscular movement, or commerce. The examples which follow in the next sections will attempt to demonstrate this.

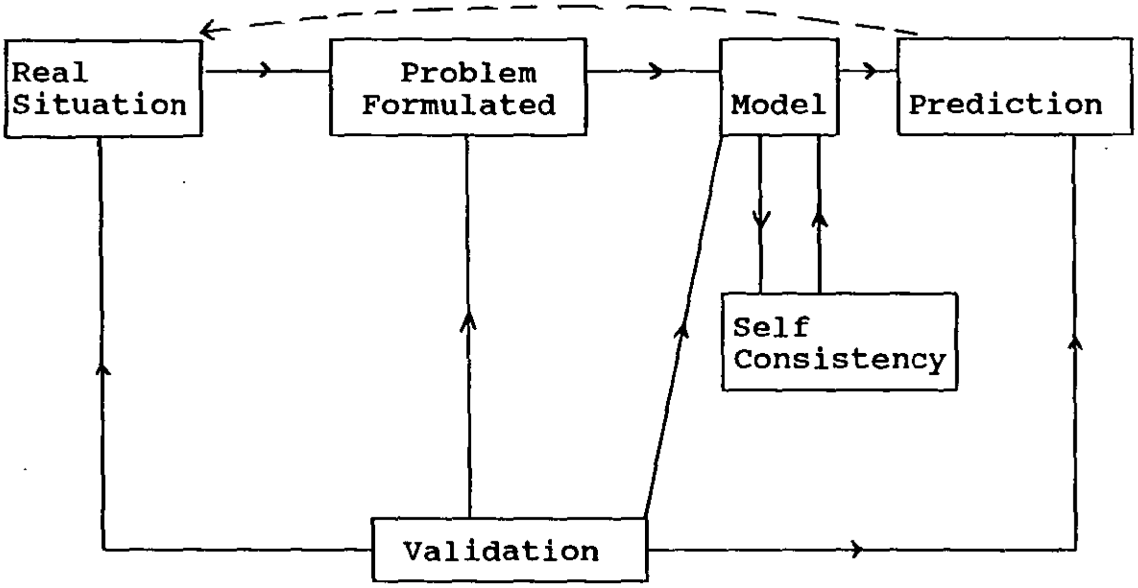
7.3 Mathematical Modelling

It is fair to say that the mathematics taught in schools and on undergraduate courses, to a greater or lesser extent, provides most students with a formal approach to their subject. The way mathematics is taught emphasises basic principles and a need for rigour. It does not appear, however, to provide students with the techniques necessary to tackle raw problems as they arise - a view with which many employers would concur. The implication here is not that there is necessarily an inherent lack of ability, but is perhaps a reflection on the formal nature of mathematics teaching in schools and colleges.

Several reports on mathematics and mathematics teaching highlight the need to treat mathematics in a less formal way and to consider the formulation of mathematical problems from real, practical situations. A broad title for this approach is mathematical modelling.

One missing ingredient in the teaching of mathematics is the representation of the so called 'real world' in mathematical terms. Students often argue that they fail to see the relevance of a piece of mathematics. Perhaps a modelling approach to real world situations would enable them to gain a more precise understanding of their significant properties, and provide them with a platform from which to deduce and predict results and events. Indeed, it would not be unreasonable for an engineer to want to know whether his bridge will withstand the loads likely to be placed on it; so he would first make a scale model of it, test it, and then build it.

The general principles of mathematical modelling requires firstly, the identification of the problem; some consideration as to whether the problem is amenable to mathematical treatment, and whether there are any skills needed which are not related to mathematics. Once the significant features have been identified, the next stage is to translate these into mathematical entities and postulate relations between entities. The process of modelling can be summarised in the following diagram.



A POSSIBLE MODELLING PROCESS

It is argued in Book 3 of Advanced Level Course of SMP that there are three stages in dealing with situations in science which are amenable to mathematical treatment. These three stages are:

i) formulation
ii) solution
iii) interpretation of results

Formulation is seen as the most difficult stage since relevant symbols and laws of science are used to produce mathematical equations for solution. The actual solution of such equations is easy compared to their formulation since, in general, there are standard methods of approach for dealing with most equations in mathematics.

A typical problem is to find the time taken for a conical funnel to empty and how this time depends on the initial height of the liquid in the funnel.

There are many factors to be considered here before any attempt is made to translate this problem into a mathematical model. Indeed there are several variables which may effect the solution and conclusion.

The time to empty, T, depends on:

i) the initial height, h_o, of the liquid;

ii) the shape of the funnel;

iii) the density of the liquid;

iv) the viscosity and surface tension of the liquid;

v) gravitational acceleration;

vi) temperature and the diameter of the funnel nozzle.

It is evident that a consideration of all these factors would result in a complex set of equations in several unknowns. One method of approach is to vary the height h_o whilst observing the change in time, T, and keeping all other factors, constant. This is the first stage of a modelling cycle.

We need now to make an assumption on the possible behaviour of the liquid under such a situation. It is reasonable to postulate that the rate of flow is proportional to the pressure difference at the orifice, and that this pressure difference is also proportional to the depth of the liquid.

Let V = the volume of liquid in the vessel at time t,

h_o = initial height,

h = height of liquid at time t,

p = pressure difference at the orifice.

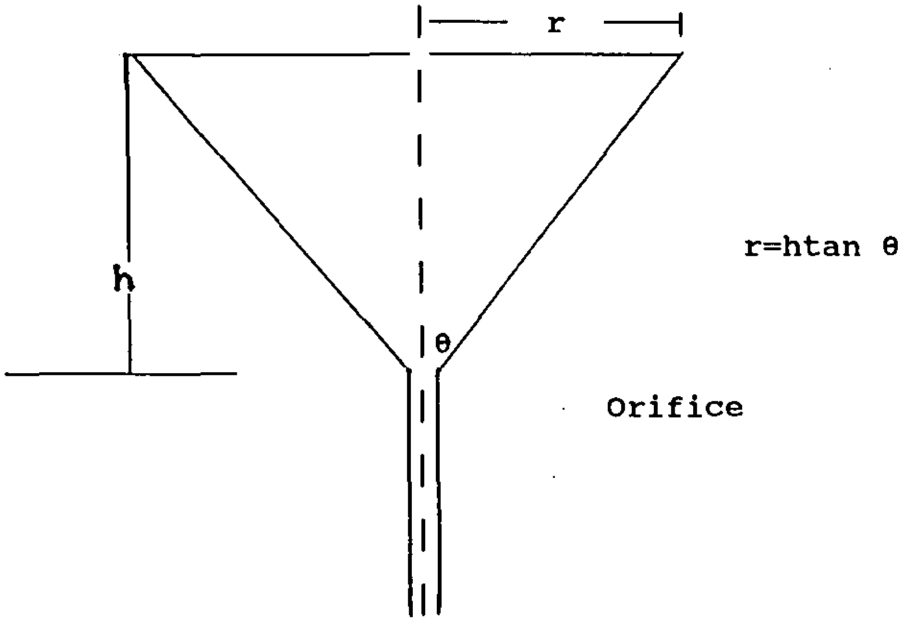


Fig.7.3(a).

The rate at which liquid flows out of the funnel is the rate at which the volume is decreasing. This is given by $\frac{dV}{dt}$ and is negative, therefore,

$$-\frac{dV}{dt} \propto P$$

and

$$p \propto h$$

Together these imply

$$-\frac{dV}{dt} \propto h$$

that is, $\frac{dV}{dt} = -kh$ ---(i) (k a positive constant)

This equation contains the variables V, t, and h and cannot be integrated in this form. We can eliminate V by using the formula for the volume of a cone,

$$V = \frac{1}{3}.\pi.\ r^2h$$

$$= \frac{1}{3}.\pi.\ (h \tan \theta)^2\ h;$$

i.e. $$V = \frac{1}{3}.\pi.\tan^2\theta.h^3$$

$$\frac{dV}{dt} = \frac{1}{3}.\pi.\tan^2\theta.3h^2.\ \frac{dh}{dt} = \pi \tan^2\theta.h^2\ \frac{dh}{dt} \quad \text{--------}(ii)$$

$$\tan^2\theta.h^2\ \frac{dh}{dt} = -\ kh \quad \text{(from (i))}.$$

$$h\ \frac{dh}{dt} = -\ A' \qquad \text{where } A' = k/\pi \tan^2\theta.$$

Integrating we get,

$$\frac{1}{2}\ h^2 = -\ A't + B$$

When $t=0,\ h=h_o \Rightarrow B=\frac{1}{2}h_o{}^2$

$$\therefore\ \frac{1}{2}h^2 = -\ A't + \frac{1}{2}\ h_o{}^2$$

$$2A't = h_o{}^2 - h^2 \ \text{——}\ (iii)$$

Equation (iii) is the equation of a parabola - See Fig.7.3(b).

-and when $h=h_o$, $t=0$; when $t=T$ and $h=0$, we have

$$T = \frac{h_o^2}{2A'}$$

i.e.
$$T = \frac{h_o^2}{2k} . \pi, \tan^2\theta$$

This supports the original assumption that $T \propto h_o^2$

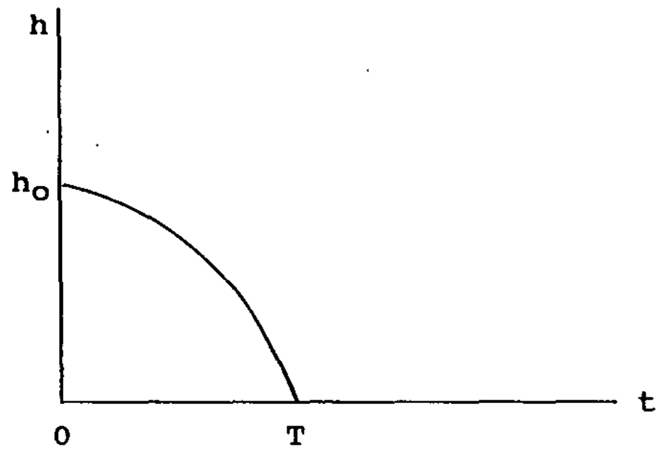


Fig.7.3(b).

The sandwich inequality discussed in section 5.3 can also be used here to analyze the problem in an alternative manner.

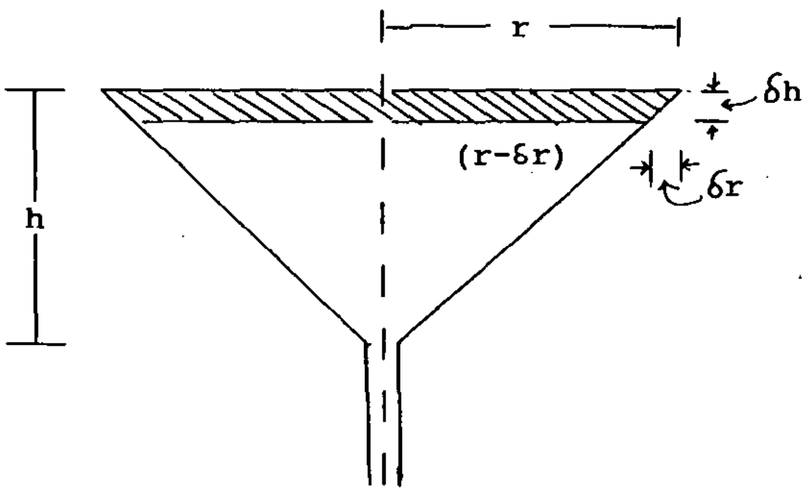


Fig.7.3(c).

If the height of the liquid is reduced by a small amount h, the volume v, shaded in Fig.7.3(c) satisfies the sandwich inequality

$$\pi r^2 \delta h > \delta v > \pi (r-\delta r)^2.\delta h$$

This gives

$$\pi r^2 \frac{\delta h}{\delta t} \geqslant \frac{\delta v}{\delta t} \geqslant \pi (r-\delta r)^2 \frac{\delta h}{\delta t}$$

Now as $\delta t \longrightarrow 0$, we get

$$\pi r^2 \frac{dh}{dt} \geqslant \frac{dv}{dt} \geqslant \pi r^2 \frac{dh}{dt} \quad (\text{since } \delta r \to 0 \text{ as } \delta t \to 0).$$

Hence,

$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt}$$

$$= \pi \tan^2\theta.\ h^2 \frac{dh}{dt} \quad \text{as in equation (ii) above.}$$

The rest of the analysis follows as before.

In most cases, questions on mathematical modelling can be reduced to a relationship between two variables in the form of a differential equation. At school level, mathematical modelling questions are usually simple, with the restrictions on variables which may affect the model, kept to a minimum. Indeed, in a number of cases, the question specifies the two variables to be considered.

At 'A' level the questions usually relate to the behaviour of populations under some controlled conditions. Malthus did a great deal of research on population models and in 1798 he proposed his 'gloomy doctrine' that the survival of the human race was only possible if periods of exponential growth were punctuated by plague and famine. The Malthusian model for the rate of population growth of an organism reared in constant conditions is given by

$$\frac{dN_t}{dt} = bN_t$$

where b is the instantaneous birth rate per head of the population. Integrating we get

$$N_t = N_o e^{bt}$$

where $N=N_o$ when $t=0$ and N_t is the instantaneous size of population at time t. Death can be included in the model by introducing the death rate, d, and putting $C=b-d$, where C is the innate capacity of the population to increase. Under natural conditions it is unusual for C to be constant even for small intervals of time. In general $C=C(t)$, and the solution of

$$\frac{dN_t}{dt} = CN_t$$

is:

$$N_t = N_0 e^{\int_0^t C(t)dt}$$

The following Further Mathematics question makes use of this model [59] (p 98-113) but is not really modelling.

> The population x of a colony of bacteria increases at a rate equal to the product of the number x of bacteria present at time t, and the capability C of the environment to support the number present at time t. The capability C is measured by the excess of the maximum number b of bacteria that can be supported by the environment over the number of bacteria actually present. Write down the differential equation governing the growth of bacteria. Solve this differential equation, expressing x in terms of t, given that the population at time t=0 is p, where $p < b$. State what happens to the population after the passage of a large interval of time.

(London: June 1980)

The wording of this question leaves a great deal to be desired, but if we accept this we can proceed as follows:

With reference to the question, let the rate of increase of the population be $\frac{dx}{dt}$. Then

$$\frac{dx}{dt} = xC \quad \text{where } C = (b-x)$$

i.e.

$$\frac{dx}{dt} = x\,(b-x)$$

is the required differential equation. The mathematical model is now completed. This confirms the point made earlier about the simplicity of the models used at this level. The solution of the resulting differential uses traditional mathematical methods. On separating the variable, we get

$$\int \frac{dx}{x(b-x)} = \int dt$$

Using the partial fractions on the left hand side of this equation, we get

$$\frac{1}{b}\int\left(\frac{1}{x} + \frac{1}{b-x}\right)dx = \int dt$$

or

$$\int\left(\frac{1}{x} + \frac{1}{b-x}\right)dx = \int bdt$$

yielding

$$\ln x - \ln(b-x) = bt + c \qquad \text{(a)}$$

when

$$t=0,\ x=p,\ \Rightarrow\ c = \ln\left(\frac{p}{b-p}\right)$$

Equation (a) becomes

$$\ln\left(\frac{x}{b-x}\right) = bt + \ln\left(\frac{p}{b-p}\right)$$

$$\ln\left[\frac{x\ (b-p)}{p\ (b-x)}\right] = bt$$

or

$$\frac{x\ (b-p)}{p\ (b-x)} = e^{bt}$$

and

$$x = pb/(p+(b-p)e^{-bt})$$

is the required solution. Clearly, for large passages of time, $(b-p)e^{-bt}$ tends to zero and x tends to b. A graph of the behaviour of the population depends on p.

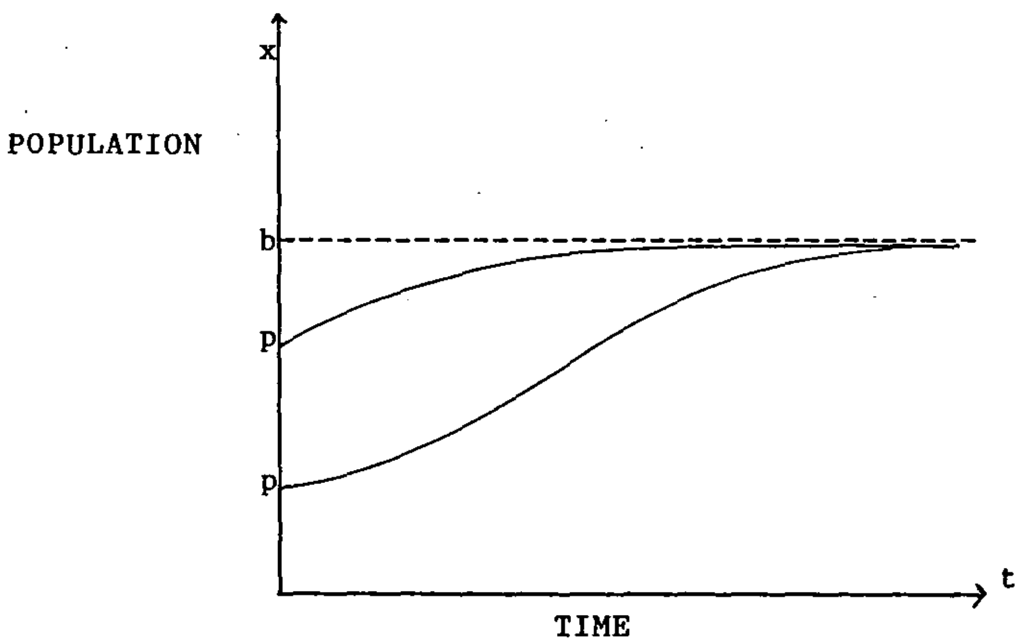


Fig.7.3(d).

In concluding, it is acknowledged that not all of the components of the modelling process referred to earlier have been discussed in the school context. In wider applications of modelling, however, it would be necessary to consider the validity of the model and, indeed, test it. In industrial applications, making predictions from mathematical models is an important exercise if the model is to be adopted.

7.4 Statistics

Statistics is one of the curricular areas which uses mathematics throughout. It embraces concepts of mathematics ranging from counting to calculus. At school level, the main application of calculus is in dealing with the probability density function (pdf). This particular application will be discussed here. We begin with the definition of the probability density function.

Definition: If X is a continuous random variable and f(x) is a function for which

i) $f(x) \geqslant 0$ for all x, and

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then f(x) is said to be a probability density function of X.

The definition requires an appreciation of the idea of a function which is non-negative for all x. In particular the second part of the definition assumes a form which is not usually discussed at any length with mathematics students at this level. Some texts prefer to use

$$\int_R f(x)\,dx = 1$$

where R is the range of possible values.

Other texts are even more specific in that they use the condition

$$\int_A^B f(x)\,dx = 1$$

where A and B are the least and greatest possible values of X respectively. The alternative definition of part (ii) above is more useful in that it is easier to relate the pdf to its mathematical context. The pdf defines the probability that x lies in the interval [a,b] which is clearly seen to be

$$\int_a^b f(x)dx$$

and is therefore represented by the area of the shaded region.

Fig.7.4(a).

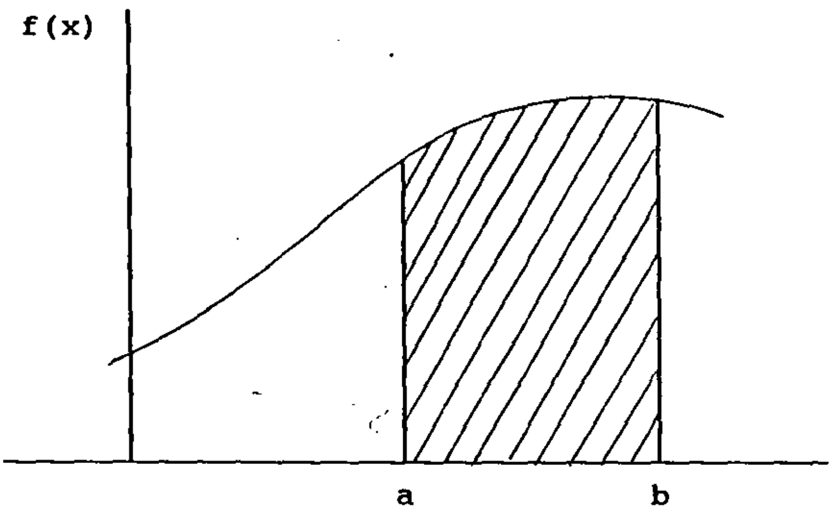


Fig.7.4(a)

The quantity

$$\int_a^b f(x)dx$$

is essentially the area under the curve y=f(x) as discussed earlier in Chapter 6.

Associated with the probability density functions are the terms mean and variance. The mean is defined as

$$E[X] = \int_R x\ f(x)dx$$

and the variance as

$$V[X] = E[X^2] - [E[X]]^2$$

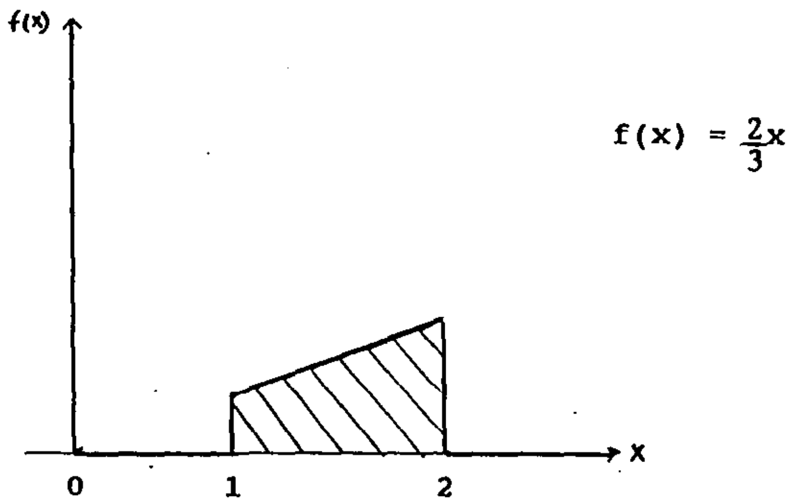
where

$$E[X^2] = \int_R x^2 f(x)dx$$

<u>Example 1</u>. Let X be a continuous variate with the following distribution:

i) $f(x) = \frac{2}{3}x$ for $1 \leqslant x \leqslant 2$

ii) $f(x) = 0$ elsewhere.



<u>Fig.7.4(b).</u>

The probability that x lies in the interval [1,2] is given by

$$P(1 \leqslant x \leqslant 2) = \int_1^2 \frac{2}{3} x \, dx$$

$$= \left[\frac{1}{3} x^2\right]_1^2 = 1$$

But also the required area is that of a trapezium given by

$$P(1 \leqslant x \leqslant 2) = \frac{1}{2}\left(\frac{2}{3} + \frac{4}{3}\right)$$

$$= 1 \quad \text{(as before)}$$

Example 2. A variate X has a pdf

i) $f(x) = Cx(2-x)$, $0 \leqslant x \leqslant 2$

ii) $f(x) = 0$, otherwise.

Find the value of the constant C and the mean and variance of x. Find also the probability that two values of x chosen at random from such a distribution will both be larger than 1.

To find C we use the basic property of pdf's

$$\int_0^2 Cx(2-x)\,dx = 1$$

$$C\int_0^2 (2x - x^2)\,dx = 1$$

$$C\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$C\,\frac{4}{3} = 1$$

$$C = \frac{3}{4}$$

The pdf is symmetrical about the point x=1 and as such the mean is one. This conclusion is facilitated by a sketch of the function. FIG 7.4.(c).

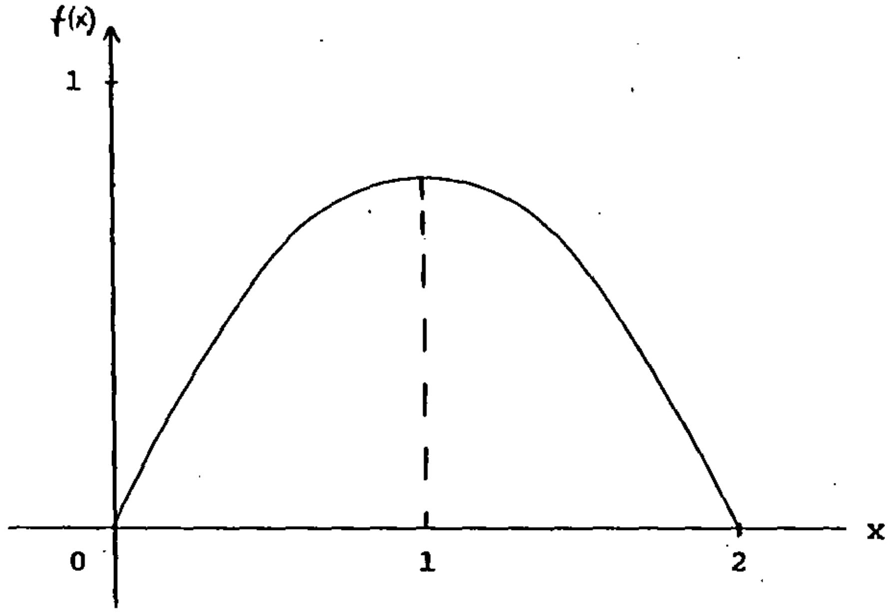


Fig.7.4(c).

It is likely that most students of statistics would arrive at this result in the more formal way.

$$\text{Mean} \qquad E[X] = \int_0^2 \frac{3}{4} x^2 (2-x)\,dx$$

$$= \frac{3}{4} \int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{3}{4} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= \frac{3}{4} \left[5\frac{1}{3} - 4 \right]$$

$$= \underline{1}$$

Now $E(X^2) = \frac{3}{4}\int_0^2 x^3(2-x)dx$

$$= \frac{3}{4}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{6}{5}$$

So variance $V[X] = E[X^2] - [E[X]]^2$

$$= \frac{6}{5} - 1$$

$$= \frac{1}{5}$$

The probability that a value of X chosen at random is larger than 1 can be found by integration as

$$P(1 \leqslant x \leqslant 2) = \int_1^2 \frac{3}{4} x(2-x)\ dx$$

$$= \frac{1}{2}$$

and hence

P (both values of x chosen at random, and hence independently, lie between 1 and 2)

$$= \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{4}$$

To conclude we look at two further applications of calculus in this context: the finding of the mode and median of a distribution. The median of a distribution occurs at M, a point of the interval on which the pdf is defined such that half the distribution lies to the left of M and half to the right. It is reasonable, therefore, to expect from the definition

$$P(A \leqslant x \leqslant B) = \int_A^B f(x)dx = 1$$

for

$$\int_A^M f(x)dx = \frac{1}{2}$$

where M is the median value. As discussed before, when the pdf is symmetrical about some particular value M then this value will be the mean as well as the median. The mode corresponds to the maximum of the function on the interval [A,B]. Differential calculus is used here. The turning point(s) of the function are examined and their nature - local maximum or minimum - determined. However, the mode may be at one end of the interval and a check must be made that the value of f(x) at the ends of the interval do not exceed that of any local maximum in the interval. The following example demonstrates the finding of the median and mode of a distribution.

<u>Example:</u> The variate X is claimed to have a pdf, such that,

i) $f(x) = 4(x-x^3)$, $0 \leqslant x \leqslant 1$

ii) $f(x) = 0$, otherwise.

Show that f(x) could indeed represent a pdf and, assuming the claim is correct, find the mode and median of X.

To show that f(x) represent a pdf we need to show that

i) $\int_0^1 4(x-x^3)\,dx=1$

and

ii) $f(x) \geq 0$ for all x

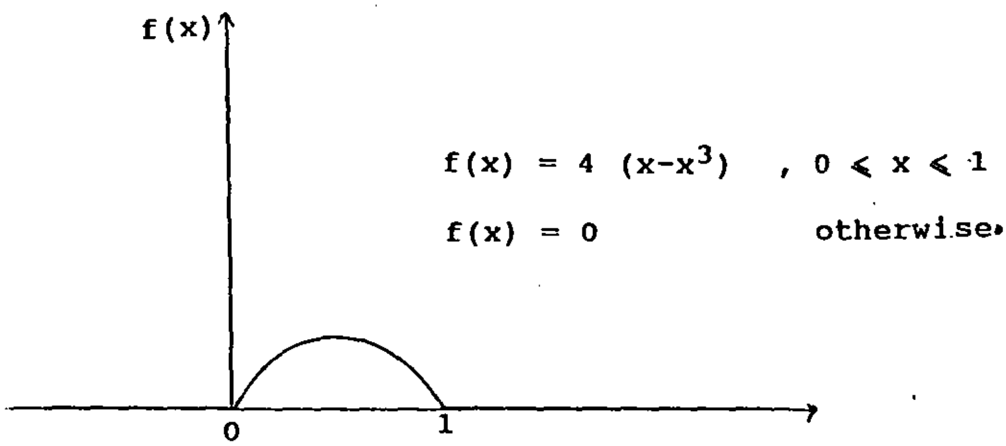
i.e. $4(x-x^3) \geqslant 0$ for $0 \leqslant x \leqslant 1$

Using integration we have

$$\int_0^1 4(x-x^3)\,dx = \left[2x^2-x^4\right]_0^1$$

$$= \underline{1}$$

A sketch of the curve is sufficient to show that $f(x) \geqslant 0$ for;
$0 \leqslant x \leqslant 1$.



To find the mode, we use calculus to find any maxima.

$f(x) = 4(x-x^3)$

$f'(x) = 4 - 12x^2$

$f''(x) = -24x$

for a turning value $f'(x) = 0$

i.e. $4-12x^2=0$

$$x = +\frac{1}{\sqrt{3}} \quad \text{or} \quad x = \frac{-1}{\sqrt{3}}$$

Since $x = \frac{-1}{\sqrt{3}}$ is of no interest here - it lies outside the interval.

We consider $f''(x)$ when $x = \frac{+1}{\sqrt{3}}$

and note that

$$f''(x) < 0$$

Therefore $x = \frac{+1}{\sqrt{3}}$ corresponds to a local maximum.

Since $f(x)=0$ at the ends of the interval the mode is at

$$x = \frac{+1}{\sqrt{3}} \simeq 0.58$$

To find the median , M, note that

$$\int_0^M 4(x-x^3)\,dx = \frac{1}{2}$$

$$\left[2x^2 - x^4\right]_0^M = \frac{1}{2}$$

Yielding a quadratic in M^2

$$2M^4 - 4M^2 + 1 = 0$$

from which we get

$$M \simeq \pm\ 1.31 \text{ or } M \simeq \pm\ 0.541$$

The only possible value of M is

$$\underline{M \simeq 0.54}$$

7.5 Economics

The study of economics is punctuated with mathematical rigour at all levels. Indeed, mathematical economics is a subject in its own right. Economics considers the relationship between variables such as profit, demand, quantity and cost. In any situation where functional relationships are considered, it is difficult not to examine how one variable in the relationship changes with another. This involves ideas of rates of change and here applications of differentiation. For example, the elasticity of demand is defined as

$$\eta = \frac{\text{\% change in quantity demanded}}{\text{\% change in price}}$$

$$= \frac{\frac{\Delta q}{q} \cdot 100}{\frac{\Delta p}{p} \cdot 100}$$

$$= \frac{\Delta q}{q} \cdot \frac{p}{\Delta p}$$

$$= \frac{\Delta q}{\Delta p} \cdot \frac{p}{q}$$

Here we look at simple applications of derivatives and integration to some of the economic variables mentioned above. The following abbreviations are used:

P = profit
Q = quantity - bought or sold
TR = total revenue
TC = total costs
AC = average costs

Application of Maxima and Minima

Example The market demand function of a firm is given by

$$4P + Q - 16 = 0$$

and the AC function takes the form

$$AC = \frac{4}{Q} + 2 - 0.3\,Q + 0.05\,Q^2$$

Find Q which gives

i) maximum revenue

ii) minimal marginal costs

Use the second derivative test in each case

---

i) Given: $4P + Q - 16 = 0$

Thus $P = 4 - 0.25Q$

Total Revenue
$$TR = P.Q = (4 - 0.25Q)Q = 4Q - 0.25Q^2$$

Revenue is a maximum when $\frac{d(TR)}{dQ} = 0$ and $\frac{d^2(TR)}{dQ^2} < 0$

$$\frac{d(TR)}{dQ} = 4 - 0.5Q$$

When $\frac{d(TR)}{dQ} = 0$, then

$$4 - 0.5Q = 0$$

i.e. $Q = 8$

$$\frac{d^2(TR)}{dQ^2} = -0.5 < 0$$

Thus revenue is a maximum when $\underline{Q=8}$

ii) Average costs $AC = \frac{4}{Q} + 2 - 0.3Q + 0.05Q^2$

Thus,

$$\text{Total costs } TC = Q.AC$$

$$= 4 + 2Q - 0.3Q^2 + 0.05Q^3$$

$$\text{Marginal Costs } MC = \frac{d(TC)}{dQ}$$

$$MC = 2 - 0.6Q + 0.15Q^2$$

Marginal Costs are a minimum when $\frac{d(MC)}{dQ} = 0$ and $\frac{d^2(MC)}{dQ^2} > 0$

$$\frac{d(MC)}{dQ} = -0.6 + 0.3Q$$

When

$$\frac{d(MC)}{dQ} = 0 \text{ , then}$$

$$-0.6 + 0.3Q = 0$$

$$Q = 2$$

i.e.

$$\frac{d^2(MC)}{dQ^2} = 0.3 > 0$$

Thus marginal costs are at a minimum when $Q=2$

The above is an example of an application of the concepts of maxima and minima discussed in Chapter 6. Most of the applications of derivatives at school level revolve around finding the maximum or minimum of the particular economic variable and as such the above example is sufficient to demonstrate this application.

Application of Area under curves

Area under curves is used here as an application of integration in economics with reference to two economic variables: Consumer's Surplus and Producer's Surplus. Economic theory asserts that in a competitive market, the price of a commodity does not reflect what consumers would be willing to pay for it rather than go without. Instead the price is a reflection on the valuation they place on their last purchase. The following diagrams look at Consumer's and Producer's surpluses and how they are measured.

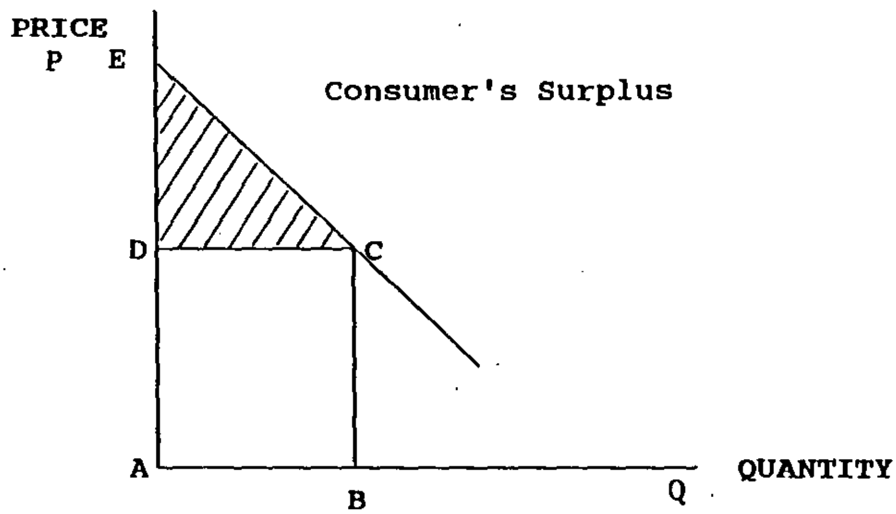
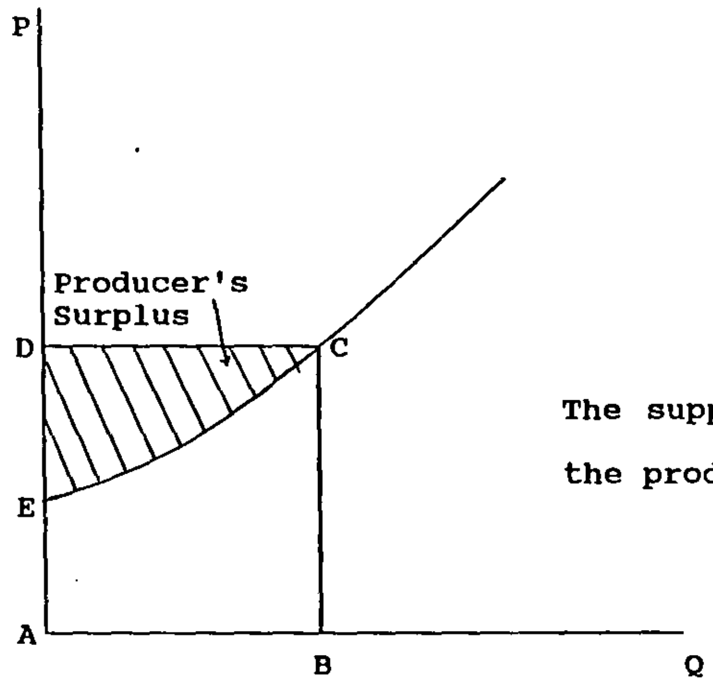


Fig.7.5(a).

A graph of price against quantity shows the demand function f . At price AD, the quantity demanded is AB. ABCD is the amount paid by consumers. However, the benefit 'derived' by consumers is the total area under the demand function over the range AB. The Consumer's surplus, therefore, is the shaded area. Evidently, the area is the area of the triangle EDC. If, however the demand function assumes the shape of a curve, then the consumer's surplus is given by

$$\text{Consumer's surplus} = \int_{0}^{q} f(Q)dQ - \text{Area of rectangle ABCD}$$

where q = AB and f is the demand function



The supply curve and the producer's surplus

Fig.7.5(b).

The producer's surplus is measured in a similar way. FIG Fig.7.5(b),

Producer's surplus = Area of rectangle ABCD - $\int_0^q g(Q)dQ$,

where q = AB and the supply function is g . The following example demonstrates this application.

Example If the demand function is given by

$p = 10 - Q - Q^2$

and the supply function is given by

$p = Q + 2$

Calculate the consumer's and producer's surplus at equilibrium price.

Demand function $P = 10 - Q - Q^2$

Supply function $P = Q + 2$

At the equilibrium price

$$10 - Q - Q^2 = Q + 2$$

$$Q^2 + 2Q - 8 = 0$$

$$(Q-2)\ (Q+4) = 0$$

thus $Q=2$ or -4

Q cannot take a negative value, therefore Q=2

When Q=2 , P=4

Fig.7.5(c) shows the Producer's and Consumer's surplus

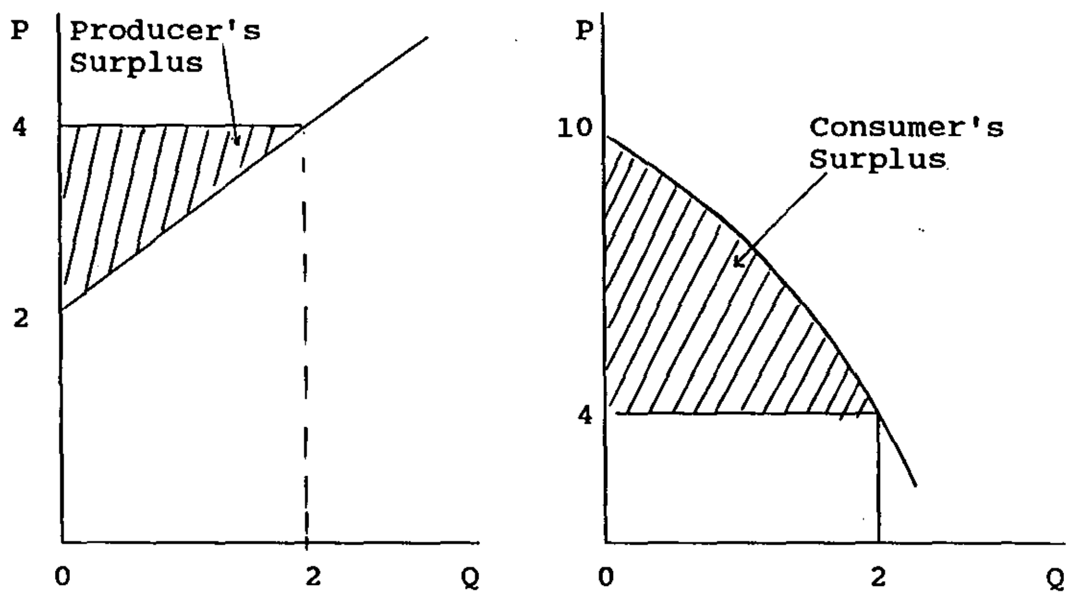


Fig.7.5(c).

Consumer's surplus:

$$\text{Area under demand curve} = \int_0^2 (10 - Q - Q^2)\,dQ$$

$$= 15\ \frac{1}{3}$$

Area of rectangle = P.Q = 8

Therefore Consumer's Surplus = $7\frac{1}{3}$

Producer's Surplus:

$$\text{Area under supply function} = \int_0^2 (Q+2)\,dQ$$

$$= \underline{6}$$

Area of rectangle = P.Q = 8

Therefore, producer's surplus = $\underline{2}$

7.6 Physics.

Physics, like statistics, uses mathematics throughout the 'A' level course, Indeed, the mechanical applications are essentially those which occur in 'A' level Applied Mathematics. Thus we discuss here other applications which would normally not occur in a mathematics course.

For example the formula for the magnetic flux density B at a point distant r from an infinitely-long straight current-carrying conductor uses calculus for its derivation.

Example

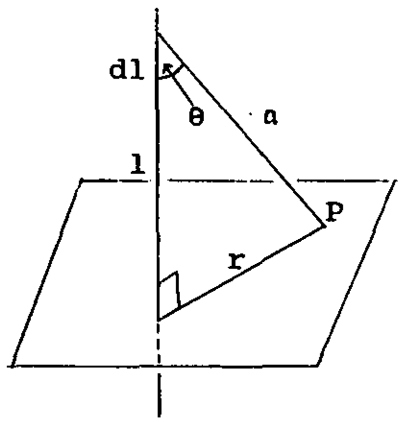


Fig.7.6(a).

Induction: From the Biot and Savart law, the induction at P Fig.7.6(a) is given by

$$B = \int \frac{\mu_0 \ I.dl.Sin\theta}{4\pi a^2}$$

Firstly, we try to express everything in terms of θ, the variable of integration.

From the geometry of the figure

$$l = r\cot\theta$$

$$dl = -r\,\mathrm{cosec}^2\theta . d\theta$$

and

$$a = r\,\mathrm{cosec}\theta$$

Thus $$B = \frac{\mu_0 I}{2\pi r}\int_{\pi/2}^{0} -\sin\theta d\theta$$

$$= \frac{\mu_0 I}{2\pi r}\Big[\cos\theta\Big]_{\pi/2}^{0}$$

$$= \frac{\mu_0 I}{2\pi r}$$

In atomic physics the rate of disintegration of radio active atoms is often discussed. The following example looks at how calculus assists that discussion.

Example: What is meant by the half-life period and decay constant?
Explain the relation between them
A radioactive source has a half-life of 4 days. Compare the initial rate of disintegration of the atoms with the rate after 3 days.

The "half life period", T, is the time taken for the number of unchanged atoms to fall to half its initial value. The "decay constant", λ , is the constant of proportionality between the rate of change, $\frac{dN}{dt}$, at any instant and the number , N, of undisintegrated atoms at the same instant.

That is $\frac{dN}{dt} = - \lambda N$, λ a positive constant

Upon integration we get

$$\underline{N = N_o e^{-\lambda t}}$$

where N_0 is the initial number of undisintegrated atoms at t=o and N is the number after a time t.

After a time T

$$\frac{N_0}{2} = N_0 e^{-\lambda T}$$

and

$$e^{\lambda T} = 2 \text{ ---------(i)}$$

or

$$\lambda T = \ln 2$$

From

$$N = N_o e^{-\lambda t}$$

rate of disintegration, r, is given by

$$\frac{dN}{dt} = - \lambda N_0 e^{-\lambda t}$$

when t=0, rate $r_0 = \frac{dN}{dt} = -\lambda N_0$

when t=3 days, rate $r_1 = \frac{dN}{dt} = -\lambda e^{-3\lambda} N_0$

From (i) above $e^{\lambda} = 2^{1/T}$

Therefore,

$$r_1 = -\lambda N_0 . 2^{-3/T}$$

$$= -\lambda N_0 . 2^{-3/4}$$

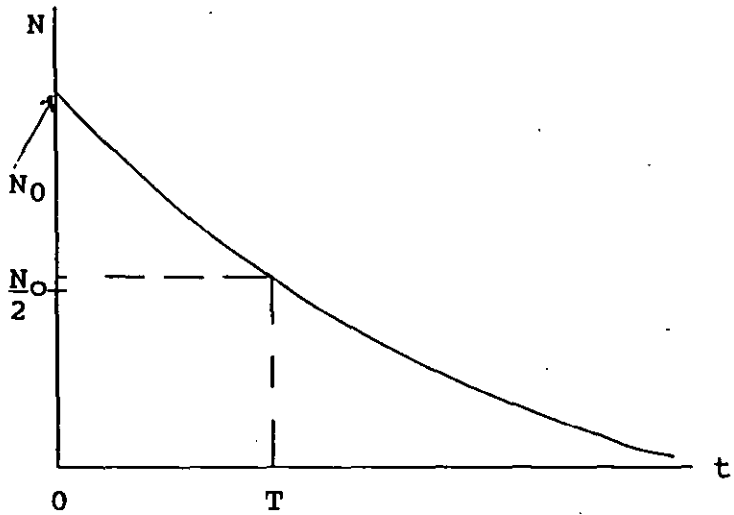
Thus the ratio of the rates required is given by

$$\frac{r_0}{r_1} = 2^{3/4}$$

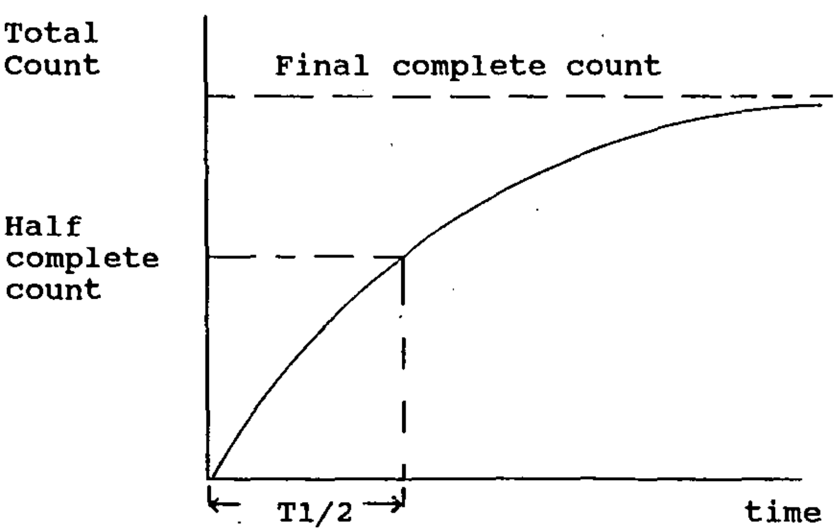
i.e.

$$\frac{r_0}{r_1} = 1.7 \text{ (approx)}$$

The following are the graphs usually associated with the half-life experiment.



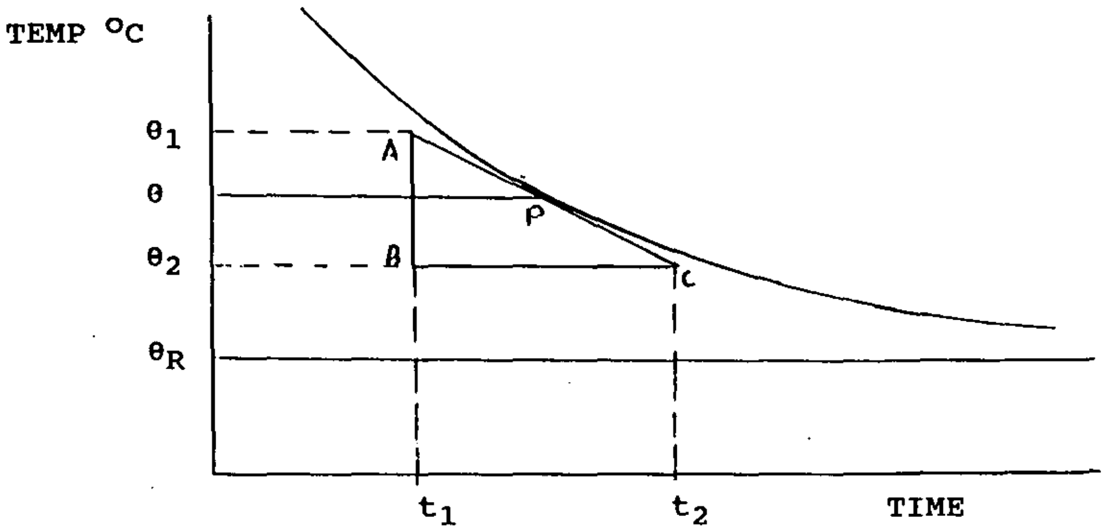
RADIOACTIVE DECAY WITH TIME



COUNT IN HALF-LIFE EXPERIMENT

Fig.7.6(b).

Newton's law of cooling is a specific application of the derivative of a function at a point.



TEMPERATURE FALL

Fig.7.6(c).

The above diagram shows the graph temperature change with time of a cooling body. The instantaneous rate of change of temperature θ at the point P is given by the gradient of the tangent to the curve at P.

The rate of fall is a measure of $\frac{AB}{BC}$, and

$$\frac{AB}{BC} = \frac{\theta_1 - \theta_2}{t_2 - t_1}$$

If Q is the quantity of heat lost per second, then experiment shows that

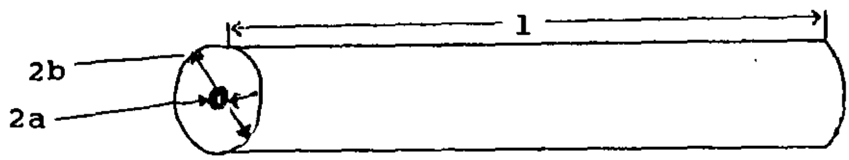
$$Q = ks\ (\theta - \theta_R)$$

which is proportional to $d\theta/dt$, the rate of cooling.

where S is the surface area of the body's surface, and θ_R is the temperature of the surroundings. k is a constant depending on the nature of the surface.

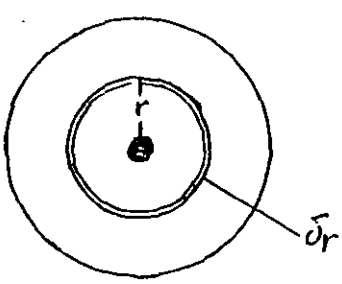
The following examples occurred as part questions at 'A' level and are chosen because they exhibit the fundamental process of summation as did the magnetic force example earlier.

<u>Example</u> A coaxial cable of length l has an inner copper conductor of diameter 2a and an outer copper conductor of internal diameter 2b separated by a plastic insulating material of resistivity ρ, as shown in the diagram.



Show that electrical resistance R between inner and outer conductors is given by the expression

$$R = \rho \frac{\ln(b/a)}{2\pi l}$$



Consider an elementary strip at a radius r. The element of resistance is given by

$$\delta R = \frac{\rho \, \delta r}{2\pi r l}$$

$$\frac{\delta R}{\delta r} = \frac{\rho}{2\pi r l}$$

$$\operatorname*{Lt}_{\delta r \to 0} \left(\frac{\delta R}{\delta r}\right) = \frac{dR}{dr} = \frac{\rho}{2\pi r l}$$

$$\Rightarrow dR = \frac{\rho \, dr}{2\pi r l}$$

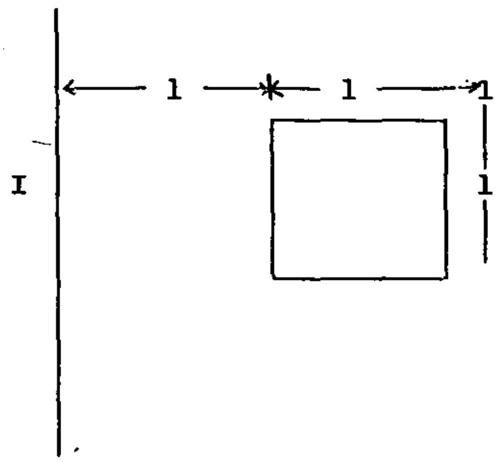
Thus

$$R = \int_a^b \frac{\rho \, dr}{2\pi r l}$$

$$= \frac{\rho}{2\pi l} \int_a^b \frac{dr}{r}$$

$$= \frac{\rho}{2\pi l} \ln(b/a)$$

Example: A square metal frame of side l and resistance R is placed with its plane parallel to a long thin wire carrying a steady current I in a vacuum. The nearer side of the frame is at a distance l from the wire as shown .



i) Show that the magnetic flux 0 through the frame is given by

$$\phi = \frac{\mu_a I l \ln 2}{2}$$

ii) derive an expression for the mean rate of heat production in the frame when the steady current I in the wire is replaced by an alternating current $I = I_0 \sin wt$.

----

i) Considering an elementary area δA of length l and width δx at a distance x from the wire.

Then $\delta A = l \cdot \delta x$

$\delta \phi = B.\delta A$ where B is the magnetic flux density

and $B = \dfrac{\mu_0 I}{2\pi x}$

Thus

$$\delta \phi = \frac{\mu_0 I}{2\pi x} . \delta A = \frac{\mu_0 I l \delta x}{2 \pi x}$$

$$\underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta \phi}{\delta x}\right) = \frac{d\phi}{dx} = \frac{\mu_0 I l}{2\pi x}$$

$$d\phi = \frac{\mu_0 I l\,dx}{2\pi x}$$

$$\phi = \int_{l}^{2l} \frac{\mu_0 l I dx}{2\pi x}$$

$$= \frac{\mu_0 I l}{2\pi} [\ln x]_{l}^{2l}$$

$$= \frac{\mu_0 I l}{2\pi} \ln 2$$

---

ii) Power dissipated as heat = $I^2R = \frac{v^2}{R}$ where v is the rate of change of flux per second $\frac{d\phi}{dt}$

$$\frac{d\phi}{dt} = \frac{d}{dt}\left(\frac{\mu_0 I l}{2\pi} \ln 2\right) \quad \text{where } I = I_0 \sin wt$$

$$= \frac{\mu_0 w I_0 l \cos wt . \ln 2}{2\pi}$$

Mean rate of heat production H is found by using the root-mean-square value I_r of the alternating current I.

$$I_r = \frac{I_o}{\sqrt{2}}$$

where Io is the maximum value of the current.

$$H = \left[\frac{\mu_o w l, (\ln 2)\ I_o}{2\ \pi}\right]^2 \cdot \frac{1}{2R}$$

i.e.

$$H = \left[\frac{\mu_o w l, (\ln 2)\ I_0}{8\pi^2 R}\right]^2$$

7.7 Chemistry

Most of the calculus used in chemistry at school level is concerned with rates of change of chemical reactions. From a particular chemical process a differential equation is set up, the solution of which depends on the standard integral

$$\int \frac{1}{x}\, dx = \ln x + c$$

Here we look at two areas: chemical kinetics - to examine the integration of the rate laws and a simple application in chemical thermodynamics .

In chemical kinetics the rate of chemical reaction is expressed as the variation in concentration of either reactants or products with time. If A is a reactant and [A] denotes the concentrations of A in moles per litre, then the rate of reaction is given by

$$\frac{-d[A]}{dt} = k_1[A]$$

Reactant A disappears as reaction proceeds, therefore, the quantity $\frac{d[A]}{dt}$ is negative.

Now $$-\frac{d[A]}{dt} = k_1\ [A] \qquad (k_1 \text{ a constant})$$

implies

$$-\frac{d[A]}{[A]} = k_1 dt$$

Upon integration

$$-\ln\ [A] = k_1 t + c$$

when t=o, the concentration of A is $[A]_o$, therefore c = $-\ln[A]_o$

Thus, we have

$$-\ln[A] = k_1t - \ln[A]_o$$

or

$$\ln \frac{[A]_o}{[A]} = k_1t$$

For a first order reaction, a graph of $\ln \frac{[A]_o}{[A]}$ against time gives a straightline with gradient k_1

Also, $\ln \frac{[A]_o}{[A]} = k_1t \qquad [A] = [A]_o\, e^{-k_1t}$

and the concentration [A] decreases with time

For a second order reaction

$$-\frac{d[A]}{dt} = k_2[A]^2$$

$$-\frac{d[A]}{[A]^2} = k_2\,dt$$

$$-\int \frac{1}{[A]^2}\, d[A] = \int k_2 dt$$

yielding

$$\frac{1}{[A]} = k_2t + \text{constant}.$$

when t=o, the concentration of A is $[A]_o$, and the constant $= \frac{1}{[A]_o}$

Thus for second order reactions

$$\frac{1}{[A]} = k_2t + \frac{1}{[A]_o}$$

Reactions of higher orders are dealt with in the same way.

In thermodynamics, it is necessary to look at heat capacities at constant pressure or volume. This analysis uses rates of change.

The heat capacity, C, of a system, can vary with temperature and is defined in the differential form.

$$C = \frac{dq}{dt}$$

where q is the quantity of heat absorbed when temperature is increased by T degrees.

The heat capacity at constant volume is equal to the rate of which the internal energy of the system increase with temperature.

i.e. $$C_V = \frac{dE}{dT}$$

when heat is supplied to a system at constant pressure expansion occurs and work is done against applied pressure. Therefore,

$$C_p = C_V + \text{work done in expansion}$$

$$C_p = C_V + p\frac{dv}{dT} \quad (p \text{ constant pressure})$$

$$= \frac{dE}{dT} + p\frac{dv}{dT}$$

But $$H = E + pv \qquad (H = \text{heat content}).$$

Thus $$\frac{dH}{dT} = \frac{dE}{dT} + p\frac{dv}{dT}$$

Therefore,

$$C_p = \frac{dH}{dT}$$

Thus, the heat capacity at constant pressure is equal to the rate of increase of heat content with temperature.

Chapter 8: Conclusion.

8. Conclusion.

Some attempt has been made to look at the way in which calculus has developed in schools and the part it has played, and will continue to play, in mathematical education. Many changes have taken place. A subject which once occupied the syllabuses of University mathematics has gradually made its way to the domains of school mathematics. This particular change was not easy as some exponents of the calculus felt that to reduce its presentation to the extent that it became presentable in schools was to trivialise the subject.

The main opponents to this change argued on the grounds that calculus would lose its rigour and be deprived of a beauty with which it has been familiar. The fact that a loss of rigour might facilitate understanding was not an appeasement. As a consequence, for many years, the resistance to change showed no signs of subsiding.

One major influence on the change was the introduction of the microcomputer. It is probably the single most influential agent of change in mathematical education [9]. It has, by its ability to demonstrate visually many conceptual ideas, introduced many students to areas of mathematics which were only dealt with in an algebraic manner. The speed with which the computer can execute mathematical calculations supersedes the process of any physical computation. This caused many teachers to be somewhat concerned about their role in the classroom. They saw the microcomputer as an invasion of the privacy of their classrooms.

Another influence to changes in approach to mathematics in general, and calculus in particular, was the way and speed with which technology was changing. This technological age brought with it new demands; it appealed to applications of mathematics rather than mathematics for its own sake.

Teachers began to realise that many students study mathematics only for the possibility of being able to use it in areas of science, engineering and commerce. It was necessary, therefore, for teachers to critically examine their approaches to many areas of mathematics, thereby responding to a change in society - the technological age - which was not a passing phase but a permanent state of affairs.

Tall [9], referring to the challenge of the new technology, argues that major problems face us in the 15 - 19 age group in mathematics education. He attributes these to the fact that applications of mathematics were becoming more technologically diverse, leading to pressure to modify the mathematics curriculum. Furthermore, changes in the technology were occurring so rapidly that individuals were required to be more flexible and capable of solving new problems as they arise. The role of the computer in changing the perception of mathematics, and indeed the nature of mathematics, for teachers and students, is very evident today. It facilitates imaginative ways of approaching mathematical concepts that are likely to create greater understanding for a new generation of students.

The prominence of the new technology in schools, however, was not in itself a solution to teachers' problems initially. Unsuitable and, in some cases, unavailable software made it difficult for teachers to transfer normal tried and tested teaching methods to a computer approach. Currently, however, there are a number of software packages available to teachers which facilitate the teaching of calculus in schools; for example, Graphic Calculus [11] and Omnigraph [64]. These enable a clear and graphical approach to many concepts of the calculus, allowing the students to become familiar with the terminology, which would otherwise have been difficult to impart: limits and the process of tending to these; numerical approaches to the derivative; and integration as a summation.

The traditional approaches to the calculus have been looked at with some discussion of the terminology used and the intrinsic difficulties associated with it. We also considered the problems teachers face in trying to teach the concepts of the calculus whilst attempting to strike the right balance between principles and processes. A further concern expressed was the restrictions on time and the need to meet syllabus requirements. Couple with this the exclusion of calculus from GCSE and what we have is a possibly critical situation, the consequences of which may only reveal themselves as the success or failure of the new examination to fulfil its aims and objectives is continually assessed.

The crosscurricular role of mathematics in schools and its ability to continue to fulfil this role in a situation which appears to change daily and makes new demands each time have been considered. Some examples of the use of calculus in other areas are included. The hope is that they show how the many basic ideas of elementary calculus are used in applications across the curriculum.

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