Statistics at A Level

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STATISTICS AT A LEVEL

by

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Abstract

In this dissertation I attempt to look at the statistics curriculum in the VIth form of the secondary school, with particular reference to the statistics content on the various Advanced Level Mathematics syllabuses.

The introduction is concerned with examining the various reasons for the inclusion of statistics on many A level Mathematics syllabuses in the mid 1960's. Also I put forward my view that statistics should be aimed at VIth form mathematicians to give them a concept of applied mathematics. The theme of my dissertation is to examine the various syllabuses to see if this was their aim as well.

I thought that it was necessary to briefly review a selection of the reports and papers that have been written on statistics at A level. I found that there has been very little written on this subject. What has been written has mainly come from professional statisticians and not from practising school teachers. Perhaps the most authoritative papers have come from the Royal Statistical Society in 1952 and 1968. This body seemed to favour statistics developing as a single subject at A level but accepted that many students would prefer to take statistics as a half-subject.

The third chapter deals with what statistics there is on the various A level mathematics syllabuses and in what forms statistics is examined. For my fourth chapter I chose to examine how one particular topic from statistics is taught. My choice was probability since

this is used in most areas of statistics. In looking at the different approaches it was necessary to examine the probability sections of several A level text-books on statistics.

A theme that I tried to incorporate all the way through the dissertation is to ask the question "is what we teach at A level, statistics or mathematics?" In chapter five I attempt to examine this question in slightly greater depth. I came to the conclusion that much of what we teach is pure mathematics in disguise, for example setting purely mechanical problems in calculating means and variances. What we should be aiming at is a course in applied mathematics and not a course in mathematics under the guise of statistics.

Chapter six looks at the new trend towards including some project or practical work as part of the statistics at A level. I examined two particular syllabuses set by the J.M.B. and the University of London Board.

The final chapter is basically a brief summary and a conclusion. I felt that much of the work which was included in the statistics sections was still too mathematical and not statistical enough,but the signs are that things are starting to change. The inclusion of project work on some syllabuses is one such sign. Essentially my recommendations are that more courses in statistics, initial and in-service, should be provided for teachers. Also it is time another report was made on the statistics curriculum in the VIth form, along with the setting up of an on-going project in statistics to offer guidance for teachers.

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CHAPTER 1

Introduction

Over the past few years our society has been making ever increasing demands on numeracy, numerical proficiency and also, statistical understanding. These demands have been reflected in our school mathematics curriculum at all levels. At A level in mathematics this has meant an expansion of many of the syllabuses to include some work on statistics. There is more than one reason why statistics is now included on many A level mathematics syllabuses and I shall now attempt to look at some of these reasons.

The "traditional" mathematics taught at secondary level usually stops with calculus of an eighteenth-century variety. But profound changes in the growth of mathematics had taken place in the nineteenth century and even by 1910 some of the more forward looking universities reflected these changes in their courses. Eventually, though it takes some time, these newer topics work their way down into the secondary school. Some of these newer topics arose out of the attempts to understand Nature which led to a construction of what are now called mathematical models, Probability and statistics had started to develop out of such things as statistical thermodynamics and mathematical genetics. So one reason that statistics has found its way on to A level mathematics syllabuses is that the nature of mathematics has changed and so the nature of the mathematics taught in schools has had to make corresponding changes so as to be representative of the subject as a whole.

In Britain, applied mathematics normally means mechanics, which is often claimed to be difficult and not a very popular subject. With a shortage of good teachers of mechanics, there results an even greater shortage, as the next generation of pupils avoids teaching the subject. As a consequence, statistics has come into the curriculum within the last decade or so, not so much as a rational choice for its own merits, but simply because teachers who are competent at pure mathematics tend to be more willing to teach statistics than mechanics. Because the tradition in statistics is so short, it is often treated in a less artificial way than mechanics, with more real and appealing applications. There is, however, considerable dispute among statisticians about what exactly should be taught.

The changing nature of some other subjects has also created a demand for statistics to be included on A level mathematics syllabuses. More and more subjects at school level, and later at University level, are having to make use of statistics. In physics, for instance, even an elementary account of radioactive phenomena must surely mention probabilistic aspects, and perhaps the student should be introduced to some properties of the Poisson process. Use of simulation and random numbers would also be of benefit to such students. Economics, geography and biology are other obvious examples where statistics is of great use. The ironical thing is that, although some of the demand to include statistics on A level mathematics syllabuses has been generated by the needs of user subjects, the statistics required by such subjects is often neglected on the mathematics syllabuses. For example, questionnaire design which is of great use in the social-sciences is hardly ever mentioned on the vast majority of A level mathematics syllabuses.

Apart from being an "examination factory", school is also supposed to cater for the student's needs in terms of his education as an all round citizen. Many students of A level mathematics do not follow a path onto a tertiary course, but leave school and enter jobs in banking, insurance and managerial positions. Even more so than the average man on the street, such people are going to have to be acquainted with simple ideas of bias, random variation, errors of measurements and so on. So, since A level examinations are no longer just a university entrance qualification but also a terminal examination, it is to be expected that the needs of the individual in society after school have also created a demand for statistics to be included on A level mathematics papers.

Until about a decade ago, there was little statistics on most of the A level mathematics syllabuses. Perhaps one of the main reasons for its inclusion in the mid 1960's was that, over the previous few years, many mathematics graduates who left university and entered the teaching profession had studied some statistics on their undergraduate courses. Also, there were some graduates of statistics who became teachers, usually of mathematics. Many of these teachers created a pressure movement with the aim of including statistics on the A level mathematics examinations. As with many subjects then, statistics first became popular at a university level and then worked its way down to the upper-end of secondary school level.

It is argued by some that, because of a lack of statisticians about ten to fifteen years ago, the universities created some pressure on the examination boards to include some statistics on the A' level syllabus. In other words, the work at A level would act as a preparatory course for students who wanted to take a degree in statistics at university. It is claimed by Professor F. Downton (1968) that since statistics has been included on A level mathematics syllabuses, he has found the students were no better prepared and in some cases remedial work has had to be done. This was some time ago, and things could have changed by now with revision of syllabuses and teachers becoming more experienced in statistical work. All the same, I would hope that no A level in mathematics which included statistics would attempt to act as a pre-university course for statisticians. The students at this level are too varied in their needs and not mature enough for such a course.

We now have the situation that statistics is included on most A level mathematics syllabuses. Many statisticians are unhappy, though, with how statistics in VIth form courses has developed. For instance, Professor D.J. Finney (1977) claims, "Statistics in schools should be presented as an aid to coherent thought about numerical matters in science and technology, and in the understanding of our society" In fact what has happened is that statistics courses tend to concentrate on the mechanics of calculating means, variances and correlation coefficients for example. This is mainly due to the facts that it is much easier to examine a student's proficiency in manipulative techniques and also many teachers of mathematics have had little or no experience in statistics. So one thing that I shall be trying to look at is the question of whether we are teaching applied statistics or applied mathematics.

There are differing views of the role that statistics should play at A level. As I have said before, I do not feel that statistics ought to be a specialist subject in its own right at VIth form level, the students at this age do not usually possess the required mathematical maturity, nor are there the required number of students who are sufficiently motivated. It is best to leave such a course until university level where there are suitably qualified teachers. There are some who say that the statistics taught at A level should be geared so as to cater for students of user subjects such as biology and geography. It would be difficult to justify setting a whole A level examination in statistics to cater for the needs of others, though. For instance, A level mathematics is not set to cater for the needs of A level physics students. The statistics required in user subjects can be taught when and where it is needed. I think, then, that statistics courses should be aimed at mathematicians in the VIth form to illustrate the usefulness and applications of mathematics. We must remember, though, that statistics is a subject in its own right and not lose sight of this fact even though the statistics will form part of a wider mathematics course. The questions on a statistics paper should not simply be pure-mathematics questions heavily disguised. There is a great overlap between mathematics and statistics and it is from this area that a worthwhile statistics section for mathematicians can be drawn. I will attempt to examine, then, this question of how the statistics fits in on the mathematics courses and what statistics is actually included on the various A level mathematics courses. This, along with the question of whether or not we are providing a satisfactory applied mathematics course in statistics for A level mathematicians will be the underlying theme throughout the various chapters.

CHAPTER 2

A BRIEF LOOK AT SOME OF THE PAST REPORTS ON STATISTICS AT A- LEVEL.

Surprisingly, there have been few authoritive papers whose brief has been concerned with the teaching of statistics in the VIth form, as related to the 'A' Level mathematics courses.

In 1952 the Royal Statistical Society produced a paper entitled, "The Teaching of Statistics in Schools". Even then, only a small part of the paper is devoted to statistics in the VIth form. In the early part of the 1950's, statistics was a relatively new subject on the syllabuses of schools, and few schools actually taught it as a formal subject. Therefore, to some extent the report was an attempt to clarify the role and purpose of statistics so that it could find its natural niche in the school curriculum.

Much of the first half of the report is concerned with discussing the function of statistics, not as preparation for university, but in its wider educative value in preparing the pupil for citizenship. The report recommends, for instance, that "the subject should be introduced into all secondary schools as part of the general education". Apart from a brief mention which says that statistics is now very important for many professions and therefore, should be a specialist subject in the upper part of the secondary school there is little else of relevance as regards VIth form statistics.

It is not until later on in the report that some six paragraphs are devoted to statistics in the VIth form although the same six paragraphs also caters for statistics in vocational training. The report makes the point that statistics of a specialized kind is important for many careers and special subjects. Therefore, the writers of the report felt that the teaching of statistics in the VIth form was a desirable development. It must be remembered that at the time, such a development had only taken place in a few schools.

The report goes on to say that the mathematical specialist would derive a satisfaction in finding that statistical theory uses concepts and methods he has learnt in his general mathematics course: there is a close analogy between the moments used in frequency distributions (the mean and the variance) and those arising in mechanics, (the centre of gravity and the moment of inertia). Further examples given are the theory of permutations and combinations as applied to probability theory and * the co-ordinate geometry of the straight line is necessary for expressing the properties of the regression line in correlation analysis. It was considered, in the report, that mathematical statistics was relatively new and rapid devolopments were taking place. Hence the budding mathematician, the report claimed, "can be inspired by the realisation that there are new and exciting possibilities for research."

It is easy to see that ideas have changed since this report was written. Most articles that you read nowadays seem to mainly conform to the idea that mathematics is the study of abstract relationships whereas statistics tends to be concerned with the analysis of data, in other words statistics is not mathematics. Some of the above mentioned topics are now criticised for their inclusion on some syllabuses since they are pieces of pure mathematics contrived and "dressed up" to appear as statistics. To a certain extent, though, I must defend the report since it categorises the above topics as mathematical statistics and not just statistics, and therefore relevant to a mathematics course.

Although it does not say so explicitly the report seems to imply that statistics as a subject in the VIth form ought to have developed in different ways. That is, mathematical statistics or the mathematics used as a tool in statistics should have formed part of the mathematics course. On the other hand, certain statistical ideas and methods are essential to quantitative biology and are required to describe biological variation and to deal with the planning of, and the drawing of conclusion from, biological experiments. Other branches of science and technology have similar requirements. Also, pupils who are going to be social scientists or business men, for instance, will require the ability to read and perhaps construct statistical tables and diagrams * to find their way around official, economic and social statistical data, and to draw sound conclusions as to causes and effects in their field of study. These aspects of statistics, the report seems to say, should develop out of the context of the subjects in which they arise plus some back up from non-mathematically biased, statistics courses.

Of course, in most VIth form courses, this just has not happened, the statistics taught is very much part of the mathematics course and the mathematics syllabus at A level. Statistics in general has not developed out of the context of user subjects. Subjects such as geography and biology use statistical techniques often only as a last resort and the statistics tends to be confined to calculation of such things as correlation coefficients and values of χ^2. These techniques require the students who are not also following a mathematics course to blindly follow a meaningless set of mathematical procedures, there is little emphasis on the student to do a great deal of interpretation of the various numerical measures once he has calculated them. The emphasis is still mathematical even when the statistics does not form part of the mathematics course.

The report also has a section on statistics in school examinations and it was felt by the report that some statistics should appear formally in the syllabuses in A level mathematics. Also, the writers of the report thought some questions with a statistical bias in subjects that use statistics, such as biology and economics, would encourage attention to the statistical aspects of these subjects in teaching. The prior recommendation has been fulfilled but the second has only been partially achieved. Perhaps this is because of teacher reluctance to make themselves familiar with a subject which many of them regard as a branch of mathematics. Many biology teachers for instance chose to study biology because it meant that they could study a science without having to achieve the mathematical proficiency required to study physics, say. Of course, this is not true nowadays since even biology has become most reliant on mathematical and statistical methods. The second recommendation should gradually become reality as more recent biology and economics graduates who have to be reasonably proficient in mathematics work their way through to senior teaching posts.

At the time the report was written it was considered by the authors that statistics should be introduced to a modest extent rather than to a specialised degree in separate papers. The reason given for this was that many teachers of mathematics are not experienced in statistics and therefore would probably hesitate to prepare pupils for specialist papers in statistics. Today, we do have specialist papers at A level and they form, in general, part of the mathematics syllabus. Many teachers of mathematics still have little experience of statistics and this could account for the heavy mathematical emphasis of statistics examinations at A level.

The only other recommendation that the report made concerning the VIth form statistics examinations was that since it was accepted practice to offer two or three alternative syllabuses for examination in A level mathematics then one of these alternative syllabuses should include statistics. This was to provide for pupils who are primarily interested in biology, economics, geography and so on. I am not sure that many such students would wish to take a paper which basically would be an 'A' level in mathematics. Apart from the fact that such students would probably prefer to study other relevant supporting 'A' level subjects and therefore would not have time to take statistics as an 'A' level subject. Also, if statistics was simply an alternative syllabus for an 'A' level in mathematics then I would suppose that such an examination would have a bias towards mathematics which would probably be well beyond the capabilities of many of the students in question.

As a whole, the report does not have a great deal to say about statistics at 'A' level and what it does say hardly offers much constructive advice concerning the actual structure of statistics courses in the VIth form. Sixteen years later, The Royal Statistical Society produced a second report, entitled The Interim Report of the R.S.S. Committee on the Teaching of Statistics in Schools (1968).

The second report, after briefly summarizing the recommendations of the first report, starts by saying that the conclusion reached by the previous committee that statistics should not be taught as a subject in its own right was no longer valid under the present conditions. This was because this Committee felt that the subject had not made enough progress in this way and therefore would be given more positive encouragement if the subject were introduced in a suitable way in its own right. It was felt that it was now practicable since mathematics undergraduates were spending an increasing amount of their time on statistics, hence more teachers of mathematics would have the necessary experience.

The report states that the most feasible stage, at that time, to introduce statistics is at 'A' level. At the time the report was written the sixth-form mathematics curriculum was undergoing quite a bit of change and one of these changes was the introduction of statistics onto the 'A' level syllabus in mathematics. The Committee approve of this change on the grounds that many students of mathematics would welcome a subject which demonstrates the application of mathematical methods in varied fields.

It seems a little strange in some ways, that earlier the report said that statistics ought to develop as a subject in its own right but here the report is giving its approval to statistics developing as part of the applied section of the 'A' level mathematics examination. Perhaps, though, the Committee were endorsing both developments with a hope that emphasis between the two developments would be different.

It was considered also, by the Committee, that the introduction of statistics to the VIth form syllabus either in the form of a whole subject or as half of a joint 'A' level course in Pure Mathematics and Statistics would go some way in meeting the needs of the VIth formers studying such subjects as biology and social sciences. Students in these sciences now need more sophisticated techniques than are contained in an O-Level mathematics syllabus. From my own experience of VIth form teaching this just has not been the case. Most syllabuses in the social sciences include the bare minimum of quantitative methods despite the nature of the subjects today. Also, the statistics on mathematics papers rarely draws its examples from the subjects mentioned. Not only that, in general, teachers of mathematics rarely discuss with their counterparts in user subjects the interface between their subjects. Hopefully, though, in the future, as 'A' level syllabuses in the social sciences reflect more truthfully the nature of their subjects, then, some discussion will have to take place.

The report goes on to suggest an approach to an elementary statistics course. The authors of the report put forward the idea that a statistics syllabus should be designed both for mathematicians and for students who wish to read experimental and social sciences. They say that this could be achieved by having a flexible syllabus in which more topics are listed than would be needed to pass the examination. I am not sure that this would work in practice, the choice of what topics would be taught would probably be decided by the likes and experience of the teacher rather than the needs of the pupils. The topics with less mathematical emphasis would tend to be neglected since the job of teaching this syllabus would be taught in the majority of cases by mathematics graduates. Even today there are few graduates of statistics in secondary school teaching. Alternatively, some teachers for the sake of examination success might attempt to teach every topic. This would lead to a terse and superficial treatment of each topic without the student achieving any real understanding of any one of them.

The Committee recommended that the mathematical level of such a course would be such that some of the topics can be taken by students having O-Level mathematics and others depend on some supporting mathematics being taught in the course. This, the report claims, there are two reasons for. One is that although many students may take both statistics and mathematics, it would make the teaching of the statistics course very difficult if it depended on progress in mathematics. Actually this difficulty does arise in 'A' level mathematics today where statistics forms a section of the total syllabus.. For example, when I teach continuous probability distributions I have to wait until colleagues have taught certain aspects of integration in pure mathematics. All the same, it is possible to plan for this. The second reason is that statistics may be taken by students who do not take 'A' level mathematics, and these students will have to be taught the necessary mathematics, rather in the way some students approach an 'A' level course in physics. The report claims that if these mathematical techniques are taught as the need arises, they can be presented as useful tools for particular statistical problems. It is my experience that there are few students of 'A' level physics who do not also take an 'A' level in mathematics. Lack of staff and time makes it impossible in many schools to give back-up lessons in mathematics to such students. I therefore think that such a system would not work in practice.

On the examination for the course, the report recommends that the setting of examinations should be referred to experienced statisticians. Also, they felt that students should be involved in their own experimental work. A notebook should be kept of such work and submitted for scrutiny by the examiners. I am not sure where the Committee expected to find enough statisticians with experience of teaching 'A' level to be examiners, though.

The report goes on to list some books and teaching aids which would be of some use in such a course before giving a course outline. It was envisaged that the course would be in two parts: the part 1 syllabus would count as a half-subject 'A' level and would form a basis for the part II options. Parts 1 and II would form a single 'A' level in statistics. With the exception of Design and Analysis of Experiments the topics suggested to form the basis of the course are much the same as found on most statistics sections of 'A' level mathematics courses today.

The remainder of the report consists of three appendices, the first is merely a copy of the various syllabuses of the statistics section of the different 'A' level boards. The second appendix gives an outline of some experiments for teaching and the final appendix is the suggested syllabus for the report's own course.

The second report is far more a sincere and constructive attempt to suggest a statistics course in the Vlth form than the first. In some ways, their report came a little late because most 'A' level boards had by then constructed their own syllabuses and these were and are very much part of the 'A' level mathematics syllabus. To some extent some of the recommendations give me the feeling that most of the members of the Committee have had little or no experience of teaching at secondary school level, with little realisation of the problems at that level.

With the exception of these two reports there has been little else published. The Mathematical Association did publish "An Approach to 'A'-Level Probability and Statistics" in 1975. This was not an attempt to suggest a course in statistics for the Vlth formers, though, but a paper on how to actually teach various statistical topics already on existing A-level courses in mathematics. Perhaps it is time that a committee consisting of professional statisticians and practising teachers of mathematics produced a more balanced and realistic paper.

CHAPTER 3

Statistics in A Level G.C.E. Syllabuses

This chapter attempts to examine the statistical content of the A level G.C.E. syllabuses offered by the examination boards. After discussing the different forms which statistics appears on the various syllabuses and a detailed critique of the statistical content, I shall concludewith a more critical appraisal of two particular syllabuses.

There are three forms in which statistics appears on various A level mathematics syllabuses: We have the single subject A level mathematics examinations in which the statistics content forms part of the applied mathematics section of the paper. For example, the mathematics paper syllabus A50 set by the Oxford Local Examination board is such an examination. Paper I is set only on the Pure Mathematics section of the syllabus and paper II is set on the Applied mathematics section of the syllabus. Equal waighting is given to mechanics and statistics on paper II. Eleven questions are set on mechanics and eleven questions are set on statistics. The candidates are required to attempt answers to eight of the questions and there is no restriction from which sections the candidate selects the questions to answer. He may, if he wishes, select all eight questions from the mechanics section or he may select all eight questions from the statistics section. On the other hand, the candidate may answer some questions from each section. So, it is possible for candidates sitting this examination to get away with not having to do any statistics at all. This fact illustrates that there are still many students of A level mathematics who leave school with little of no knowledge of statistics. Of course, it has always been possible for students to opt to do only Pure mathematics and so avoid doing any mechanics, but if VIth formers opt to do a single subject in mathematics then they ought, in my opinion, to be required to study all those aspects of applied mathematics which happen to be fashionable in school mathematics at the time.

Quite a popular option now is an A level in Pure mathematics with Statistics as set by the University of London Board, for example. This particular paper was first examined in June 1977 so it is a fairly recent innovation. The syllabus for Paper I has much in common with that of the A level Pure Mathematics Paper I set by the Board. However, there are some questions from the statistics section of the syllabus included in Paper I. It is claimed by the board that sufficient Pure Mathematics topics have been maintained to justify the title of the subject. Having examined the speciman paper of this examination though, the bias seems to be towards the statistics. Paper II contains ten questions on statistics of which the candidate must attempt to answer seven. It is expected that, during their course of study, the candidates will carry out three projects in statistics. The students are required to take the projects into the examination with them and refer to them if they so wish. No marks are awarded for doing the projects but the examination questions are geared so as the students need to use the experience that they have gained in doing their projects.

The other form in which statistics is examined at A level is in the form of a single subject in statistics. One such board who set this form of examination is the Associated Examining Board. Ten papers are set by the Board in mathematics, and papers 9 and 10 form the statistics examination. It is possible with this Board to also take paper 9 with paper 1, which is a pure mathematics paper, and hence sit an examination in Pure mathematics and Statistics.

Both papers in statistics are three hours in duration and require the candidate to attempt six questions out of a possible ten. As might be expected from a single subject examination in statistics some of the content examined goes further than the content examined in single subject mathematics. For instance, this is the only examination at A level which requires the use of probability paper and sets questions on analysis of variance. With this examination a prescribed amount of background mathematics is laid down which is considered necessary to cope with some of the statistical techniques. I would imagine that it would be possible to teach the pure mathematics as and when it is required. Such topics as hyperbolic functions and determinants are required and it would seem to me that someone with a certain amount of mathematical ability would be best suited for this particular syllabus in statistics.

Considering that this was a single subject examination in statistics and therefore is likely to attract non-mathematics specialists, the questions, in many cases, seemed highly manipulative in the mathematical sense. This is perhaps more understandable of questions on statistics sections of single subject mathematics examinations which is being set for more mathematically orientated students. One would expect, at least I would, to find more questions of an applied statistics nature as opposed to an applied mathematics emphasis on a paper of this kind. The emphasis should be as much on the interpretive nature of statistics as on the manipulative aspects of the mathematics which is used as a tool in statistics. I am not trying to say, one way or the other, that this is a good or bad paper but I do question whether this truly could be called an examination in statistics. Perhaps a better title would be statistical mathematics.

I considered 18 different syllabuses and the statistical content of these syllabuses was analysed according to a list of 61 categories. It would have been possible to use a more detailed list but as syllabuses are often not very specific I decided that a comprehensive, fairly general, list would convey a more accurate picture. It was occasionally difficult to classify phrases in a syllabus and sometimes a syllabus is a bit vague. Moreover a syllabus only provides a basic framework, teachers are free to develop the subject in their own way.

The analysis is presented in tables 1 and 2. Probability is mentioned in every syllabus, in fact this is the only topic which is mentioned in every syllabus, which is not too surprising since it is pretty basic to most mathematical statistics courses. Also it is fairly straightforward to set and mark questions in probability. Very little emphasis seems to be placed on the more descriptive side of the statistics content, but it could be that the different syllabuses assumed that this would be covered anyway, without these topice being explicitly stated. The Oxford Local Examination Board, for example, in their syllabus say that a knowledge of such topics is expected but no actual question will be set specifically on them. I get the impression from looking at the syllabuses and past papers of different boards that this is a fairly typical attitude of those people who write the syllabuses.

It is perhaps a little surprising, though, that there is such a low coverage of such topics as questionnaire design, time series, index numbers and control charts since all of these examinations are supposed to be in statistics. This seems to reflect an aversion to topics with a lower mathematical content that all the papers have. I am not sure whether this is due to an unwillingness on the part of mathematics teachers to teach them or it could be that these topics are not considered mathematical enough for students specialising in mathematics. It is likely to be both reasons. Incidently, it is important to note that, in practice, the mathematical and non-mathematical aspects cannot be separated, e.g. in order to calculate confidence limits for some quantity, we must first have collected some data and this may well involve questionnaire/survey design. So, might it not be such a bad idea if the boards were to include such topics on their syllabuses?

Most syllabuses include some form of significance testing of one type or another. I was somewhat surprised not to find many boards who included the t distribution (3 out of 18) and the χ^2 distribution (5 out of 18). I am surprised because fairly mathematically manipulative questions can be set on these topics which makes them easy to mark and set. Also these topics are greatly used in such subjects as Biology and Geography and an increasing number of people now study these subjects at A level as well as A level mathematics.

There are one or two special topics, for example, Markov chains and generating functions, which appear on only a few of the syllabuses. These tend to be confined to Further Mathematics type syllabuses and are really only suitable for the mathematics specialists since these topics demand a more sophisticated mathematical approach than normal. Even more so than the A level mathematics papers, the Further and Higher Mathematics A level papers include topics which are basically testing the manipulative ability of candidates in topics which are essentially mathematics but have application in certain statistical topics. There is little statistical content in terms of an interpretive emphasis.

In general I feel that the statistical content of most of the mathematics syllabuses is somewhat narrow. There is a marked emphasis on the numerical manipulative side, which is easier to examine, there is not a balance with the qualitative aspects of statistics. Though the latter is harder to teach and examine, it is argued by some it is the more important side as, without it, one is faced with the charge that statistics is simply the manipulation of data and, as such, irrelevant to most people.

I have chosen two particular syllabuses to take a slightly closer look at ; the A50 syllabus set by Oxford Local Examinations and the Statistics syllabus set by the Joint Matriculation Board. I hope to achieve this by considering a question from each of their papers both on regression.

This first question was set by Oxford in 1977 and was question 21 on paper II:

The state of Tempora demands that every household in the country shall have a reliable clock; inspectors are being introduced throughout the country to implement the policy. The Chief Inspector has the following data on the population size of towns, where Inspection Units have been set up, and the number of man-hours spent on inspection.

Population (thousands)	3	4	5	9	13	15	18	20	21	22
Man-hours spent on inspection	8	11	13	18	24	26	31	32	34	33

(i) Calculate the regression line for predicting the number of man-hours from the population size (note that the mean value of each variate is a whole number).

(ii) Predict the man-power required (in man-hours) for a new Inspection Unit to be installed in a town with a population of 17,000.

The second question was set by the J.M.B. in a speciman paper in 1976. (This particular examination will be first sat in 1978).

A textile firm manufactures a particular fabric in a variety of different qualities at corresponding different prices. In 1970 various amounts of the different qualities were sold; the amounts, y (in thousands of metres), and the prices, x (in pence per metre), are shown below.

Price (x)	37	55	25	20	47	62	24	52
Amount sold (y)	75	62	71	61	70	59	65	69

(i) Estimate the equation of the regression line of y or x.

(ii) Draw a scatter diagram of the data and draw the regression line on this diagram. Comment on the appropriateness of the linear model in this case.

Both questions are fairly typical of the papers that they are taken from and neither are particularly difficult at this level. What strikes me most about the first question is that the context from which it is taken is unreal and contrived. Somewhat of a contradiction to the nature of statistics.

The numbers in the first question are even contrived to have their appropriate means working out conveniently as whole numbers, thus making the calculation of the standard deviations and the co-variance a much easier task than usual. Though the point must be made that calculators are not allowed in Oxford's examination whereas they are in J.M.B's examination.

The bias in the first question is definitely mathematical, first calculating the regression line and then by substition into this finding another quantity. It is easy to see that this is where the bias lies, even the numbers have been rigged to give a convenient numerical answer. Once these numerical quantities have been calculated there is no requirement on the candidate to make any qualitative or interpretive judgements.

On the other hand, the second question looks as if it might be using realistic data and does ask the candidate to go another stage further and actually draw a scatter-diagram of the data. This is so that the candidate can make a judgement of whether or not he thinks that the appropriate model has been used in this case, a statistical decision rather than a mathematical one.

I deliberately chose two questions on the same topic so that such a comparison could be made. Of course, it is impossible to make any valid conclusion on the basis of just two isolated questions but this comparison does go someway to indicate the difference in emphasis between the two syllabuses. In some ways it is perhaps an unfair comparison because one question is taken from a statistics examination and the other from a single-subject mathematics paper. I justify my comparison on the grounds that, rightly or wrongly, statistics comes under the general regulations for mathematics examinations on all boards and also a comparison is made easier if the questions are at opposite ends of the spectrum.

I am not attempting to make any judgement at this stage which type of question is the best. What I have tried to show is that the main differences between syllabuses are really differences in emphasis between the manipulative aspects of statistical mathematics and the interpretive aspects of statistics. Teachers ought to be aware of which emphasis their syllabus has. From looking at past papers it seems that most syllabuses tend to emphasise the quantitative and manipulative aspects of the subject, though. It is difficult also to say which is the best form of how to include statistics on the A level curriculum. A mathematical approach to statistics on single subject mathematics examination excludes non-mathematical students from taking statistics courses, but an applied statistics approac may be difficult or uninteresting for mathematicians to teach and may not appeal to mathematically inclined students who are not yet motivated to use their mathematics in the solutions of real problems. Perhaps in future the boards should attempt to examine the possibility of setting up two different courses to suit the different needs of students. I would hope that the mathematical aspects of statistics would remain as part of the applied mathematics papers but new courses could be devised to cope with the applied statistics. The greatest problem is, who is going to teach such a course though?

Analysis of Syllabuses

(Table 1a)

	A1	A2	A3	C	OC	O1	O2	W	L1	L2	L3	L4	L5	L6	J1	J2	J3	J4
Statistical enquiries		1	1		1													
Collection of data	1	1	1															
Census methods		1	1															
Classification of data		1	1															
Sampling method		1	1											1			1	
Tabulation		1	1															
Questionnaires		1	1		1													
Bias in sampling		1	1															
Diagrammatic presentation of data	1	1	1											1				
Frequency distribution	1	1	1			1	1	1						1			1	1
Histograms and frequency polygons	1	1	1			1	1							1			1	1
Descriptive properties of distributions	1	1	1															
Cumulative frequency distribution	1	1	1			1	1										1	1

(Table 1b)

	A1	A2	A3	C	OC	O1	O2	W	L1	L2	L3	L4	L5	L6	J1	J2	J3	J4
Estimation of medians etc.	1	1	1		1	1	1	1						1	1	1	1	1
Calculation of measures of average	1	1	1	1	1	1	1	1	1	1				1	1	1	1	1
Calculation of measures of dispersion	1	1	1	1	1	1	1	1	1	1				1	1	1	1	1
Probability theory	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Permutation and Combinations	1		1		1	1	1	1									1	
Addition law for mutually exclusive events	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Generalised addition law	1						1	1	1	1	1	1	1	1	1	1	1	1
Conditional probability	1	1	1	1	1	1	1	1		1	1	1	1	1			1	1
Independence	1	1	1	1	1	1	1	1	1	1	1	1	1	1			1	1
Bayes' Theorem							1	1										
Markov Chains							1						1					
Discrete probability distribution	1	1	1	1	1	1	1	1	1	1		1	1	1				1
mean and variance of above	1	1	1	1	1	1	1	1	1	1		1	1	1				1

(Table 1c)

	A1	A2	A3	C	Oc	O1	O2	W	L1	L2	L3	L4	L5	L6	J1	J2	J3	J4
Binomial distribution	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1
Poisson distribution	1	1	1	1	1	1	1	1		1		1	1	1		1	1	1
Poisson $\sim$ Binomial	1			1	1	1	1	1										1
Continuous distributions (probability density functions)		1	1	1	1	1	1	1		1		1	1	1	1	1	1	1
Cumulative distribution function				1		1	1	1										
Mean and variance of continuous distribution		1	1	1	1	1	1	1		1		1	1	1	1	1	1	1
Expectation	1	1	1	1		1	1	1		1			1		1	1		
The uniform distribution		1	1	1				1		1			1	1			1	1
The Normal distribution	1	1	1	1	1	1	1	1	1			1	1	1		1	1	1
Moment generating functions								1										
Probability generating functions													1		1	1		
Normal $\sim$ Binomial	1			1	1	1	1	1						1			1	1

(Table 1d)

	A1	A2	A3	C	OC	O1	O2	W	L1	L2	L3	L4	L5	L6	J1	J2	J3	J4
Normal ~ Poisson					1	1	1	1						1			1	1
Probability paper	1	1	1															
Sampling distribution		1	1	1	1	1	1	1		1			1	1		1		1
The central limit theorem	1	1	1	1		1	1	1						1				1
Estimation of parameter using a statistic			1	1		1	1	1		1			1	1			1	1
Confidence Limits	1		1	1	1	1	1	1		1			1	1	1	1	1	1
Two tailed significance tests			1	1		1	1							1	1	1	1	1
One tailed significance tests				1		1	1							1	1	1	1	
The t test			1				1											1
The χ^2 test		1	1			1	1											1
The F test to show 2 samples from same pop'n			1															
Wilcoxon Signed Rank test							1							1				
Mann Whitney U test							1											

(Table 1e)

	A1	A2	A3	C	OC	O1	O2	W	L1	L2	L3	L4	L5	L6	J1	J2	J3	J4
Regression (linear)		1	1	1	1	1	1	1		1			1	1			1	1
Confidence intervals for regression quantities								1										
Rank correlation	1	1	1		1					1			1	1			1	
Product moment correlation		1	1	1	1	1	1			1							1	
$\tanh^{-1} r$ test for equality of correlation coefficients		1	1															
Random numbers and simulations	1				1	1								1			1	
Time series etc.		1	1											1			1	
Index numbers		1	1											1			1	
Control charts			1															
Analysis of variance			1															

Key to Examination Syllabus in Table.

A1	Associated Examining Board	-	Alternative Mathematics Paper
A2	Associated Examining Board	-	Mathematics paper 9
A3	Associated Examing Board	-	Mathematics paper 10
C	Cambridge Local Examinations	-	Mathematics
OC	Oxford and Cambridge	-	Mathematics
O1	Oxford Local Examinations	-	Mathematics
O2	Oxford Local Examinations	-	Further Mathematics
L1	University of London	-	Mathematics (C381)
L2	University of London	-	Further Mathematics (C382)
L3	University of London	-	Mathematics (D391)
L4	University of London	-	Further Mathematics (D392)
L5	University of London	-	Higher Mathematics (410)
L6	University of London	-	Pure Mathematics with Statistics (420)
J1	Joint Matriculation Board	-	Mathematics (syllabus 8)
J2	Joint Matriculation Board	-	Further Mathematics (syllabus B)
J3	Joint Matriculation Board	-	Pure Mathematics with Statistics
J4	Joint Matriculation Board	-	Statistics

(Table 2 a)

Frequency of Topics in Syllabuses.

Topic	Number of syllabuses (out of 18)
Statistical enquiries	3
Collection of data	3
Census methods	2
Classification of data	2
Sampling methods	4
Tabulation	2
Questionnaires	3
Bias in sampling	2
Diagrammatic presentation of data	4
Frequency distributions	9
Histograms and frequency polygons	8
Descriptive properties of distributions	3
Cumulative frequency distributions	7
Estimation of medians etc.	12
Calculation of measures of average	15
Calculation of measures of dispersion	15
Probability theory	18
Permutations and combinations	7
Addition law for mutually exclusive events	18
Generalised addition law	13
Conditional probability	15
Independence	16
Baye's theorem	2
Markov chains	2

(table 2 b)

Topic	Number of syllabuses (out of 18)
Discrete probability distributions	15
Mean and variance(of above)	15
Binomial distribution	17
Poisson distribution	15
Poisson $\sim$ Binomial	7
Continuous distributions	15
Cumulative distributions functions	4
Mean and variance of continuous distributions	15
Expectation	12
The uniform distribution	9
The Normal distribution	15
Moment generating functions	1
Probability generating functions	3
Normal $\sim$ Binomial	9
Normal $\sim$ Poisson	7
Probability paper	3
Sampling distributions	12
The central limit theorem	9
Estimation of parameter using a statistic	10
Confidence limits	14
Two tailed significance tests	9
One tailed significance tests	7
The t test	3
The χ^2 test	5
The F test to show 2 samples from same pop.	1

(table 2 c)

Topic	Number of syllabuses (out of 18)
Wilcoxon Signed Rank test	2
Mann - Whitney U test	1
Regression (linear)	12
Confidence intervals for regression quantities	1
Rank correlation	8
Product moment correlation	8
$\tanh^{-1} r$ test for correlation equality	2
Random numbers and simulation	5
Time series etc.	4
Index numbers	4
Control charts	1
Analysis of variance	1

CHAPTER 4

SOME APPROACHES TO TEACHING PROBABILITY AT A LEVEL.

Probability appears on all of the statistics sections of the A level Mathematics syllabuses and is an essential ingredient of all statistics courses at this level. Some pedantic statisticians argue that probability is itself a branch of pure mathematics and not statistics, hence some textbooks claim to be a course in probability and statistics, indicating that there is a distinction between the two. However, in my opinion, probability is a vital part of statistics in the same way as arithmetic is vital to mathematics.

Accepting, then, that probability is an important part of the statistics course it is the job of every teacher who is responsible for teaching statistics at A level to examine the different approaches to, and methods of, teaching probability.

1. "Equally Likely" Approach.

Using the equally likely theory, a numerical probability is obtained using the following sort of argument. Suppose we want to find the probability of obtaining a head when a coin is spun. We would assume that the coin was fair, that is, either face is just as likely to land uppermost by symmetry. Then there is a one in two chance of obtaining a head when a coin is spun. This means that the probability of obtaining a head is $\frac{1}{2}$.

Historically, the study of probability had its birth in gambling. In the seventeenth century the French gambler the Chevalier de Méré, set a wager that within a sequence of 24 tosses of a pair of dice, he would roll at least one 12. He learnt from bitter experience that he could only afford to make such a bet with the odds in his favour, at even money he was a loser. What de Mére wanted to know was, what odds should he offer?.

A letter to a friend, Blaise Pascal, initiated the study of probabilty as a serious mathematical study. The "equally likely approach", then, has its origins in the science of gambling.

Below are a few definitions of probability, from a selection of A level texts which use this approach.

(I) "Suppose an event E can happen in h ways out of a total of n possible equally likely ways. Then the probability of occurrence of the event (called its success) is denoted by
$p = Pr(E) = h/n$" Theory + Problems of statistics: Murray R. Spiegel p.99

(II) "If a random process can result in n equally likely and initially exclusive outcomes and if a of these outcomes have an attribute A then the probability of A is the ratio a/n and we write

$$P(A) = a/n$$"

Statistics: A Second Course (2nd ed'n) - R. Loveday p.31.

(III) "If an event can happen in a ways out of n equally possible occurrences, then the probability of the event happening is a/n" Advanced General Statistics - B.C.Erricker p. 92

(IV) "If E is the set of outcomes of an experiment, all of which are assumed to be equally likely, and an event A satisfied by a subset A of them, then $P(A) = n(A)/n(E)$. "
Statistics and Probability - S.E. Hodge + M.L. Seed p.10.

(V) "If for a certain trial, there is a finite set E of possibilities, all equally likely on grounds of symmetry and if a subset A of these possibilities is associated with an event A, then the probability of the event A is $n(A)/n(E)$ " S.M.P. Advanced Mathematics Book 2 page 553.

These definitions are typical of the books which use the "equally likely" approach, and really all say the same thing. Some of the more modern textbooks tend to use set language which only adds to the difficulties of those students who have not followed a Modern Mathematics course. Unlike the other definitions though, the S.M.P. definition attempts to elaborate on the meaning of "equally likely" events by using the idea of symmetry. The other books, by the use of examples, imply that symmetry is the criteria for determining the number of "equally likely events", but it is not actually stated.

As has already been mentioned, most of the early work on probability was tied up with gambling of one form or another, e.g. throwing dice, card games or roulette. Gamblers wanted some sort of idea what their chance or liklihood was of obtaining a particular result.

This background led to the above definitions of probability, and [illegible] books which use this type of definition, use the following sort of argument in determining numerical probabilities.

Loveday, in his book, introduces the idea of probability by discussing the outcomes when an ordinary die is cast. Each of the numbers one, two, three, four, five, six has an equal chance of falling uppermost. He then makes the claim that because the six outcomes of trial are exhaustive and mutually exclusive, there is, therefore, 1 chance out of 6 of obtaining any particular number. Hence we say that "the probability of throwing a six = 1/6" The author then uses further illustrations to demonstrate the idea of probability: An ordinary pack of 52 playing cards contains 4 aces and hence, if a card is selected at random from a well-shuffled pack, there are 4 chances out of 52 of it being an ace. Thus "the probability of selecting an ace = $4/52 = 1/13$". Also Loveday discusses the probability of obtaining a head when a coin is tossed. The coin may fall either as a head or a tail and thus, "the probability of a head = $\frac{1}{2}$".

In all of this discussion, Loveday implies that he is using the physical symmetries of the various outcomes but does not actually convey this message to the student. He also states that the different outcomes are equally likely, but fails to mention that this is an assumption that we need to make so as to make the calculation of numerical probabilities possible. In fact it is unlikely that there is absolute physical symmetry and there could be a slight bias for a coin to turn up heads, for example. The bias is slight but the reader should be made aware of it all the same.

However it is quite reasonable to assume that the equally likely approach is quite suitable in coin tossing situations. The point which should definitely be made in class is that real applications of probability, the outcomes are not remotely equally likely. This point is hardly ever mentioned in text books and this is bad since many pupils, even of 'A' level standard, could leave school under the impression that determining probabilities in real life situations is as easy as determining probabilities in games of chance where the events are usually equally likely.

Although basically the same philosophy in approach, some books, such as S.M.P. Advanced Mathematics Book 2, elaborate the ideas by resorting to set language, to be more precise they use what is known as the possibility space.

It is sometimes argued that the use of set notation is not necessary in any discussion of probability and in fact those students who have followed a traditional mathematics course need to master two new concepts instead of one. i.e. probability and the idea of set. Also valuable lesson time is taken up introducing the idea of set. I would counter this by saying that although the idea of a set is not required for teaching probability, set language is a convenient and neat way of listing all the possible outcomes in a particular experiment. Later on in the course, by using Venn diagrams, the multiplication and addition laws of probability can be adequately justified to mathematically oriented students. Not a lot of set theory is required and this can be easily introduced as a matter of course, and does not really take up any great amount of lesson time. I would therefore support the use of set-theory in any course in probability that was part of a mathematics course. Also some A Level mathematics syllabuses now include Bayesian probability in which the use of sets is vital.

As with Loveday's book, the initial examples in S.M.P. are confined to games of chance, i.e. spinning a coin, rolling a die, drawing a card from a pack and so on. The word trial is used to describe the combination of an incident and making a record of the result. For example, a coin is tossed and then the result "head" or "tail" is recorded. S.M.P. subscribe to the view that the idea of a possibility space is central in the theory of probability. Of course a listing of possible outcomes is essential to any approach to probability. What I think varies in the emphasis given by S.M.P. to the set formulation, i.e. to the abstract structure. For each trial there is a number of possibilities and this set of all possibilities is called the possibility space for the trial. All the examples given in the S.M.P. Advanced Mathematics Book 2 are confined to situations which produce a finite number of possibilities, so that the possibility space is a finite set; this constitutes the universal set for discussion of the trial.

In an attempt to make it easier for the student to grasp the concept of probability, this book resorts to using diagrams in which the elements of the possibility space are represented by points. For example, when the trial is to toss a coin and record the result, the possibility space {Head, Tail} has two elements and is represented as follows:

+	+	Fig.1
Head	Tail	

Somewhat a trivial example, but the book does then consider a slightly more sophisticated trial. This trial consists of rolling a pair of dice (one green, one red) and recording the number of spots on the top faces of the green and red dice respectively. This has a larger possibility space and S.M.P.'s book suggests a more orderly fashion for displaying the point. The trial yields a possibility space of 36 elements; each is an ordered pair of numbers (g,r) where g is the score on the green die, and r that on the red die, S.M.P. uses this ordered pair as the Cartesian co-ordinates of the corresponding point and the possibility space is drawn as follows:

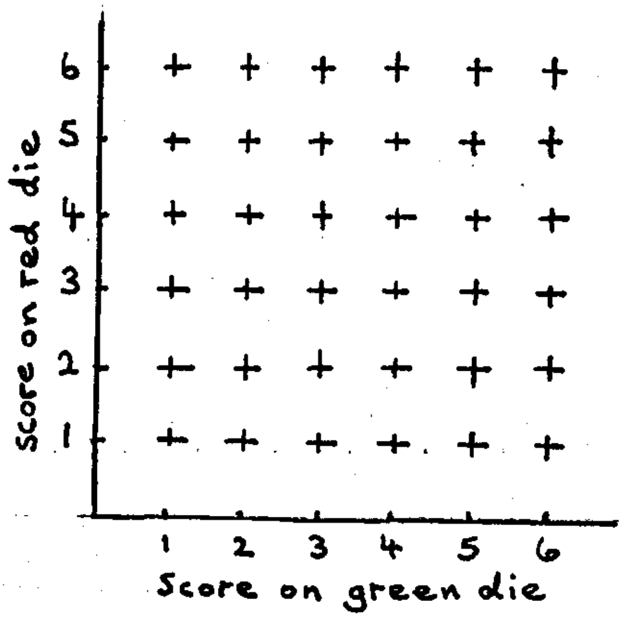


fig.2

I shall come back at a later stage to demonstrate how the book uses such a diagram to calculate numerical probabilities.

Once S.M.P. has introduced the concept of a possibility space they move on to discuss what they mean by the idea of equally likely possibilities. This idea is used in all of the "equally likely" definitions of probability but the S.M.P. book was the only A level text that I found which even attempted to discuss this concept. Somewhat surprising when the whole definition of probability hinges on this idea and it is very much the weak link of the definition.

In their consideration of "equally likely possibilities" they use the following two examples to illustrate the discussion:

" (1) I roll a pair of dice (one red and one green) and record the number of pips on the top faces of the green and red die respectively.

(II) I roll a pair of dice and record the larger of the numbers of the pips on the top two faces (or either one if the two numbers are equal).

These two trials could be carried out simultaneously, the same roll will serve for both and they will differ only in what is recorded. For (1) the possibility space has 36 elements:

$$\left\{(g, r): \quad g, r. \text{ integers}, \quad 1 \leqslant g \leqslant 6, \ 1 \leqslant r \leqslant 6\right\}$$

The possibility space for (II) however, has just six elements:

(1, 2, 3, 4, 5, 6,)" *

The book then records the results that were allegedly obtained when the two trials were actually carried out twenty times in succession.

trial (1).

(1, 6)(3,1) (6, 4) (1,5) (4, 6) (4, 2) (1, 4) (1, 6) (4, 5)(1, 4)
(3, 1)(6,5) (1, 1) (4,5) (2, 6) (3, 4) (4, 2) (1, 1) (1, 5)(5, 1)

---

* (S.M.P. Advanced Mathematics, Book 2 page 549)

trial (2).

6 3 6 5 6 4 4 6 5 4

3 6 1 5 6 4 4 1 5 5

The results are used to indicate to the reader that there is an important difference between the trials. S.M.P. claims that although the results of trial (1) include certain repetitions, and some scored have occurred more frequently than others, it does not cause them to doubt what they expected before they started. I am not sure that the average Vlth former is going to be convinced by the evidence of such a small sample however. Anyway, apart from that criticism, the book then goes on to indicate to the student that if successive results are traced on figure 2, it will appear that the corresponding points are scattered randomly over the diagram. With trial (II) the situation is different; there would seem, according to this limited survey, that there is a strong bias towards the higher numbers of the possibility space. For example. the outcome 6 occurs six times, 5 and 4 five times each, whilst 1 occurs only twice and 2 not at all.

It is pointed out that the reason for this is quite easy since the only roll which will give the possibility 1 in trial (II) is (1, 1); but the possibility 6 will arise from any of the eleven rolls;

(1, 6) (2, 6) (3, 6) (4, 6) (5, 6) (6, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5)

It is then indicated that we expect that all the possibilities for trial (1) are "equally likely" then we might say that in trial (II) 6 is "eleven times as likely" as 1.

The arguments put forward here to try to explain why the higher numbers are predominant, are to my mind, pre-supposing the truth of "equally likely possibilities" the very idea that S.M.P. are trying to demonstrate with the experimental result. In other words their argument is circular. But at least, to some extent S.M.P. overcomes the circularity by using a series of trials to suggest that while the events in (1) are equally likely the events in (II) are not. As I said before, though, I am not sure that such a short series of trials is likely to be very convincing to the reader. It might have been better either (a) simply to show that the equally likely definition gives conflicting results if applied to both (1) and (II), or (b) to take two much simpler situations, for example take a coin and a drawing pin, and carry out a large number of throws with each to indicate that in one case the events may be roughly equally likely, but in the other case, they definitely are not.

However, S.M.P. do offer the caution to the reader that whether or not all the possibilites for a trial can be regarded as equally likely depends on the choice of what is recorded as well as on the apparatus with which the trial is conducted. As far as S.M.P.'s approach is concerned where there is no evidence to show otherwise then all of the elements of the possibility space are taken to be equally likely on the grounds of physical symmetry.

As an introduction to the idea of probability this is perhaps not a bad idea, more complicated possibility spaces can be considered at a later stage.

S.M.P. now develop their argument further by introducing the idea of an event because they say that in the theory of probability we are concerned to assign numbers, called probabilities, to events. It must be remembered that at this stage in their book, S.M.P. still have not formally defined probability.

To give the student an idea of what they mean by an event, S.M.P. give the example of the event "I win a prize" when the monthly draw of the Premium Bonds is carried out. In other words an event is something which may or may not occur as the result of a particular action. "This event will occur only if one of the numbers produced by E.R.N.I.E. is the same as one of the numbers held by the bond-holder. Now E.R.N.I.E. can produce a very large number of different possible winning lists for a particular draw; the set of all such lists constitutes the possibility space, and if the draw is fair all the list should be equally likely. Some of these lists will include one or more of the bond-holders bond numbers, and if one of these is selected then the event "I win a prize" occurs." *

By considering this example, S.M.P. are able to declare more formally that with the occurrence of an event is associated a subset of elements of the possibility space. This correspondence forms the basis of the definition of an event by S.M.P. Some other

* (S.M.P. Advanced Mathematics, Book 2 page 551)

A level text-books in fact define an event to be a subset of the possibility space, they do not even mention the word correspondence in the context of the definition. I prefer S.M.P. 's definition, it has a subtle difference to most of the other definitions of an event since in reality an event is not a subset. Sets are abstract entities, part of a mathematician's language; probability should be made meaningful to the Vlth former by always relating it to reality. Hence the word correspondence implies that the subset itself is not the event but merely an abstract representation of it.

To try and reinforce the idea of an event the book reintroduces the example: "I roll a pair of dice and record the number of pips on the top faces of the green and red die respectively." The event they consider is: "The sum of the scores is least 10." "Thus the possibility space is $\{(g, r): g, r \text{ integers}\}$

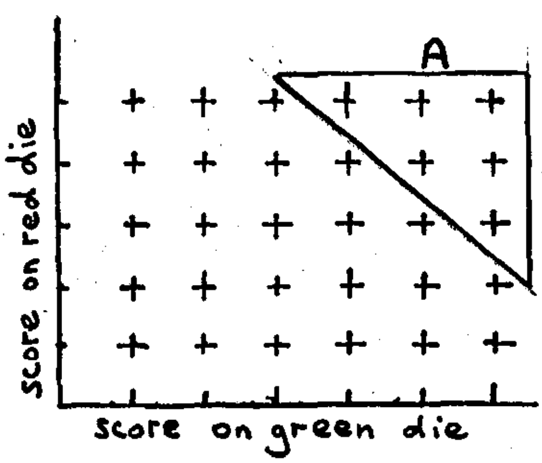


fig. 3

In this example the event occurs if a double six, a six and a five and so on, is rolled; that is, if the scores belong to the subset.

$$\{(4,6), (5,6), (6,6), (5,5), (6,5), (6,4)\}$$

This subset which is denoted by A is shown diagrammatically in Figure 3.

Denoting the possibility space by E, we find that n(E) = 36 n (A) = 6." *

Thus the event occurs for 6 out of the 36 possibilities for the trial. This implies that the probability of obtaining a score of at least 10 is 6/36 = 1/6. The formal definition of probability based on this reasoning appears soon after this example but not before S.M.P. have considered what they call an alternative analysis.

" Instead of recording the separate scores on the green and red dice, we could have recorded the sum of the scores directly. We would still be looking at the same event but the trial would be different:

Trial: I roll a pair of dice and record the sum of the scores on the two top faces.

Event: The sum of the scores is at least 10

Possibility space: $\{2,3,4,5,6,7,8,9,10,11,12\}$

The event now corresponds to the set $\{10,11,12\}$ which has just three numbers, we can denote this subset by A'. Denoting the possibility space by E' we see that n(E') = 11 and n(A') = 3, so that the event occurs for 3 out of the 11 possibilities for the trial." **

Of course, unlike the student who would probably be reading this account of probability for the first time, we know from the experience of hindsight that the first analysis of the situation is the correct analysis because the definition of

* (S.M.P. Advanced Mathematics, Book 2 pages 551-552)

** (S.M.P. Advanced Mathematics, Book 2 page 552) probability depends on the elements of the possibility space being equally likely. In the first analysis the possibility space contains equally likely elements but the second possibility space does not. Deciding on whether or not the elements are equally likely is a subjective judgement and being able to make a correct decision, when it is possible to do so, comes from experience. In this instance the decision is not too difficult since the trial is not all that complicated.

The important thing is that S.M.P. have made the students aware that it is possible to consider the experiment in different ways and by writing down a different possibility space obtains a totally different numerical probability. It is important to make A level students think about these sort of things. All credit to S.M.P. because only they of the A level texts that I referred to attempted to discuss the problem of choosing the appropriate possibility space. They make the student aware of which is the correct analysis by using the "frequency" approach to probability in their next section. I will discuss this section when I deal with the "frequency"approach at a later stage.

The trouble with the"equally likely" theory is that there are a number of problems:

Problem 1

All definitions which subscribe to the "equally likely" theory have the disadvantage in that the words "equally likely" are vague. In fact, since these words seems to be synonymous with "equally probable", the definition is circular because we are essentially defining probability in terms of itself.

Despite the weakness of the definition, most Vlth formers accept it as what might be called a working definition, since in terms of experience and in terms of determining chance in gambling situations it does seem to have a certain amount of intuitive appeal to them. Like all branches of mathematics we have to make a starting point somewhere and accept the truth of certain basic assumptions. Perhaps then we should just accept the truth (as far as 'A' level is concerned) of the definition of "equally likely" probability without attempting to be too rigorous.

The definition does at least allow the average 'A' level student to calculate numerical probabilities in certain circumstances and facilitate a fairly easy introduction into the subject. Apart from that we, as teachers, would be failing in a much wider educational sense to teach our heritage since it is the "equally likely" theory which historically started the "ball rolling".

Problem 2.

As S.M.P. show in their 'A' level Book 2, which I mentioned earlier, certain experiments can be analysed in two different ways. When two dice were rolled it was possible, as the book shows, to write down two different possibility spaces for the same experiment and hence two different probabilities were found.

When first introducing probability I find a common mistake made by some Vlth formers is that they cannot see that the results of tossing a coin twice are the same as tossing two coins at once. No great disgrace, since even the great eighteenth century French mathematician Jean le Rond d'Alembert could not see that the results are the same. It is important to see that the results obtained from the two different considerations are really the same, because if the Vlth former realises this then it is easier for him to see that the result "head - tail" is a different outcome to "tail - head". This is not so easy if we consider two coins being spun simultaneously. If the student does not realise this then he will formulate the wrong possibility set.

To explain what I mean, suppose we wish to determine the probability of obtaining one head when two coins are tossed. We can argue in two different ways:

(a) There are four possible outcomes:

head - head, head - tail, tail - head, and tail - tail.

We can write this in the form $E = \{(H,H),(H.T),(T.H),(T.T).\}$

Each outcome is equally likely and the required event corresponds to the subset $\{(H.T),(T.H)\}$, hence the required probability in $2/4 = 1/2$

(b) There are three possible outcomes:

0 heads, 1 head, 2 heads.

We can write this in the form E' = $\{0, 1, 2\}$
Each outcome is equally likely and the required event corresponds to the subset $\{1\}$ hence the required probability is 1/3.

So we have the awkward situation of having two different probabilities (1/2 and 1/3) for the same event i.e. one head when two coins are tossed simultaneously.

The difficulty that arises from the definition is that the definition relies on us to make a subjective judgement of deciding what actually constitutes an "equally likely" outcome. It is experience of dealing with these types of problems that eventually allows one to choose the correct possibility space, that is, provided the possibility space is not too complicated.

If we try to apply the "equally likely" theory definition we first have to draw up a possibility space and make the decision that each outcome is equally likely to happen. The example of tossing a coin twice and the example in S.M.P.'s book of rolling a pair of dice show that it is possible for two different people to analyse an experiment differently and write down two different outcome spaces for the same event. Both people would assume that they have written down an outcome space which contains equally likely outcomes but their differing analysis of the event will lead them to obtaining probabilities of the event occurring which are not equal. Hence the equally likely definition cannot be a very sound one if differing answers for probabilities are found depending on who calculates the probability. The problem lies in the fact that it is up to each individual to decide what constitutes an equally likely outcome, so therefore this definition gives subjective probabilities for an event.

Unless we accept that probability is not objective, which I feel is not acceptable to most mathematicians, we need to look elsewhere for an alternative definition.

Problem 3.

To apply the "equally likely" definition of probability we must have a set of equally likely outcomes. This is why the definition is best suited for determining numerical probabilities in games of chance, which would be expected, since it is from games of chance that this definition of probability is derived. Most games of chance give us a set of equally likely outcomes, for example, the two sides of a coin when it is tossed, the 52 cards of a pack when one card is drawn or six faces of a die when it is rolled. The trouble is, that games of chance only provide a few every day applications of probability. For the majority of applications, there is no obvious set of equally likely outcomes. For example, suppose that someone is considering building a self-service petrol filling station. The would be garage owner is obviously going to be interested in how many pumps he ought to have on his fore-court. He does not want to have too many pumps because that would be a waste of money and space. Also, he does not want to have too few pumps because that would possibly mean that he would have too many cars queuing in any fixed period of time. Suppose that at any particular time there can be a certain number of cars waiting to fill up with petrol. The possibility space then, would have the following outcomes: 0 car, 1 car, 2 cars, 3 cars, ... and so on. However, it is most unlikely that these outcomes are equally likely, unless the filling-station is a busy motor garage. For instance it is very unlikely that there will be 20 cars, say, waiting for petrol. Although it is impossible to use the "equally likely" theory to determine the probabilities of the different outcomes in this sort of situation, it is of great practical importance to the station owner to attach some sort of probabilities to the number of cars that can be expected to any particular time. This is so that a rational decision of how many pumps should be installed can be made.

This inapplicability of the equally likely definition is briefly mentioned in S.M.P. Advanced Mathematics Book 2 (Page 565) and I mean briefly. They say that in the situation of wanting to assign a probability to the event, " on any particular day it will be cloudy" the assignment of any such probability cannot be determined by considering possibilities which are "equally likely on the grounds of symmetry." They do not discuss any further how such probabilities can be found except to say that in certain cases such probabilities can be found by sampling and using the relative frequency approach.

In their book, Hodge and Seed discuss that the equally likely approach is of little use if the probability of obtaining a four from a biased die is required. They suggest an approximate value for the probability can be found by throwing the die 100 times and obtaining a proportion for the number of times that the four occurs. This proportion would then be an estimate of the required probability. Again, their discussion is brief and is not put into the context of a real life situation which is far more suitable for such a discussion.

2. The"Frequency"approach.

This particular approach tends to be less popular if popularity is measured by the number of books which use this approach. Even then the books which use the frequency definition tend to put it forward as an alternative to the equally likely approach.

Below are just a selection of the definitions of probability which subscribe to the frequency theory:

(1) "If an event happens a times in n trials then

$$\operatorname*{Lt}_{n \to \infty} (a/n) = p$$

This reads "the limit of a/n as n tends towards infinity equals p." "n tends towards infinity" means a large number of trials should take place. What is "large" must be determined by the nature of the problem." (Advanced General Statistics - B.C. Erricker page 92).

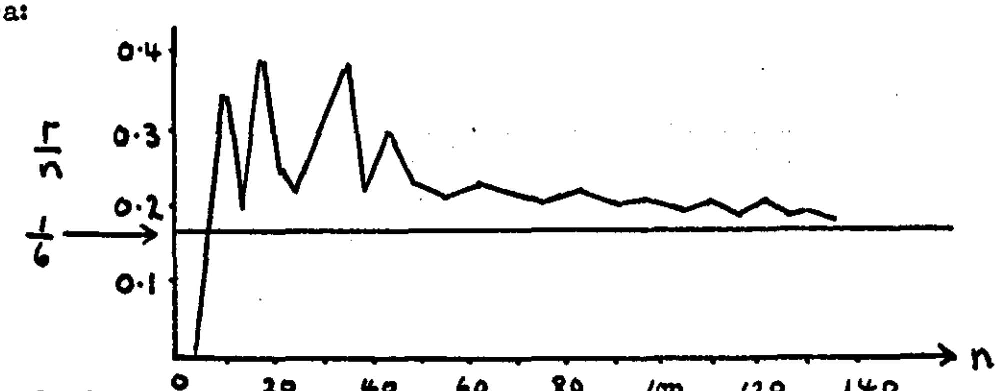
(II) "if a trial is repeated a large number of times, the relative frequency of the set of possibilities associated with a particular event will be approximately equal to the probability of the event".
(S.M.P. Advanced Mathematics, Book 2 page 553)

(III) "If an event A is satisfied by a certain outcome (or outcomes) of an experiment and, when the experiment is repeated n times under exactly similar conditions A occurs r times out of n then $P(A) = \lim_{n \to \infty} (r/n)$"

(Statistics and Probability, S.E.Hodge and M.L.Seed page 4

They attempt to explain this further by saying "that for any value of n, the fraction r/n will be an estimate will approach some value which we will agree to call probability".

The following diagram is also given to further illustrate the idea:



Graph of r/n as n increases for the event of scoring a 4 on a di

(Statistics and Probability, S.E.Hodge and M.L.Seed page 4)

S.M.P. use the frequency idea of probability to try to clear up the problem of which sample space should be used in the experiment in which two dice (one green and one red) are rolled. They seem to use the frequency approach as a back up to the equally likely approach rather than offering it as a better or preferred definition of probability.

As I mentioned earlier, S.M.P., by considering two differing analyses of the dice rolling experiment, arrive at two different values for the numerical probability of obtaining at least ten as the sum of the two scores on the top faces of the two dice. The first analysis produced 1/6 as the answer and the second analysis produced 3/11 as the answer. We were faced with the problem of deciding which one, if any, of them is the correct answer.

S.M.P.'s text supposes that a pair of dice have been rolled 900 times and the separate scores recorded as in the first analysis. The results would give a population of 900 outcomes such as:

$$\left\{ (1,6), \quad (3,1), \quad (6,4), \ldots\ldots, (2,5) \right\}$$

Now as the possibility space consists of 36 equally likely possibilities so we would expect each of these to appear about 900/36, or 25 times in the population. Since the event "the sum of the scores is at least 10" is associated with six of these possibilities, we would expect this to occur about 6 x 25 or 150 times. Thus we are prepared to make decision in the belief that:

The number of the occurrences of the event = 6/36 X 900

In other words, if A is the event then

$$\text{fr}\ (A) \approx \frac{n\ (A)}{n\ (E)} \times \quad \text{fr}(E)$$

$$\frac{\text{fr}\ (A)}{\text{fr}\ (E)} \approx \frac{n\ (A)}{n\ (E)}$$

This means the second analysis of the dice rolling situation is unsatisfactory. Here the possibilities are not all equally likely: we would expect to get a total of 7 far more frequently than a total of 2 for example.

Apart from S.M.P., which uses the frequency approach as a back-up to the equally likely approach, the other 'A' level texts which include the frequency definition of probability do not pursue the approach much futher than the definition and a brief discussion of it. Obviously the frequency approach is far more suitable for the empirical situation, so apart from trivial experiments it is not practicable to carry out meaningful experiments in the classroom situation: hence the textbooks tend to concentrate on problems in which the equally likely approach is applicable. Therefore, the treatment of the frequency idea of probability tends to be somewhat terse in 'A' level texts on statistics and probability.

It is worth mentioning, before I go any further that there is a slight difference between the frequency definition of probability given by S.M.P. and the other text books. Most 'A' level texts that use this definition of probability use a "limiting frequency" definition (i.e. probability is defined as a limit of the proportion of successes as n approaches infinity). S.M.P.'s definition avoids talking about limits. In practice we always have a finite set of trials, the idea of a limit is alright in the mathematical model, but not in the real world. The definition put forward by S.M.P. ties in with the idea of probability being measured by frequency in a long series of trials, but is not defined by a limiting relative frequency. The distinction between these two forms of the definition is worth discussing in class, since it is a useful illustration of how we needto often use a mathematical model as the limit of a practical situation. In other words the practical situation will often approximate to a certain mathematical model. Some Vlth formers do not always make the distinction unless they are told.

As with the equally likely approach to probability there are also a number of problems with the frequency approach to probability.

Problem 1

One of the major problems with this definition is that how do we know that the ratio approaches some value at all. And how large do we make n before we get reasonably close to it.

Suppose that we investigate this further, and this is something which needs to be discussed in class, and we actually toss a coin some number of times. By actual experimentation can we actually say anything about the number of heads and tails we are likely to get?

When a coin was tossed four times, on four occasions the following results were recorded:

(1) 1 head and 3 tails.

(11) 2 heads and 2 tails.

(111) 4 heads and 0 tails.

(1V) 3 heads and 1 tail.

There appears to be no pattern and the results are scattered over the range 1 to 4 heads.

We can also see what happens if we toss a coin 8 times. Again on four occasions:

(i)	3 heads and 5 tails
(ii)	4 heads and 4 tails
(iii)	5 heads and 3 tails
(iv)	7 heads and 1 tail.

Again, there does not seem to be any pattern at all. This time the number of heads varies between 3 and 7.

If we toss a coin a larger number of times, 50 say, then we shall see that the pattern becomes more obvious.

(i)	22 heads and 28 tails
(ii)	25 heads and 25 tails.
(iii)	19 heads and 31 tails.
(iv)	26 heads and 24 tails.

In terms of the size of the samples the spread of results is much less than it was before and the results seem to be clustering around half heads and half tails. We would expect then, the larger we make our sample, the proportion of heads to stabilise. Under the frequency theory, the value to which this proportion converges is defined as the probability of a head.

Problem 2.

Perhaps one of the greatest advantages that the frequency definition has over the equally likely definition is that the frequency view is far more easily applied in real world situations where there are no sets of equally likely outcomes. In a factory, for example, producing certain components it is usually very desirable to estimate the probability of a defective item coming off the production line. Suppose that in a run of 500 articles we count 5 defectives, an estimate of the probability of obtaining a defective component would be 5/500 = 1/100.

The problem is that it is not always possible to apply the frequency view. Suppose that it is necessary to test the effectiveness and possibly the safety of a new drug. We could calculate the probability that the drug will cure a certain illness by expressing the number who recover to the number who have the drug administered to them. The major difficulty is that how do we know that the people who recovered or even just some of them might have recovered despite being administered the drug.

It might also be required to calculate the amount of the drug which constitutes being an overdose. One way would be to administer a certain dose to a number of people and calculate the probability that it was lethal by counting the number of people who died. Although this would give some numerical measure of the required probability, this method is inhuman apart from being illegal.

In this situation it is impossible to apply the frequency definition of probability since we cannot contrive to find a suitable set of trials.

Problem 3.

When attempting to estimate the probability of a head when we toss a coin using the frequency method it could be seen that we need to take a fairly sizable sample before we obtain anything like a reasonable result. Even with the samples of size 50 there was still some amount of variation; the results were not settling down as much as would be desired to come to a definite conclusion concerning a worthwhile estimate of the required probability. In a great many real world situations which require the applications of probability 50, or desirably more, trials are far too time consuming.

For example, if we wished to calculate the probability that Concorde's engine contained a fault, it would be very difficult since it takes a long time to make one engine and also not many have been made. Not only that, since the engines are made over a period of time and in different places then different technicians and engineers will have been responsible for the manufacture of the engine, hence the probability of an engine having a fault will not necessarily be constant for each engine.

In the main then, as far a 'A' level tests are concerned, the frequency view of probability is usually discussed only after first defining probability in terms of the equally likely view. The discussion tends to be brief and quickly dismissed. As far as problems go, they seem to be, without exception, ones which are based on games of chance or situations which are made unreal by contriving them to consist of equally likely outcomes. I suppose the main reason for this is that the questions set in actual A level examinations also tend to be of this type. Also, people who write these text books tend to have little imagination and little experience of practical situations in which probability is used. The only time that a question requires some knowledge of the relationship between probability and frequency is when it is required to calculate a set of theoretical frequencies from a probability distribution.

3. The Subjective Approach to Probability.

"Rovers could either win, draw or lose their next match against United, so that the probability of their winning is 1/3".

Of course there is a very big flaw in this argument, in fact it is impossible to calculate the theoretical probability since the different alternatives are not equally likely. Also it is not possible to obtain a relative frequency value of the probability since we cannot possibly repeat a particular match many times in exactly the same, or even similar circumstances. Any probability attached to the event happening would be entirely subjective.

The only 'A' level text book which even mentions the idea is Statistics and Probability by Hodge & Seed on page 7. Even then their terse discussion is restricted to eleven lines and is dismissed fairly quickly. The last sentence in their consideration of this aspect of probability reads, "..... though it is not impossible mathematically to consider such a probability as valid, we shall prefer in general to restrict our attention to events whose probabilities can be obtained either theoretically or experimentally".

This last statement seems to sum up the impression that 'A' level text-book writers must have about subjective probability, that is, it is not worth teaching at this level. Perhaps this is because the people who write 'A' level statistics text-books tend to be mathematicians who are not really trained to cope with subjective forms of measurement. It is a pity really because I feel that Vlth formers should be at least made aware of the existence of the idea of subjective probability and should not just think that probability can always be calculated in a clear cut manner.

As far as teaching subjective probability at 'A' level is concerned there are three problems.

Problem 1.

It is difficult to actually set problems which involve subjective probabilities, since the Vlth former is hardly likely to have a sufficient background to be able to cope. Intuition plays an important part in determining subjective probabilities and intuition in probability requires an experience over a wide range of probability situations few of us are likely to obtain. To judge from the report by Wood and Brown (1976) this could explain the sex differences in probability understanding. Just as more boys than girls are likely to play with "Meccano" and mend bicycles, giving them an advantage in Mechanics, so games of chance may be a commoner activity among boys than girls to give them an advantage in Probability. It would be ironic if this were the case, since one argument for the introduction of Probability and Statistics in school was the difficulty girls have had in Mechanics.

Problem 2.

Mathematics by nature, at A level is a very objective type of subject and the idea of introducing the concept of a subjective numerical measure on to the syllabus might seem, to some, a contradiction of the subject's ideals. Also, how do you mark questions involving the determination of subjective quantities? Teacher's of mathematics have not got the background and how do you tell if the answer is acceptable? I realise that other subjects such as Sociology and English literature, for instance, involve giving subjective answers and value judgement but in these subjects the issues involved are wide whereas in probability a precise and definite answer is required.

Problem 3.

Intuition, which plays an important part in determining subjective probabilities, has a nasty knack of letting us down in certain cases.

What, for example, is the probability that, of 30 persons chosen at random, at least two have the same birthday? Reason tells us that (making one or two assumptions) the probability is about 0.7, but it is doubtful if many of us"feel" that this is correct however.

Well, we have looked at the different approaches there are to probability so which one is best? The answer to this will be different for different people depending on who is answering the question at the time. The answer that they will give will depends very much on what type of mathematical background that they have and also how they see the nature of mathematics. For instance, the applied mathematician who views mathematics as a tool will want a clear-cut well defined statement of how to calculate probability so that he can apply his answer. On the other hand, the pure-mathematician who perhaps has done some mathematical philosophy will be more happy to accept the subjective view of probability.

I think that, as teachers, it is our job to make our students aware of all three approaches and point out the relative merits and de-merits of all three views. Also different situations and problems require different models to suit each special need. The classical approach might be perfectly suitable for one situation but another situation might be better served by using the frequency view, and so on. As teachers, we must point out different examples which use each of the three approaches.

In terms of actual suitability for teaching, despite its theoretical misgivings, the classical approach still has its merits.

Most children have had some experience of games of chance and it makes sense, for instance, that the probability of obtaining a five when a die is rolled is 1/6. We must use any acceptable means of teaching with understanding and if the classical approach allows Vlth formers to understand the concept of elementary probability, then I feel that this is sufficient justification to carry on using this view as the main approach for teaching probability at advanced level. Of course, once the concept of probability has been introduced using the equally likely theory the other ideas should be brought in. Of course, one of the possible weaknesses of the equally likely approach is that one cannot really distinguish between the real world and the mathematical model. It is the same problem that arises with 'A' level mechanics, with its "frictionless slopes" for example.

The only time that it might be acceptable to use an alternative introduction is if a school has a computer terminal with a sufficient number of visual display units. Then it may be possible by means of simulation to introduce probability by means of the relative frequency idea. The only trouble is that not many school possess such equipment.

Combination of Events.

So far we have only considered the probability of one event. We sometimes require to know not just the probability of event A happening but the probability of A or B happening, say, A and B are each single events in their own right, but we can also consider the compound event A or B which will have a corresponding probability. It is to be remembered that in mathematics, though, the word "or" can be taken to mean one of two things:

(1) either A or B or both.

(11) either A or B but not both.

This distinction is not always made in some text-books, especially the not so recent ones.

For example, if I want to know how many people are wearing a hat or gloves, then I should include those wearing both a hat and gloves. On the other hand, if I want to know how many people are wearing a pair of brown or a pair of black shoes, I should not expect to meet someone wearing both a pair of brown and a pair of black shoes.

Apart from the compound event A or B we are also interested in the other compound event, that is A and B.

There are two approaches that text-books take for finding these required probabilities. The first is to state, without proof, the results and the second is to use set theory in conjunction with the equally likely theory. The first approach does not, obviously, need any description or elaboration from me.

The second approach requires that we represent the event A and the event B by set A and set B, so that the event A and B is represented by the set $A \cap B$ and the event A or B is represented by $A \cup B$.

The Addition Rule.

We can illustrate the situation more clearly with a Venn diagram (Fig. 4).

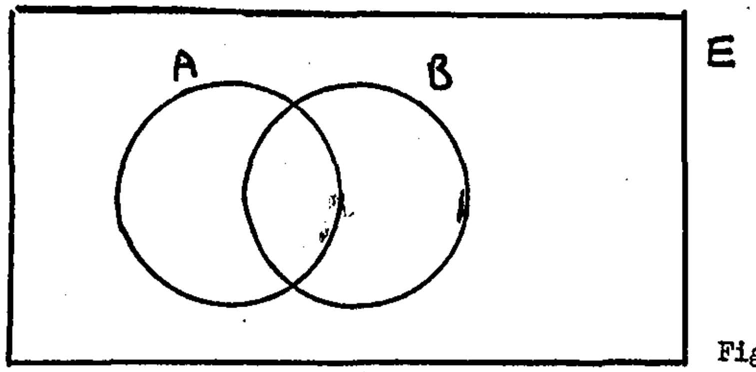


Figure 4.

Using the "equally likely" definition of probability we have:

$$P(A \cup B) = \frac{n(A \cup B)}{n(E)}$$

The number of elements in the union of A and B is not the number in A added to the number of elements in B, because the union contains elements common to both sets. If we did this then we would be adding the number of elements in the intersection of the 2 sets twice.

$$\text{Hence} \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Dividing both sides of the equation by n (E) gives:

$$\frac{n(A \cup B)}{n(E)} = \frac{n(A)}{n(E)} + \frac{n(B)}{n(E)} - \frac{n(A \cap B)}{n(E)}$$

and this, from the definition of probability, gives:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is known as the addition law.

Mutually Exclusive Events.

It is quite possible that in a particular situation $A \cap B = \phi$ where ϕ denotes the empty set, and so $P(A \cap B) = 0$. For example, if a coin was tossed, and A represented the event "Head" and B represented the event "Tail" then $A \cap B = \phi$. This is self evident since it is impossible to obtain both a head and a tail when a single coin is tossed once.

The Venn diagram would look like Fig. 5.

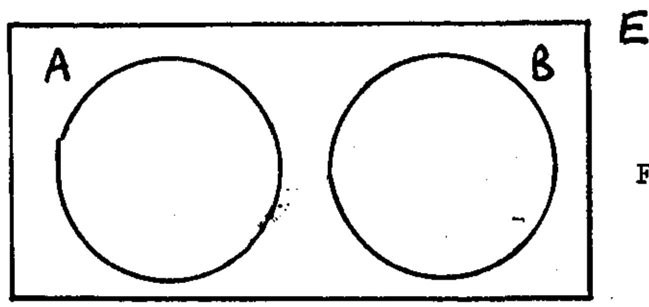


Figure 5.

When we have a situation like this we say that the events are mutually exclusive. A and B cannot happen simultaneously, A occurs prohibits B from happening and B occurring prohibits A from occurring.

In this situation the addition law becomes:

$$P(A \cup B) = P(A) + P(B)$$

A particular case of mutually exclusive events is when one event is the complement of the other. In this case we have:

$$P(A \cup A') = P(A) + P(A')$$

Therefore:

$$1 = P(A) + P(A')$$ which gives:

$$P(A') = 1 - P(A)$$

For example, if the probability of winning a game is 2/3, then the probability of losing is 1/3.

Conditional Probability

When considering the probability of A and B occurring we need to consider also the concept of conditional probability. A typical definition is : the probability of an event A occurring when it is known that event B has occurred is said to be the "probability of A given B". In symbolic form this is written

$P(A|B)$, where the bar is read as "given".

So we have:

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

We divide both the numerator and the denominator of the fraction by $n(E)$, where $n(E)$ is the number of elements in the original possibility set.

So we have:

$$P(A|B) = \frac{n(A \cap B) \;/\; n(E)}{n(B) \;/\; n(E)}$$

$$= \frac{P(A \cap B)}{P(B)}$$

This gives:

$$P(A \cap B) = P(A|B) \times P(B)$$

which can also be written as

$$P(A \cap B) = P(A) \times P(B|A)$$

<u>Independent Events</u>.

Two events A and B are said to be independent when:

$$P(A \cap B) = P(A) \times P(B)$$

By comparing this with:

$$P(A \cap B) = P(A) \times P(B|A)$$

we see that this condition is satisfied when $P(B|A) = P(B)$

In words, when event A takes place, it has no bearing on the result of the probability of event B. An example is any sort of draw in which replacement occurs after each draw.

The above treatment is typical of those text-books which use set-theory to find the probabilities of compound events. Most mathematically minded A level students seem to respond well to this sort of treatment, but it is worthwhile demonstrating the results hold true with empirical probabilities. For many students, statement of the results coupled with empirical demonstrations is usually enough to convince them of the truth of the rules, in other words they make sense.

The idea of independence always seems to be the most difficult concept for the average Vlth former to grasp hold of. Some text-books give the mathematically precise definition of independence that is, A and B are independent events if P (A and B) = P (A) X P(B) The problem is that although it is possible to calculate P (A) and P (B) it may not be possible to calculate P (A and B) so how do we prove A and B are independent. It is more likely in fact, that knowing P (A) and P (B) we wish to calculate P (A and B) using the multiplication rule. The trouble is that we cannot use the rule correctly unless A and B are independent in the first place and we cannot prove that unless we know P (A and B). The whole argument is circular and where do we break into the circle?

Some text-books give a description of what is meant by independence rather than giving a precise definition which does not seem to have a great deal of practical use. They say that A and B are independent if A happening does not effect the probability of B occurring. Obviously this is not a definition but by making a subjective decision we can at least use the multiplication rule when it seems feasible to do so. I think that it is acceptable to do this, on the grounds that Statistics is applied mathematics and if we wish to solve certain problems in real life, we often have to make such assumptions, based on sound reasoning of course. We can easily make mistakes by making incorrect assumption as the following example illustrates.

Part of a recent C.S.E. question had the following information and request:

A class of thirty has twenty pupils taking statistics and five prefects. Find the probability that a pupil at random is both a prefect and studies statistics.

Presumably the candidates were expected to argue that doing statistics and being a prefect were independent (how could one effect the other?) so that the required probability is

$$\frac{20}{30} \times \frac{5}{30} = \frac{1}{9}$$

But if we draw a Venn diagram (fig. 6) we get

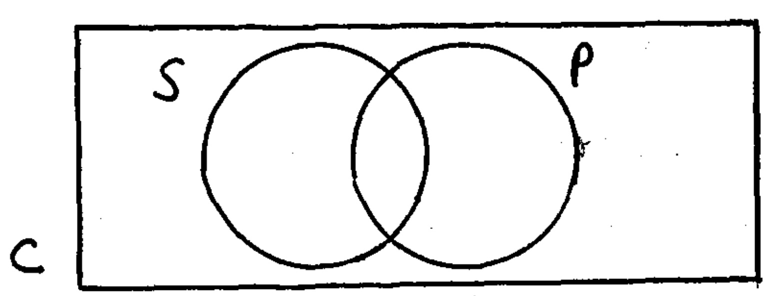


Figure 6

Since $n(C) = 30$, $n(S) = 20$ and $n(P) = 5$

it follows $n(S \cap P)$ could be 0,1,2,3,4 or 5. So that the required probability is 0, 1/30, 2/30, 3/30, 4/30 or 5/30 and none of these is 1/9.

Apart from this being a very bad question, in this situation it is impossible for the two events to be independent statistically. So if events appear to be independent it does not mean that they are if we apply the more precise mathematical definition. Hopefully no 'A' level cadidate will have to answer such a question, but this example does illustrate that we do make awfully big assumptions to get an answer sometimes. We may be asked, for instance, to assume that seeds germinate independently with given probability. Is this a realistic assumption? What happens if we do not assume this?

We cannot get away from having to make such assumptions, but at least our VIth form students should be made aware that it is an assumption we are making very often. I did not find any of the 'A' level statistics text-books that really spelt this out to the student, though.

I have attempted to give a brief synopsis of some of the approaches that 'A' level statistics text-books take on probability. Probability is an important topic and a great deal of statistical work requires a sound knowledge of the topic. Most of the text-books that I looked at devoted just one chapter of their book to probability, but it would be quite possible to write a book on probability alone. It is a pity that statistics books, from my experience, do not discuss all approaches to probability rather than just one or sometimes two. And finally, the last main criticism that I have of their treatment of probability is that none of them discuss in detail or at length, any serious applications of probability. As I said earlier, most 'A' level mathematics teachers have had little or no training in statistics and the books would be doing a great service to pupils to include a section on applications.

It must be remembered, however, that statistics is still a relatively new subject at school level and it is going to take time before teachers and text-book writers decide on an approach which includes all the right things and is aimed at the right level.

CHAPTER 5

IS IT STATISTICS OR MATHEMATICS?

When producing solutions to questions from past 'A' level papers, it often occurs to me that perhaps what I am doing is not really statistics but mathematics. The question of what constitutes being mathematics and what constitutes being statistics is an emotive one. The dividing line, if it exists, is purely arbitary and determined subjectively. Some mathematicians and statiticians might argue that statistics and mathematics are totally separate subjects, others are of the view that the subjects are like two intersecting sets, whilst there are those who would say that statistics is merely a branch or subset of mathematics. I believe that subscribing to either the first or last of these views displays a high degree of subject chauvinism and that the middle view is the more realistic of the three, especially at school level.

It seems to me, when I read through a great many 'A' level statistics text-books and look at the statistics questions set at 'A' level, that in our hurry to get to the mathematical models underlying our statistics we can easily forget the source and inspiration of the problems, the statistics itself. We are in grave danger of finishing up with a box of idealised equipment in our statistics similar to our box of equipment in mechanics. We have replaced our light rods, inextensible strings and perfectly elastic spheres with unbiased dice, perfectly random procedures and independent events. It is all too easy to put our emphasis on the model rather than the reality with too little emphasis on how reasonable our model is. Just as physics is more than the mathematics it uses, so is statistics. If we continue to divorce our statistics from real data and real life situations then we are in danger of making the subject as sterile and academic as much as theoretical mechanics has become. Much of the statistical work done in mathematics seems irrelevant to the geographer, biologist and economist because it concentrates on techniques of calculation and not on the context which leads to the need for these calculations. To a certain extent I would defend this emphasis since at 'A' level, statistics is usually part of a mathematics examination and hence the examination must be designed to examine the mathematical ability. Even so, mathematical ability is surely not confined to how well one can cope with "number crunching" situations which many 'A' level statistics questions seem obsessed with. What I mean by "number crunching" is the sort of problem which involves little more that substituting numbers into rote learnt formulae and carrying out a series of tedious arithmetical procedures. By slightly altering the context, many questions put on to the statistics sections of 'A' level mathematics papers would be highly suitable for the pure-mathematics section. We must attempt, if it is possible, to set questions which are drawn from a statistical context but require some sort of mathematical analysis to solve the problem.

What sort of impression does the VIth former get of statistics when completing his 'A' level course in mathematics with statistics? With our emphasis on dice, cards, balls in urns etc. what sort of view does he get of the role of statistics in society? Does he view it purely as something to be used when gambling? We largely omit the use of official statistics, the vast amounts of data that are collected and published almost daily. How many leave school at the end of the VIth form feeling that statistics is important in society today? How many realise the breadth of applications from insurance, decision making in industry, drawing conclusions about smoking and lung cancer, seeing whether speed limits reduce road accidents, whether having more police cuts the crime rate, whether legislation on race is having the desired social effect, quality control in industry, drug testing in medicine and so on?

The trouble is, that in mathematics in schools we tend to teach only that which can be quantified. The problem is as Engel (1971) says "... at this stage almost every interesting problem is beyond the analytical skills of the student."

Whilst defending the mathematical emphasis on the statistics sections of 'A' level Mathematics examinations, I do feel that we do need to do much more basic work in schools. We need to establish statistics more as an experimental science with its own concepts and ideas. A need exists to use experiments, games, simulations and many other devices to give an intuitive feel for these ideas. Not enough time is spent just looking at data and asking simple questions about it; where does it come from, who collected it and how and why, how accurate is it likely to be, what does it show? Can we develop a simple idea of inference through an appreciation of data?

I think that it is desirable that we have more realism both in our statistics teaching and in the statistics questions set at 'A' level. We must attempt to keep our examples connected with everyday life and in the context in which they arise. Before getting down to calculations we ought to try to keep in mind that most calculations will be done for a purpose, other than satisfying an examiner, and that we should try to take a closer look at the actual data first.

D.V. Lindley (1977) makes the point that "The basic fact that should never be forgotten in any teaching of statistics is that the subject is concerned with data, with numerical information about something of interest in its own right. The mean is not 20, but 20 tons per acre, or £20 per week. The pupil must be perpetually exposed to data, he must dirty his hands and not take an entirely mathematical view of statistics. Much of our teaching is geared to those aspects of statistics which are easily examinable: to the mechanics of the t-test or χ^2" This would seem so on looking at some examination questions, for example one that requires the pupil to calculate a correlation coefficient without, at the same time, making sure that he also understands why a correlation coefficient is meaningful in the context, or what the value of the coefficient tells us, is not really a very good applied mathematics question to set.

I shall attempt to examine the problem of whether it is statistics or mathematics by considering a selection of questions, with their solutions, from some past Oxford 'A' level papers. I choose Oxford since this is the board with which I am most familiar. All the following questions are taken from the paper II of the A 50 syllabus.

1975 Question 14

A random variable X has probability density function f(x) given by

$$f(x) = Kx(1 - x) \qquad (0 \leqslant x \leqslant 1)$$
$$= 0 \qquad \text{otherwise}$$

Show that the mean and variance of X are 0·5 and 0·05 respectively.

Find the probability that an observation chosen at random from the distribution is more than two standard deviations from the mean.

Solution : The first thing that must be found is the constant K. The student must realise that it is the area under a probability curve that represents the probability. Hence the total area beneath the curve is equal to unity. For the moment that is all the statistics that the student need know. The rest is pure mathematics since every 'A' level mathematics student should know that the area under the curve is given by

$$\int_{-\infty}^{\infty} f(x) \; d\,x$$

this must be unity

$$\int_{0}^{1} kx\ (1-x)\ dx = 1$$

By integrating and substituting the limits we obtain K = 6.

The second part of the question requires the examinee to evaluate the mean of X. Using expectation algebra we have

$$\text{the mean } \mu = E(X)$$

$$\text{where} \quad E(X) = \int_{-\infty}^{\infty} x\ f(x)\ dx$$

$$= \int_{0}^{1} 6x^2\ (1-x)\ dx$$

Again by the usual procedures we do in fact find

$$\mu = 0\cdot5$$

Similarly by using the formula $\int_{-\infty}^{\infty} x^2\ f(x)\ dx \ - \mu^2$ to find the variance, we also verify its value to be 0·05.

Finally, we come to the last part of the question and again the examinee must realise that probability is represented by the area under the probability curve.

Since the variance is 0·05 then the standard deviation is $\sqrt{0\cdot05}$. Hence the required probability is given by

$$\Pr\left(0\cdot5 - 2\sqrt{0\cdot05} > x > 0\cdot5 + 2\sqrt{0\cdot05}\right)$$

$$= 2\int_{0}^{0\cdot0513} f(x)\ dx$$ since the curve is symmetrical about a vertical axis through the mean.

Well, to my mind, the only piece of actual statistical knowledge that any one needs to know to answer that question was that area under a probability curve is numerically equivalent to the probability of obtaining X between the required limits. Most of the solution tested how well a candidate can integrate and substitute in limits. In fact it was quite possible, and this is something from my experience that weaker 'A' level candidates do, to simply remember as formulas and set procedures how to find probability in questions of this type.

There was no attempt to put this question into the context of a practical situation and even the random variable X had no units assigned to it, to my mind an essential minimum requirement of any applied mathematics question. This question, in my opinion, would have been better placed on the pure-mathematics paper, but I doubt that it would have been accepted since the arithmetic involved in the last part of the problem was tedious and involved, especially since the use of calculators are forbidden in the mathematics papers set by Oxford.

It would have been much better to set a question which examined the role of a probability distribution as a model to describe practical situations which exhibit random variation.

For example, a question could be set in which a set of observations would be given in the question. These observations could, for example, consist of the times at which vehicles passed an observation point on a stretch of road over a period of time. It could then be suggested that an exponential probability distribution might constitute a reasonable model for inter-vehicular times in a given direction. The candidate could be asked why this was a reasonable model and also by calculating estimates, from the actual data, of the required parameters, use the model to predict expected frequencies. It would then be reasonable to ask the student to carry out a test of goodness of fit to see whether the model is a reasonable approximation of reality. This way the question is at least in the context of physical reality but at the same time there should be enough mathematics to justify the inclusion of such a question on a statistics paper of a mathematics examination. I feel that any question on a statistics paper of a mathematics examination ought to be statistical to some extent, else why call it a statistics paper? At the same time, there should be some mathematics in the question since there is little point in setting it in a mathematics examination.

1975 Question 15

The assembly of an article in a factory demands the use of 10 screws. For the convenience of the assembly workers the screws are put into packets of 10 by the storekeeper. The manager buys a new stock of screws equal in number to half the current stock, and tells the storekeeper to mix the new and old screws together thoroughly. Shortly after the mixing has taken place the manager picks up a packet of 10 screws and discovers only one new screw in the pack. What is the probability of one or fewer new screws in a pack?

The manager suspects that some of the new screws may have been stolen. He inspects 9 further packs and finds the following numbers of new screws:

3, 1, 2, 2, 1, 5, 2, 4, 3.

Does this evidence confirm the manager's suspicion?

Solution: To calculate the required probability we must first decide on what mathematical model we need. To do this we have to make the assumption that the stock of screws is large enough to take one new screw from the stock without affecting the probability of obtaining a new screw. In other words we need to assume that we have independent probabilities. Also assuming that each screw is equally likely to be chosen we can see that the probability of a screw chosen at random being new is equal to 1/3 since the ratio of new to old screws is 1 : 2

I just wonder how many VIth formers who manage to arrive at this correct answer really make these considerations when arriving at the answer. They obviously make the assumptions but not consciously and they are unaware that they are doing so.

Since a sample of ten screws really consists of the repeated trial (10 times) of choosing a screw at random and also since the probabilities are independent we are led to suppose that the Binomial model with n = 10 and p = 1/3 is to be used to obtain the probability.

Therefore Pr (1 or fewer new screws) = Pr (0 new screws)

+ Pr (1 new screw)

$= \left(\frac{2}{3}\right)^{10} + 10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{9}$

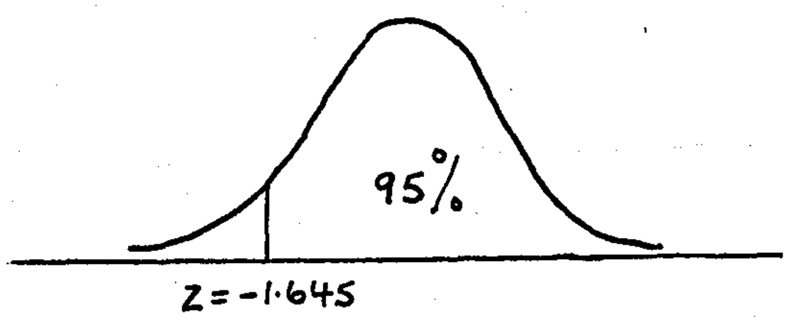
= 0·104

The examinee would therefore need to be aware of when to apply the Binomial model, how to apply the Binomial model and also how to use the addition rule of probability. The emphasis here is statistical rather than mathematical, but then again it could be argued this is one of those situations in which we are in such a great hurry to get to the mathematical model that underlies the statistics, that we have made more assumptions than we ought. In other words we have a similar situation to questions in mechanics when we assume that we have weightless rods and the liks. I am not sure that I would go along with this since I would regard these mathematical models as essential tools of the statistician and therefore part of statistics. In this problem I feel that the assumptions made were justified otherwise the problems would be unsolvable at this level.

Anyway, to return to the solution of this question, it is now necessary to write the number of new screws per pack as proportions and then regard these proportions as values from the sampling distribution of proportions which is a normal distribution with mean 1/3 and standard deviation $\sqrt{\frac{\frac{1}{3}\cdot\frac{2}{3}}{10}} = \sqrt{5}/15 = 0.15$

Since the manager requires to know whether or not some of the screws have been stolen he needs to get some minimum number of screws per pack below such a level he would be pretty certain that his suspicions were correct. In other words we need a one tailed significance test.

The above values of the mean and standard deviation are arrived at using the null hypothesis that the mean is 1/3 i.e. that no stealing has taken place. The alternative hypothesis is that the mean proportion is less than 1/3 i.e. that stealing has been going on.



By considering the area under the standardised normal curve, it can be seen that due to random chance the probability of obtaining a proportion less than 1·645 standard deviations below the mean is 0·05 which is quite remote. Now the smallest number of screws per pack is 1 which is equivalent to 0·1 as a proportion. Standardising this score we obtain

$$Z = \frac{0{\cdot}1 - 0{\cdot}333}{0{\cdot}15} = -1{\cdot}56$$

In other words the lowest proportion lies in the acceptance region so we accept the null hypothesis. This means that we would reject the manager's suspicions.

From the answers given by the examining board, this was the intended approach. Personally, I think that it would have been better to take the ten packs together as one sample of 100 screws. 24 of these screws are new and the question is then, are 24 screws out of 100 suspiciously low if the overall proportion of new screws is 1/3? As we are now dealing with a sample of 100 screws (rather than 10), the Normal assumption is quite acceptable.

There is quite a bit of statistics in this problem and many VIth formers might find the solution difficult because they are not told which mathematical models they have to use. Therefore, it is not simply a question which involves statistical mathematics but involves the student calling on his knowledge of statistics so as to decide which model he needs to use. At least someone has attempted to put the question into the context of a physical real world situation, even though it is contrived.

One of my main criticisms of this problem is that the size of the samples is only equal to ten, yet we are taught from theory that the size of the samples should be large before the proportions become normally distributed. I would not regard 10 as being large! This might penalise some of the more able 'A' level candidates who might suppose that strictly they should not use the Normal model.

Overall though, taking into consideration that the candidate only has about twenty minutes to answer a question and also a really worthwhile question would be beyond the skills of the candidate, this is not a bad attempt at setting a question which is statistical but at the same time contains some mathematics.

1975 Question 20

A manufacturer wishes to estimate the proportion of cars fitted with radios in a small town and the surrounding district. The following schemes for choosing a sample are suggested. State the criteria you would use for assessing these schemes and discuss the relative merits of the schemes. Suggest improvements if any occur to you, and state the scheme you would prefer.

(I) Choose at random 200 names from the telephone directory. Ring the corresponding numbers and ask about the cars belonging to the household.

(II) Take every n th name from the electoral register, choosing n so that the total sample size is 200.
Visit the homes of the people chosen and ask about the household's cars.

(III) Stop cars as they enter the High Street on a weekday morning and check whether or not they have radios. Inspect 200

Solution: The solution to this particular question was discussed in some detail at a meeting between the 'A' level examiners and a number of teachers who are responsible for teaching the statistics section of the A50 paper. The meeting was held at Oxford on Friday, 12th September 1975 and the following solution is the one the examiners used in that marking scheme.

I was told that this question was to test if candidates could apply the principles of sampling to realistic cases.

The following criteria for assessing the scheme were expected:

(a) lack of bias (Is the population sampled identical with the target population? Is the sample a random one?)

(b) practicality of the method.

The following comments on relative merit were expected:

(i) Cars owned by telephone subscribers rather than cars as a whole are being sampled. (There is good reason to think that this sample may differ from the general population). Even among telephone subscribers, those who are out a lot are likely to be missed, and hence their cars have less chance of inclusion. The scheme is cheap and easy to carry out.

(ii) Most car owners are likely to be on the electoral register. A car owned by a couple has twice the chance of being chosen as one owned by a single person. The scheme is expensive of time and money because of the visits and the large number of electors who do not have cars.

(iii) Cars likely to be on the High Street on a weekday morning form only a part of the whole population and some cars, e.g., of those who shop in the morning, are more likely than others. The method is relatively easy and cheap, if allowed by the police.

Candidates were then expected to choose a scheme, with possible modification, and say why they chose it. No method is obviously the best.

This question seems to have no mathematics in it at all. There is certainly no quantitative element in this question. In fact when I included this question in a trial 'A' level examination some time ago, I found that all of the solutions were in the form of formal essays rather than a series of points. The question was very badly answered and if I had not known better I would have believed that I was marking answers to a question from a sociology examination. For instance, most of the candidates who answered this question were obsessed with all telephone subscribers being middle class and therefore not a representative sample. All very true, telephone subscribers are probably not representative but a sociological analysis was not required.

Having had a mathematical training myself I found it difficult, before attending the meeting with the examiners, to write down a solution. This is because the emphasis of this question is perhaps too statistical and the mathematical models that underlie the statistical theory are too implicit rather than being explicit. Since this question was part of the statistics section of a mathematics paper I feel that it was not a fair question to set. Both teachers and pupils need experience of dealing with sampling techniques before being able to cope with answering a question with such statistical emphasis.

I think that it would have been better to set a question on sampling theory which at least allowed the candidates to relate their answer to some mathematical model, perhaps even a question which allowed them to use random numbers to detect a sample or even a simulation problem. We cannot expect mathematics teachers or mathematics students to cope with questions in statistics which involve little mathematics, but on the other hand they should be expected to know enough statistics to cope with statistical problems which require mathematical models to help solve them. This question was not such a situation. There needs to be some sort of compromise of the sort of question that can be set on the statistics section of mathematics paper.

Two criteria must be satisfied: The questions should be statistical in nature and also the problem should involve the use of some sort of mathematical model as part of the solution. The mathematics should naturally arise from the practical context of the problem. Any numerical solution such as a coefficient of correlation for instance, should then be interpreted with respect to the statistical problem.

The three questions fell into three categories: question 14 was totally mathematical since only a knowledge of pure-mathematics techniques was required to solve the problem. Question 15 involved both subjects since both a knowledge of statistics and the use of mathematical models was wanted. This is the type of question which I feel should be on a statistics paper of a mathematics examination since the numerical solution (i.e. the z score) requires a statistical decision to be made by the candidate. The last question I considered (that was number 20) required no mathematical analysis and could be best answered by someone who had experience of actually carrying out a survey, though a reasonable solution could be made by using common sense. Not the best question to set on a mathematics paper.

I do not believe, as D.F. Kerridge (1976) obviously does, that, to quote him, "mathematics is a menace" to statistics. There are obviously some parts of statistics which do require the use of mathematical models. This is, I feel where the two subjects are like two overlapping sets, and I believe that the syllabus and examination on statistics sections of 'A' level mathematics should be set with this overlap in mind. By design, it should be a syllabus in mathematical statistics and not in statistical mathematics (like question 14) nor should it be a syllabus just on statistics (as is typified by question 20).

Some questions are very obviously one thing (mathematics) and some questions the other (statistics). It seems pointless to try to distinguish whether some questions set at 'A' level are mathematics or statistics, because very often the question is both. Much of statistics is a branch of applied mathematics and therefore we have a right, indeed a duty, to include certain aspects of statistics on the 'A' level mathematics syllabus. By emphasis much of this is very mathematical and is best taught by statistically inclined mathematicians. There are many aspects of statistics which have no relevance at all to a mathematics syllabus, e.g. questionaires, and these are best left to be taught by the statistician.

On our 'A' level courses in mathematics we are not trying to prepare people to become statisticians, most of us do not have the competence. We are simply trying to show the pupils that mathematics is not just a subject in its own right, but is a useful tool as well; it is our duty as teachers to teach this aspect of our subject. I therefore do not agree with many statisticians who claim that we are poaching from their subject and have no right to teach any statistics. This is because it is not always a question of "is it statistics?" or "is it mathematics?" because much of it is both at the same time. Any dividing line would be artificial. A mathematics examination should test uses as well as pure manipulative skills.

CHAPTER 6

Project Work in Statistics at A Level.

One of the more recent innovations in statistics on A level mathematics syllabuses has been the introduction of project work. Two notable examples of this new trend are the A level statistics paper set by the Joint Matriculation Board as from June 1978 and the A level Pure Mathematics with Statistics set by the University of London Board which was first examined in June 1977.

For the examination set by J.M.B., "candidates will be required to submit a personal study of statistical (or probabilistic) methods applied to one specific area of practical enquiry. The length should not be more than 15 pages excluding the presentation of raw data. Projects will be assessed by the candidates' teacher and submitted to the Board together with the teacher's assessment report for use in moderating the centre's assessments".

The project is considered to be a vital part of the syllabus giving each candidate a chance to gain experience of an in-depth study of statistical methods applied over a substantial period to a particular area of practical enquiry. A single topic is chosen by the candidate, in consultation with the teacher, for the project.

The Board issues the warning to keep the project feasible in terms of:

"(i) access to relevant data of an appropriate scale (sufficient to provide useful information whilst remaining manageable within the time scale of the project).

(ii) the candidate's experience: the required methods of probability modelling, of data presentation, summary and analysis, and of valid interpretation, need to be largely within the syllabus (although relevant extra syllabus topics at a reasonable level are not excluded).

(iii) the time available within the work programme for the design and execution of the work and for the preparation of a project report (bearing in mind that 15% of the marks will be allocated to the project work)"

It is expected that the candidates will spend about 40 hours on the project work, during their second year of study, although this time is not rigid since it will depend on the nature of the project topic chosen. The project is concluded with a report, by the candidate, and this report should include the aims of the project, probability models used, interpretation of results, details of data, representation of data, statistical methods used and conclusions drawn from the work.

The teacher has quite a role to play, he helps the candidate choose the topic, acts as a consultant throughout the time that the project is being undertaken and then assesses the project. The Board sets out the following four qualities to be assessed:

" A. Ability to select an appropriate probability model and set of statistical techniques for the investigation of the topic of the project and to design a plan of procedure.

B. Ability to carry out a practical investigation in accordance with a plan of procedure.

C. Ability to present the results of practical investigations in a valid and informative manner.

D. Ability to draw conclusions from practical experience and evidence."

On actual choice of the topics the Board says they "may arise from any area of activity as experienced in the classroom, at home, in leisure activity, in society or from general reading. Data may refer for example, to experiments in E.S.P., surveys of pupil's attitudes to controversial issues or behavioural characteristics; local community matters (shopping, political views, the new by-pass, etc.), population statistics of a demographic nature, experimental results on problems in Physics, Chemistry, Biology, economic indicators, road traffic flow, measurements on manufactural products; results of examinations or tests of various types, simulation of practical situations.

As a single specific example on the last-mentioned item, a candidate may decide to investigate the advantages of the introduction of an appointments system by the local doctor for his evening surgery, by simulating the condition of the surgery a few times both with, and without, an appointment system. Assumptions about arrival pattern of patients, and consultation time, will need to be backed up by observations of the real-life situation."

The project work required for the University of London's A level in Pure Mathematics with Statistics requires, perhaps, a less in-depth approach than that needed for J.M.B's A level. Also, the project work is not marked by the examiner nor the teacher but candidates take their projects into the examination and are expected to refer to them in their answers. It is considered that doing project work is an essential part of a study of statistics, though.

It is expected that a candidate will carry out three projects relating respectively to three areas, sampling, chance variations (this includes tests of hypothesis) and experimentation.

This Board list four parts that they would expect each completed project to consist of:

(i) Statement of Aims - precise, and of about fifty words.

(ii) Data - collected by the candidate, with notes of difficulties and precautions, and other relevant information.

(iii) Analysis of the data.

(iv) Conclusions and Observations - usually about 200 words in length.

The syllabus also gives a list of suggested projects of which I reproduce a selection.

Sampling.

Weather in holiday resorts.

Favourite winning horse races.

Shoe and collar sizes.

Colour of cars.

Chance Variation.

Journey times.

Length of grasses.

Votes in General Election.

Differences in authorship

Experimentation.

Heights and weights

Effect of temperature on cress seeds

Income and expenditure amongst pupils

Does extra sensory perception exist?

As already stated the candidates will be required to bring their completed projects to the examination room. They are allowed to quote, if necessary, from their projects in answering the appropriate questions of Paper 2, which as the specimen paper indicates, are designed so that they can be satisfactorily answered only by a candidate who has properly carried out the projects with a due realisation of the practical difficulties inherent in any statistical investigation.

The following question is taken from Paper II of a speciman paper issued by the University of London Board in Sept.1974:

Describe briefly how you made use of the normal distribution in one of your projects and give reasons for any assumptions that it was necessary to make.

A port authority has a large labour force with distribution of I Q which has mean I Q of 103 and standard deviation of 12. If 100 employees are selected at random, estimate how many would be suitable for a job requiring an I Q between 100 and 110.

It can be seen that the first part of the question explicitly requires an answer involving the candidates first hand experience of dealing with an experiment in which the normal distribution was made use of. It is somewhat unfortuate if the candidate has not made use of the normal distribution in his projects of course. This is perhaps unlikely though considering how many times the normal distribution crops up in the sorts of situation that an A level candidate will draw his projects from. The second part of the question is more typical of the questions set at A level and really only requires theoretical considerations, not least of all making the assumption that I Q approximates to a normal distribution.

The one major agreement that both Boards have,is concerning the philosophy of why project work is important to students studying statistics at A level. Both feel that project work is a necessary component because statistics is basically an experimental subject. It seems that they also have common objectives in that they both see the role of the practical work is to reinforce the theoretical work, in other words it gives the theory some sort of meaning. Also the practical work is meant to give to students some sort of intuitive feel for the subject.

The Boards differ in respect to assessment, one gives a possible 15% of the marks to the project whereas the other does not assess the project, but requires it to be used in Paper 2 of their examination. Assessing project work has all the usual arguments against it, for instance, the teacher could be responsible for doing a lot of the candidate's work for him. Also there is the problem of the halo effect and standardisation of results. Here we have the added problem of whether all teachers of mathematics are in fact suitably qualified to mark a practical statistics project.

The University of London's regulations get over this to some extent by not attempting to assess the practical work. In some ways this seems to be somewhat of a contradiction to what the purpose of an examination is, but it is hoped by the Board that their examination questions are so geared as to make the carrying out of a project absolutely necessary. I just wonder if the candidates themselves feel as much of an incentive to carry out a project under this system than if they were actually given marks for their practical work. I admire the thinking and trust in the University of London's philosphy but much will depend on the enthusiasm of the teacher. For all its difficulties I think that all project work should be assessed, but I also think that J.M.B. should follow London's example and allow the students to take their projects into the examination with them. I feel that it is right to assess the project work since it is incentive to do well. Most students, if asked, would probably wish for their work to be marked on the grounds that if they have put a lot of effort into it, then they want to know what it is worth. A mark is a convenient yardstick and acts as a suitable reward for those who have done well.

The Boards also differ in the emphasis that they place on the depth they feel the candidates should go. J.M.B. require a single project which will obviously go deeper and cover more qualities than the three projects required by the Univeristy of London's syllabus. Perhaps this difference is understandable though because the J.M.B.'s syllabus is a single subject in statistics, and therefore the candidates should have longer time to spend on practical work, than those candidates who only are taking statistics as half a subject. Also you would expect students who are taking statistics as a single subject to have a deeper understanding of the subject than VIth formers who are taking the Pure Mathematics with Statistics examination.

As yet, practical projects are new at A level and only a few A level candidates who take statistics will be involved. It is to be hoped that even if a practical project is not a requirement of the syllabus, more and more teachers of mathematics will still include some practical work in their courses. Of the arguments levelled against including any practical work in an A level course, lack of time is perhaps the greatest. Whatever the arguments though, one cannot ignore the benefits of including such work. First ideas of histograms, their gradual build-up and a heuristic idea of the shape of the underlying probability distributions can be obtained from at least a hundred or more observations of some simple experiment.

Also, personal participation in obtaing observations brings to the statistical analysis the zest of discovery. Therefore, hopefully all teachers of A level statistics will attempt to include some practical work in their courses in the future. I have included some ideas of practical work in the appendix.

Project work forces a student to (1) think about the appropriateness of a certain mathematical model in a particular situation, and (2) to interpret the results of a statistical analysis in terms of the real world question that he is trying to answer. These aspects of statistics are not reflected in most examination questions, so the introduction of project work suggests that a greater emphasis is now being placed on model choice and interpretation of results. This indicates a swing away from a purely mathematically manipulative type of examination in statistics towards one which tests other aspects of statistics. Perhaps, it will not be long before the majority of Boards will follow suit and also set some form of project work in an attempt to create examinations in applied statistics as opposed to applied mathematics.

CHAPTER 7

Conclusion

How can things be improved? Perhaps, the greatest difficulty arises from statistics still being a relatively new subject in schools. There must be today, a gradually increasing number of new graduates going into school teaching, who have had University courses in statistics. For some, though, these will have been mainly theoretical and concerned with statistical mathematics because the course is part of a mathematics degree, or a joint degree in mathematics and statistics. For others their course ha[illegible] been of a more applied character associated with other disciplines.

At school level in the VIth form, the statistics that is taught tends to emphasise the mathematical side of the subject and forms part of the mathematics syllabus. Because the subject, at school level, has developed within the context of the mathematics syllabus, this has meant examinations are set by mathematicians and the subject taught by mathematicians. Also, the questions in examinations which involve some sort of mathematically manipulative procedure are far easier to set and mark than questions which require some sort of interpretive analysis. Perhaps as more graduates who have received statistics courses of an applied nature take jobs as teachers in schools this situation may start to change.

Professor D.J. Finney makes the point "Statistics is no more a branch of mathematics than is engineering: it is a discipline in its own right, ultimately concerned with the collection and interpretation of data rather than with mathematical relationships." (1977)

To a point I agree with him, but there is a large grey area in which it is not possible to distinguish whether the context is mathematics or statistics. It is as if we have two intersecting sets, the sets being mathematics and statistics. I do not feel that at school level we have the time or facility to cope with statistics as a subject in its own right. For a start, we do not have the required number of suitably qualified teachers, and only a very large comprehensive school could support a fully qualified statistics teacher. Also, most students do not have the motivation nor the maturity at VIth form level to be able to cope with a statistics course in its owm right. I think that such a course is best left until university level. At school level then, I think that we should confine ourselves to the area which is both mathematics and statistics. It is our job as mathematics teachers not just to teach mathematics as a subject in its own right, but also to teach mathematics as an applied subject. This is why it is extremely desirable to teach some aspects of statistics. On the other hand, the statistics that we do teach in A level mathematics must bear some resemblance to statistics and should not just be an exercise in statistical mathematics. Very often, in practice, though, it is extremely difficult to separate the mathematical and non-mathematical aspects of statistics. For example, to test some particular hypothesis we first have to collect some sort of data and this may well involve the use of a survey. At present though, this sort of statistical content is neglected by the majority of A level mathematics syllabuses. What we must try to do is set examinations which test these aspects of statistics because it is not possible to fully appreciate the purpose of the mathematics without doing so. It should always be remembered, though, that the statistics sections form part of the mathematics syllabuses and therefore should remain as a test of applied mathematics and not applied statistics. In other words, we must not swing too much the other way.

At present, the balance between the interpretive and manipulative aspects of the subject is not right. The trend is now starting to change however, with the recent inclusion of project work on two of the statistics syllabuses. It is to be hoped that this trend will continue and that more examination boards will include some sort of project work on their respective syllabuses. This would at least give A level students the chance to learn a bit more about the non-mathematical side of statistics which would help them to understand, and give them insight into, the mathematical aspects of the subject. The time factor is one great difficulty if project work is included and, to allow for this, I feel that it would be possible, and indeed desirable, to cut down on some of the content that exists on the syllabuses at the moment. For instance, it is possible to give students an idea of what a test of hypothesis is without having to teach him four or five different types. Surely, just the inclusion of one of two types of test of hypothesis would be sufficient to impart the necessary concepts. Therefore, I would recommend a complete rationalisation of the content of the statistics syllabuses and to compensate for this include some sort of project work.

I would like to see the statistics section contain questions which tests both the manipulative and interpretive aspects of the subject. Also, in much the same style as the University of

London's Pure Mathematics with Statistics examination, I would like the questions to relate in some way to the work carried out by students in their project work. For example, part of a question might require the candidate to explain what sorts of data approximate to certain kinds of probability distributions or under what circumstances they have had to make use of mathematical models in their projects. This sort of approach, I feel, would at least give the students a much better idea of what applied mathematics is all about and is not just a disguised set of pure-mathematics problems.

Apart from changes in the syllabuses, it is necessary that we have enough teachers cabable of teaching statistics in the VIth form. For obvious reasons this job is going to be left to teachers of mathematics to do. It is to be hoped that colleges of education and university departments of education will include courses in the teaching of statistics in the initial training of teachers. Also it is highly desirable that the Department of Education and Science will sponsor the setting up of far more in-service courses in statistics for teachers. Not just three or four day courses but full-time courses which last a term or alternatively one day a week for a year. A fairly long course is required because there is much for the teacher to learn and assimilate, especially if he is unfamiliar with statistical concepts and techniques.

Help for the teacher could also be provided by setting up an on - going project on statistics in the VIth form. This could be similar in structure to the Schools Council Project on Statistical Education (P O S E) which deals with statistics in schools for pupils up to the age of 16 years. I would hope that such a project would be university backed but would be run by suitably experienced school-teachers in conjunction with university lecturers in statistics. Such a project could provide suitable work-sheets with a view to creating a text-book after receiving feed-back on how the work sheets went down in the classroom. Perhaps, even a problem clinic for teachers could be set up by the project as well. What I think would be unsatisfactory, is if a special examination was set in the same way that a special examination is set for the School Mathematics Project. I think that it is better that the project caters for existing syllabuses, taking into account the trend towards project work.

Finally, I feel that it is time that we had a report from a committee set up by the Royal Statistical Society, or by the Mathematical Association, for instance. It is nearly a decade since the R.S.S. last published their report on the Teaching of Statistics in Schools. Things have changed since then and this time I would like to see the report specifically on A level statistics. It would be best that the committee was not just made up of professional statisticians, but school teachers of mathematics and even teachers of user subjects such as biology and economics. This way a more balanced and realistic set of recommendations would be made. Of course, if such a committee was set up, it is to be hoped that the examination boards will take notice.

Therefore, it is to be hoped that over the next few years, statistics, which is still a relatively new subject at school level, will become firmly established and accepted as a meaningful branch of applied mathematics at A level.

APPENDIX

Practical Experiments in Statistics - A Few Suggestions.

1. Coin-tossing. There are numerous possibilities here, apart from the obvious ones of demonstrating how the relative frequency of heads tends to stabilise to $\frac{1}{2}$.
   For example, divide the tosses up into groups of, say 10, generating observations from a Binomial distribution. Record the length of the gaps between successive heads - geometric distribution. As a variant, try tossing a drawing pin.

2. Traffic observation. Under suitable conditions, data conforming approximately to the Poisson distribution (number of vehicles per minute) and exponential distribution (intervals between vehicles) can be obtained. In addition, if the type (car, bus, commercial vehicle, etc.) of vehicle is recorded data suitable for fitting Binomial or geometric distributions can be obtained. See Scott (1976).

3. Draw a grid on the bottom of a box. Shake some very small seeds (e.g. carrot seeds) in the box and count the numbers that fall in each grid square (Poisson distribution). See Scott (1976).

4. Throw darts at a dart-board, aiming at the centre. Record the x and y co-ordinates of the points of impact. Are these Normally distributed? Independent? Are the variances the same?

5. Students are presented with straight lines and asked to bisect them by eye. Subsequently the errors are measured and recorded. For a particular student and length of line,

these should be approximately Normally distributed. Questions which might be investigated include the following. Is there any evidence of bias and, if so, is there a connection with the right or left handedness of the students? Does the variance differ from one student to another? How does the length of the line affect the variance?.

6. Pebble sampling experiment. Random and judgement sampling methods are compared for estimating the mean weight of a set of pebbles.

7. Word length comparisons.

<u>Aim</u> To examine whether the length of words used differs from one author to another, or with time, or according to subject matter (various statistical aspects of literary style have been studied in the hope of finding some which discriminate between authors).

<u>Equipment.</u> Choose two authors of similar date and two works by each.

<u>Data.</u> From each work take samples of at least 250 words. Count the number of letters in each word. Samples should represent the whole text but need not be completely random. Rules need to be decided on how to deal with abbreviations, hyphenated words etc.

<u>Analysis.</u> Summarize the four sets of data by suitable tables and/or histograms and by finding the mean and variance of word length in each set.

Compare in a large sample test, the mean word length in the two works by the same author and also compare the two authors. See Scott.

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