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THE USE OF OPEN UNIVERSITY TELEVISION PROGRAMMES IN THE TEACHING OF SIXTH FORM MATHEMATICS

by

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Abstract

Although very little is known about the process of learning from television it is possible to identify a number of possible uses of television in teaching/learning situations in mathematics. However, in order that television should even stand a chance of being a suitable medium for use in the classroom there are a number of obstacles to be overcome.

The Open University has made a significant contribution towards increasing the effectiveness of television as a teaching/learning medium although it has not been relevant to consider this work in a classroom context. Nevertheless, several television programmes, in particular those from the mathematics foundation course, do seem to have potential for use in the teaching of A level mathematics in schools. This dissertation describes a project to investigate whether or not this is the case.

After an initial discussion on the use of television and the obstacles to its successful implementation in the classroom, during which the project itself is introduced, the first step is to examine whether the programmes in the Open University mathematics foundation course are relevant to A level mathematics and acceptable to teachers.

Next, the suitability of the programmes for use in the classroom is considered and this involves a pilot study in which a number of programmes, together with draft support materials, are developmentally tested in schools. The response of teachers is indicated and some of the draft materials are subsequently revised.

The outcome, albeit based on a small sample of schools, suggests that the programmes do indeed have potential as a sixth form classroom resource - although there is still some way to go before the programmes are likely to be available for widespread use.

Acknowledgements are made to the members of the Open University course team of
M101 Mathematics: a foundation course.

I declare that this dissertion is entirely my own work.

Contents

1. Television in the Mathematics Classroom

(i) Why use television?

Perhaps the most obvious reason for using television in the classroom is simply that television exists as a source of knowledge and it is here to stay. To ignore it would be just like ignoring any other resource such as books, the blackboard, experimental apparatus, micro-computers and so on. Moreover, since almost every home in the United Kingdom possesses a television, it is a very familiar resource, more familiar in some households than books. It is therefore perhaps true to say that television, if not the only source, is certainly a major source of knowledge for all of us, adults and children alike, and the advance in information technology suggests that it will have an increasingly important role to play in the future. This argument has long been recognized. For example, over ten years ago V.S. GERLACH and D.P. ELY (1971 p 368) wrote

> Preconditioned learners come to school as confirmed TV consumers. The use of TV in instruction can capitalize on this acceptance...

As a classroom resource television certainly has some attractions: it is relatively cheap in comparison to other resources, it is easy and convenient to use, it has the power to be entertaining, and it can hold the viewer's interest, voluntarily, for long periods of time. But is it a suitable medium for teaching or, more important, learning? Well, these two activities are very different and, as A.W. BATES (1981) points out, they should really be considered separately.

In rather simplified terms then, teaching is concerned with the selection, presentation, representation and prior structuring of knowledge and experience. A medium can be defined as a means by which something is communicated. Television is a means of communicating information and is therefore just one way of presenting and representing knowledge. Thus it would seem that television is indeed a suitable medium for teaching.

It is not so obvious though that television is a suitable medium for learning, an activity which is concerned with the perception, assimilation, interpretation and relating of knowledge and experience. Although there has been plenty of research on the use of television, very little seems to be known about its effectiveness as a learning medium. A.J. ROMISZOWSKI (1974 pp 223-225) summed up the position in the early seventies as follows

> There are hundreds and hundreds of reported studies comparing TV with traditional methods but as yet no conclusive results... Much of the research is subjective and poorly designed. Those studies which are properly controlled generally repeat the pattern of no significant difference...

and the position seems to be about the same today! This means that at the moment it is really only possible to speculate on the suitability of television as a learning medium from a teaching point of view. Of course, such speculation is valuable in itself, since it can only lead to more thought being given to the ways in which various ideas are taught, and this should at least result in more effective teaching - if not more effective learning.

It is only relatively recently that educationalists have begun to seriously consider the question of multi-media teaching and it is within this context that most of the discussion on the suitability of various media has taken place. For example, D. OLSUN and J. BRUNER (1974) suggest that although content can be presented through any medium, some mental skills (which they associate with the process of assimilating and using knowledge, although these are only vaguely defined) are better developed through one medium than another. Research at the Open University, which is probably the largest multi-media teaching institution in the world - using print, TV, audio-tape, face-to-face tuition etc., supports this hypothesis in that A.W. BATES and M. GALLAGHER (1977) have shown that, where the same content is presented through several media, some media seem to be better than others in developing certain skills in using or elaborating on that content.

It might therefore be reasonable to suppose that each medium has certain qualities that make it suitable for use in various teaching/learning situations. For example, consider the concept of differentiation which is a central topic in the study of the calculus. Students need to become familiar with the terminology, they need to be able to manipulate algebraic expressions by applying various rules, and, ideally, they need to understand what is meant by the concept. Familiarity with the terminology is probably best achieved by hearing someone talk about it and this suggests that face-to-face tuition is required. Learning to manipulate could be achieved by a variety of media: some face-to-face tuition might again be appropiate but perhaps the most suitable medium here is print since this can provide worked examples and exercises for students to work through at their own pace. Reaching an understanding of what is meant by differentiation is a more complicated process and indeed is often never properly achieved. A variety of media is probably needed here, for example, print and face-to-face tuition - although these often are only able to convey a 'static' image. Moving pictures could help students to understand better what is meant by (say) a limit and this suggests that television might also have a part to play.

Ideally then, as this example demonstrates, knowledge and experience should be presented through a number of alternative media since this can only increase the amount of learning that takes place, and perhaps this in itself provides the best answer to the question

Why use television?

However, the discussion above also suggests that in order for television to be suitable as a teaching/learning medium its use must be carefully planned, and this raises the following, perhaps more important, question

When is television appropriate?

<u>(ii) When is television appropriate?</u>

For a start there are certainly many situations where television is <u>not</u> the most appropriate means of conveying knowledge yet unfortunately, as D. LEECH (1980) points out, it is often used in place of another, more useful medium. The problem is that there is very little guidance provided on selecting the most appropriate medium for a given task. What is really needed is a taxonomy of instructional media, but although there is evidence of plenty of thought in this area (for example, V.S. GERLACH and D.P. ELY (1971) in <u>Teaching and Media: a systematic approach</u>, D. LEECH (1980) in <u>What makes television the right medium?</u> and the rather empirical classification that has been adopted by the Open University) there is still no accepted authoritative work on the subject. So how should the decision be made about when to use television? Perhaps the best approach is to examine the <u>particular qualities</u> of television and hence to draw up a list of its possible uses, and it is this approach that is adopted here (the only uses to be considered being those that are relevant in mathematics).

The most obvious characteristic of television is that it is highly visual and undoubtedly this is the key to its potential use in the classroom. T.S. ALLAN (1973. p 54) stressed this when he wrote

> The great strength of television is the iconicity of its symbols. We do not insist enough on the translation of verbal generalization into experience...

Television can certainly do many things that the teacher cannot (at least not without great effort); it can show movement, it can bring the outside world into the classroom and it can present up-to-the-minute information not readily available. The Hayter Report (C.G. HAYTER (1973)) refers to all these points. It also draws attention to the more nebulous qualities of television which might affect student attitudes to learning, i.e. television can be especially exciting, entertaining, stimulating and motivating.

G. SALOMON (1979) discusses television's qualities in a more formal context. He distinguishes between the following three kinds of symbol systems employed by the various media:

digital systems where meaning is conveyed by written language, musical notation, mathematical symbols
analogic systems made up of continuous elements which can be reorganized into various meanings and forms (such as voice quality, music, dance)
iconic systems which use pictorial representation with a variety of possible visual experiences and meanings.

The great strength of television is that, unlike other media, it can combine all three symbol systems; it is therefore an extraordinarily rich medium.

Thus, the medium of television does have a number of useful qualities. Next, the ideal way to proceed would be to examine each teaching/learning situation carefully and then to identify those where television might be a suitable medium due to one or more of these qualities. But even within the restricted domain of mathematics this would take a long time, and it is likely that the conclusions would be subjective and contentious. For example, this approach was recently applied to the single topic area of hypothesis testing in statistics and it took months to even identify various teaching/learning situations, let alone reach a consensus of opinion as to which medium was most suitable in each situation. This example demonstrates that the approach is unpractical. (Perhaps it also explains why there is as yet no definitive taxonomy of instructional media!) So, rather than stumble through this rather laborious process, it is probably just as useful to consider the particular qualities of television and then to draw up a list of its possible uses within mathematics based on experience and judgement alone. The results will again be subjective and contentious, but probably no more so than a list drawn up in any other way. A.W. BATES (1981) provides the list in Table 1.1 which is probably as good as any (see p.10).

Table 1.1 The possible uses of television in mathematics

1.	to illustrate principles involving dynamic change or movement
2.	to illustrate abstract principles through the use of specially constructed models
3.	to illustrate principles involving 1, 2, 3 or n-dimensional space
4.	to demonstrate principles involving approximations
5.	to use animated, slow motion or speeded up film to demonstrate changes over time
6.	to teach advanced concepts without the students having to master the mathematical techniques by using methods in 1, 2, 3 and 4 above
7.	to demonstrate how mathematics is used in the real world to solve problems (mathematical modelling) or where visualisation of the application in its total environment is necessary to understand the way in which the principle is applied.

This list suggests that television could be a suitable medium to use in many teaching/learning situations in mathematics. Indeed, the BBC goes one step further than this in the report BROADCASTING and MATHEMATICS (1979) when they state

> It can be said with some confidence that no mathematical topic is incapable of being dealt with on the screen provided sufficient ingenuity is shown...

although this might be a little <u>too</u> optimistic!

<u>(iii) The obstacles</u>

However, although television <u>could</u> be a suitable medium to use in many teaching/learning situations in mathematics this does not necessarily mean that it is a suitable medium to use <u>in the classroom</u>. There are a number of obstacles that need to be overcome if television is to be a successful classroom resource and these are indicated below.

(a) <u>The passive nature of television</u> Because students can be so used to television as a source of entertainment they may not be able to adapt to its use in a learning situation. V.S. GERLACH and D.P. ELY (1971, p 369) recognized this limitation when they wrote

> The very familiarity of TV sometimes contributes to habits of inattentiveness and passivity. Most students do not know how to learn from TV and often reject it...

This suggests that it is not enough for the teacher just to switch on the television and sit back. Suitable activities need to be provided in order to ensure that the students are active rather than passive viewers (see (b) and (d) below).

(b) <u>The limitations of broadcast television.</u> It is now generally recognized that live broadcast television is of restricted value. It must be watched at fixed times, and it is not possible to stop the programme or replay it. Thus it can be difficult to fit the programme into the school timetable. And since the programme proceeds at a fixed pace which is not in the control of the teacher there is no opportunity to cater for individual differences in the class - with the result that if a student does not follow one particular point he may lose the thread of the entire programme. These major drawbacks perhaps explain why television did not become a major teaching resource in the late sixties and early seventies.

However, the arrival of the video-recorder has effectively removed these obstacles and has greatly enhanced the flexibility and adaptability of the medium. The BBC in their report BROADCASTING and MATHEMATICS (1979) reported that in 1979 over 75% of secondary schools had video-recorders, and the figure is undoubtedly higher in 1983. This means that in almost all secondary schools today recorded material should be available just when required, and it can be stopped and replayed whenever necessary. Thus teachers now have complete control over the use of the resource and can decide for themselves when and how they wish to use it with a class of students.

(c) The limitations of video Of course, there are still some obstacles to using television even when the school does have a video-recorder (for example: do teachers have access to the equipment when it is needed? can they use the equipment?), and although these obstacles can be overcome by suitable planning they do need to be considered.

(d) Television on its own is often ineffective The discussion in Chapter 1(i) suggests that very little is known about the process of learning from television. However, there is good reason (based on experience at the Open University) to believe that television on its own is ineffective unless it is designed as a video-tape in the first place - with specific activities built into it. Few existing television programmes have this property, the result being that, whereas television might provide an excellent means of demonstrating a concept, the programme is soon over and all that is left is an image. For the student to gain maximum benefit from the programme it is necessary to build on that image. This suggests that, for television to even stand a chance of being effective, it needs to be integrated closely with other media - such as print, face-to-face tuition, experimental apparatus and so on.

(e) The need for adequate preparation time Furthermore, in order for television to be effective as a classroom resource its use needs to be carefully planned. Ideally the teacher should watch the programme in advance and then devise a strategy that will incorporate it (and the support materials) into his/her teaching. This requires time,a scarce commodity for the hard pressed teacher and not usually allowed for in the school timetable.

(f) The teacher should be able to use the resource The success of television as a teaching/learning medium depends to some extent on the effectiveness of the user in handling the resource. Thus teachers need to be trained in using television. J. NEWSOM (1963) first drew attention to this requirement almost 20 years ago in the context of providing a suitable education for the less able but, of course, his observation applied more generally. Yet by the mid-seventies F. GRANT (1976) maintained that there were few training institutions which had heeded this advice. R. LEFRANC (1978) also refers to this need for suitable training: he reports that teachers often do not follow the advice that is provided on how to use a programme and often just use it in isolation without its accompanying support materials. Such bad practice might not exist if suitable training was provided for teachers.

(g) The pedagogical drawbacks In the past a teacher has often chosen not to use a television programme because the approach conflicted with his/her own views on what to teach and how to teach it. This attitude was perhaps most marked in Britain where teachers had a certain amount of freedom to make their own decisions on content and approach because of the variety of syllabuses that were available and because of the vague way in which these syllabuses were laid out. (A report by the CENTRE OF EDUCATIONAL TECHNOLOGY at the UNIVERSITY OF SUSSEX (1968) suggests that this was not such a problem in other countries where education was more centrally controlled.) Hence, until fairly recently, television has tended to be more concerned with rather general topics for use as enrichment material rather than direct teaching - in this country at least. This in itself has been an obstacle, for the demands of examination syllabuses often mean that there is no time for enrichment material. However, shifting trends in educational thinking, along the lines of a core curriculum and a reduction in the number of examination syllabuses, should mean that there is more scope for using television as a direct teaching resource in the future. Attitudes towards the activities of teaching and learning are also changing and this too suggests that television could become a much more acceptable resource in the classroom - from a pedagogical point of view.

Probably because of all these obstacles television has not made as much impact in the classroom as it might have done (F. GRANT (1976)). Hopefully though, now that people are more aware of the obstacles, programme makers will be in a better position to plan the programmes and their accompanying support materials and users will be more willing to prepare better for the use of such materials. So although television may not have been particularly successful in the past this does not necessarily mean to say that it will not be a useful classroom resource in the future.

(iv) Television and mathematics

The history of television in mathematics reflects the points made above. Table 1.2 (see p.15) indicates all the mathematics series that have been produced by the BBC in the past 25 years. (The reason for concentrating on the provision of programmes by the BBC is that this has been by far the most active network in schools broadcasting.)

Table 1.2 Mathematics Series broadcast by the BBC since 1958

Date	Series	Target Audience	No. of Programmes	Weekly/ Fortnightly	Ancillary Material
1958	Mathematics	11-12	5	W	Teacher's notes (TN)
1961	Mathematics and Life	11-12	10	W	TN
1962-65	Pure Mathematics	6th Form	96	1/2 W	Student's notes (SN)
1963-67	Middle School Mathematics	13-14	28	W	TN
1964-65	Modern Mathematics	6th Form	16	1/2 W	SN
1965-70	Mathematics (Radio)	11-12	8	W	Pupil's pamphlet, TN
1965-70	Mathematics in Action	6th Form	28	W	SN
1965-66 / 1965-66	Primary School Mathematics	9-11	20	W	TN
1967	Mathematics around you	7-9 (backward)	8	W	TN
1967-72	Maths Today - Year 1	11-12	14	F	TN, Workcards (W)
1968-73	Maths Today - Year 2	12-13	14	F	8 mm film loops
1969-77	Maths Workshop - Stage 1	9-10	14	F	TN with W
1970-77	Maths Workshop - Stage 2	10-11	14	F	TN with W
(1971: Open University broadcasts started on radio and television)					
1970-71	People and Computers	6th Form	6	W	SN
1972-78	Countdown	14-16 (slow learners)	28	F	TN, Pupil's workbook
1975-80	Mathshow	11-13	14	F	TN with W
1976-	Keep up with the times	7-9	8	W	TN
1977-81	It's Maths!	9-10	14	F	TN with W
1977	Focus - on Numeracy	FE Students	10	W	Booklet
1978-82	Everyday Maths	14-16 (slow learners)	14	F	TN with W
1978-79	Prospect (Radio)	16-18	5	W	TN
1979	It Figures	(C.E.)	10	W	Book
1979-	Maths Topics	13-16	10	W	TN
1982-	Maths Help - Part 1	(C.E) O Level	12	W	Book
1983-	Maths Help - Part 2	(C.E) O Level	12	W	Book

Source: Broadcasting and Mathematics 1979

Note The table does not include the recent series on micro computers.

Table 1.2 demonstrates that most of the programmes that have been made up till now have been aimed at primary/middle schools or have been concerned with fairly basic mathematics. This is perhaps a direct result of obstacle (g)(p.13), and simply demonstrates the fears of programme-makers to provide programmes that were directly related to any particular examination syllabus. Or it could reflect obstacle (b)(p.11), since timetabling tends to be more of a problem in secondary schools - particularly for examination orientated classes. Whatever the reasons, there have been very few television series aimed at CSE, O Level and A Level mathematics, and it is only recently, with Maths Topics and Maths Help, that programmes have been concerned with topics that appear specifically on various examination syllabuses.

The table also demonstrates that, although the BBC have always recognized the need for some form of guidance on how to use the programmes (ancillary material being provided with all series), the form of this ancillary material has changed over the years. This perhaps reflects the change in attitude towards obstacles (a)(p.10), and (d)(p.12) in that the materials have gradually involved more and more student activity.

The BBC's own attitude towards television as a resource for teaching mathematics in the classroom has certainly evolved with experience as might have been expected. In the early days there was an implicit assumption that television could be used on its own as an alternative means of teaching mathematics and that it might even replace the teacher - at least in understaffed schools (J. CAIN (1965)). Some years later it was recognized that television could only be effective in the classroom if suitable activities were provided for students and if these were carried out under the guidance of a teacher (D. ROSEVEARE (1972)). Also, over the last 10 years the limitations of live broadcast TV (obstacle (b)(p.11)) have been widely recognized with the result that all schools programme are now transmitted on the understanding that they will be recorded, for use as and when required. Thus, the programme-makers have become more and more aware of the need to overcome all the obstacles listed in Chapter 1(iii), and this must surely have resulted in an improvement in the effectiveness of the programmes as a teaching resource.

As for the 'success' of the various uses of television in the mathematics classroom, it is extremely difficult to comment. In the end - as for any other resource - this can only be judged by the teachers and students who have actually used the materials. Suffice to say perhaps that the BBC themselves operate an extensive feedback system designed to improve the effectiveness of their TV provision to schools, and it is reasonable to suppose that the 'success' of television as a classroom resource has improved over the years and will continue to do so in the future.

But what of the future? Will TV be used more or less? How will it be used? Perhaps the best clue to the possible answers to these questions lies in the way that up till now the use of television in the classroom has tended to reflect current thinking towards the teaching of mathematics in general. For example, the series 'Modern Mathematics' was introduced just when this was a subject of major discussion; today there is no real distinction between 'modern' and 'traditional' mathematics and in recent series, such as Maths Topics and Maths Help, a mixture of topics and approaches is included. Also, in the mid-seventies there was particular emphasis in the Great Debate on the need for basic mathematics and this attitude was reflected in series such as 'Countdown', 'Focus - on Numeracy' and 'Everyday Maths'. And in recent years there have been numerous programmes on computing, and particularly, on micro-computers; such series reflect the huge increase in interest in this area.

The following extract from the Cockroft Report (W.H. COCKROFT (1982) p 71) provides some idea of the current attitude towards mathematics teaching:

Mathematics teaching at all levels should include opportunities for

- exposition by the teacher
- discussion between teacher and pupils and between pupils themselves
- appropriate practical work
- consolidation and practice of fundamental skills and routines
- problem solving, including the application of mathematics to everyday situations
- investigational work.

In the past it was certainly recognized that all these activities were beneficial to younger children and for the less able, but the general feeling seemed to be that perhaps not all of them were necessary for examination classes. The rather limited use of television in secondary mathematics has perhaps simply reflected this attitude, for the medium of television is appropriate in many of the types of activity listed here. However, if in the future all these activities are to be encouraged at all levels of mathematics teaching, then from now on there might be more scope (and enthusiasm) for using television in all mathematics classrooms for a greater variety of purposes.

(v) The project

The above discussion on the history of television in mathematics made no specific mention of the programmes that have been produced by the Open University. But this multi-media teaching institution, by its very nature, has probably made the most significant contribution towards increasing the effectiveness of television as a teaching/learning medium in the last ten years. Why? Because all the courses that are produced by the University are designed from the outset to be multi-media, and the extensive debate and discussion on what constitutes the most appropriate medium for any given teaching/learning situation has resulted in carefully selected television programmes designed to reflect the unique qualities of television. The mathematics courses that have been produced in recent years are no exception to this and the course teams responsible feel that they have been especially successful in making effective use of television. This view is confirmed by student feedback.

The reasons for the particular success of television in recent mathematics courses at the Open University would seem to be as follows:

- programmes are only used for the purposes listed in Table 1.1
- all the programmes are tightly integrated with other materials and these are designed so that the student is unlikely to gain maximum benefit <u>without</u> watching the television component
- television programmes are transmitted at regular intervals and this helps to pace the students through the course.

The first two points suggest that the Mathematics Faculty has effectively overcome obstacles (a) and (d) on pp.10-12. And the last point partly explains why the limitations of live broadcast TV do not apply here (obstacle (b) on p.11), although Open Unversity students <u>are</u> highly motivated and will put up with a considerable degree of inconvenience. The remaining obstacles ((e) (f) and (g) on pp.12-13) are irrelevant since the programmes are not shown in a classroom situation and there is no need for a teacher.

Now the interesting thing is that some of these Open University television programmes may also be relevant to students of A level mathematics, and there is therefore a possibility that they could be used in the mathematics classroom. This would go some way towards filling the gap in the provision of suitable schools' programmes (see Table 1.2 on p.15). It could also go some way towards meeting the specific need for more stimulating teaching materials in sixth form mathematics, a need identified by the Cockroft

Report as the following extract indicates (W.H. COCKROFT (1982) pp. 170-171).

> It is also possible for mathematics at A level to be presented as a very technical and somewhat arid subject with little relation to other school subjects except, perhaps, physics or to the activity of the world at large. It is therefore important that teachers should seek to counteract this impression by making use of opportunities which arise to emphasize the broader role of mathematics. The very varied applications of mathematics should be stressed and illustrations of these applications drawn from as wide a range as possible.
>
> The increasing use of 'mathematical modelling' in, for example, the social sciences provides many possibilities for an enterprising teacher and many more traditional applications are to be found in the physical sciences. Reference to the historical background of some of the topics which are being studied can both help to explain their importance and also add interest and depth to the A level course. A micro-computer can provide a stimulus to adventurous thinking, very often initiated by the students themselves; the investigative work which can arise in this way should be encouraged. Occasional discussion of some of the assumptions underlying mathematics and of the nature of the knowledge it provides is necessary if students are to be enabled to talk about mathematics in ways which others will understand. <u>There is at present a lack of teaching materials which assist sixth-form teachers to work in these ways and more are required.</u>

This project investigates the potential classroom use of television programmes from the Open University mathematics foundation course. As has already been indicated, there is reasonable evidence to suggest that the Mathematics Faculty has already successfully overcome some of the obstacles in using television (namely obstacles (a) and (d) on pp.10-12), although some modification will be needed if only parts of the materials are to be used. Also, since most schools have video-recorders, it will be assumed that obstacle (b)(p.11) is no longer a problem. However, it remains to be seen whether

- the programmes are relevant to sixth form mathematics and acceptable to teachers (obstacle (g) on p.13)
- the programmes can be used effectively in the classroom (obstacles (c), (e) and (f) on pp.12-13).

In Chapter 2 the relevance of the programmes to various A level syllabuses is considered and the question of teachers' attitudes is first tackled. The investigation continues in Chapters 3, 4 and 5 with a description of a pilot study which looks at the suitability of the programmes for use in the classroom.

2. Sixth Form Mathematics: the Possibilities of using TV Programmes from the Open University Foundation Course in Mathematics M101

(i) M101: an outline

The Open University mathematics foundation course is an introductory course in mathematics lying somewhere between O level, A level and the first year of an undergraduate course. The entry behaviour, i.e. the level of mathematics which students are expected to have attained before starting the course, is well defined and corresponds roughly to the mathematics covered in the first four years of secondary school. The age of Open University students can range from 20 to 80, with a large proportion in their thirties or forties and it is therefore assumed that students have only encountered a traditional approach to mathematics before embarking on the course.

M101 is divided into six blocks.

Block I builds upon the assumed entry behaviour and introduces some basic ideas such as iteration, transformations, trigonometry and functions. It is recognized that students tend to be weak in algebraic manipulation and one of the main aims of the early material is to provide appropriate experience in this area.

Block II extends the work on functions to cover logarithms, exponential functions and the idea of sketching a graph with the help of transformations. The iteration thread is picked up again and in the discussion of whether an iteration process converges or diverges the concept of a 'scale factor' emerges. Other topics covered in this block are inequalities, an intuitive look at limits and line-fitting using the least squares method.

In Block III the 'scale factor' of Block II is seen to be the derivative and this leads to a study of the calculus. The connection between differentiation and integration is explored and various techniques of differentiation and integration are introduced. Some applications are included such as areas, speed and acceleration, Taylor polynomials and the Newton-Raphson method of solving equations.

Block IV provides a change of direction. Here the theme is algebraic and centres on the properties and applications of matrices. Topics include probability, equivalence relations and the connection between geometric transformations and their matrix representation.

The title of Block V is 'Mathematical Modelling' and this block provides an introduction to applied mathematics. Students are encouraged to think about the application of mathematics to some real life problems. Stock control at the National Coal Board provides an introduction to the steps involved in attacking a problem using elementary mathematics. Then the administration of drugs and the problems posed by pollution in the River Thames provide the setting for a study of how to set up differential equations. Other topics covered are statics and statistical sampling.

Block VI introduces students to some aspects of pure mathematics and the emphasis is on mathematical structure. The topics covered include complex numbers, groups, proof by mathematical induction, axiom systems and isomorphisms and homomorphisms.

In addition to the six blocks there is a 'floating unit' in which students are encouraged to reflect on the approach that they adopt when tackling mathematical problems.

Throughout the course students are encouraged to be active rather than passive learners. Many ideas are introduced via the use of an electronic calculator and exercises are structured in order to promote understanding and to make students think about what they are doing. The approach is multi-media: most of the course is contained within printed texts but television and audio-tape are also used. Audio-tapes are designed to be used alongside the printed texts and 'talk' students through new ideas or techniques. The television programmes, transmitted weekly, help to pace students through the course and provide another way of looking at important concepts; computer animations and models are used quite extensively and it is felt that these help to stimulate students' understanding.

The course has been extremely successful in that it is fairly popular with students (at least it is as popular as one might hope a mathematics course to be!) and has a pass rate which is comparable to that of the other foundation course at the Open University.

(ii) <u>The potential of the M101 programmes as a classroom resource</u>

It was recognized at the outset that M101 might have something new to offer to teachers of A level mathematics (or equivalent). The content of the foundation course was comparable to that of many A level syllabuses and the materials involved a variety of media: print, audio-tape and television. More important, it was felt that the approach adopted towards the development of key ideas was fresh and stimulating. Initially there was some hope of selling the complete package to schools and colleges on a block basis. However, the sale of Open University materials is regarded as a profit making exercise and the cost of one block of M101 (including the associated television programmes) was set at about £1000. Excluding the television programmes, the cost was more modest at about £10 per block, but the teaching texts and audio tapes alone were unlikely to be of interest to conventional teaching institutions since they were designed for self study. Undoubtedly, if there <u>was</u> any potential for using M101 in the classroom it was centred on the use of the television programmes. Used on their own, these were less likely to interfere with the teaching strategy adopted by the teacher, and they had the definite attraction of offering something which the teacher might not be able to achieve using a blackboard.

Thus it was decided to concentrate only on making this component of the course available to teachers, provided it could be made a viable proposition. There was, of course, one major obstacle - that of making the programmes available to teachers. As implied above, the cost of the programmes was prohibitive. And there were licensing problems involved if teachers were to be encouraged to record the programmes off-air. (Unfortunately, Open University television programmes do not fall under the auspices of the Schools Broadcasting Council and a licence fee must be paid if they are recorded.) However, there was no point in tackling this problem before investigating further

- how the TV programmes fitted in to various A level syllabuses
- whether they would be well received by teachers
- how much support material would be needed.

Comparing the topics covered in the TV programmes with some commonly used A level syllabuses provides an initial indication of the relevance of the Open University materials for use in sixth form teaching. Table 2.1 (pp.26-32) demonstrates that there does seem to be a significant overlap - especially with those syllabuses where a more 'modern' approach is adopted. The left hand column of the table outlines the topics covered in the 33 M101 television programmes. A cross in the table indicates that a topic is specifically mentioned in an A level syllabus. A cross in parentheses indicates that an implicit reference is made to a topic; perhaps the topic is covered in the preceding O level syllabus or it may be assumed that the topic is included under a broader heading.

The overlap is most marked in the early programmes corresponding to Blocks I to III. This is not surprising since the calculus and the work leading up to it plays an important role in any introductory mathematics course. The material in Block IV is less commonly covered at A level but programmes TV 18, TV 19 and TV 20 could certainly provide enhancement material. The programmes in Block V (TV 21 - TV 25) provide what was in 1978 a rather new

Ctd. on p.33

Table 2.1 OVERLAP BETWEEN M101 TV AND SOME COMMONLY USED A LEVEL SYLLABUSES

CONTENT OF M101 TELEVISION PROGRAMMES	CONTENT OF A LEVEL SYLLABUSES L (B)	L (B) F	O	O F	JMB (A)	JMB (A) F	JMB (B)	JMB (B) F	AEB (P/A)	AEB (AL)	C (A)	C (A) F	C (B)	C (B) F
BLOCK I														
1. Symbols and Equations														
Tackling a practical problem	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)
Solving quadratics	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Iteration processes	X	X	X	X			X	X		(X)			X	X
Numerical method for finding $\sqrt{2}$	X	X	(X)	(X)			X	X		X			X	X
2. The Binomial Theorem														
Expanding brackets	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)
The Binomial Theorem			X	X	X	X	X	X	(X)	X	X	X	X	X
The notation ${}^nC_r$	(X)	(X)	X	X							X	X	(X)	(X)
3. Trigonometric formulas														
Rotations	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	(X)
Sin $(\alpha+\beta)$	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Cos $(\alpha+\beta)$	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Translations	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	(X)
The meaning of tan θ	X	X	X	X			X	X	X	X	X	X	X	X
4. Inverse Functions														
Domain, codomain, rule	X	X	X	X			X	X		(X)			X	X
One-one functions	X	X	(X)	(X)			X	X		(X)			X	X
Inverse functions	X	X	X	X			X	X		X			X	X
Arccos, arctan	X	X	X	X			X	X	X	X			X	X
5. Tackling an Assignment Question														
Revision of techniques in Block I														
How to tackle a solution														
BLOCK II														
6. Rational Numbers and $\sqrt{2}$														
Computational view of numbers							(X)	X		(X)				
Geometrical view of numbers							(X)	X		(X)				
Idea of proof by contradiction		X	X	X			X	X	(X)	(X)		X	X	X
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: an exploration	X	X	(X)	(X)			X	X	X	X	(X)	(X)	X	X
Geometrical constructions	(X)	X	(X)	(X)	(X)	(X)			(X)		(X)	(X)		
Iteration	X	X	X	X			X	X		X			X	X

Ctd.

CONTENT OF A LEVEL SYLLABUSES

CONTENT OF M101 TELEVISION PROGRAMMES	L (B)	L (B) F	O	O F	JMB (A)	JMB (A) F	JMB (B)	JMB (B) F	AEB (P/A)	AEB (AL)	C (A)	C (A) F	C (B)	C (B) F
7. <u>Functions and Graphs</u>														
Translations	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	(X)
Scalings	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	(X)
Reflections	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	(X)
The effects of these transformations on graphs of functions	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	(X)
8. <u>Inequalities</u>														
Increasing and decreasing functions														
Inequalities	X	X	X	X			X	X		X	X	X	X	X
Exponential/logarithm functions	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Finding $\log_e (1.5)$		X	(X)	(X)	X	X	X	X	(X)	X			X	X
9. <u>Iteration and Convergence</u>														
Scale factors													(X)	(X)
Iteration	X	X	X	X			X	X		X			X	X
Convergence and divergence of iterative processes.	X	X					X	X		X				X
10. <u>$x \mapsto \frac{1}{x}$: An Area for Revision</u>														
Area under $x \mapsto \frac{1}{x}$	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)
Rules of logarithms	X	X	X	X	X	X	X	X	X	X	X	X	X	X
$\log_e x$	X	X	X	X	X	X	X	X	X	X	X	X	X	X
<u>BLOCK III</u>														
11. <u>The Derived Function</u>														
Scale factors														
Definition of the derivative	(X)	(X)	(X)	(X)	(X)	(X)	X	X	(X)	(X)	X	X	X	X
Differentiation from First principles	(X)	(X)	(X)	(X)	(X)	(X)	X	X	(X)	(X)	X	X	X	X
Tangents: a geometrical interpretation of derivatives	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Ctd.

CONTENT OF A LEVEL SYLLABUSES

CONTENT OF M101 TELEVISION PROGRAMMES	L (B)	L (B) F	O	O F	JMB (A)	JMB (A) F	JMB (B)	JMB (B) F	AEB (P/A)	AEB (AL)	C (A)	C (A) F	C (B)	C (B) F
12. <u>The Behaviour of Functions</u>														
Techniques of differentiation	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Maxima, minima	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Point of inflection	X	X	X	X	X	X	X	X	(X)	X	X	X	X	X
Sketching polynomials using calculus techniques	X	X	X	X	X	X	X	X	X	X	X	X	X	X
13. <u>The fundamental Theorem of Calculus</u>														
Integral as a limit	X	X	(X)	(X)	X	X	X	X	(X)	X	(X)	(X)	X	X
Integral as an area	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Idea of approximate sum	X	X	(X)	(X)	X	X	X	X	X	X	(X)	(X)	X	X
Integration as inverse of differentiation	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Techniques of integration	X	X	X	X	X	X	X	X	X	X	X	X	X	X
14. <u>Taylor Polynomials</u>														
Taylor polynomials		(X)	X	X				X		X		X	X	X
Taylor (Maclaurin) series		X	X	X				X		X		X	X	X
Expansion of sin x, cos x, $(1+x)^r$		X	(X)	(X)		X		X	X	X		X	X	X
Idea of convergence		X						X		X				X
15. <u>Why e?</u>														
Idea of a^x and $\log_a x$	X	X	X	X	X	X	X	X	X	X	(X)	X	X	X
$\frac{d}{dx}(e^x)$ and $\frac{d}{dx}(\log_e x)$	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Expansion of e^x and $\log_e (1 + x)$		X	(X)	(X)		X		X	X	X		X		X
<u>BLOCK IV</u>														
16. <u>Networks and Matrices</u>														
Using networks														
Matrix representation of networks														
Matrix multiplication	(X)	(X)	(X)	(X)			(X)	(X)	(X)	(X)			X	X

Ctd.

CONTENT OF A LEVEL SYLLABUSES

CONTENT OF M101 TELEVISION PROGRAMMES	L (B)	L (B) F	O	O F	JMB (A)	JMB (A) F	JMB (B)	JMB (B) F	AEB (P/A)	AEB (AL)	C (A)	C (A) F	C (B)	C (B) F
17. <u>An Introduction to Relations</u>														
The idea of a relation			X	X				X					X	X
The use of dots and arrows to depict a relation			(X)	(X)				(X)						(X)
Application to a problem involving linked and unlinked rings														
18. <u>Conics</u>														
A Greek approach to the conic sections														
The geometry of conics		X	X	X	X	X						X		
Focus-distance property		X	(X)	(X)	X	X						X		
Focus-directix property		X	(X)	(X)	X	X						X		
19. <u>Transformations and Matrices</u>														
Transformations in $\mathbb{R}^2$	(X)	X	X	X			(X)	X		X			X	X
Matrix representation	(X)	X	X	X			(X)	X		X			X	X
Image grids of matrix transformations	(X)	X	X	X			(X)	X		X			X	X
20. <u>Dilation in Skew Directions</u>														
A scaling with two invariant lines (also mentioned in TV 19)				X									(X)	X
Matrix representation by QDQ^{-1} where D is a diagonal matrix		X		X						X				(X)
Implicit idea of eigenvalues, eigenvectors		X	X	X						X				
Powers of matrices		X								X				
<u>BLOCK V</u>														
21. <u>Modelling Stock Control</u>														
Introducing mathematical modelling														
Making assumptions									(X)	(X)				
Formulating and solving mathematical problems	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)
Interpreting the solution									(X)	(X)				

Ctd.

CONTENT OF A LEVEL SYLLABUSES

CONTENT OF M101 TELEVISION PROGRAMMES	L (B)	L (B) F	O	O F	JMB (A)	JMB (A) F	JMB (B)	JMB (B) F	AEB (P/A)	AEB (AL)	C (A)	C (A) F	C (B)	C (B) F
22. <u>Modelling Drug Therapy</u>														
Case study concerning the use of the drug theophyline														
Setting up a simple differential equation	X	X	X	X	(X)	(X)	(X)	(X)		X	X	X	X	X
Solution by separation of variables	X	X	X	X	X	X	X	X	X	X	X	X	X	X
23. <u>Modelling Surveys</u>														
Binomial distribution		(X)	X	X			X	X		X	X	X	X	X
Idea of estimation based on a sample			X	X			X	X		X	X	X	X	X
24. <u>Modelling Cranes</u>														
Designing cranes using modelling methods														
Parallelogram of forces	X	X	X	X	(X)	(X)			X	X	X	X	X	X
Forces in equilibrium	X	X	X	X	(X)	(X)			X	X	X	X	X	X
25. <u>Modelling Pollution</u>														
The development of a mathematical model applied to pollution in the River Thames														
Setting up differential equations	X	X	X	X	(X)	(X)	(X)	(X)		X	X	X	X	X
Solution by separation of variables	X	X	X	X	X	X	X	X	X	X	X	X	X	X
<u>BLOCK VI</u>														
26. <u>Complex Numbers</u>														
Notion of $i = \sqrt{-1}$	X	X	X	X	X	X	X	X		X	X	X	X	X
Geometric treatment of the complex roots of unity	X	X	X	X	X	X	X	X		X		X	X	X
Modulus, argument	X	X	X	X	X	X	X	X		X	X	X	X	X
27. <u>Group Theory</u>														
Symmetries of a rectangle		X	(X)	(X)			(X)	(X)		(X)			(X)	X
Symmetries of an equilateral triangle		X	(X)	(X)			(X)	(X)		(X)			(X)	X
Combination table of symmetries		X	X	X			X	X		(X)			(X)	X
Groups axioms.		X	X	X			X	X		X			X	X

Ctd.

CONTENT OF A LEVEL SYLLABUSES

CONTENT OF M101 TELEVISION PROGRAMMES	L (B)	L (B) F	O	O F	JMB (A)	JMB (A) F	JMB (B)	JMB (B) F	AEB (P/A)	AEB (AL)	C (A)	C (A) F	C (B)	C (B) F
28. <u>A Non-Euclidean Universe</u>														
Axioms for Euclidean geometry														
Consistency and independence in axiom systems														
A non-Euclidean axiom system														
29. <u>Mister Klein looks at geometry</u>														
A catalogue of geometries:														
Projections														
Affinities														
Similarities	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)		(X)	(X)	(X)	(X)	X
Isometries			(X)	(X)			(X)	(X)					(X)	X
Symmetries	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	X
Rotations	(X)	(X)	(X)	(X)			(X)	(X)		(X)			(X)	X
30. <u>The Fundamental Theorem of Algebra</u>														
Complex functions $f:\mathbb{C}\rightarrow\mathbb{C}$		X												X
Winding numbers														
Application to the fundamental theorem of algebra														
5(a). <u>Mathematical Induction</u>														
Proof by induction		X	X	X			X	X	(X)	(X)		X	X	X
31. <u>Problem Solving</u>														
Reflecting on the processes involved in problem solving	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)	(X)
32. <u>Catastrophe Theory</u>														
An introduction to catastrophe theory via some case studies														

Ctd.

<u>Notes</u> (i) The following abbreviations are used in the table

L	(B)		London (Syllabus B)
L	(B)	F	London (Syllabus B), Further Mathematics
O			Oxford Delegacy of Local Examinations
O		F	Oxford Delegacy of Local Examinations, Further Mathematics
JMB	(A)		Joint Matriculation Board (Syllabus A)
JMB	(A)	F	Joint Matriculation Board (Syllabus A), Further Mathematics
JMB	(B)		Joint Matriculation Board (Syllabus B)
JMB	(B)	F	Joint Matriculation Board (Syllabus B), Further Mathematics
AEB	(P/A)		Associated Examining Board (Pure and Applied)
AEB	(AL)		Associated Examining Board (Alternative Syllabus)
C	(A)		Cambridge Local Examinations Syndicate (Syllabus A)
C	(A)	F	Cambridge Local Examinations Syndicate (Syllabus A), Further Mathematics
C	(B)		Cambridge Local Examinations Syndicate (Syllabus B)
C	(B)	F	Cambridge Local Examinations Syndicate (Syllabus B), Further Mathematics

(ii) The information in the table is taken from 1983 syllabuses

(iii) In cases where a syllabus contains a number of options the contents of <u>all</u> options have been considered.

approach to mathematics. However, now that the role of mathematical modelling is more widely recognized, these programmes are particularly relevant to today's sixth form students even though they may not be particularly mentioned on any syllabus. Similarly, although the programmes in Block VI do not necessarily fit into the syllabuses particularly well, they nevertheless provide some stimulating material. Each of these programmes is designed as an introduction to some topic in pure mathematics and so requires little preparation on the part of the student. These programmes can also lead on to more specialized work and so are particularly suitable for more able students. Finally, the last two programmes, TV 31 and TV 32, are designed to enable students to reflect on the nature and purpose of mathematics: TV 31 is concerned with the processes involved in problem solving, TV 32 takes a look at the place of mathematics in society. These programmes should therefore be suitable for all students of mathematics.

Thus the content of the programmes does appear to be relevant but it is difficult to draw any firm conclusions from the table. The main problem is that examination syllabuses tend to list topics vaguely and the level of understanding required is not made very clear. So, although the content of the programmes appears to overlap, the approach may not be suitable. It is possible to get a better idea of the level of understanding required at A level by studying the actual examination papers, but even that would not be particularly conclusive because there still remains plenty of scope for an individual teacher to decide at what level he or she wants to tackle a topic.

But perhaps this is exactly where the attraction of the M101 TV programmes might lie for they can provide treatment of topics at various levels depending upon the guidance given by the teacher. They can provide an introduction to an idea without becoming bogged down in mathematical symbolism and they can also provide an alternative approach to a topic that has already been encountered, making them suitable for revision purposes or for the stimulation of further ideas. A good example of this flexibility may be found in TV 10 '$x \mapsto \frac{1}{x}$: an Area for Revision'. This programme could be used to introduce the idea of an area under a curve. It could also be used to remind students of the properties of logarithms or of the nature of the graph of $x \mapsto \frac{1}{x}$, as well as revising integration, for the programme draws all these ideas together. It also demonstrates how the function $x \mapsto \frac{1}{x}$ may be manipulated using scalings and translations, and this could lead to work in other areas.

In addition to this flexibility in approach the TV programmes certainly help to liven what is often thought of as a rather 'dry' subject. Extensive use is made of graphics, models and computer animations, and all programmes are made in colour. However, they do tend to be unconventional and they are fully integrated with the accompanying teaching texts (often specific references to the texts are included in the programmes). This means that, although the programmes appear to be relevant and certainly have some attractive features, it is impossible to assess their likely potential in the classroom in the absence of any feedback from teachers themselves.

Thus in May 1978 teachers were invited to view some television programmes at a one-day conference at Chelsea College, London (R.M. REES. (1979)). Members of the course team outlined the philosophy and contents of M101 and a maths advisor from the London area described how the TV material related to core areas of various A level syllabuses. Participants then had the opportunity to watch some of the programmes and to discuss their reactions in small groups. The response suggested that many of the programmes were eminently suitable for A level work. Some participants also pointed out that the materials had potential as a basis for the discussion of teaching methods in teachers' centres and teacher training courses. Particular reference was made to the liveliness of the programmes. One of the conference speakers remarked that some of the programmes reminded him of his favourite mathematics lessons. Another speaker referred to the fact that the Open University materials were aimed at generating genuine pleasure and thrill in working with mathematics, an experience which he considered to be rather rare these days. And the author of the conference report, Ruth Rees, stated that she was particularly impressed by the way that the medium of television helped to make mathematics 'come alive'. Encouragement indeed!

But the conference also highlighed the fact that the programmes would not be effective unless they were supported by appropriate back-up materials. Synopses of the programmes were required, along with a clear description of the background that students would need before watching the programme and some suggestions for follow-up work. This was not unexpected for the programmes were never designed to stand on their own. The course team recognized that effective learning cannot be brought about by television alone; television may provide the stimulation but students can only gain understanding by working through the ideas themselves at their own pace. That is why the television programmes were closely integrated with the teaching texts in the first place.

The next step was therefore to create a suitable package of support materials based on the teaching texts. However, because of the problems involved in ensuring that the project was financially viable and because of pressure of work on other projects, there was no immediate follow up.

Then in 1980 John Richmond, Executive Producer in Mathematics at the BBC Open University Production Centre, prepared some video-tapes on various aspects of the calculus using edited excerpts from the M101 TV programmes - along with suitable back-up materials. These were demonstrated at a teachers' meeting at Brunel University. Again the response was favourable and several teachers borrowed the tapes to use in their own classrooms - with reported success. But again this project was not followed up, partly because of pressure of work and partly because it was at this stage that it was necessary to consider the following practical aspects of making the M101 television programmes a viable proposition for use in the classroom.

- The cost of making either the complete programmes or edited tapes available to teachers was still the biggest obstacle to be overcome
- It was necessary to consider how the materials might be made available to schools and colleges throughout the country
- If all the programmes were to be edited this would require some manpower input from the BBC which was unlikely to be forthcoming at that time because of the requirements of other courses.

It was at this stage of development that the project was adopted as the basis for this dissertation and instead of continuing John Richmond's investigation into the potential of edited tapes it was decided to concentrate once again on the use of complete programmes as broadcast. The reasons were as follows.

- It avoided the necessity of obtaining manpower input from the BBC thus reducing the likely cost of the operation.
- Tenuous discussion with the Schools Broadcasting Council in Autumn 1981 suggested that it might be possible to overcome the licensing problems involved in recording off-air. This meant that there was a possibility that teachers might be able to record the programmes (for later use) as they were transmitted by the BBC in the University year - without incurring any costs. There was also some hope that the University might be able to use some additional off-peak slots to transmit the M101 programmes again during school hours.
- The edited tapes produced by John Richmond were very tightly integrated with the support material. Teachers were directed when to show an excerpt from the tape and when to work through specific exercises. This meant that there was very little scope for working with the material in different ways and at different levels. It was felt that offering the complete programmes would lead to more flexibility, especially if

decisions such as when to use the programme, how to view it and when to work through accompanying exercises were left to the individual teacher.

(iii) A preliminary workshop

The next step was to investigate further the feasibility of working with complete programmes. Draft support materials were therefore prepared for TV 10 '$x \mapsto \frac{1}{x}$. An Area for Revision' using Block II Unit 5 of M101. These materials are reproduced in Appendix 1.

Teachers were then invited to a workshop in December 1981 in order to

- view the television programme and discuss its suitability for use in the classroom
- comment upon the appropriateness of the draft support materials and make suggestions about the format.
- view the M101 course materials and discuss reactions to the project as a whole.

It was also hoped that, by attending the workshop, teachers might become interested in the project and would offer to assist with the developmental testing of the materials. Unfortunately the date chosen for the workshop coincided with one of the worst blizzards of the winter and several participants phoned to say that they could not attend because of blocked roads. In spite of this, 12 teachers from Milton Keynes and Leighton Buzzard did manage to get there. All were involved in teaching A level mathematics.

The workshop began with a discussion about TV 10 and its back-up materials. The enthusiasm shown by all teachers present for using both the programme and the support materials was particularly encouraging. There was a surprising range of suggestions as to when and how this material might be used. One teacher said that he would use it with his fifth form in order to convey the idea of integration at an intuitive level - prior to A level study, another commented on the suitability of the material for his brightest students - giving them 'something to think about'. Some teachers thought that they would use the entire programme as it stood, others felt that they would only show excerpts and not necessarily all on the same occasion. This seemed to confirm that this programme at least had the desired flexibility.

Teachers were then asked about the format of the draft support materials and the following points emerged.

- The summary of the television programme should be given first and this should include an indication of the times of various sequences.
- The exercises should be laid out in such a way as to require the minimum amount of copying.

Another issue raised at this stage was the use of Open Unviersity jargon and terminology. (This was one of the reasons why John Richmond prepared edited tapes: the tapes cut out all references to other M101 materials.) However, the teachers felt that this would not cause any problems because such references were unlikely to interrupt the flow of the programme.

Then followed the most agreeable part of the workshop, a lively discussion on the various ways of teaching integration. One of the problems with television is that the teacher is expected to use someone else's approach, and responses such as 'I wouldn't teach it that way' were certainly anticipated. This type of response was indeed given at the workshop but not in a negative sense, rather the reaction seemed to be 'I <u>couldn't</u> teach it that way!' The teachers drew attention to the fact that it was not always possible to show things clearly on the blackboard (in this case the idea of approximating an area using sequences of under- and over-estimates) and so they usually <u>had</u> to adopt a different approach. Ideally, they would prefer to use a variety of approaches and television would provide a valuable alternative if it were available.

Finally there was a brief discussion of the other topics covered by the M101 TV programmes. Particular interest was expressed in the programmes covering pre-calculus work and the calculus (TV 1 to TV 15), and those associated with Mathematical Modelling (TV 21 - TV 25) and Mathematical Structures ((TV 26 - TV 30).Several teachers also mentioned the need for similar materials in mechanics and statistics. (These two subjects are not covered in any depth in M101 but they are included in other Open University courses and there are possibilities of adapting other suitable television programmes for use in the sixth form. Indeed, another member of staff is at this time independently working on the adaption of mechanics programmes from a second-level course.)

Thus the clear message conveyed by teachers at the workshop was

> Carry on with what you are doing. We are very
> interested and would like to use anything that
> you can provide.

It was therefore decided to go ahead with the production of draft support materials for other M101 programmes with a view to developmental testing and a pilot study was set up as described in Chapters 3, 4 and 5.

3. The Pilot Study: an Outline of the Contents

It was decided that the pilot study should concentrate only on the television programmes from Blocks I to III of M101, covering pre-calculus work and some aspects of the calculus. The following thirteen programmes were involved (see Table 2.1 on pp. 26-32):

1. Symbols and Equations
2. The Binomial Theorem
3. Trigonometric Formulas
4. Inverse Functions
6. Rational Numbers and $\sqrt{2}$
7. Functions and Graphs
8. Iteration and Convergence
10. $x \mapsto \frac{1}{x}$: an Area for Revision
11. The Derived Function
12. The Behaviour of Functions
13. The Fundamental Theorem of Calculus
14. Taylor Polynomials
15. Why e?

(Note Television programmes 5 and 8 were omitted from the pilot study: TV5 is primarily concerned with the assessment procedure in M101, TV8 contains a number of gimmicks which meant that it was difficult to prepare suitable back-up materials.)

A draft package of support materials based on the teaching texts in Blocks I to III of M101 was prepared to accompany these television programmes, taking into account the suggestions made at the December workshop. The package was intended to:

- enable teachers to assess the content of each television programme and to decide if and when that programme (and its accompanying work) might be fitted into his/her teaching

- provide suitable exercises for students to work through before and after the programme
- help teachers to prepare for lessons involving the M101 television programmes in the shortest possible time. Unfortunately, as with all television, there is no substitute for watching the M101 programmes in order to plan for their use in the classroom, but it was hoped that the support materials would help to minimize the time required.

The package of support materials essentially comprises thirteen separate sections, one for each of the selected programmes, and, wherever possible, each section is designed to be complete within itself. Occasionally one section may be directly related to the work in an earlier section and whenever this occurs appropriate references are provided.
The introduction to the package outlines the background of the project, lists the various themes running through the programmes and provides guidance on how to use the materials.
Each section begins with a summary of the associated television programme and includes the times for the various sequences. Then follows a list of pre-requisites which students will need if they are to gain maximum benefit from watching the programme and working through the accompanying material. The rest of the section mainly consists of exercises for students to work through before and after the programme. Not all the exercises are essential but the questions marked with an asterisk should certainly be tackled before watching the television programme: such questions may be directly referred to in the programme or they may demonstrate a particular approach or a type of notation that is used which students may not have encountered before. The exercises are laid out in such a way to facilitate copying for students' own use and full solutions are provided. Finally, each section ends with some suggestions for extending the work covered by the programme. These include references to other relevant sections.

The draft support materials are reproduced in their entirety as Appendix 2.

4. Developmental Testing

(i) The aims

As outlined in Chapter 3, the pilot study involved the use of thirteen complete M101 television programmes together with supporting written material. The materials were tested by a number of schools in the period January - July 1982. Throughout this period of developmental testing the television programmes themselves were regarded as fixed (i.e. there was no intention to alter the programmes in any way). This meant that the aims of developmental testing were simply to

- determine the likely response of teachers to the use of the Open University materials in the classroom
- identify those programmes which were unsuitable for use in the classroom
- improve the written support materials.

Teachers were especially requested to make use of television programmes TV1-TV4 because for the purpose of this dissertation only Sections 1-4 of the written materials were to be revised. However, feedback was of course welcomed on all aspects of the pilot study.

(ii) The organization

The draft package of support materials was circulated to 10 local schools in January 1982. (Restricting the circulation in this way helped to keep the cost and administration of the pilot study to a minimum. However, the project did attract a certain amount of attention and copies were sent to other schools on request.)

Two copies were made of each of the thirteen programmes on VHS cassettes. The cassettes were available from the Open University on request, either by letter or telephone. Every effort was made to dispatch the cassettes promptly on receipt of the request and it was suggested that in order to facilitate this, each loan should be for a maximum period of two weeks. Details of all loans were recorded and if necessary teachers were informed when a loan was overdue. The limited availability of video-tapes was another reason for encouraging only a small number of schools to take part in the developmental testing and even then there were problems in supplying cassettes on demand (see p.44). It would certainly have been desirable to have had more copies but, as has already been mentioned, it was necessary to keep the costs of the pilot study as low as possible.

In June 1982 a questionnaire was sent to all teachers who had received the draft package of support materials. The first part of this questionnaire requested the following information:

. the A level syllabus used in the school
. which (if any) of the M101 television programmes had been used in the period up to June 1982
. which (if any) of the M101 television programmes the teacher planned to use in the future
. the reasons for not using any of the materials
. general comments on the suitability of the materials.

The second part of the questionnaire requested more detailed feedback on the suitability of the support materials for those programmes which the teacher had already used. For each section in this category the teacher was asked to comment on

. the relevance of the work to the A level syllabus
. preparation time
. the length of time students spent watching the programme and working through the accompanying material
. the suitability of the TV summary
. the appropriateness of the pre-requisites
. the suitability of the suggested pre-programme and post-programme work
. errors, omissions and suggested additions to the written materials.

The questionnaire is reproduced as Appendix 3.

Teachers were requested to reply to the questionnaire by the end of July and a second workshop was planned in order to discuss their reactions more fully. However, this workshop did not take place, partly as a result of other commitments on the part of the author and partly because of the understandable lack of interest shown by the teachers at the end of the summer term. An alternative more popular proposal was that the author should visit local teachers who had used the Open University materials and who had responded to the questionnaire, and a series of such follow-up visits was therefore arranged for September.

(iii) The response from teachers

(a) Take-up of programmes

As reported in Chapter 2(iii) the initial response of local teachers at the December workshop was very enthusiastic. However, there was a disappointingly low take-up of the programmes in the early months of 1982 as shown in Table 4.1 (p.45) which provides details of requests for loans. This may have been due to lack of interest or it could have resulted from the fact that teachers were very much bound by the exam syllabus in the months leading up to June and just could not fit the progammes into their teaching. Then in June and July there was a very heavy demand for programmes and it was extremely difficult to respond to all requests promptly. Indeed it was impossible to supply all schools with all the programmes they requested because of the limited number of copies available.

5 of the 6 schools which borrowed cassettes at some time were local. (Verdin Comprehensive is situated in Chester and was not originally included in the developmental testing. However the teacher was extremely keen to use the Open University materials and one does not refuse offers of help.) Thus, of the 10 local schools originally involved, 5 actually used some of the materials.

Table 4.1 Record of loans January-July 1982

Programme Nos	Date	School
2, 4, 7, 15	12.3.83	Vandyke Upper
3, 6	12.3.82	Cedars Upper
9, 10, 11	30.3.82	Aylesbury Grammar
15	5.4.82	Vandyke Upper
1-9, 12-15	7.4.82	Aylesbury Grammar
1-9	19.4.82	Aylesbury Grammar
9-15	9.5.82	Verdin Comprehensive
11, 12, 13, 14	12,5,82	Denbigh
9, 10, 15	15.6.82	Aylesbury Grammar
10, 11	23.6.82	Denbigh
12, 13, 14	1.7.82	Vandyke Upper
1, 4	1.7.82	Cedars Upper
2, 3, 4, 6 ,7	1.7.82	Verdin Comprehensive
1, 9	1.7.82	Lord Gray
2	5.7.82	Cedars Upper
9, 12	7.7.82	Denbigh

(b) Response to the questionnaire

The questionnaires were sent to all ten local schools plus Verdin Comprehensive. Replies were received from the six schools which had used the programmes but only one reply was received from the remaining five.Here the reason given for not using the materials was the fact that staff shortages meant that A level classes were only receiving a fraction of the teaching that they might otherwise expect and that this was strictly centred on the examination itself. Also, the teacher in question was leaving and therefore did not plan to use the Open University materials at a later date. The author then contacted the other four schools. In two schools the teachers were interested but had not been able to use the programmes up till then, although they did indicate that they planned to use the materials in the future as and when the syllabus allowed. In the other two schools the teachers were just not sufficiently interested in the project.

(iv) Feedback

The comments from teachers who had used some of the programmes are summarized below. This summary is based both on the completed questionnaires and on the follow-up visits to the 5 local schools (the author did not visit Verdin Comprehensive).

(a) General comments

A number of A level syllabuses were in use in the six schools: Oxford, AEB, JMB and MEI, and all the teachers responded very favourably. They felt that the materials provided a useful resource for the sixth form no matter what syllabus was being followed. The use of computer graphics and animations was particularly welcome. Indeed the general impression conveyed seemed to be

> The sixth form alway tends to be low in the priority stakes for resources and so any material that might be provided is worthwhile - especially if it can be provided at a reasonable cost.

There was surprisingly little criticism either of the programmes or of the support materials. None of the programmes was regarded as unsuitable for use in the classroom. Even where a teacher had experienced difficulties because the approach differed from that used in the syllabus, or had used the materials in a different way to that indicated, there was no suggestion that the materials should be substantially altered. Of course, there was some criticism and this usually concerned the television programmes themselves rather than the associated written materials. For example, one teacher reported

> Too much time was spent explaining less complex work and the style of presentation was rather condescending.

But such criticism was not necessarily discouraging, as demonstrated by the following comment - from the same teacher!

> Students found the manner of the presenters rather entertaining at times and this may have distracted them from the purpose of the programme.

Only two teachers reported difficulties in obtaining access to a suitable video-recorder at the appropriate time. In one school the only video-recorder available was a Philips machine which would not take VHS cassettes. However, a suitable video-recorder was borrowed specifically for the project. (This problem would not arise if schools were to record their own tapes and, in fact, need not have arisen during the developmental testing period; had the teacher reported the problem at an earlier stage it would have been possible to supply Philip's cassettes.) In another school the teacher reported that the video recorder was already booked out when he had wanted access to it because classes lower down the school had more priority in the use of audio-visual resources. However, the teacher did plan to argue his case more thoroughly in the following school year.

Understandably, all the teachers reported that there were bound to be some organizational problems in using television in the classroom (partly because students were unused to this medium). But these did not discourage them from planning to use the materials further. On the whole, sixth formers tend to be more responsible than younger pupils and so it was felt that the organizational problems would not be as acute as they might be if television were to be used lower down the school. Also, the attitude seemed to be that such problems were a small disadvantage in comparison to the livelier mathematics lessons that might result when television was used.

(b) Future use

All the teachers reported that they planned to use the materials again. Many of the programmes had been shown experimentally in the post-examination period: some were used for revision purposes, others were used in induction courses for fifth formers who planned to take A level mathematics. The teachers felt that the programmes were particularly suitable for such activities and would use them again in the same way. In addition, now that they had seen the materials in use they felt more confident about fitting them into more direct teaching activities as and when appropriate.

(c) Cost and availability

It was felt that in order to encourage widespread use of the materials the cost should be kept as low as possible. It was recognized that the problem of how to obtain the programmes was as yet unresolved but there was no adverse reaction to the proposition that schools might be asked to record the programmes off-air using their own video-tapes. The cost of providing these tapes was regarded as a reasonable outlay for a useful resource. As for the written materials, the teachers reported that a cost of about £10 would be acceptable. The author pointed out that in order to achieve such a low cost the amount of material would need to be reduced. It was then suggested that the best way to cut the material would be to omit the solutions and to provide only final answers, possible exceptions being for those questions marked with an asterisk.

(d) Format of the written support materials

The written materials were regarded very favourably and the general feeling seemed to be that they helped to minimize the amount of preparation time that was required on the part of the teacher. Some changes in the format were proposed by various teachers and these were discussed in more detail in the follow-up visits. The proposed changes are listed below.

- The sections should be re-ordered in sequences of related programmes In the draft package the sections are simply presented in order of

transmission. It is certainly true that ignoring the transmission order would mean that several programmes fit into nice sequences (these are already indicated in the introduction to the draft support materials). However, even if the suggestion were implemented, a certain amount of cross-referencing would still be required because some programmes fit into several such sequences.
For the moment no decision has been taken on this point. If the programmes are simply to be recorded off-air then the order of presentation of the written materials should ideally be related to the transmission order of the programmes. If special arrangements can be made to have the programmes broadcast separately for schools then it might be possible to alter the transmission order and hence to re-sequence the written materials, otherwise the programmes will just be broadcast as part of the Open University year and the order of presentation should perhaps remain unchanged.

. Each section should include a list of discussion points on various aspects of the programmes The intention of this would be to draw teachers' attention to difficult or interesting parts of the programme which might be worthy of further explanation or exploration.
It was agreed to implement this suggestion.

. There should be more references to computing under the heading 'Possible Extensions' Computing was not included in the syllabus in any of the six schools concerned but there was general approval for this suggestion. It was recognized that many sixth formers are very interested in micro-computers and that it might be useful to channel some of this enthusiasm into work that is more directly concerned with A level mathematics. Indeed, some teachers suggested that detailed computer programs might be included here. On the other hand, the author felt that

it was unwise to be too specific because of the variety of hardware and software that is currently available.
In the end it was therefore decided to extend the references but to continue to keep them at a very general level.

. The exercises in each section should be printed separately and in the smallest possible space in order to facilitate copying It was also suggested that the exercises should ideally be provided as spirit-masters. The author agreed to look into this possibility, although it is not likely to be a feasible proposition because of cost implications.

(e) Detailed comments on programmes

Table 4.2 (see p.51) indicates the numbers of teachers who commented on the various programmes and the associated written materials.

Table 4.2 Programmes for which detailed comments were received

TV programme	Number of teachers who commented
1	3
2	5
3	3
4	3
6	2
7	3
9	2
10	4
11	1
12	2
13	1
14	2
15	2

It was agreed that for the purposes of this dissertation the author should consider only the detailed comments on TV1-TV4, for which the support materials are to be revised, and these are outlined in Chapter 5. However, although the detailed comments on the remaining programmes were not required here they will not be ignored and will be taken into account when the appropriate written materials are revised at a later date.

(v) Conclusions

Although the developmental testing was only on a small scale it does appear to have achieved its aims (see p.42).

- The response of teachers seemed to be that the materials would be suitable for use in the classroom - and very welcome.
- None of the programmes was identified as unsuitable for use in the classroom (although some were less suitable than others).

. Various suggestions were made which would lead to improvements in the written materials. These suggestions were of two types:

comments on the general format

detailed comments on various sections.

As a result of developmental testing, Sections 1-4 of the written materials were revised and the revised materials are reproduced in Chapter 5.

5. The Revised Support Materials for TV1-TV4

The following suggestions from Chapter 4 (iv) on format and layout were incorporated in the revised written materials.

- Each section should include a list of discussion points. Such a list was added to each section and it was decided that this new heading should also incorporate those points which had previously been included under the heading 'Possible Extensions'.
- The package should include more references to computing activities Such references were added where possible.
- The exercises in each section should be printed separately In the revised material the exercises are set out on separate pages and it is intended that eventually these exercises will be set using a smaller print size.
- There is no need for full solutions to the exercises. It is intended that only the final answers should be provided although these answers have not been included at all here, partly to save space and partly because they do not really differ from those in the original draft materials (except in length !).

Detailed comments, received from teachers during the developmental testing period as a result of the questionnaires and visits, were also considered and these are indicated under the various section headings. The author did not necessarily follow up all these comments and suggestions. Much of the feedback just seemed to confirm that the existing draft materials were satisfactory and it was felt that some comments simply reflected the range of reactions that might be expected. Also, several comments related to the television programmes themselves and these were rather difficult to deal with since there were no plans to alter the programmes in any way. However, teachers will be reminded in the revised introduction to the package that there is always the option of not watching the programme for a few minutes if they want to handle some aspect slightly differently. It is also intended that the introduction should include some reference to the importance of the pre- and post-programme work. (The introduction has not been rewritten at this stage because it is so dependent on the way in which the programmes will be made available to schools and this is still to be decided.)

In addition the draft materials were discussed with a number of colleagues and their suggestions were also considered at this stage.

(i) Section 1: Symbols and Equations

The completed questionnaires indicate that:

- students enjoyed the programme
- the amount of time spent on this section ranged from 1 - 2 hours
- the teachers would use the material again
- in each case the complete programme was watched in one viewing
- the TV summary provides a reasonable idea of what is involved in the programme
- the pre-requisites are realistic
- the pre-programme work, if not essential, helps to promote understanding
- the post-programme work is essential.

In addition, the following detailed comments were received.

> This work is eminently suitable at the beginning of an A level course or for use in an induction course for fifth formers.
>
> Although the material is not directly covered on many A level syllabuses (i.e. those which do not include iteration) the programme is neverthless very useful.
>
> The first few minutes of the television programme are rather slow and repeat some of the pre-programme work.
>
> The algebraic manipulation in substituting $r = \frac{x}{y}$ definitely needs to be covered before the programme. ALso, the post-programme work should include another question which involves this type of manipulation. (This teacher had however, omitted Question 5 from the pre-programme work (see Appendix 2).)

The revised written materials are reproduced on pp.56-64 and these indicate how the various suggestions were incorporated.

SYMBOLS AND EQUATIONS

This section is concerned with the setting up and solving of quadratic equations. The solution of quadratic equations by factorization and by the formula method is revised, then an iterative method of solution is introduced which students are invited to explore with the help of a calculator.
This section does not require any mathematics above 0 level standard. It therefore provides a good introduction to sixth form mathematics.

PROGRAMME SUMMARY

The programme essentially tackles two problems about international paper sizes. One purpose is to show the process of changing a given practical problem into a mathematical one in the form of an equation, the other is to introduce an iterative method of solving equations.

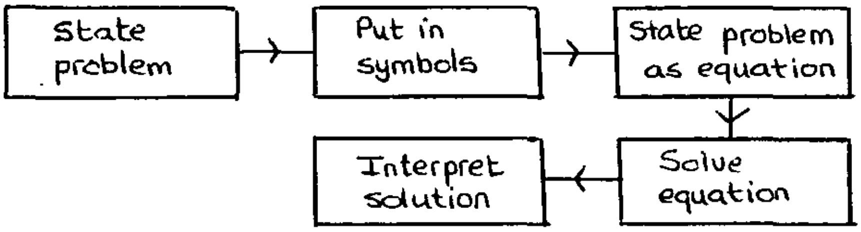
The programme begins with a brief reference to a problem about organizing an exhibition which students should have tackled beforehand (see Question 1 in the suggested pre-programme work).

3 mins Then a problem is posed about international paper sizes (A3, A4, A5 etc): what is the ratio of sides of a piece of A size paper? The first step is to explore the problem in a practical sense.

3 mins Symbols are introduced and the problem is expressed algebraically as an equation - and consequently solved.

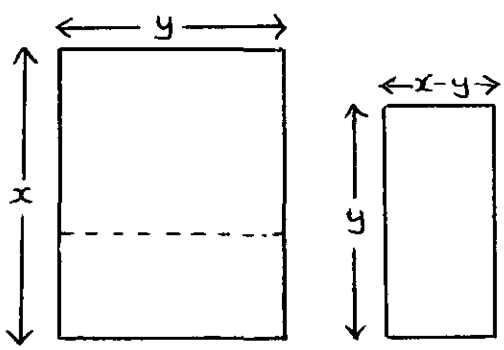
2 mins The presenters reflect on the process of problem solving with reference to the framework in Figure 1.1.

Figure 1.1



5 mins A second problem is introduced: what is the ratio of a rectangle such that after cutting off a square of side equal to the width of the rectangle the resulting shape is the same as the original (i.e. the shapes are similar)? See Figure 1-2.

Figure 1.2



The problem is tackled with reference to the framework for problem solving in Figure 1.1 and is solved using the formula method.

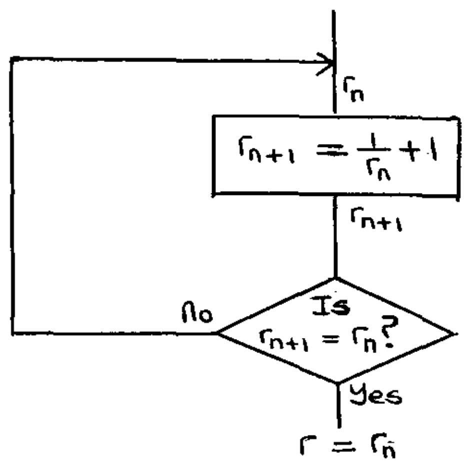
5 mins Such methods of solution are all very well for quadratic equations. But what about situations where there is no formula? The programme goes on to discuss the essential ingredients of an iterative process. (The bisection method is mentioned although only as an aside reference; for more details of this see SECTION 6: RATIONAL NUMBERS AND $\sqrt{2}$.) A method of formula iteration is introduced in order to solve the equation $r^2 - r - 1 = 0$ (from the problem above).

$\boxed{\text{3 mins}}$ The notation for the iteration formula

$$r_{n+1} = \frac{1}{r_n} + 1$$

is explored with reference to the flow diagram in Figure 1.3.

Figure 1.3



$\boxed{\text{2 mins}}$ Finally a third problem is introduced which is to be tackled immediately after the programme. This concerns the overall stopping distance of a car travelling at various speeds as found on the back of the Highway Code.

PRE-REQUISITES

Before working through this section students should be familiar with the following:

(i) the solution of quadratic equations of the form $ax^2 + bx + c = 0$ by factorization

(ii) the solution of quadratic equations of the form $ax^2 + bx + c = 0$ using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(iii) ratio and proportion

(iv) the use of a calculator to evaluate expressions of the form

$$\frac{1}{r} + 1, \qquad \frac{1}{r - 1}, \qquad \sqrt{\frac{100 - v}{0.05}}$$

(v) subscript notation.

DISCUSSION POINTS

1. The algebraic manipulation in the programme centres on the replacement of the term $\frac{x}{y}$ by the symbol r. It would be useful to discuss this type of manipulation before the programme (see Question 3 in the suggested pre-programme work).

2. Do students understand how their own calculator works, i.e. do they know the type of algebraic precedence it uses? can they use the memory correctly?

3. Practical mathematics <u>always</u> involves a certain degree of approximation. The idea of Mathematical Modelling might be discussed further.

4. Numerical methods of solving equations <u>are</u> necessary. Although the iterative process introduced in the programme is only for a quadratic equation the method also works for higher order equations for which there <u>are</u> no formulas.

5. The flow diagram in Figure 1.3 could lead to a study of the more conventional flow charts used in computing. It could also be extended to incorporate a specified degree of accuracy in the answer.

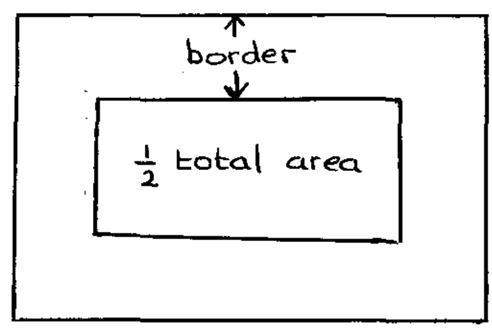
6. Usually there are many ways of rearranging an equation into a form suitable for iteration: for some rearrangements the iterative process will converge, for others it will diverge. This could lead to further exploration. For example, is it possible to <u>predict</u> in advance whether the iteration based on a given rearrangement will converge or diverge? (See SECTION 9: ITERATION AND CONVERGENCE.)

7. What is so special about numbers like $\sqrt{2}$? SECTION 6: RATIONAL NUMBERS AND $\sqrt{2}$ might provide an interesting follow-up here.

STUDENT EXERCISES

Pre-programme work

*1. (a) An exhibition organizer for a local craft exhibition is offered a room with floor dimensions 12 metres by 16 metres. He wishes to arrange exhibits around the sides of the room, leaving space for people to circulate in the centre. The local regulations state that at least half the floor area must be left clear.



Suppose the exhibits take up a border of equal width all around the room. This exercise investigates the width of this border if exactly half the total floor area is to be left clear in the centre.

(i) Using x metres to stand for the width of the border write down expressions for
- . the length of the clear floor space
- . the width of the clear floor space
- . the area of the clear floor space.

(ii) The value of x is restricted. Between which whole numbers must the value of x lie?

(iii) The total floor area is $16 \times 12 = 192\text{m}^2$. Use this to write down an equation which expresses the fact that the clear space should have an area equal to half the total floor area.

(iv) Solve the resulting equation by factorization. What does the solution mean in terms of the original problem?

(b) How wide should the border be if the room has floor dimensions 15 metres by 20 metres and exactly half the total floor area is to be left clear in the centre?

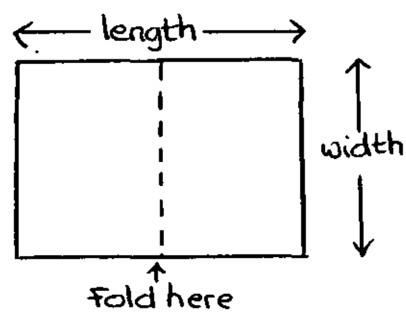
(c) How wide should the border be if the room has floor dimensions 14 metres by 14 metres and exactly half the total floor area is to be left clear in the centre?

(Hint: you will need to use the formula method here.)

2. (a) Measure the length and width of a piece of A4 paper and calculate the ratio

$$\frac{\text{longer side}}{\text{shorter side}}.$$

(b) Fold the paper in half and repeat the above calculation.



*3. The following equations involve the two symbols x and y. In each case replace $\frac{x}{y}$ by r in order to obtain an equation just involving r.

(a) $\frac{x}{y} = \frac{x + y}{x}$ (b) $x = \frac{x - y}{2x} + y$.

Post-programme work

4. The stopping distance d feet of a car travelling at v mph is given (approximately) by the formula

$$d = 1.0v + 0.05v^2.$$

To find out how fast you can drive and still stop in 100 feet therefore requires you to find the solution to

$$100 = v + 0.05v^2.$$

(a) Use the rearrangement

$$v_{n+1} = 100 - 0.05v_n^2$$

with starting value $v_1 = 40$ to try to solve the equations.

(b) Use the rearrangement

$$v_{n+1} = \sqrt{\frac{100 - v_n}{0.05}}$$

and a starting value of $v_1 = 40$ to try to solve the equation.

5. (a) Try to solve the equation $x^2 = 2$ using the rearrangement

$$x_{n+1} = \frac{1}{2} x_n + \frac{2}{x_n}$$

with starting value $x_1 = 1$.

(b) Suggest a rearrangement that might help you to solve $x^2 = a$ as accurately as you wish for any positive value of a. Check your suggestion by trying it out for a few different values of a. What happens if a is negative?

6. Consider the equation

$$x = \frac{x + y}{x} .$$

(a) Use the substitution $r = \frac{x}{y}$ to rewrite this equation in terms of r only.

(b) Solve the resulting equation using the rearrangement

$$r_{n+1} = \frac{r_n + 1}{3r_n}$$

with starting value $r_1 = 1$.

(By the way, you should of course check that the equation that you obtained in Part (a) <u>can</u> be rearranged into the form given in Part (b).)

7. The original rearrangement of the equation $r^2 - r - 1 = 0$ as obtained in the television programme was

$$r = \frac{1}{r - 1} .$$

However this rearrangement was not used for the formula iteration method. This exercise explains why.

(a) Use the rearrangement

$$r_{n+1} = \frac{1}{r_n - 1}$$

with starting values

(i) 1 (ii) 0.5 (iii) 2

to try to solve $r^2 - r - 1 = 0$. What happens?

(b) Investigate what happens with the rearrangement

$$r_{n+1} = r_n^2 - 1$$

with starting values

(i) 1 (ii) 0.5 (iii) 2.

(ii) Section 2: The Binomial Theorem

The completed questionnaires indicate that:

- those students who were at the beginning of an A level course found the programme very useful, those who watched it after one year in the sixth form found the programme too easy and a bit boring
- the amount of time spend on this section ranged from 25 mins (i.e. only watching the programme) to 1 hour
- the teachers would use the material again
- in each case the complete programme was watched in one viewing
- the TV summary provides a reasonable idea of what is involved in the programme
- the pre-requisites are realistic and should certainly include familiarity with the idea of coefficients
- the amount of pre- and post-programme work that is required depends upon students' past experience: for students at the beginning of an A level course the the work is essential, where the programme is used for revision purposes the work could be omitted.

In addition, the following detailed comments were received.

> In our syllabus this is a major topic and will be dealt with in greater detail with numerous more examples. However I would certainly use this as an introduction - especially in the induction course for next year's intending lower sixth. It was an excellent choice of programme; it gives them a chance to look at A level work and yet follow the ideas involved, even though it is very different to what they have met at O level.

> The multiplication of brackets at the beginning of the

programme was rather laboured. A reasonable A level student is too familiar with brackets to need to spend time on this. (Here the programme was used for revision at the end of the first year sixth.)

The section of the programme regarding routes from Charlotte Square to the Scott Monument was too long. I would edit that section and do combinations by listing. (Here the programme was used at the end of the first year sixth.)

The notation $\binom{n}{r}$ should be mentioned since this is used in many A level syllabuses.

More repetitive exercises should be included.

Some reference should be made to the Binomial Series.

The revised written materials are reproduced on pp.67–75 and these indicate how the various suggestions were incorporated.

2. THE BINOMIAL THEOREM

This section investigates the expansion of $(a + b)^n$ by considering the number of different ways of obtaining terms involving a^r $(0 \leqslant r \leqslant n)$ from the n sets of brackets. In the course of the investigation the construction of Pascal's Triangle is explored in terms of the equation

$$^{n}C_{r} = {}^{n-1}C_{r} + {}^{n-1}C_{r-1}.$$

This section provides an excellent introduction to A level mathematics since it does not require any mathematics above 0 level standard. However, students who are already familiar with the ideas may find that the pace is rather too slow.

PROGRAMME SUMMARY

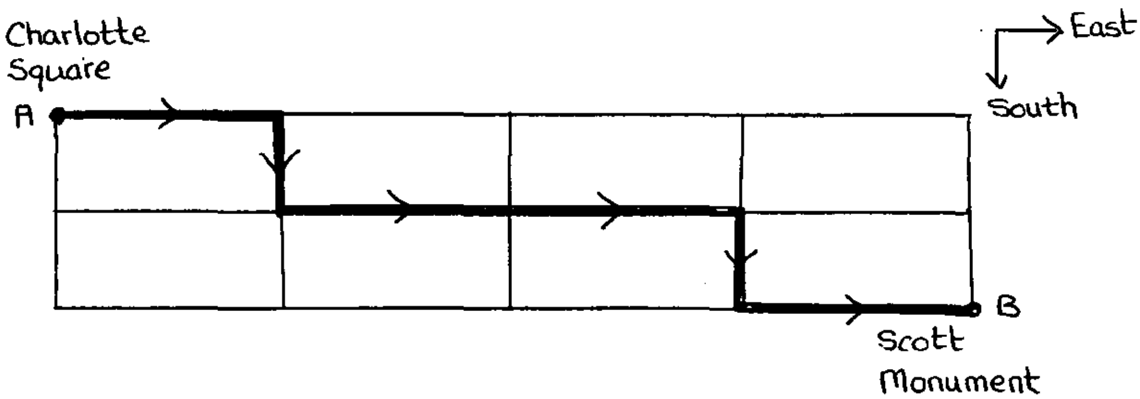
The expansion of $(a + b)^6$ is investigated. What does it mean? How is it obtained? The problem is reduced to a route finding exercise which is easier to visualize, and which leads directly to Pascal's Triangle. The link between Pascal's Triangle and the Binomial Theorem is then explored further.

2 mins How to expand $(a + b)^n$? Before the programme students should have seen how to expand $(a + b)^2$ and $(a + b)^3$ (see Questions 1 and 2 in the suggested pre-programme work). The special case $(a + b)^6$ is introduced.

6 mins The meaning of $(a + b)^6$ is investigated by referring back to the simpler cases $(a + b)^2$ and $(a + b)^3$. The expansion of $(a + b)^6$ is seen to involve the same principle - choosing symbols from each of the brackets in turn. The coefficient of each term (e.g. the coefficient of a^4b^2) indicates the number of different ways of choosing those particular symbols.

$\boxed{\text{5 mins}}$ The idea of choosing symbols is compared with the number of ways of getting from one grid point to another (in a minimum number of steps). This is illustrated by considering the number of different routes from Charlotte Square to the Scott Monument in Edinburgh as shown in Figure 2.1.

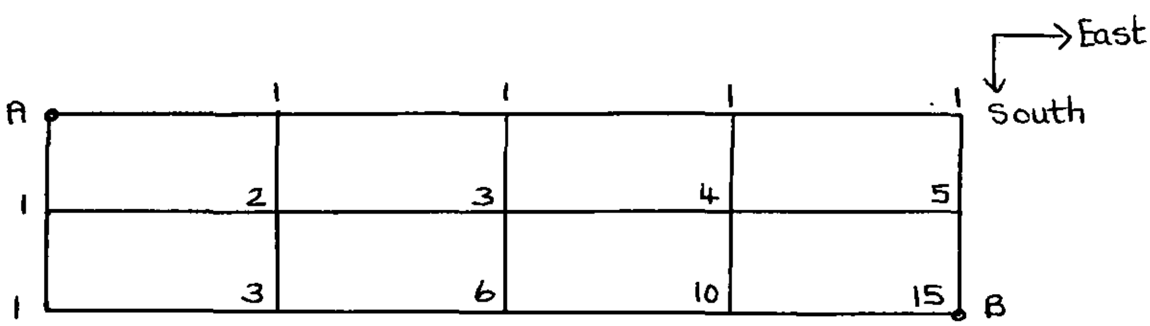
Figure 2.1



$\boxed{\text{1 min}}$ The coefficient of a^4b^2 and the number of different routes from A to B in Figure 2.1 both involve the number of different ways of choosing 2 things out of 6.

$\boxed{\text{2 mins}}$ The route problem is easier to visualize. The number of routes to any grid point is obtained by adding together the numbers of routes to the two adjacent preceding grid points. This leads to Pascal's Triangle as shown in Figure 2.2.

Figure 2.2



$\boxed{\text{3 mins}}$ Moving one place South corresponds to choosing the symbol b, moving one place East corresponds to choosing the symbol a. All the coefficients in the expansion of $(a + b)^6$ can be obtained just by extending the grid.

$\boxed{\text{1 min}}$ The link between Pascal's Triangle and the expansion of $(a + b)^n$ for various values of n is explored further by looking at the diagonals on the grid.

$\boxed{\text{3 mins}}$ The notation ^nC_r is introduced to stand for a general point on the grid system on the nth diagonal and the rth row. The structure of Pascal's Triangle can then be completely explained in terms of the equation

$$^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$$

since all the numbers on the borders are ones.

PRE-REQUISITES

Before working through this section students should be familiar with the following:

(i) the expansion of brackets of the form $(a + b)^2$ and $(a + b)(a^2 + 2ab + b^2)$

(ii) the meaning of a coefficient

(iii) some idea of symmetry.

DISCUSSION POINTS

1. Students might find that the Edinburgh sequence in the programme is rather slow and longwinded. You may therefore prefer to omit this sequence and to deal with the idea in some other way.

2. In the programme Pascal's Triangle is presented in a somewhat unusual format which may require some explanation if students have already met the idea before.

3. This section of work could be used as a starting point from which to discuss <u>other</u> number patterns in Pascal's Triangle.

4. It might be worthwhile spending some time discussing the Binomial Theorem more formally. In particular it might be appropriate to discuss the fact that it is restricted to expansions of the form $(a + b)^n$ where n is a positive integer. This could lead to a discussion of the Binomial Series for $(1 + x)^r$ $(|x| < 1,\ r \in \mathbb{Q})$ (see SECTION 14: TAYLOR POLYNOMIALS).

5. The notation ${}^nC_r$ may need to be discussed further. Also, the connection between this notation and the notation $\binom{n}{r}$ might be pointed out.

6. The relationship
$${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$$
can be used to show that
$${}^nC_r = \frac{n!}{r!(n - r)!}$$
and hence various binomial coefficients can be evaluated.

7. Further repetitive work might be set on using the Binomial Theorem. In particular, its application to the evaluation of numerical expressions of the form $(1.01)^n$, $(0.99)^n$ etc might be further explored.

STUDENT EXERCISES

Pre-programme work

*1. (a) Write down the expansion of $(a + b)^2$.

(b) Write down the expansion of $(a + b)^3$. You might find that it helps to write $(a + b)^3$ as $(a + b)(a^2 + 2ab + b^2)$.

*2. One way of obtaining the expansion of $(a + b)^2$ is to write it as $(a + b)(a + b)$ and to multiply out the brackets, and that is probably what you did in Question 1. Alternatively, you can think of the expansion in the following way: every contribution to the expansion is the product of two symbols - an a or b from the first bracket multiplied by an a or b from the second. The expansion may then be found by considering the number of ways of obtaining all the possible combinations of symbols that might be selected. Thus in $(a + b)^2$

$(a + b)(a + b)$ the term involving a^2 can be obtained in only one way, by choosing an a from each bracket. This means that the coefficient of a^2 in the expansion of $(a + b)^2$ is 1.

$(a + b)(a + b)$ Similarly the coefficient of b^2 is 1.

$(a + b)(a + b)$ On the other hand, the product ab can be obtained in two ways, either by choosing an a from the first bracket and a b from the second or by choosing a

$(a + b)(a + b)$ b from the first bracket and an a from the second. So the coefficient of ab is 2.

Hence $(a + b)^2 = a^2 + 2ab + b^2$.

(a) Use a similar argument to explain the expansion of $(a + b)^3$.

(b) The expansion of $(a + b)^4$ may also be obtained using this alternative method of choosing symbols.

(i) Use this method to write down the coefficients of ab^3 and a^2b^2 in the expansion of $(a + b)^4$.

(ii) Check your answers by multiplying out the expression $(a + b)(a + b)^3$.

(Hint: Use your solution to Question 1(b)).

<u>Post-programme work</u>

3. The expansion of $(a + b)^6$ is

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

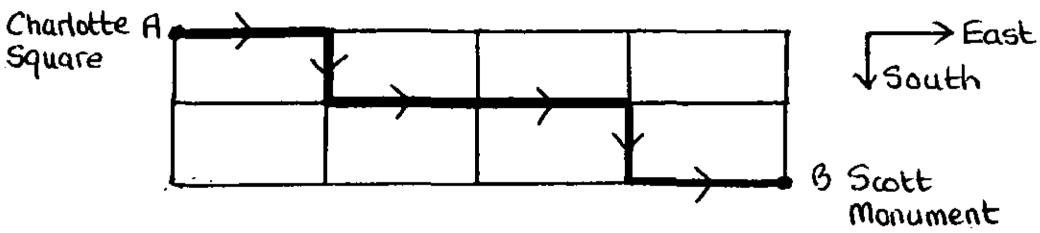
Use this expression to find the number of ways of choosing

(a) three b s

(b) four b s

from six brackets (and a s from the remaining brackets).

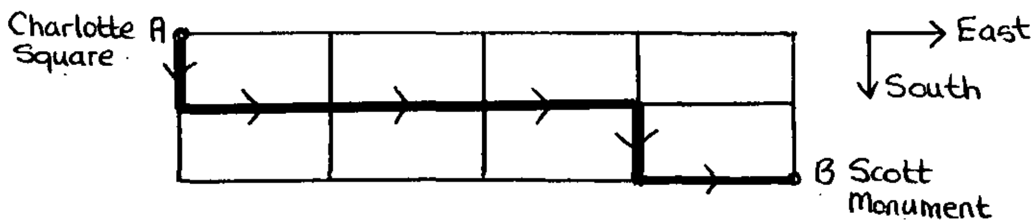
4. The route chosen in the programme from Charlotte Square to the Scott Monument was



This corresponds to the following choices of direction at each junction.

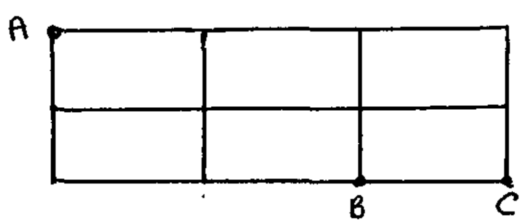
E	S	E	E	S	E

What choices correspond to the following route?

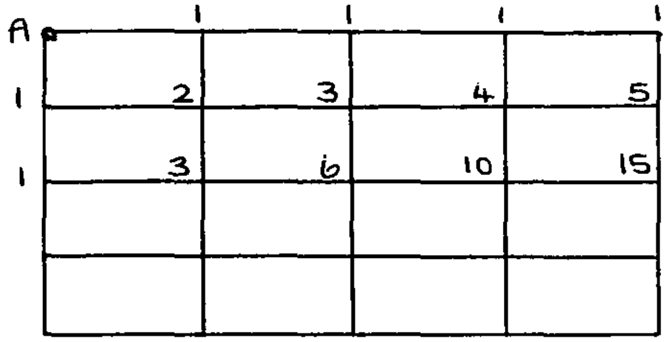


5. In the following diagram count the number of (different) shortest routes from

(a) A to B (b) A to C.



6. (a) Complete the following diagram by giving the number of shortest routes from A to each of the remaining junctions on the grid.



(b) Use your answer to Part (a) to write down

(i) the coefficient of a^3b^2 in the expansion of $(a + b)^5$

(ii) the coefficient of a^3b^4 in the expansion of $(a + b)^7$

(iii) the coefficient of a^4b^4 in the expansion of $(a + b)^8$.

7. Use Pascal's Triangle to write down the expansion of $(a + b)^5$.

8. (a) Work out Pascal's Triangle up to and including the seventh diagonal on the grid.

(b) Hence write down the expansion of $(a + b)^7$.

9. (a) (i) By substituting $a = 2c$ obtain the expansion of $(2c + b)^5$.

(ii) By substituting $b = 3d$ obtain the expansion of $(a + 3d)^5$.

(iii) Now find the expansion of $(2c + 3d)^5$.

(b) Use a similar approach to find

(i) $(3s + 4t)^4$ (ii) $(2t + 5u)^3$.

10. (a) (i) Write $x - y$ as $x + (-y)$ and so obtain the expansion of $(x - y)^6$.

(ii) By writing $2v - u$ as $2v + (-u)$ obtain the expansion of $(2v - u)^6$.

(b) Use a similar approach to find

(i) $(m - 5n)^4$ (ii) $(3s - 4t)^4$ (iii) $(5t - 2u)^3$.

11. (a) By writing 1.001 as $(1 + 0.001)$ use the first three terms of the expansion of $(a + b)^7$ to find an approximation to $(1.001)^7$.

(b) By writing 0.999 as $(1 - 0.001)$ use the first three terms of the expansion of $(a + b)^8$ to find an approximation to $(0.999)^8$.

Compare your answers with those given by a calculator.

(iii) Section 3: Trigonometric Formulas

The completed questionnaires indicate that:

- those students who watched the programme at the end of the lower sixth found the programme interesting and helpful, but when the programme was used in order to introduce the ideas students found it too hard
- the TV summary provides a reasonable idea of what is involved in the programme but parts of it might be extended a little and more diagrams included
- the pre-requisites should mention that students will gain more from the work if they have met the formulas before and it will help if they have met them in the context of matrix multiplication
- the pre-programme work helps to promote understanding of the programme
- The post-programme work is not essential but it does indicate the sort of work that should be done.

In addition the following detailed comments were received.

> I felt on reflection that I used the programme poorly and did not really prepare the students properly for it. I had intended that it would provide a 'nice' introduction to sin $(\alpha + \beta)$ etc. but the students did get very lost during the programme. It is the one programme that I did not feel happy about and the student reaction confirmed this.

> The work is relevant to those syllabuses which include work on transformations but it is unlikely to be of interest

where the syllabus is more traditional.

I found myself wanting to use matrices. The programme provides a nice geometrical approach but it would help if the connection between this and the matrix method (which I usually use to introduce these formulas) could be made more explicit.

The revised written materials are reproduced on pp.78-85 and these indicate how the various suggestions were incorporated.

3. TRIGONOMETRIC FORMULAS

This section takes a geometric look at the formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

and shows how they can be obtained by considering the point (1, 0) rotated through α then rotated through β.

This section is more appropriate for use with those syllabuses which have a more modern slant. It is best used for revision purposes, particularly if students have already met the formulas in the context of matrix multiplication, in which case the programme provides a good illustration of the connection between matrix algebra and the geometry of transformations.

PROGRAMME SUMMARY

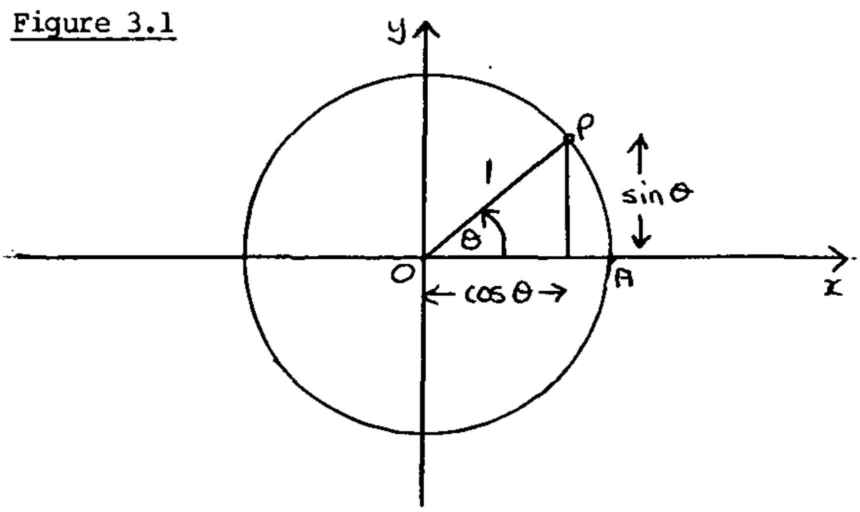
The programme derives the formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

using rotations and translations.
The meaning of $\tan\theta$ is investigated using another transformation - a dilation (or scaling).

$\boxed{\text{5 mins}}$ The link between rotations and trigonometry is reviewed by examining the definitions of $\sin\theta$ and $\cos\theta$ as shown in Figure 3.1, and their graphs. (Note: θ is measured in radians.)

Figure 3.1



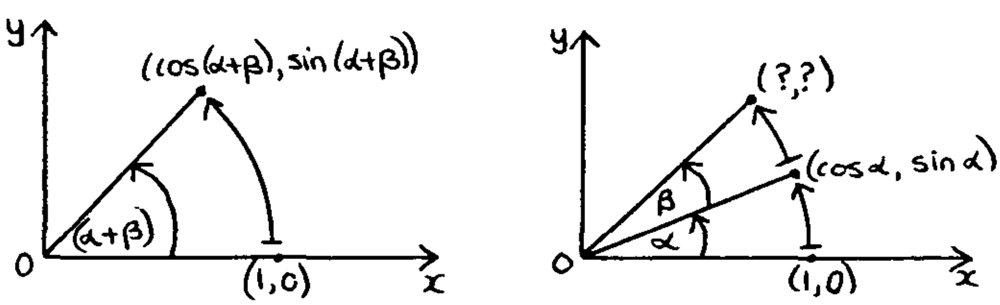
$\boxed{\text{3 mins}}$ Using the original definitions, and the fact that the graphs of $\sin\theta$ and $\cos\theta$ are very similar, the properties

$$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$

are obtained.

$\boxed{\text{3 mins}}$ The presenters now consider what is meant by a rotation through $(\alpha+\beta)$. This is demonstrated by looking at the point (1, 0). The rotation may be looked at in two ways: either as a rotation through the angle $(\alpha+\beta)$ or as a rotation through α followed by a rotation through β (see Figure 3.2).

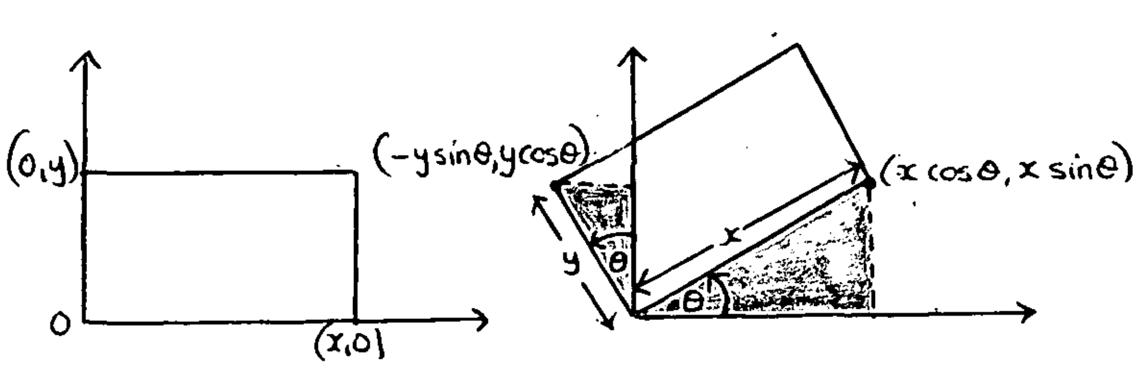
Figure 3.2



The formulas are derived by considering <u>both</u> approaches. It transpires that it is necessary to find out what happens to the point $(\cos\alpha, \sin\alpha)$ when it is rotated through β. The notation r_θ is introduced to denote rotation through θ.

$\boxed{\text{5 mins}}$ A rectangle with corners (0, 0) (x, 0) (0, y) and (x, y) is rotated through θ as shown in Figure 3.3.

Figure 3.3

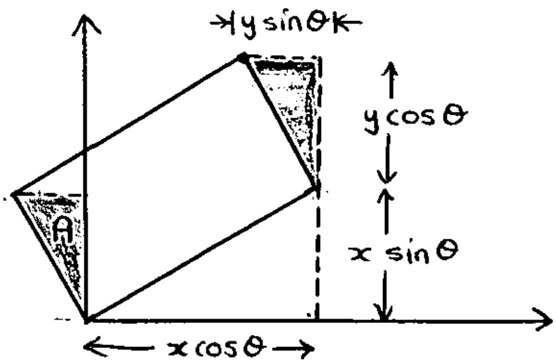


The dimensions of the shaded right-angled triangles suggest that under the rotation r_θ

$$(x,\ 0) \mapsto (x \cos\theta,\ x \sin\theta)$$
$$(0,\ y) \mapsto (-y \sin\theta,\ y \cos\theta).$$

The next step is to consider the general point (x, y).

Figure 3.4



By translating the shaded triangle marked A as shown in Figure 3.4 it is shown that under the rotation r_θ
$(x,\ y) \mapsto (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta)$.

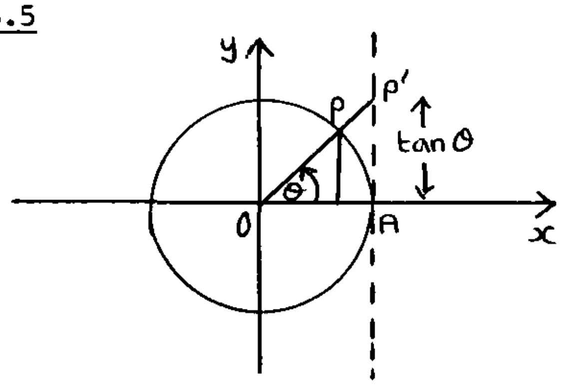
2 mins Hence under the rotation r_β
$(\cos\alpha,\ \sin\alpha) \mapsto (\cos\alpha \cos\beta - \sin\alpha \sin\beta,\ \cos\alpha \sin\beta + \sin\alpha \cos\beta)$
and this gives the required formulas.

5 mins Finally the programme looks at the meaning of $\tan\theta$. $\tan\theta$ is commonly introduced via the definition

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

but it may also be expressed geometrically as a single length with the help of another transformation - a dilation (or scaling) as shown in Figure 3.5.

Figure 3.5



This leads to an exploration of the graphical properties of tan θ .

PRE-REQUISITES

Before working through this section students should be familiar with the following:

(i) the definitions of cos θ and sin θ as the x- and y-coordinates of the point P on the unit circle obtained by rotating the point (1, 0) through θ

(ii) the graphs of $x \mapsto \sin x$ and $x \mapsto \cos x$

(iii) the idea of radian measure

(iv) the use of a scientific calculator

(v) the effect of a translation of (a, b) on the point with coordinates (u, v) as movement to the point with coordinates (u + a, v + b)

(vi) the effect of a dilation (or scaling) by a factor λ with centre the origin on the point with coordinates (u, v) as movement to the point with coordinates $(\lambda u, \lambda v)$.

In order to gain most benefit from the programme students should previously have met the formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. (These formulas are derived rather quickly in the programme, perhaps too quickly for students who have not seen the ideas before.) It will also help if students are familiar with the algebraic and geometric interpretation of expressions such as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

DISCUSSION POINTS

1. The properties of the graphs of $x \mapsto \sin x$ and $x \mapsto \cos x$ might be discussed further with the help of the computer graphics used in the programme.

2. The programme investigates the effect of a rotation through α followed by a rotation through β in terms of what happens to the coordinates of the point (x, y) in the Cartesian plane. You might like to pause at this stage in order to look at this in terms of matrix algebra; the result is the same since

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

This could lead to further discussion on the connection between matrix algebra and geometric transformations.

3. The geometric interpretation of a tangent is not usually discussed at the same time as the trigonometric definition; further discussion might be worthwhile.

4. The post-programme work can be extended to cover other standard exercises on trigonometric formulas.

5. The various geometric effects of transformations might be explored further (see SECTION 7: FUNCTIONS AND GRAPHS which looks at the effects of various transformations on graphs of functions).

6. Reflections were not mentioned in the programme. Nevertheless they can be explored using a similar approach. In particular, if q_θ represents a reflection in the line through the origin making an angle θ with the x-axis, then the effect of q_θ on the general point (x, y) may be described as

$$(x, y) \mapsto (x\cos 2\theta + y\sin 2\theta,\ x\sin 2\theta - y\cos 2\theta)$$

or $$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\sin 2\theta + y\sin 2\theta \\ x\sin 2\theta - y\cos 2\theta \end{pmatrix}.$$

STUDENT EXERCISES

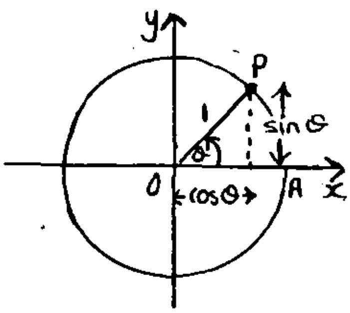
Pre-programme work

1. Two translations f and g are specified by the rules

$$f:(x,\ y)\mapsto(x + 3,\ y - 1)$$
$$g:(x,\ y)\mapsto(x - 2,\ y + 4)$$

(a) Apply f to (0, 0) and then apply g to the result.
(b) Apply f to (2, 3) and then apply g to the result.
(c) What point do you obtain by applying f to (x, y) and then applying g to the result?
(d) Describe the overall effect of applying f then g.

2.



On this diagram P is obtained by rotating the line OA through an angle θ . Thus, since OA has unit length, P is a point on the circumference of a circle of unit radius.

(a) Use this definition to obtain each of the following

(i) $\cos 0$ (ii) $\sin\frac{\pi}{2}$ (iii) $\cos\frac{\pi}{2}$ (iv) $\cos\pi$ (v) $\sin\frac{3\pi}{2}$.

(b) $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$.

Hence write down

(i) $\cos\frac{2\pi}{3}$ (ii) $\sin\frac{5\pi}{6}$.

3. (a) Use your calculator to complete the table on the next page. (Give your answers to one decimal place.)
(Note: x is measured in radians so make sure your calculator is in the appropriate mode.)

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin x																	
cos x																	

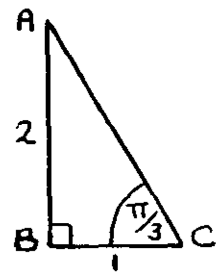
(b) Use the table to draw the graph of $x \mapsto \sin x$.

(c) Use the table to draw the graph of $x \mapsto \cos x$.

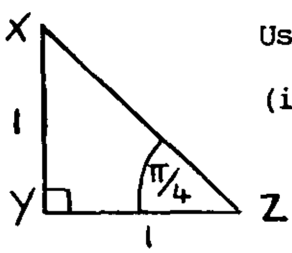
<u>Post-programme work</u>

4. (a) Use this triangle to write down

(i) $\cos\frac{\pi}{3}$ (ii) $\sin\frac{\pi}{3}$.



(b) Use this triangle to write down



(i) $\cos\frac{\pi}{4}$ (ii) $\sin\frac{\pi}{4}$.

(c) Use Parts (a) and (b) and the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ to evaluate

(i) $\cos\frac{7\pi}{12}$ (ii) $\sin\frac{7\pi}{12}$.

5. The formula for $\cos(\alpha+\beta)$ is

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

(a) (i) Replace β by α to obtain a formula for $\cos 2\alpha$.

(ii) Use the result

$$\cos^2\alpha + \sin^2\alpha = 1$$

to obtain a formula for $\cos 2\alpha$ in terms of $\cos\alpha$ alone.

(iii) Find a formula for $\cos 2\alpha$ in terms of $\sin\alpha$ alone.

(b) Treat $(\alpha - \beta)$ as $(\alpha + (-\beta))$ to obtain a formula for $\cos(\alpha - \beta)$.

6. The formula for $\sin(\alpha + \beta)$ is

$$\sin(\alpha + \beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta .$$

(a) Replace β by α to obtain a formula for $\sin 2\alpha$.

(b) Treat $(\alpha - \beta)$ as $(\alpha + (-\beta))$ to obtain a formula for $\sin(\alpha - \beta)$.

7. Tan θ is defined as $\dfrac{\sin\theta}{\cos\theta}$ $(\cos\theta \neq 0)$.

Hence $\tan(\alpha + \beta) = \dfrac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$ $(\cos(\alpha + \beta) \neq 0)$.

(a) Write down a formula for $\tan(\alpha + \beta)$ involving $\cos\alpha$, $\sin\alpha$, $\cos\beta$ and $\sin\beta$.

(b) Divide top and bottom throughout by $\cos\alpha \cos\beta$ to get a formula involving $\tan\alpha$ and $\tan\beta$ only.

8. (a) Find a formula for $\cos 3\alpha$ in terms of $\cos 2\alpha$, $\sin 2\alpha$, $\cos\alpha$ and $\sin\alpha$.

(b) Use your solutions to Question 5 and 6 to express $\cos 3\alpha$ in terms of $\cos\alpha$ only.

9. (a) Use the formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$ (see Question 6) to show that

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos\alpha \sin\beta .$$

(b) By writing A for $(\alpha + \beta)$ and B for $(\alpha - \beta)$ show that

$$\sin A - \sin B = 2 \cos\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right).$$

(iv) Section 4: Inverse Functions

The completed questionnaires indicate that:

- the work is directly relevant to those A level syllabuses which include some work on functions but it is not really appropriate where the course is more traditional (although the work on arccos and arctan might be useful)
- students who were already familiar with functional notation enjoyed the programme and found that it helped them to understand the mathematical concepts
- the only indication of the amount of time spent on the section was about $1\frac{1}{2}$ hours but here the students had not worked through the post-programme work
- the teachers would use the materials again
- in each case the complete programme was watched in one viewing
- the TV summary provides a reasonable idea of what is involved in the programme
- the pre-requisites are realistic
- the pre- and post-programme work aids understanding but is not essential.

In addition, the following detailed comments were received.

> The graphics used are good - especially as an illustration of the graph of a function. The idea of the inverse function comes over particularly well.

> I did not like the flow diagram approach to finding the inverse of a function.

The easier functions could have been dealt with less slowly so that more time could be spent on the more complex ones.

The revised written materials are reproduced on pp.88-94 and these show how the various suggestions were incorporated.

4. INVERSE FUNCTIONS

When does a function have an inverse? This question is investigated in the context of the formal definition of a function in terms of the domain, codomain and rule.

Computer graphics are used to demonstrate the need for the function to be one-one. But even when a function is not one-one it can be split into a number of parts, each of which has an inverse, and this leads to the definitions of arccos and arctan as evaluated by a calculator.

This section is particularly appropriate for use with those syllabuses which have a more modern slant. It is probably best used in the first year of an A level course.

PROGRAMME SUMMARY

The programme investigates the 'reverse' of the function

$t: [0, 4]$

$t: x \mapsto 4x - x^2.$

A similar approach is used in order to define the meaning of arccos and arctan.

| 1 min | The programme begins by reminding students of the definition of a function in terms of the domain, codomain and rule. It is essential to know the domain and codomain in order to find the inverse of a function.

| 2 mins | A Ballista is a weapon used by the ancient Romans; it fires a ball in a fixed trajectory given by the function

$t: [0, 4] \rightarrow \mathbb{R}$

$t: x \mapsto 4x - x^2.$

$\boxed{\text{2 mins}}$ So, given the distance x to a fixed post, it is possible to calculate the height of the post by finding $t(x)$. But in practice the ancient Romans knew the height of the post and wanted to determine where to position the Ballista. This involves reversing the effect of the function t, but there is a problem because t is not one-one.

$\boxed{\text{2 mins}}$ What about the simpler function

$$g: \mathbb{R} \to \mathbb{R}$$
$$g: x \mapsto x^2 \, ?$$

Again g is not one-one so it does not have an inverse. However, the graph of g suggests that the function <u>can</u> be split into two one-one functions

$$l: N \to \mathbb{R} \quad \text{and} \quad r: P \to \mathbb{R}$$
$$l: x \mapsto x^2 \quad \text{and} \quad r: x \mapsto x^2$$

where N = {Negative Reals + zero}

P = {Positive Reals + zero},

and each of these functions <u>does</u> have an inverse.

$\boxed{\text{3 mins}}$ The next step is to find an inverse for r. The domain of r^{-1} is the image set of r. Hence

$$r^{-1}: P \to P$$
$$r^{-1}: x \mapsto \sqrt{x}$$

where $\sqrt{x}$ is the positive square root of x.

Similarly

$$l^{-1}: P \to N$$
$$l^{-1}: x \mapsto -\sqrt{x}.$$

$\boxed{\text{2 mins}}$ The same approach is used for the Ballista function t since this too can be split into two one-one functions.

$\boxed{\text{2 mins}}$ The technique for reversing the rule is demonstrated with the function

$$h: \mathbb{R} \rightarrow \mathbb{R}$$
$$h: x \mapsto 5 - 4x.$$

$\boxed{\text{5 mins}}$ The Ballista problem is solved.

$\boxed{\text{5 mins}}$ A similar approach leads to the idea of arccos and arctan as evaluated by a calculator.

PRE-REQUISITES

Before working through this section students should be familiar with the following:

(i) the Real line $\mathbb{R}$ and intervals of the form [a, b].

(ii) the manipulation of functions expressed in the form

$$f: \mathbb{R} \rightarrow \mathbb{R}$$
$$f: x \mapsto 2x^2 - 3.$$

(In particular the terminology domain, codomain, rule and image set should have been met before.)

(iii) the properties of a one-one function

(iv) the graphical representation of a function (linear and quadratic)

(v) the sine, cosine and tangent functions and their graphs (using radian measurement)

(vi) the use of a scientific calculator.

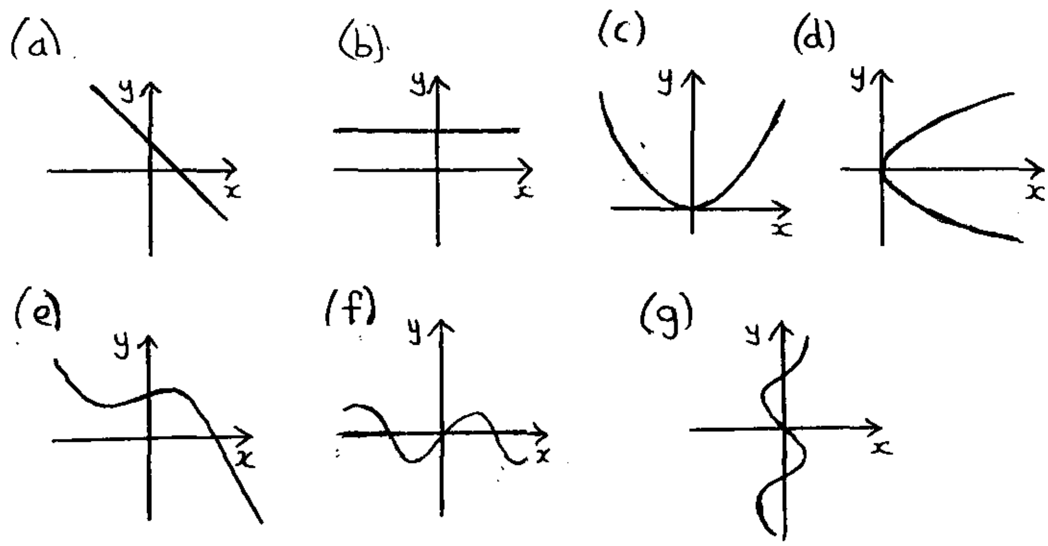
DISCUSSION POINTS

1. Students may need to be reminded of how to find the image set of a function, especially when the function is quadratic.
2. When sketching the graph of a function it is important not to forget the domain. And what is so special about the graph of a one-one function?
3. There are many methods of reversing the rule in order to find the inverse; the method used in the programme may not be familiar to students.
4. The formal argument used to show that a function is one-one may need further discussion - or it could be completely omitted.
5. Students might explore the graphs of the trigonometric functions in order to find other suitable definitions for the functions arccos x etc. They can always check their findings using a calculator.
6. This section could lead to further work on the properties of functions; in particular the composition of functions might be explored. When is it possible to find an inverse for such a function? (SECTION 7: FUNCTIONS AND GRAPHS looks at the effect of transformations on the graphs of functions and may be worth looking at if students wish to explore the more general properties of functions.)

STUDENT EXERCISES

Pre-programme work

*1. Determine which of the following are graphs of functions.



*2. Which of the graphs in Question 1 above are graphs of one-one functions?

*3. A function is a one-one function if, whenever $f(a) = f(b)$, then $a = b$ (where a and b are elements of the domain).

(a) Prove that the function f defined by

$$f: \mathbb{R} \to \mathbb{R}$$
$$f: x \mapsto 2x - 1$$

is one-one.

(b) Show that the cosine function defined by

$$\cos: \mathbb{R} \to \mathbb{R}$$
$$\cos: x \mapsto \cos x$$

is not one-one.

*4. (a) Sketch the graphs of the functions

(i) $l : \{x \in \mathbb{R} : x \leqslant 0\} \to \mathbb{R}$ (ii) $r: \{x \in \mathbb{R}, x \geqslant 0\} \to \mathbb{R}$

$l : x \mapsto x^2$ $r: x \mapsto x^2$.

(b) Prove that each of these functions is one-one using the definition in Question 3 above.

*5. The function t is defined by

$t: [0, 4] \to \mathbb{R}$

$t: x \mapsto 4x - x^2$.

(a) Solve the equations

(i) $t(x) = 0$ (ii) $t(x) = 4$ (iii) $t(x) = 5$.

(b) Show that t is not a one-one function.

(c) Multiply out $4 - (x - 2)^2$ and so show that the rule for t can be written as

$x \mapsto x^4 - (x - 2)^2$.

(d) a and b are distinct numbers in the domain of t for which $t(a) = t(b)$. Use the result of Part (c) to find an equation connecting a and b.

(e) Draw the graph of the function t.

(f) What is the image set of the function t?

(g) Suggest two intervals lying within the interval [0, 4] which, when used as domains for the rule $x \mapsto 4x - x^2$, produce one-one functions.

Post-programme work

6. k is a one-one function such that

$k: 2 \mapsto 4$ and $k(4) = 7$.

Find (a) $k^{-1}(4)$ (b) $k^{-1}(7)$.

7. The function m is defined by

$$m: [0, 4] \to \mathbb{R}$$

$$m: x \mapsto 2x - 1.$$

(a) Sketch the graph of m. Is m a one-one function?

(b) What is the image set of m?

(c) Define the function m^{-1}.

8. The function v is defined by

$$v: [2, 4] \to \mathbb{R}$$

$$v: x \mapsto 4x - x^2.$$

Obtain a definition of the function v^{-1} and sketch its graph.

9. (a) Show that the sine function is not one-one.

(b) Suggest three possible intervals which, when taken as domains for the rule $x \mapsto \sin x$, produce one-one functions.

(c) Experiment with the 'inverse sine' key on your calculator to determine which interval is used for the domain of the rule $x \mapsto \sin x$ in order to produce the inverse function arcsin.

6. Summary and the Way Forward

(i) Summary

Although very little is known about the process of learning from television it _is_ possible to identify a number of possible uses of television in teaching/learning situations in mathematics. However, in order that television should even stand a chance of being a suitable medium for use _in the classroom_ there are a number of obstacles to be overcome. For example, it is very difficult to use _live_ broadcast TV: it is much better if a video-recorder is used. And the use of a television programme in isolation is not particularly suitable: it is essential that it is backed up by adequate support materials. It is also essential that teachers feel enthusiastic about the materials and that they prepare adequately for their use.

The project set out to investigate the potential use in the sixth form of the television programmes from the Open University mathematics foundation course. These programmes already went some way towards overcoming some of the obstacles listed above. It remained to be shown whether the programmes were _relevant_ and _acceptable_ to teachers and whether they could be used effectively _in the classroom_. The first step was to determine whether the programmes _were_ relevant and acceptable to teachers and this was tackled in Chapter 2. In theory, as Table 2.1 (pp.26-32) indicates, most of the programmes, at least in Blocks I, II, III, and V, appear to cover topics specifically mentioned in a number of commonly used A level syllabuses. And the Chelsea Conference (pp.34-35) demonstrated that teachers were very enthusiastic about the programmes - in principle anyway.

Next, were the programmes _suitable_ for use _in the classroom_? At this stage it was necessary to consider the ways in which the programmes might be used and to provide appropriate support materials - using the existing teaching texts. A pilot study, based on the first 15 programmes (excluding TV5 and

TV8), was set up in order to investigate how the Open University materials worked in the classroom, and the results, albeit based on a rather small sample, suggest that these programmes at least, can be used effectively (see Chapters 3 and 4).

Naturally there were some reservations (see Chapter 4(iv) and Chapter 5); and these concerned both the programmes and the support materials. However, since there were no plans to alter the programmes at all, the only way of improving the materials was to revise the written component. Such revision was duly implemented as demonstrated by Chapter 5.

The pilot study suggests therefore that it will be possible to provide a package of Open University materials for use in the sixth form that overcomes almost all the obstacles that have been identified in using television. The only remaining obstacles concern the time that is required by the teacher in preparing to use the materials and the lack of training in using the resource. However, the reactions of teachers suggest that the materials do help to minimize the time required and that this preparation is worthwhile even if it does take time. As for the lack of training, it can only be hoped that such provision will be improved in the future.

(ii) The way forward

The pilot study demonstrates that the M101 programmes - plus support materials - do have potential as a teaching resource for sixth form mathematics but there is still some way to go before they can be made available on a large scale.

First, the support materials require further attention. The pilot study just involved TV1-TV15, and at the moment it is only the materials for TV1-TV4 that have been revised. The remaining draft materials in the pilot study need to be similarly revised. It is also necessary to extend the work that has been done to include other programmes from the course. Draft materials do now exist for some of the programmes in Blocks IV, V and VI but these have yet to be developmentally tested. It might also be a good idea to retest the revised materials for TV1–TV15. It is intended that such work should be carried out over the next year with a view to publishing the package in 1984.

Second, it is necessary to find some mechanism for making the programmes available to teachers. There are essentially two options here.

- Teachers might be asked to record the programmes as they are transmitted during the University year. Alternatively, special arrangements might be made to transmit the programmes during school hours.
- Some means might be found of selling the programmes.

There are difficulties with both options: if teachers are asked to record the programmes then it is necessary to find some way around the licensing problem, if the programmes are to be sold then it is necessary to find some way of selling them at a reasonable cost. Discussions are under way but as yet this problem is unresolved. This is another reason for not making the package available until 1984.

Finally, it is necessary to find some way of making the printed support materials available to teachers at a reasonable cost. Again there is as yet no decision here although there is good reason to suppose that the cost of the package can be kept to about £10. It might be more difficult though to find a suitable means of informing teachers and distributing the materials. This issue is also currently under discussion.

The fact that the materials cannot be made available for widespread use until 1984 should not however be viewed as discouraging since the extra thought and effort put into them at this stage can only be beneficial in the long run. Indeed a recent article on the use of educational television by M. GALLAGHER (1982) suggests that the method adopted for developing the Open University materials for use in the classroom is the best way of ensuring that the resulting package will be useful to teachers and will be implemented succesfully.

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LOGS AND THE AREA UNDER $f:x \mapsto \frac{1}{x}$

This section deals with the link between $\log_e x$ and the corresponding area under $x \rightarrow \frac{1}{x}$. And the rule

$$\log ab = \log a + \log b.$$

can be demonstrated by adding together the corresponding areas. No knowledge of integration is needed. Indeed, this TV programme could well be used as an introduction to integration.

PRE-REQUISITE KNOWLEDGE

Before working through this material students should be familiar with the following:

(i) The idea of logs and knowledge of the rule

$$\log ab = \log a + \log b.$$

Preferably students should know about $\log_e$ - although this *could* be introduced via the programme.

(ii) The idea of a closed interval on the Real line of the form

$$[a, b].$$

(iii) Inequalities of the form $a < b < c$.

(iv) Scaling graphs in the x- and y-directions. Alternatively, in some syllabuses this is known as one-way stretch. Students should have some idea of the fact that an x-scaling of λ transforms $f(x)$ to $(\frac{1}{\lambda}x)$, and a y-scaling of μ transforms $f(x)$ to $\mu f(x)$ (although not necessarily in this form).

Section ??? of this package provides a good illustration of these ideas.

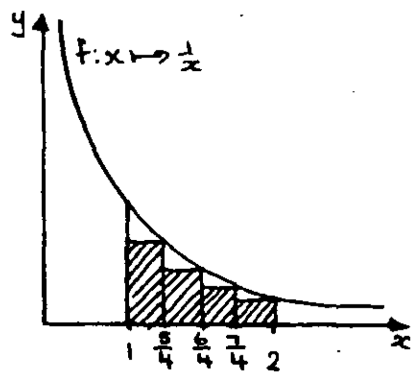
PRE WORK

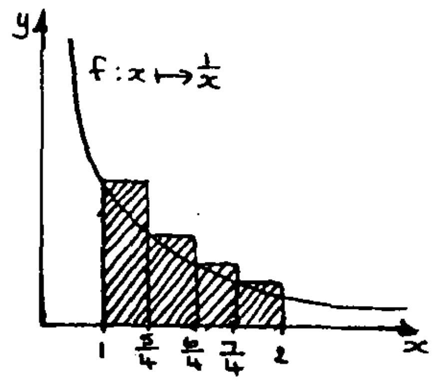
1. (i) Complete the table below and so plot the graph of $f: x \mapsto \frac{1}{x}$.

x	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	6
$\frac{1}{x}$											

(ii) (a) $f(\frac{5}{4})$, (b) $f(\frac{6}{4})$, (c) $f(\frac{7}{4})$.

(iii) Use your answers to Parts (i) and (ii) to calculate the following shaded areas.





2. This question involves logarithms to the base 10.

(i) Use your calculator (log-tables) to find
(a) $\log_{10} 3$, (b) $\log_{10} 6$, (c) $\log_{10} 2$

(ii) Hence find (a) $\log_{10} 18$, (b) $\log_{10} 12$.

(iii) Now write down $\log_{10}(ab)$ in terms of $\log_{10} a$ and $\log_{10} b$.

3. Your calculator may also have a key which gives logarithms to the base e. (e is just a number, its value is about 2.72). These logs are often called Naperian or Natural logarithms and are frequently denoted by Ln. - So the key on your calculator may be labelled Ln.

(i) Find $\log_e 2$ using your calculator. You should get a result in the interval [0.693, 0.694].

(ii) Use your calculator to find
(a) $\log_e 3$, (b) $\log_e 6$.

(iii) From parts (i) and (ii) calculate
(a) $\log_e 3 + \log_e 6$, (b) $\log_e 6 + \log_e 2$.

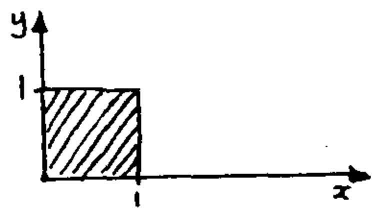
(iv) Use the inverse key on your calculator to check
that (a) $\log_e 3 + \log_e 6 = \log_e 18$
and (b) $\log_e 6 + \log_e 2 = \log_e 12$.

$\boxed{\text{INV}}$ $\boxed{\text{Ln}}$

This demonstrates that the addition rule also works for $\log_e$. That is $\log_e ab = \log_e a + \log_e b$.

In fact this rule works for logs to <u>any</u> base.

4. The shaded square below has area 1.



(i) The square is scaled in the x-direction by a factor of 2. Draw a diagram to represent the resulting shape. What is the area of the resulting rectangle?

(ii) The square is scaled in the y-direction by a factor of $\frac{1}{2}$. Draw a diagram to represent the resulting shape. What is its area?

(iii) The square is first scaled in the x-direction by a factor 2, then scaled in the y-direction by a factor $\frac{1}{2}$. Draw a diagram to represent the resulting square. What is its area?

(iv) The square is now scaled in the x-direction by a factor r, then in the y-direction by a factor s. Draw a diagram to represent the shape. What is its area?

SUMMARY OF THE PROGRAMME

The graph of $x \mapsto \frac{1}{x}$ has the following property. If A(r) denotes the area under the graph from x = 1 to x = r. Then A(r) + A(s) = A(rs). This is the same (apart from the notation), as the logarithm rule

$$\log r + \log s = \log rs.$$

In fact A(r) is the logarithm of r to base e, $\log_e r$. The programme deduces some evidence that A(r) is indeed a logarithm. The logarithm property is demonstrated by weighing pieces of card cut to fit the area and we explain how the area A(2) can be calculated. The result of such a calculation is compared with the tabulated value of $\log_e 2$. The method used involves repeatedly calculating under estimates and over estimates of the area by dividing it into rectangles. Each calculation gives an interval which includes the area. By dividing the area into more rectangles the interval becomes smaller. So we can trap the exact area in a nest of intervals.

The proof that A(r) has the logarithm property uses two results about scaling in the x and y-directions. First, a scaling by a factor s in either direction multiplies the area by s. Second, a scaling by s in the x-direction, followed by a scaling of $\frac{1}{s}$ in the y-direction leaves the graph of $x \mapsto \frac{1}{x}$ unchanged.

POST WORK

No specific post work is necessary for this programme, although it may be helpful to go over the part on scaling at the end of the programme, relating this to the scaling of shapes as in the pre-work. Also, some explanation may be needed as to why a scaling of s in the x-direction followed by a scaling of $\frac{1}{s}$ in the y-direction, leaves the actual graph of $x \mapsto \frac{1}{x}$ unchanged. Perhaps, try it with some specific examples.

e.g. when x = 2, s = 3. etc.

POSSIBLE EXTENSIONS

This programme could clearly lead in to a discussion of integration. Specifically, it could be used to show that

$$\int_1^r \frac{1}{x}dx = \log_e r.$$

with a neat demonstration that $\log_e 1 = 0$.

It draws attention to the fact that, when integrating, one should always use Naperian (Natural) logs rather than $\log_{10}$. The programme also provides a very visual demonstration of the logarithm rule:

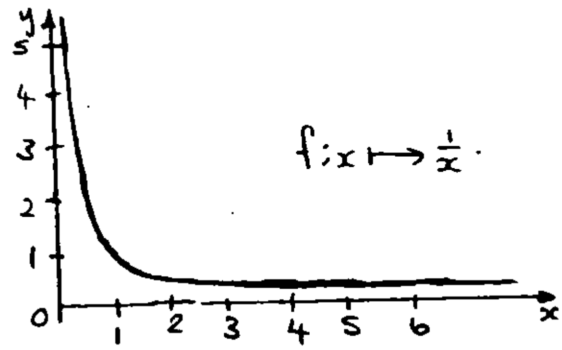
$$\log ab = \log a + \log b.$$

Hence the programme could lead into further work on logs.

SOLUTIONS

1. (i)

x	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	6
$\frac{1}{x}$	5	3	2	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$



(ii) (a) $f(\frac{5}{4}) = \frac{4}{5}$, (b) $f(\frac{6}{4}) = \frac{4}{6}$, (c) $f(\frac{7}{4}) = \frac{4}{7}$.

(iii) (a) Area $= (\frac{1}{4} \times \frac{4}{5}) + (\frac{1}{4} \times \frac{4}{6}) + (\frac{1}{4} \times \frac{4}{7}) + (\frac{1}{4} \times \frac{4}{8}) = 0.634$

(b) Area $= (\frac{1}{4} \times 1) + (\frac{1}{4} \times \frac{4}{5}) + (\frac{1}{4} \times \frac{4}{6}) + (\frac{1}{4} \times \frac{4}{7}) = 0.760$

2. (i) (a) To 4 decimal places $\log_{10} 3 = 0.4771$.
(b) $\log_{10} 6 = 0.7782$
(c) $\log_{10} 2 = 0.3010$.

(ii) (a) $18 = 3 \times 6$, so $\log_{10} 18 = \log_{10} 3 + \log_{10} 6 = 1.2553$
(b) $21 = 6 \times 2$, so $\text{lon}_{10}\ 12 = \log_{10} 6 + \log_{10} 2 = 1.0792$

(iii) $\log_{10} ab = \log_{10} a + \log_{10} b$.

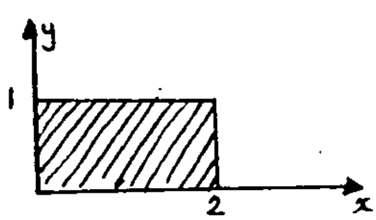
3. (i) $\log_e 2 = 0.6931$ (to 4 decimal places)

(ii) (a) $\log_e 3 = 1.0986$; $\log_e 6 = 1.7918$

(iii) (a) $\log_e 3 + \log_e 6 = 2.8904$
(b) $\log_e 6 + \log_e 2 = 2.4849$

(iv) From part (iii) (a) $2.8904 = \log_e 18$
(b) $2.4849 = \log_e 12$.

4. (i)

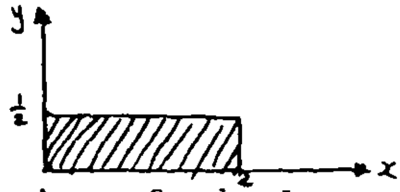


Area $= 2 \times 1 = 2$ sq units.

(ii)

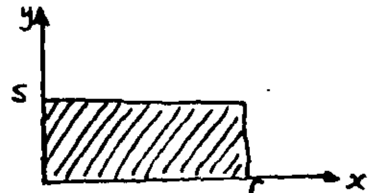
Area = $1 \times \frac{1}{2} = \frac{1}{2}$ sq units.

(iii)



Area = $2 \times \frac{1}{2} = 1$ sq units.
So the resulting area is the same as the original.

(iv)



Area = rs sq units.

APPENDIX 2

USING TELEVISION IN THE

TEACHING OF

SIXTH FORM MATHEMATICS

CONTENTS | PAGE

INTRODUCTION

The Open University Mathematics Foundation Course, M101, was first presented in 1978. It was immediately apparent that there was significant overlap with parts of many A-level syllabuses. Indeed, at the outset there were intentions of making the material more generally available to sixth form teachers. But the costs involved in buying an Open University course are prohibitive for most schools and anyway the teaching texts are designed for self study and would not be suitable for classroom teaching.

However, M101 does incorporate extensive use of television and it was felt that this aspect of the course would be particularly relevant to schools. Needless to say there were still problems - the 32 television programmes were designed as an integral part of the teaching texts and referred specifically to work covered in the written material so it would be difficult to use them out of context. In addition Open University programmes are not broadcast as schools programmes and recording them off air would introduce licence problems. Moreover the cost of buying the television programmes is quite unrealistic (£205 per programme!)

However initial discussion confirmed that sixth form teachers would welcome the opportunity of using television to demonstrate important ideas, so members of the University continued to investigate the feasibility of making the programmes available. Special tapes were made - extracting parts of several programmes, concentrating on the use of computer graphics and computer animations, and forming them into resource packages with accompanying material. But again there are problems in making these special tapes generally available.

Last year we decided to investigate the possibility of using the programmes as they stand, together with supporting written material. It is intended that we should publish the written material as a book and that the price should include a licence fee which will allow schools to record the programmes when they are broadcast in the University year.

This package forms a pilot study for the project. It covers the first half of M101 and mainly involves work leading up to and including the calculus. Tapes of the television programmes may be borrowed from the University (details of obtaining them are given in HOW TO USE THIS

MATERIAL). We hope that you can find time to look at some of the material and use it in your classroom. Any feedback you can supply will be greatly appreciated.

If all goes well we hope to make the material generally available in 1983.

To report feedback or to obtain further information please contact

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MILTON KEYNES

HOW TO USE THIS MATERIAL

This package is designed to be very flexible and so we do not recommend specific ways of using it. However, we have tried to organize the contents into a suitable format which allows you to determine quite quickly when and how you can fit the material into your teaching.

There are thirteen sections:

1. *Symbols and Equations*
2. *The Binomial Theorem*
3. *Trigonometric Formulas*
4. *Inverse Functions*
6. *Rational Numbers and* $\sqrt{2}$
7. *Functions and Graphs*
9. *Iteration and Convergence*
10. $x \mapsto \frac{1}{x}$*: an Area for Revision*
11. *The Derived Function*
12. *The Behaviour of Functions*
13. *The Fundamental Theorem of Calculus*
14. *Taylor Polynomials*
15. *Why* e?

Note: The section numbers correspond to the programme numbers as broadcast. Unfortunately programmes 5 and 8 do not fit easily into the format and have therefore been omitted - hence the gaps).

Whenever possible we have tried to ensure that each section stands on its own. Occasionally though, a section builds specifically on the work of an earlier section, and whenever this happens we suggest students should work through all or part of that earlier section first.

Indeed there are several themes running through the material. These are summarised below for easy reference.

1. ITERATIVE PROCESSES OF SOLVING EQUATIONS. SECTIONS 1, (6), 9.

Iterative processes are introduced in *Section 1: Symbols and Equations* and the issue of whether or not an iterative process converges is followed up in *Section 9: Iteration and Convergence*. Also, *Section 6: Rational Numbers and* $\sqrt{2}$, although not directly related to these sections, does involve one particular iterative method of finding $\sqrt{2}$.

2. GEOMETRIC TRANSFORMATIONS. SECTIONS 3, 7, (10).

In *Section 3: Trigonometric Formulas* we derive the sine and cosine formulas by considering the effect of successive rotations on a point in the Cartesian Plane. The effects of translations, scaling and reflections on various graphs are investigated in *Section 7: Functions and Graphs* and we show how this knowledge can help in graph sketching. Then in *Section 10: $x \mapsto \frac{1}{x}$: an Area for Revision* we examine the area under the graph of $x \mapsto \frac{1}{x}$ and this involves looking at the effect of successive scaling on areas under the curve.

3. FUNCTIONS. SECTIONS 4,7,(12).

Section 4: Inverse Functions investigates when a function has an inverse and, if it has one, how to find it. *Section 7: Functions and Graphs* explores the graphs of various functions in terms of the effects of geometric transformations such as scaling, translations and reflections. Furthermore we show how these transformations can be used to sketch the graphs of quadratics and functions such as $x \mapsto \frac{ax + b}{cx + d}$. Finally *Section 12: The Behaviour of Functions* investigates how to sketch the graph of a polynomial function using calculus techniques.

4. GRAPH SKETCHING. SECTIONS 7, 12.

In *Section 7: Functions and Graphs* we show that scaling, reflections and translations are very useful in sketching graphs of the form $x \mapsto ax^2 + bx + c$ and $x \mapsto \frac{ax + b}{cx + d}$. Then in *Section 12: The Behaviour of Functions* we use calculus techniques to sketch the graphs of various polynomial functions.

5. DIFFERENTIATION: SECTIONS 9, 11, 12, 15.

The idea of a scale factor is introduced in *Section 9: Iteration and Convergence*. This definition of scale factor is all very well for polynomial functions but it breaks down with functions such as sin x. In *Section 11: The Derived Function* we consider a slightly different definition of the scale factor which holds for all functions and which turns out to be the derivative. In the same section we go on to show how the derived function can be constructed geometrically by considering the tangent to the original function. *Section 12: The Behaviour of Functions* discusses how the analysis of the first and second derivatives is used in sketching the graph of a function. Finally, in *Section 15: Why e?*, we look at the significance of the functions e^x and $\log_e x$ by examining the derivatives of a^x and $\log_a x$.

6. INTEGRATION: SECTIONS 10, 13

The idea of the area under a curve is first introduced in *Section 10: $x \mapsto \frac{1}{x}$: An Area for Revision* where we show that the area under $x \mapsto \frac{1}{x}$ between $x = 1$ and $x = r$ is $\log_e r$. *Section 13: The Fundamental Theorem of Calculus* builds upon the idea of areas under a curve to demonstrate diagrammatically the Fundamental Theorem of Calculus.

SECTION FORMAT

At the beginning of each section you will find a summary of the associated TV programme with an indication of the time spend at each stage. Each TV programme has a total length of about 23 minutes. Immediately after the Programme Summary we indicate the pre-requisites which students will need in order to gain maximum benefit from the work in that section. Occasionally we suggest here that students should have worked through a previous section. Usually this is because a direct reference is made back to the work in that particular section.

The rest of the section consists of exercises for students to work through before and after the programme. To save time, these have been laid out to facilitate copying. Not all the questions are essential but those marked with an asterisk should certainly be tackled before watching the television. Sometimes this is because the question is specifically referred to, or it may be because we adopt a particular approach to a topic which could differ from that already encountered in the classroom. And occasionally students need to be familiar with a particular type of notation.

(Note At the end of each section full solutions are included for all exercises.)

Finally we suggest possible ways in which the work may be extended. These include references to other sections which may be relevant. Of course we recognise that our suggestions are by no means exhaustive; they are included just to set you thinking.

REFERENCES TO M101

The TV programmes were designed as an integral part of M101: Maths a Foundation Course and specific references are made to the teaching texts associated with that course. (Such references take the form Block A: Unit B, Section N). This is usually to remind Open University students of where they have seen an idea before. In many such instances students in your class will have met this idea earlier in the syllabus and sometimes, if the reference is more specific, we have tried to ensure that there is an opportunity for them to become familiar with it immediately beforehand in the pre-programme work. Thus we feel that the odd reference to an Open University text should not affect the general flow of the programme and should not detract from its value.

OBTAINING THE PROGRAMMES

It is intended that eventually teachers will be able to tape the television programmes as they are broadcast. This package is only a pilot study and VHS cassettes (or whatever you require) can be borrowed from

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We shall try to ensure that you receive the tape(s) promptly. It would be appreciated if tapes could be returned immediately after use and preferably no more than two weeks after they are borrowed.

WHERE DOES THE MATERIAL FIT IN TO THE SYLLABUS?

We recognize that M101 does not correspond to any particular A-level syllabus, but it does cover areas which are common to all sixth form courses (e.g. in calculus) and other topics which are now found in many syllabuses. Unfortunately, in order to determine exactly what each programme has to offer and how it can be used, it will be necessary to watch the programme yourself. Nevertheless we hope that the TV summary, together with the pre-programme and post-programme exercises, give you a good indication of what each section involves.

You will then be in a position to decide which ones fit into the syllabus which you are currently teaching and where and when they can be used in your work in the classroom.

1. SYMBOLS AND EQUATIONS

PROGRAMME SUMMARY

This programme tackles two problems about international paper sizes. The main purpose is to show the process of changing a given practical problem into a mathematical one in the form of an equation. The programme also introduces an iterative method of solving equations which can be further explored in the follow up work.

The programme begins with a reference to a problem about an exhibition organizer, which students should have tackled beforehand. (See the suggested Prework.)

| 3 mins | The first problem is to find the ratio of the

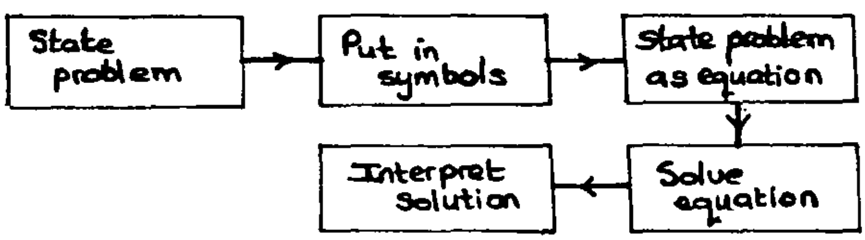
sides of a piece of A size paper (A3, A4, A5 etc.). We begin by investigating this in a practical sense.

| 3 mins | Symbols are introduced and the problem is expressed

algebraically as an equation - and consequently solved.

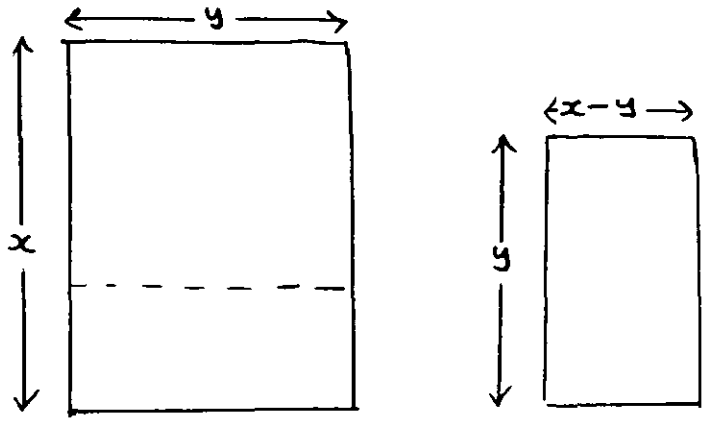
| 2 mins | We reflect on the process of problem solving with

reference to the framework below.



| 5 mins | The second problem is to find the ratio of sides of a

rectangle such that after cutting off a square of side equal to the width of the rectangle, the resulting shape is the same as the original one.



This leads to the golden section number ($\simeq 1.618$). The problem is tackled with reference to the same framework and is solved using the quadratic formula.

5 mins

But what happens in situations where we can't use a formula? Well, there are other methods, for example, the Bisection method. (Note: this is only an aside reference; it is not needed in any of the associated work. For more details see *SECTION 6: RATIONAL NUMBERS AND $\sqrt{2}$*). The method of formula iteration is introduced. It is demonstrated by solving (again) the quadratic equation $r^2 - r - 1 = 0$ (from the problem above) using the re-arrangement

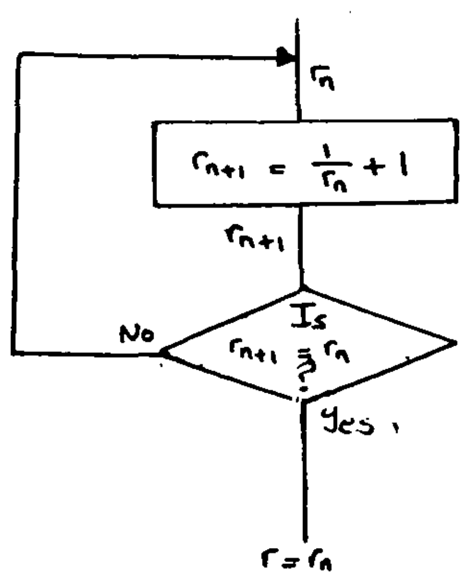
$$r = \frac{1}{r} + 1.$$

3 mins

The notation for iteration formula

$$r_{n+1} = \frac{1}{r_n} + 1$$

is explored with reference to the flow diagram



$\boxed{\text{2 mins}}$ Finally we introduce a problem to be tackled after the programme. This concerns the overall stopping distances of a car travelling at various speeds - to be found on the back of the High way Code.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) solution of quadratic equations of the form $ax^2 + bx + c = 0$ by factorization and using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(ii) ratio and proportion

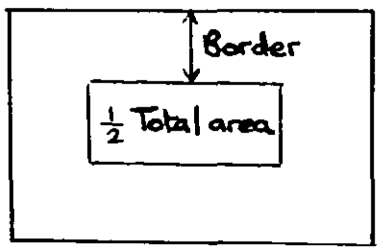
(iii) use of a calculator to evaluate expressions of the form

$$\frac{1}{r} + 1, \quad \frac{1}{r - 1}, \quad \sqrt{\frac{100 - v}{0.05}}, \text{ etc.}$$

(iv) subscript notation.

PRE-PROGRAMME WORK

*1. An exhibition organizer for a local craft exhibition is offered a room with floor dimensions 12 metres by 16 metres. He wishes to arrange exhibits around the sides of the room, leaving space for people to circulate in the centre. The local regulations state that at least half the floor area must be left clear.



How wide must he make the border so that half the total area is clear in the centre?

(i) Using x-metres as the width of the border write down expressions for
   (a) the length of the clear floor space
   (b) the width of the clear floor space
   (c) the area of the clear floor space.

(ii) The value of x is restricted by physical considerations. Between which whole numbers must the value of x lie?

(iii) The total floor area is (16 × 12) sq metres = 192 sq metres. Use this to write down an equation expressing the fact that the clear space must have an area equal to half the total floor area.

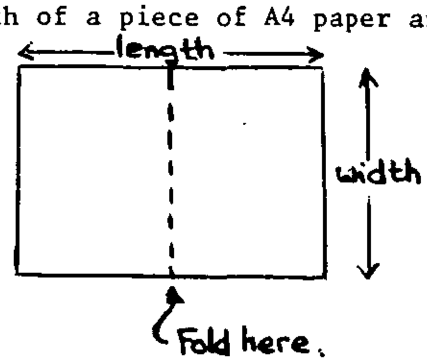
(iv) Solve the resulting problem by factorizing. What does this mean in terms of the original problem?

2. How wide should the border be if the room has floor dimensions 15 metres by 20 metres?

3. How wide should the border be if the room has floor dimensions 14 metres by 14 metres?
(Hint: you will need to use the quadratic formula here.)

4. (i) Measure the length and width of a piece of A4 paper and calculate the ratio $\frac{\text{longer side}}{\text{shorter side}}$.

(ii) Fold the paper in half and repeat the above calculation.



5. The following equation involves two symbols x and y:

$$\frac{x}{y} = \frac{x + y}{x}$$

Replace $\frac{x}{y}$ by r to obtain an equation just involving r.

POST-PROGRAMME WORK

6. The stopping distance, d feet, of a car travelling at v miles an hour is given by the formula

$$d = 1.0v + 0.05v^2.$$

To find out how fast you may drive and still stop in 100 feet requires a solution of

$$1.0v + 0.05v^2 = 100.$$

(i) Use the rearrangement

$$v_{n+1} = 100 - 0.05v_n^{\ 2}$$

with starting value

$$v_1 = 40$$

to try to solve the equation.

(ii) Use the rearrangement

$$v_{n+1} = \sqrt{\frac{100 - v_n}{0.05}}$$

and a starting value of

$$v_1 = 40$$

to try to solve the equation.

7. (i) The original form of $r^2 - r - 1 = 0$ in the television programme was

$$r = \frac{1}{r - 1}.$$

This rearrangement was not used in the formula iteration method. This exercise explains why.

Use $r_{n+1} = \frac{1}{r_n - 1}$ with starting values

(a) 1 (b) 0.5 (c) 2

to try to solve $r^2 - r - 1 = 0$.

(ii) Investigate what happens with the rearrangement

$$r_{n+1} = r_n^{\,2} - 1$$

with starting values (a) 1 (b) 0.5 (c) 2.

POSSIBLE EXTENSIONS

1. This section provides an introduction to iterative methods of solving equations. Usually there are many ways of rearranging the equation into a form suitable for iteration. For some rearrangements the iterative process converges; for others it diverges. Such processes could be explored further. For example is it possible to predict in advance whether an iteration process converges or diverges for a given rearrangement?

   NOTE SECTION 9: ITERATION AND CONVERGENCE investigates further the variation in behaviour of formula interations.

2. This section could also lead to work on computer programs to find solutions of equations using iterative methods.

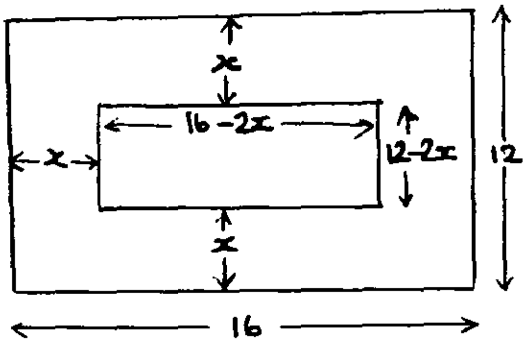
3. You may also wish to discuss the need for numerical methods of solving equations. Although we illustrated the method for a quadratic equation it also works for higher order equations where there are no formulas to make things easier.

4. This section shows that some equations do not have exact solutions. However it is always possible to find the solution as accurately as we please. What's special about numbers like these? This could lead to a discussion of irrational numbers.

   NOTE SECTION 6: RATIONAL NUMBERS AND $\sqrt{2}$ might be useful here. This section also introduces another iterative method of finding $\sqrt{2}$; this involves repeatedly subdividing the interval [1, 2] into 10 equal intervals.

SOLUTIONS

1. (i)



(a) Length = $16 - 2x$

(b) Width = $12 - 2x$

(c) Area = $(16 - 2x)(12 - 2x) = 192 - 56x + 4x^2$ sq m.

(ii) From the diagram x must be positive and x must be less than half the width, 12. Thus the value of x must be between 0 and 6.

(iii) $192 - 56x + 4x^2 = \frac{1}{2}(192) = 96.$

This can be rearranged as

$$4x^2 - 56x + 96 = 0.$$

(iv) The left-hand side factorizes to give

$$4(x - 2)(x - 12) = 0.$$

This has solutions $x = 2$ or $x = 12$. But the value of x must lie between 0 and 6. So the solution must be $x = 2$. That is, the border width should be 2 metres. (Check by substitution.)

2. Here the value of x must lie between 0 and $\frac{15}{2}$ (= 7.5).

The area of the clear floor space is given by $(20 - 2x)(15 - 2x)$ or $300 - 70x + 4x^2$ sq m.

Half the total floor area is $\frac{1}{2}$ (300) = 150, so the equation is

$$300 - 70x + 4x^2 = 150 \quad \text{or} \quad 4x^2 - 70x + 150 = 0.$$

The left-hand side factorizes to give

$$2(2x - 5)(x - 15) = 0$$

which has solutions $x = 2.5$ or $x = 15$. Since the value of x must lie between 0 and 7.5, the solution we want is $x = 2.5$. This means that the border width should be 2.5 metres. (Check this result by substitution.)

3. Here, the value of x must lie between 0 and $\frac{14}{2}$ (= 7).

Using the same method we get the equation

$$4x^2 - 56x + 98 = 0$$

or $\quad 2(2x^2 - 28x + 49) = 0.$

The solutions to $2x^2 - 28x + 49 = 0$ are given by

$$x = \frac{28 \pm \sqrt{(28)^2 - (4)(2)(49)}}{4}.$$

Hence $x = \dfrac{28 \pm 14\sqrt{2}}{4} = 7 \pm \dfrac{7}{2}\sqrt{2}$.

(Note: You may like to draw attention to the fact that using a calculator to evaluate this expression gives only an approproximate value for the solution. It is not possible to obtain an exact numerical solution, but it is possible to find the solution as accurately as we please - using iterative methods like the one introduced in the programme.)

Again, the physical constraints on x mean that the solution we require is

$$7 - \frac{7}{2}\sqrt{2} \simeq 2.05 \text{ (to 2 decimal places)}$$

so that the border width must be 2.05 metres.

(Note: This is accurate enough for practical purposes.)

4. (i) Length = 29.6 cm (to 1 decimal place)

Width = 20.9 cm (to 1 decimal place)

Ratio $= \dfrac{29.6}{20.9} = 1.4$ (to 1 decimal place).

(ii) Length = 20.9 cm (to 1 decimal place)

Width = 14.8 cm (to 1 decimal place)

Ratio $= \dfrac{20.9}{14.8} = 1.4$ (to 1 decimal place).

5. On the right-hand side divide top and bottom by y to obtain

$$\frac{x}{y} = \frac{\frac{x}{y} + \frac{y}{y}}{\frac{x}{y}} = \frac{\frac{x}{y} + 1}{\frac{x}{y}} .$$

Now put $r = \dfrac{x}{y}$ to obtain

$$r = \frac{r + 1}{r} \text{ or } r = 1 + \frac{1}{r} .$$

(Note: The television programme involves manipulation of this type and also uses the fact that $r = 1 + \frac{1}{r}$ is a rearrangement of $r^2 - r - 1 = 0$.)

6. (i) You should have found that the values did not settle down.

$v_1 = 40$

$v_2 = 20$

$v_3 = 80$

$v_4 = -220$ (!) etc.

This process seems unlikely to lead to a solution.

(ii) This time the results are more helpful:

$v_1 = 40, v_2 = 34.64, v_3 = 36.15, v_4 = 35.73, v_5 = 35.85,$

$v_6 = 35.82, v_7 = 35.83, v_8 = 35.83.$

v_7 and v_8 agree to two decimal places. Thus we have an approximate solution of 35.83 mph.

(Check this result by substitution.)

7. (i) The results for $r_{n+1} = \frac{1}{r_n - 1}$ are given below.

(a) $r_1 = 1$. Error calculating r_2 (since it involves dividing by zero).

(b) $r_1 = 0.5$, $r_2 = -2$, $r_3 = -0.3333333$, $r_4 = -0.75$,

$r_5 = 0.5714286$ ... $r_9 = -0.6170213$,

$r_{10} = -0.6184211$, $r_{11} = -0.6178862$

r_{10} and r_{11} agree to three decimal places so $r = -0.618$ is an approximate solution.

(c) $r_1 = 2$, $r_2 = 1$. Error calculating r_3. (since it involves dividing by zero).

This rearrangement has found the negative solution to

$$r^2 - r - 1 = 0.$$

But in the programme we were interested only in positive values of r. So this rearrangement is not very useful.

(ii) The results for $r_{n+1} = r_n^2 - 1$ are given below.

(a) $r_1 = 1$, $r_2 = 0$, $r_3 = -1$, $r_4 = 0$, $r_5 = -1$ and so on:

The values continue to alternate between 0 and -1.
(Note: These are not solutions to $r^2 - r - 1 = 0$.)

(b) $r_1 = 0.5$, $r_2 = 0.75$, $r_3 = -0.4375$, $r_4 = -0.8085938$

$r_5 = -0.3461761$ ... $r_{15} = -1$, $r_{16} = 0$, $r_{17} = -1$ ...

After several iterations the values alternate between -1 and 0.

(Note: These are not solutions to $r^2 - r - 1 = 0$.)

(c) $r_1 = 2$, $r_2 = 3$, $r_3 = 8$, $r_4 = 63$, $r_5 = 3968$, $r_6 = 1574502$ ..

The values do not settle down and so the process is unlikely to lead to a solution.

2. THE BINOMIAL THEOREM

PROGRAMME SUMMARY

Two alternative methods of obtaining the expansion of $(a + b)^6$ are investigated. One of these methods involves the use of Pascal's triangle. The notation ^nC_r is introduced and the construction of Pascal's triangle is represented by the equation

$$^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}.$$

| 2 mins |

How can we expand $(a + b)^n$? Before the programme students should have seen how to expand this expression for small values of n. We look at the special case where n = 6. What does this mean?

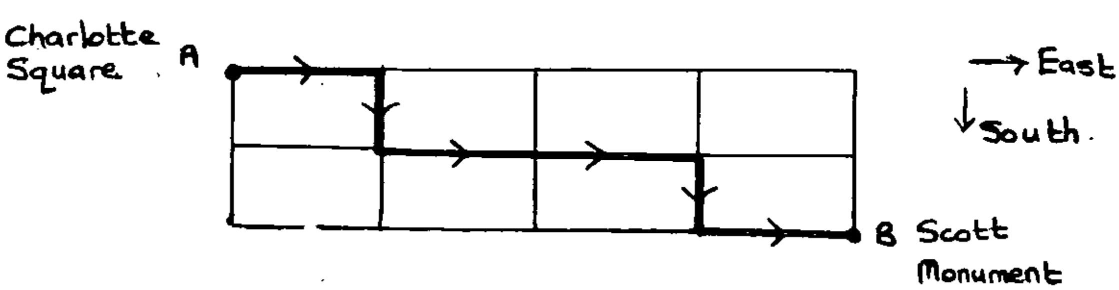
| 6 mins |

We investigate the simpler cases where n = 2 and n = 3. The expansion of $(a + b)^6$ involves the same principle - choosing symbols from each of the brackets in turn. The coefficient of each term (for example a^4b^2) indicates the number of different ways of choosing those particular symbols.

| 5 mins |

The idea of choosing symbols is compared with the number of ways of getting from one grid point to another (in a minimum number of steps).

This is illustrated by means of streets in Edinburgh.

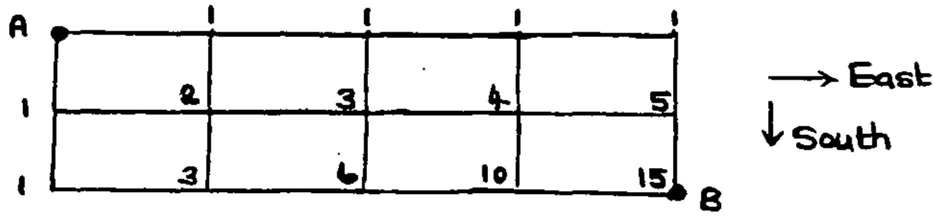


| 1 min |

The coefficient of a^4b^2 and the number of routes from A to B in the diagram above both involve the number of ways of choosing 2 things out of 6.

| 2 mins |

The route problem is easier to visualize. We show that the number of routes to any point is obtained by adding together the number to the two adjacent preceding grid points. This leads to the following diagram.



3 mins

Moving south corresponds to choosing the symbol b; moving east corresponds to choosing the symbol a. By expanding the grid we can obtain all the coefficients in the expansion of $(a + b)^6$.

1 min

The link between Pascal's triangle and the Binomial Expansion of $(a + b)^n$ for various values of n is explored further by looking at the diagonals on the grid.

3 mins

The notation ${}^nC_r$ is introduced to stand for a general point on the grid system on the nth diagonal and the rth row. The structure of Pascal's triangle can then be explained using the equation

$${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$$

and the fact that the numbers on the border of the triangle are all 1s.

<u>PRE-REQUISITES</u>

Before working through this section students should be familiar with the following:

(i) expansion of brackets of the form

$$(a + b)^2 \text{ and } (a + b)(a^2 + 2ab + b^2)$$

(ii) writing down coefficients of algebraic terms

(iii) some idea of symmetry.

PRE-PROGRAMME WORK

*1. (i) Find the expansion of $(a + b)^2$.

(ii) Find the expansion of $(a + b)^3$ by writing

$$(a + b)^3 = (a + b)(a + b)^2.$$

From this you may guess that the expansion of $(a + b)^n$ has the form

$$(a + b)^n = a^n + \boxed{}\, a^{n-1}\, b + \boxed{}\, a^{n-2}\, b^2 + \ldots + \boxed{}\, a\, b^{n-1} + b^n.$$

The general formula for the coefficients in this expression is called the Binomial Theorem.

*2. One way of obtaining the expansion of $(a + b)^2$ is to write it as $(a + b)(a + b)$ and multiply out the brackets. Alternatively, every constribution to the expansion is the product of two symbols — an a or b from the first bracket multiplied by an a or b from the second bracket. The term involving a^2 can be obtained in only 1 way by choosing an a from each bracket, so the coefficient of a^2. is 1. Similarly the coefficient of b^2 is 1. On the other hand, the product ab can be obtained in two ways, either by choosing an a from the first bracket and a b from the second or by choosing a b from the first bracket and an a from the second. Hence the coefficient of ab is 2.

(i) Use an argument similar to this to explain the expansion of $(a + b)^3$.

(ii) The expansion of $(a + b)^4$ may also be obtained using two methods.

(a) Use your solution to Question 1 (ii) to expand

$$(a + b)^4 = (a + b)(a + b)^3.$$

(b) Use the method of choosing symbols to obtain the coefficient of a^2b^2 in the expansion of $(a + b)^4$.

WORK TO BE TACKLED DURING OR AFTER THE PROGRAMME

3. The expansion of $(a + b)^6$ is

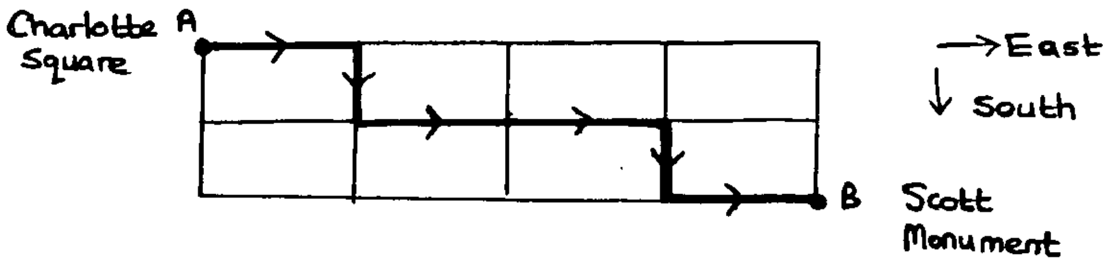
$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

Use this expression to find the number of ways of choosing

(i) three b s (ii) four b s

from six brackets (and a s from the remaining brackets).

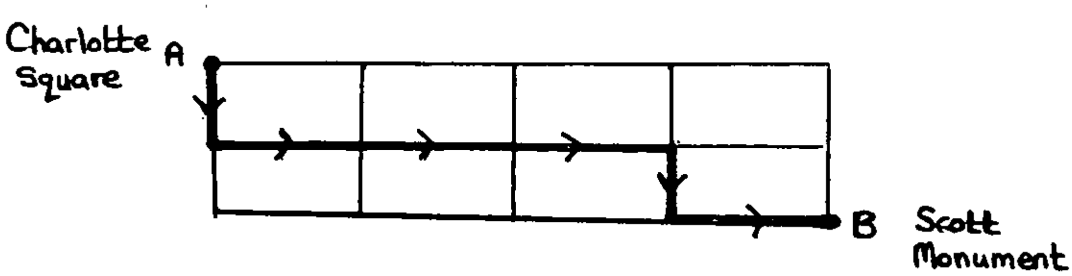
4. The route chosen in the programme from Charlotte Square to the Scott Monument was



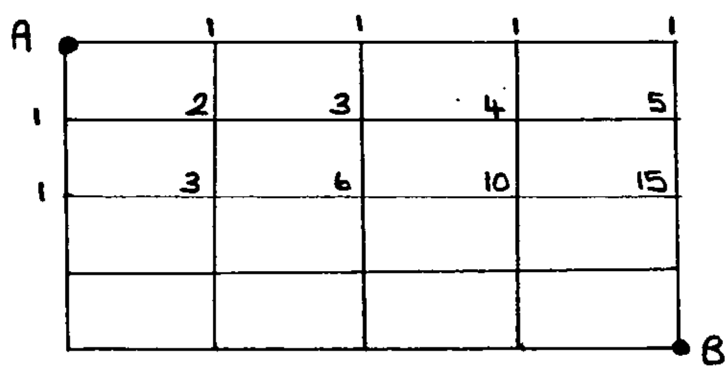
This corresponds to the following choices of direction at each junction.

E	S	E	E	S	E

what choices correspond to the following route?



5. (i) Complete the following diagram by giving the number of shortest routes from A to each of the remaining junctions.



(ii) Use your answer to Part (i) to write down

(a) the coefficient of a^3b^2 in the expansion of $(a + b)^5$

(b) the coefficient of a^3b^4 in the expansion of $(a + b)^7$

(c) the coefficient of a^4b^4 in the expansion of $(a + b)^8$

6. Use Pascal's Triangle to write down the expansion of $(a + b)^5$.

POST-PROGRAMME WORK

7. Work out Pascal's triangle up to and including the seventh diagonal.

8. Use Pascal's triangle to write down the expansion of $(a + b)^7$.

9. (i) By substituting $3t = a$ and $4s = b$ obtain the expansion of $(3t + 4s)^5$.

   (ii) By writing $2v - u$ as $2v + (-u)$ obtain the expansion of $(2v - u)^6$.

The symmetry of Pascal's triangle means we can write the expansion of $(a + b)^n$ as

$$^nC_0\, a^n + {}^nC_1\, a^{n-r}\, b + \ldots + {}^nC_r\, a^{n-1}\, b^r + \ldots + {}^nC_{n-1}\, a\, b^{n-1} + {}^nC_n\, b^n.$$

This expansion is called a Binomial Expansion and the numbers, ^nC_r, are called Binomial Coefficients. The result which enabled us to write down the expansion is called the Binomial Theorem.

POSSIBLE EXTENSIONS

1. The section may be used as a starting point to discuss other number patterns in Pascal's triangle, which is more usually written as

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \end{array}$$

. . . .

2. ${}^nC_r$ represents the number of ways of choosing r things from n. The programme provides only an introduction to this concept and, expressions of this form could be further investigated. In particular, using the relationship

$$ {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} $$

it can be shown that

$$ {}^nC_r = \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r(r-1)(r-2)\ \dots\ \times 3 \times 2 \times 1} $$

or $${}^nC_r = \frac{n!}{r!(n-r)!}$$

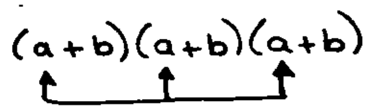
3. The Binomial Expansion may be used to find approximate values for expressions of the form $(1.01)^n$ or $(0.99)^n$.

SOLUTIONS

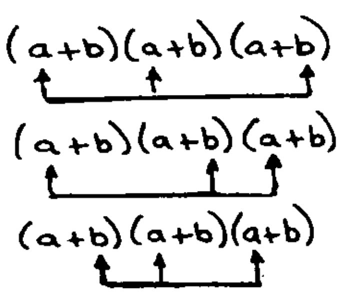
1. (i) $(a+b)^2 = a^2 + 2ab + b^2.$

(ii)
$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a(a^2+2ab+b^2) + b(a^2+2ab+b^2)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

2. (i) Every term in the expansion involves the muliplication of three symbols - an a or b from the first bracket, an a or b from the second bracket and an a or b from the third bracket. The term involving a^3 can be obtained in only one way, namely by choosing an a from each bracket.



Hence a^3 has a coefficient of 1. The term involving a^2b is obtained by choosing an a from two brackets and a b from one.

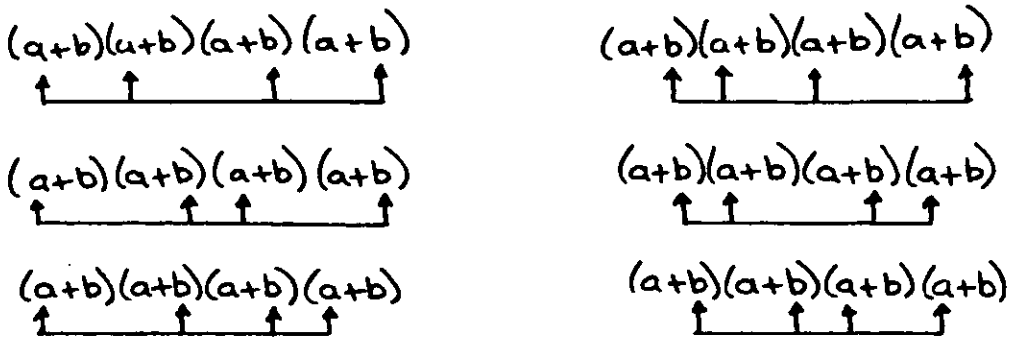


There are three ways of doing this so the coefficient of a^2b is 3.

Similary, the coefficient of ab^2 is 3 and the coefficient of b^3 is 1.

(ii) (a)
$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)^3\\ &= (a+b)(a^3+3a^2b+3ab^2+b^3)\\ &= a(a^3+3a^2b+3ab^2+b^3)\\ &\quad + b(a^3+3a^2b+3ab^2+b^3)\\ &= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\end{aligned}$$

(b) We can choose a's from two brackets and b's from the remaining two brackets in six ways.



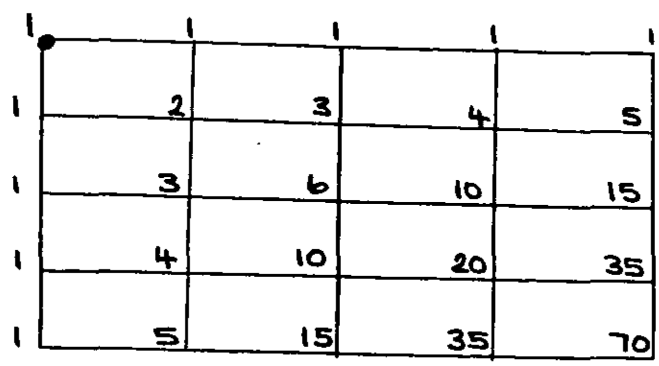
3. (i) The number of ways of choosing three b s and 3 a s is the coefficient of a^3b^3, 20.

(ii) The number of ways of choosing four b s and two a s is the coefficient of a^2b^4, 15.

4.

S	E	E	E	S	E

5. (i)



(ii) (a) 10 (b) 35 (c) 70

6. Taking the coefficients from the fifth diagonal in Pascal's triangle

1	1	1	1	1	1
1	2	3	4	5	
1	3	6	10		
1	4	10			
1	5				
1					

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

7.

1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	
1	3	6	10	15	21		
1	4	10	20	35			
1	5	15	35				
1	6	21					
1	7						
1							

8. $(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$

9. (i) $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$

So $(3t + 4s)^5 = (3t)^5 + 5(3t)^4(4s) + 10(3t)^3(4s)^2 + 10(3t)^2(4s)^3 + 5(3t)(4s)^4 + (4s)^5$

$= 243t^5 + 1620t^4s + 4320t^3s^2 + 5760t^2s^3 + 3840ts^4 + 1024s^5.$

(ii) $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

Putting $2v = a$ and $-u = b$,

$$(2v - u)^6 = (2v)^6 + 6(2v)^5(-u) + 15(2v)^4(-u)^2 + 20(2v)^3(-u)^3 + 15(2v)^2(-u)^4 + 6(2v)(-u)^5 + (-u)^6$$

$$= 64v^6 - 192v^5u + 240v^4u^2 - 160v^3u^3 + 60v^2u^4 - 12vu^5 + u^6$$

3. TRIGONOMETRIC FORMULAS

PROGRAMME SUMMARY

The formulas

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta.$$

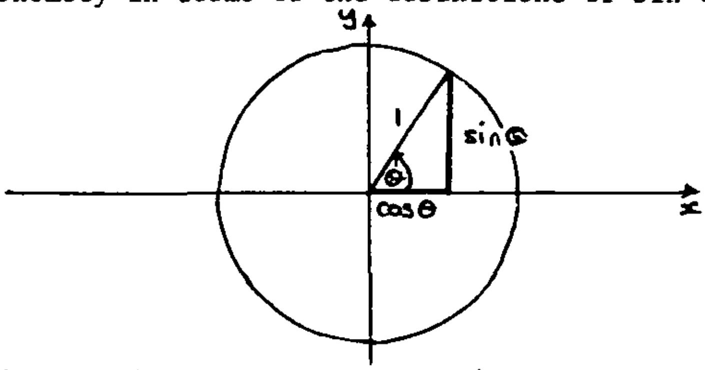
$$\sin(\alpha + \beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta.$$

may be obtained by considering a rotation of the point (1, 0) through α followed by a rotation through β.

. . .

| 5 mins | We begin by reviewing the link between rotations and

trigonomety in terms of the definitions of $\sin\theta$ and $\cos\theta$.



and in relation to the graphs of $\sin\theta$ and $\cos\theta$.

(Note: θ is measured in radians.)

. . .

| 3 mins | The graphs look very similar and together with the original

definition this leads to the properties.

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

and $$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta.$$

. . .

| 3 mins | What happens when the point (1, 0) is rotated through

$(\alpha + \beta)$? We can look at this in two ways - first as a rotation through $(\alpha + \beta)$; second as a rotation through α followed by a rotation through β. This is how we derive the formulas we want. The notation r_α for rotation through α is introduced.

| 4 mins | We consider the effect of the rotation r_θ on the coordinates

(1, 0), (x, 0), (0, 1) and (0, y). With the help of translations we can determine the effect of r_θ on the general point (x, y) .. as

$(x, y) \mapsto (x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$.

3 mins

.... And so we get the required formulas.

5 mins

We turn our attention to tan θ. What's the meaning of tan θ? It comes from the definition

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

But we can also express tan θ as a single length with the help of another transformation - a dilation (or scaling). This allows us to explore the graphical properties of tan θ.

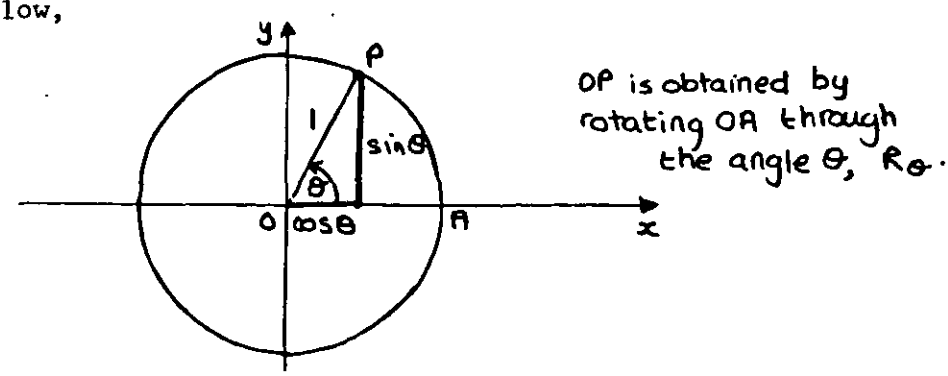
PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) the effect of a translation of (a, b) on the point with co-ordinates (u,v) as movement to the point with coordinates $(u+a, v+b)$.

(ii) the effect of a dilation (or scaling) with centre the origin on the point with coordinates (u, v) as movement to the point with coordinates $(\lambda u, \lambda v)$.

(iii) the definition of cos θ and sin θ respectively as the x- and y-coordinates of the point P on the unit circle illustrated below,



(iv) the graphs of $x \mapsto \sin x$ and $x \mapsto \cos x$

(v) the definition of the radian measure of an angle as arc length over radius

(vi) the use of a scientific calculator.

PRE-PROGRAMME WORK

1. A first guess at formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ might be

$$\cos(\alpha + \beta) = \cos\alpha + \sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha + \sin\beta.$$

Find suitable values of α and β to show that these formulas do not hold.

2. Two translations f and g are specified by the rules

$$f: (x, y) \mapsto (x + 3, y - 1)$$

$$g: (x, y) \mapsto (x - 2, y + 4).$$

(i) Apply f to (0, 0) and then apply g to the result. What are the coordinates of the final point?

(ii) Apply f to (2, 3) and then apply g to the result. What are the coordinates of the final point?

(iii) (a) What point do you obtain by applying f to (x, y) and then applying g to the result.

(b) Describe the overall effect of applying f then g.

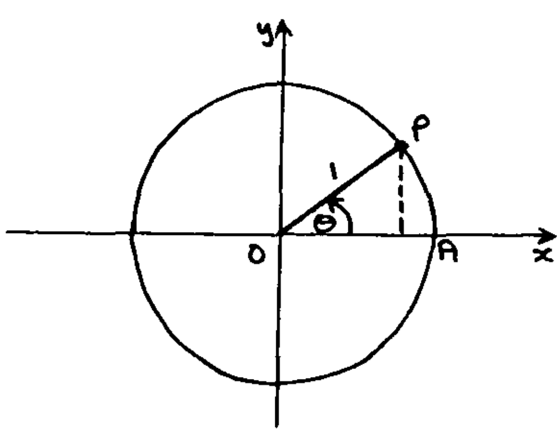
3. (i) Use your calculator to complete the table below. (Note: θ is measured in radians so make sure your calculator is in the appropriate mode.)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$																	
$\cos\theta$																	

(ii) Use the table to draw the graph of $\theta \mapsto \sin\theta$.

(iii) Use the table to draw the graph of $\theta \mapsto \cos\theta$.

4.



OA = 1

On this diagram P is obtained by rotating the line OA through an angle θ. Thus, since OA has unit length, P is a point on the circumference of a circle of unit radius.

Use this definition (and Pythagoras' Theorem) to obtain $\cos\theta$ and $\sin\theta$ for each of the following angles.

(i) (a) $\theta = 0$ (b) $\theta = \frac{\pi}{2}$ (c) $\theta = \pi$ (d) $\theta = \frac{3\pi}{2}$.

(ii) (a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{3\pi}{4}$.

WORK TO BE TACKLED DURING OR AFTER THE PROGRAMME

5. (i) Check the formula for rotation through an angle θ,

$$(x, y) \mapsto \big((x\cos\theta - y\sin\theta), (x\sin\theta + y\cos\theta)\big)$$

by finding the image of the point $(1, 0)$.

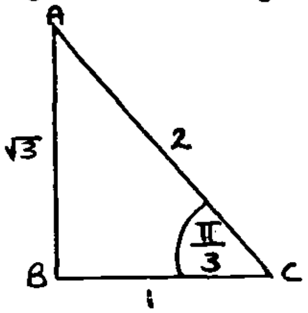
(ii) Use Part (i) to find the effect of a rotation through $\frac{\pi}{2}$ on the point $(1, 0)$.

6. The programme yields the two formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

(i) Use Part (ii) of Question 4 to write down (a) $\cos\frac{\pi}{4}$ and (b) $\sin\frac{\pi}{4}$.

(ii) Use the triangle below to write down (a) $\cos\frac{\pi}{3}$ and (b) $\sin\frac{\pi}{3}$



(iii) Use Parts (i) and (ii) and the formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ to evaluate

(a) $\cos\frac{7\pi}{12}$ and (b) $\sin\frac{7\pi}{12}$.

POST-PROGRAMME WORK

7. The formula for $\cos(\alpha + \beta)$ is

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

(i) (a) Replace β by α to obtain a formula for $\cos 2\alpha$.

(b) Use the result

$$\cos^2\alpha + \sin^2\alpha = 1$$

to obtain a formula for $\cos 2\alpha$ in terms of $\cos\alpha$ alone.

(c) Find a formula for $\cos 2\alpha$ in terms of $\sin\alpha$ alone.

(ii) Treat $(\alpha - \beta)$ as $(\alpha + (-\beta))$ to obtain a formula for $\cos(\alpha - \beta)$.

8. The formula for $\sin(\alpha + \beta)$ is

$$\sin(\alpha + \beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta.$$

(i) Replace β by α to obtain a formula for $\sin 2\alpha$.

(ii) Treat $(\alpha - \beta)$ as $(\alpha + (-\beta))$ to obtain a formula for $\sin(\alpha - \beta)$.

9. Tan θ is defined as $\dfrac{\sin\theta}{\cos\theta}$ $(\cos\theta \neq 0)$.

Hence $\tan(\alpha + \beta) = \dfrac{\sin(\alpha + \beta)}{(\alpha + \beta)}$.

(i) Write down a formula for $\tan(\alpha + \beta)$ involving $\cos\alpha$, $\sin\alpha$, $\cos\beta$ and $\sin\beta$.

(ii) Divide top and bottom throughout by $\cos\alpha \cos\beta$ to get a formula involving $\tan\alpha$ and $\tan\beta$.

10. (i) Find a formula for $\cos 3\alpha$ in terms of $\cos 2\alpha$, $\sin 2\alpha$, $\cos\alpha$ and $\sin\alpha$.

(ii) Use your solution to Questions 7 and 8 to express $\cos 3\alpha$ in terms of $\cos\alpha$ alone.

11. (i) Use the formula for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$ (Question 8) to prove that

$$\sin(\alpha + \beta) - \sin(\alpha + \beta) = 2\cos\alpha \sin\beta.$$

(ii) By writing A for $(\alpha + \beta)$ and B for $(\alpha - \beta)$ prove that

$$\sin A - \sin B = 2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right).$$

POSSIBLE EXTENSIONS

1. The post programme work could be extended to cover other formulas such as.

   $\sin A + \sin B$; $\cos A + \cos B$, $\cos A - \cos B$;

   $\tan(\alpha - \beta)$; $\tan 2\alpha$; and so on.

   You may wish to use the formulas to obtain the exact values of expression such as.

   $\sin \frac{\pi}{12}$; $\cos \frac{\pi}{12}$;

   $\sin \frac{\pi}{8}$; $\cos \frac{\pi}{8}$ $\quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = 2\cos^2(\frac{\pi}{8}) - 1$.

   etc., and so on.

2. This section could be used as a basis for further work on rotations. For example, the links between a rotation through α (centre the origin) and the matrix

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

   might be explored.

3. Also - the effect of a rotation through θ, centre (a, b), on a general point (x, y) could be explored.

4. Reflections were not mentioned in the programme. Nevertheless they can be explored using a similar approach. In particular, reflection in any line passing through the origin making an angle θ with the x-axis can be represented by the transformation.

$$(x, y) \mapsto (x\cos(2\theta) + y\sin(2\theta),\ x\sin(2\theta) - y\cos 2\theta)$$

   or the matrix

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

NOTE *SECTION 7: FUNCTIONS AND GRAPHS looks at the effect of various transformations on the graphs of various functions.*

SOLUTIONS

1. Putting $\alpha = \beta = \frac{\pi}{2}$ we get

$\cos(\alpha + \beta) = -1$ whereas $\cos\alpha + \cos\beta = 0$
and $\sin(\alpha + \beta) = 0$ whereas $\sin\alpha + \sin\beta = 2$.
(There are many values of α and β which you could have chosen).

2. (i) f: $(0, 0) \mapsto (3, -1)$

g: $(3, -1) \mapsto (1, 3)$.

(ii) f: $(2, 3) \mapsto (5, 2)$

g: $(5, 2) \mapsto (3, 6)$.

(iii) (a) f: $(x, y) \mapsto (x + 3, y - 1)$

g: $(x + 3, y - 1) \mapsto (x + 1, y + 3)$.

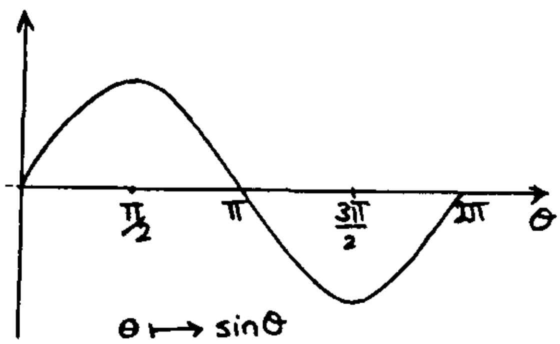
(b) The overall effect is the translation h, where

h: $(x, y) \mapsto (x + 1, y + 3)$.

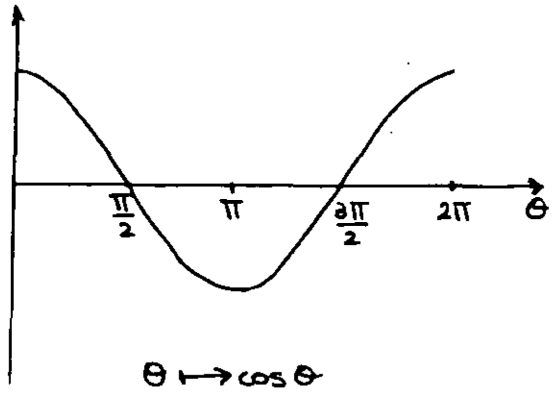
3. (i) To one decimal place:

θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0
$\cos\theta$	1	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0	0.5	0.7	0.9	1

(ii)



(iii)



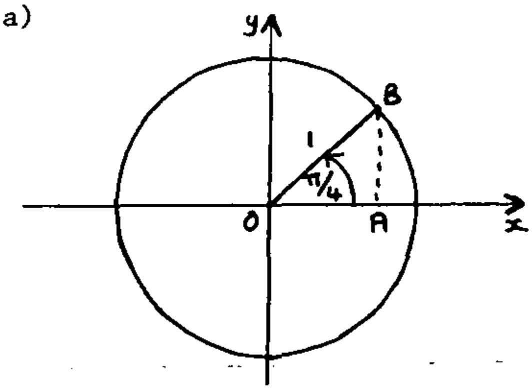
4. (i) (a) $\cos(0) = 1$; $\sin(0) = 0$

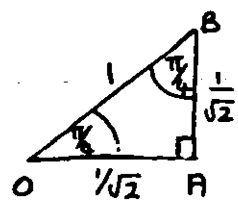
(b) $\cos(\frac{\pi}{2}) = 0$; $\sin(\frac{\pi}{2}) = 1$

(c) $\cos(\pi) = -1$; $\sin(\pi) = 0$

(d) $\cos(\frac{3\pi}{2}) = 0$; $\sin(\frac{3\pi}{2}) = -1$

(ii) (a)

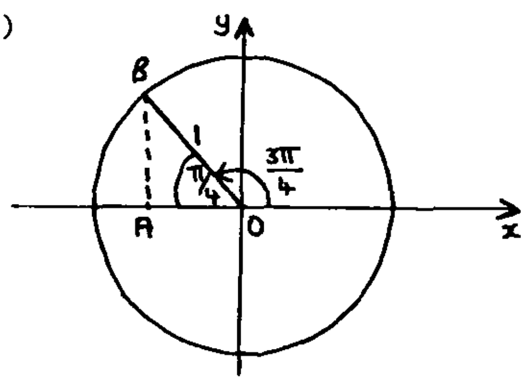




OAB is the right angled triangle where $OA = AB = \frac{1}{\sqrt{2}}$.

Hence $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$; $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

(b)



A similar argument to Part (i) shows that

(a) $\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$; $\sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$

5. (i) $(1, 0) \mapsto (\cos\theta, \sin\theta)$.

(ii) Putting $\theta = \frac{\pi}{2}$ we get $(1, 0) \mapsto (0, 1)$.

6. (i) (a) $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; (b) $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

(ii) From the triangle

(a) $\cos\frac{\pi}{3} = \frac{1}{2}$; (b) $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(iii) (a) $\cos(\frac{7\pi}{12}) = \cos(\frac{\pi}{3} + \frac{\pi}{4}) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$$= (\tfrac{1}{2})(\tfrac{1}{\sqrt{2}}) - (\tfrac{\sqrt{3}}{2})(\tfrac{1}{\sqrt{2}})$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}}.$$

(b) $\sin \frac{7\pi}{12} = \sin (\frac{\pi}{3} + \frac{\pi}{4}) = \cos \frac{\pi}{3} \sin \frac{\pi}{4} + \sin \frac{\pi}{3} \cos \frac{\pi}{4}$

$$= (\frac{1}{2})(\frac{1}{\sqrt{2}}) + (\frac{\sqrt{3}}{2})(\frac{1}{\sqrt{2}})$$

$$= \frac{1 + \sqrt{3}}{2\sqrt{2}} .$$

7. (i) (a) $\cos 2\alpha = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$

$$= \cos^2 \alpha - \sin^2 \alpha .$$

(b) $\sin^2 \alpha = 1 - \cos^2 \alpha$.

Hence $\cos 2\alpha = \cos^2 \alpha - (1 - \cos^2 \alpha)$

$$= 2 \cos^2 \alpha - 1.$$

(c) $\cos^2 \alpha = 1 - \sin^2 \alpha$.

Hence $\cos 2\alpha = (1 - \sin^2 \alpha) - \sin^2 \alpha$

$$= 1 - 2 \sin^2 \alpha.$$

(ii) $\cos (\alpha - \beta) = \cos \alpha \cos (-\beta) - \sin \alpha \sin (-\beta)$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

(since $\cos (-\beta) = \cos \beta$ and $\sin (-\beta) = - \sin \beta$).

8. (i) $\sin 2\alpha = \sin (\alpha + \alpha) = \cos \alpha \sin \alpha + \sin \alpha \cos \alpha$

$$= 2 \sin \alpha \cos \alpha .$$

(ii) $\sin (\alpha - \beta) = \cos \alpha \sin (-\beta) + \sin \alpha \cos (-\beta)$

$$= - \cos \alpha \sin \beta + \sin \alpha \cos \beta.$$

(since $\sin (-\beta) = - \sin \beta$ and $\cos (-\beta) = \cos \beta$).

9. (i) $\tan (\alpha + \beta) = \dfrac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)}$

$$= \frac{\cos \alpha \sin \beta + \sin \alpha \cos \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}.$$

(ii) Dividing throughout by $\cos \alpha \cos \beta$ we get

$$\tan (\alpha + \beta) = \frac{\dfrac{\cancel{\cos} \alpha \sin \beta}{\cancel{\cos} \alpha \cos \beta} + \dfrac{\sin \alpha \cancel{\cos} \beta}{\cos \alpha \cancel{\cos} \beta}}{\dfrac{\cancel{\cos} \alpha \cancel{\cos} \beta}{\cancel{\cos} \alpha \cancel{\cos} \beta} - \dfrac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} .$$

$$= \frac{\tan \beta + \tan \alpha}{1 - \tan \alpha \tan \beta} \left(\text{or } \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}\right)$$

10. (i) $\cos 3\alpha = \cos (2\alpha + \alpha)$

$$= \cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha$$

(ii) $\cos 2\alpha = 2 \cos^2 \alpha - 1$, $\sin^2 \alpha + \cos^2 \alpha = 1$, and

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$, hence $\cos 3\alpha = (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha$

$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$

$$= 4\cos^3\alpha - 3\cos\alpha.$$

11. (i) $\sin(\alpha + \beta) - \sin(\alpha - \beta)$

$= (\cos\alpha\sin\beta + \sin\alpha\cos\beta) - (-\cos\alpha\sin\beta + \sin\alpha\cos\beta)$

$= 2\cos\alpha\sin\beta.$

(ii) If $A = \alpha + \beta$ and $B = \alpha - \beta$,

$\alpha = A - \beta = B + \beta.$

So $2\beta = A - B$ and $\beta = \dfrac{A - B}{2}$.

Then $\alpha = A - \beta = A - \left(\dfrac{A - B}{2}\right) = \dfrac{A + B}{2}$.

Hence $\sin A - \sin B = 2\cos\left(\dfrac{A + B}{2}\right)\sin\left(\dfrac{A - B}{2}\right)$.

4. INVERSE FUNCTIONS

PROGRAMME SUMMARY

In order to have an inverse a function must be one-one. Also, if a function is not one-one it can be split into a number of one-one parts each of which has an inverse.

| 1 min |

We begin by reviewing the properties of a function in terms of the domain, codomain and rule. The importance of the domain and codomain is illustrated when we try to find the inverse of the function.

| 2 mins |

A Ballista is a weapon used by the Ancient Romans. It fires a ball in a fixed trajectory, which is given by

$t: [0, 4] \to \mathbb{R}$

$t: x \mapsto 4x - x^2.$

| 2 mins |

Suppose x is the distance to a fixed post, then t(x) is the height of the post. Thus knowing the function of the trajectory we can calculate t(x) for any value of x. But, in practice, the Romans knew the height of the post and wanted to determine where to position the Ballista. This means reversing the effect of the function t. But there is a problem because t is not one-one.

| 2 mins |

We look at the simpler function

$g: \mathbb{R} \to \mathbb{R}$

$g: x \mapsto x^2.$

Again g is not one-one so it doesn't have an inverse. But looking at the graph of g suggests that g can be split into two one-one functions:

$\ell: N \to \mathbb{R}$ and $r: P \to \mathbb{R}$

$\ell: x \mapsto x^2$ $\quad$ $r: x \mapsto x^2.$

where N = {Negative Reals + zero}

P = {Positive Reals + zero}.

and each of these functions has an inverse.

$\boxed{\text{3 mins}}$ We begin by finding an inverse for r. This demonstrates that we need to consider the image set of r as the domain of r^{-1}. We show that

$$r^{-1}: P \rightarrow P$$

$$r^{-1}: x \mapsto \sqrt{x}.$$

Similarly

$$\ell^{-1}: P \rightarrow N$$

$$\ell^{-1}: x \mapsto -\sqrt{x}.$$

$\boxed{\text{2 mins}}$ We return to the Ballista function. A similar approach indicates how to find the inverse of the two one-one functions.

$\boxed{\text{2 mins}}$ The actual technique of reversing the rule is demonstrated with the function

$$h: x \mapsto 5 - 4x.$$

$\boxed{\text{5 mins}}$ This allows us to solve the Ballista problem.

$\boxed{\text{5 mins}}$ Finally we look at the cosine and tangent functions. Using similar procedures we arrive at the ideas of arccos and arctan - as evaluated by a calculator.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following

(i) the Real line $\mathbb{R}$, and intervals of the form [a, b]

(ii) the sine, cosine and tangent functions and their graphs (in terms of radians)

(iii) manipulation of functions expressed using the notation

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f: x \mapsto 2x^2 - 3.$$

In particular the terminology domain, codomain and image set should have been met before.

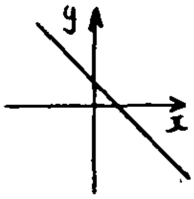
(iv) properties of one-one functions

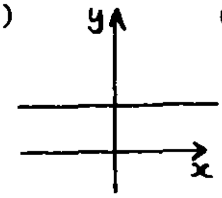
(v) graphical representation of functions

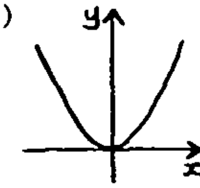
(vi) use of a scientific calculator.

PRE-PROGRAMME WORK

*1. Determine which of the following are graphs of functions.

(i)


(ii)


(iii)


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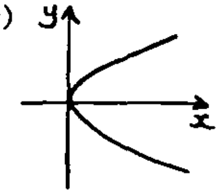

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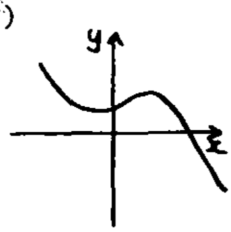

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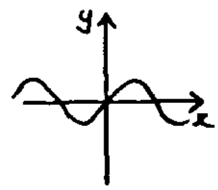
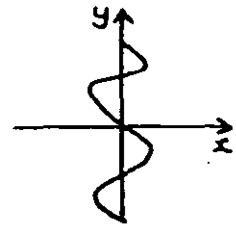


(vii)


*2. Which of the graphs in Question 1 above are graphs of one-one functions?

*3. A function is a one-one function if, whenever $f(a) = f(b)$, then $a = b$ (where a and b are elements of the domain.)

(i) Prove that the function f, defined by

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$
$$f: x \longmapsto 2x - 1$$

is one-one.

(ii) Show that the cosine function defined by

$$\cos: \mathbb{R} \longrightarrow \mathbb{R}$$
$$\cos: x \longmapsto \cos x$$

is <u>not</u> one-one.

*4. (i) Draw the graphs of the functions

(a) $\ell: \{x \in \mathbb{R}: x \leq 0\} \longrightarrow \mathbb{R}$ and $r: \{x \in \mathbb{R}: x \geq 0\} \longrightarrow \mathbb{R}$

$\ell: x \longmapsto x^2$ $\qquad$ $r: x \longmapsto x^2$.

(ii) Prove that each of these functions is one-one using the definition in Question 3 above.

*5. The function t is defined by

$$t: [0, 4] \longrightarrow \mathbb{R}$$
$$t: x \longmapsto 4x - x^2$$

(i) Solve the equations

(a) $t(x) = 0$ (b) $t(x) = 4$ (c) $t(x) = 5$.

(ii) Show that t is not a one-one function.

(iii) Multiply out $4 - (x - 2)^2$ and so show that the rule for t can be written as

$$x \mapsto 4 - (x - 2)^2.$$

(iv) a and b are distinct numbers in the domain of t for which $t(a) = t(b)$. Use the result of Part (iii) to find an equation connecting a and b.

(v) Draw the graph of the function t.

(vi) What is the image set of the function t.

(vii) Suggest two intervals lying within the interval $[0, 4]$ which, when used as domains for the rule $x \mapsto 4x - x^2$, produce one-one functions.

POST-PROGRAMME WORK

6. The function ℓ is defined by

$$\ell: \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{R}$$

$$\ell: x \mapsto x^2.$$

Obtain a definition of the inverse function ℓ^{-1} and a graph of ℓ^{-1}.

7. Given that k is a one-one function such that

$$k: 2 \mapsto 4 \text{ and } k(3) = 7$$

Find (a) $k^{-1}(4)$ and (b) $k^{-1}(7)$.

8. The function m is defined by

$$m: [0, 2] \to \mathbb{R}$$

$$m: x \mapsto 2x - 1$$

(i) Prove that m is a one-one function.

(ii) Draw the graph of m.

(iii) What is the image set of m?

(iv) Define the function m^{-1} and draw its graph.

9. The function v is defined by

$$v: [2, 4] \to \mathbb{R}$$

$$v: x \mapsto 4x - x^2$$

Obtain a definition of the function v^{-1} and draw its graph.

10. (i) Prove that the sine function is not a one-one function.

 (ii) Suggest three possible intervals, which, when taken as domains for the rule $x \mapsto \sin x$, produce one-one functions.

 (iii) Experiment with the 'inverse sine' function on your calculator to determine which interval is used for the domain of the rule $x \mapsto \sin x$ in order to produce the inverse function arcsin.

POSSIBLE EXTENSIONS

1. You may wish to explore further the graphs of the trigonometric functions and the use of a calculator to find other values of arcos x and so on.

2. This section could of course lead to further work on functions and their properties. In particular, the composition of functions might be a next step. When is it possible to find an inverse for such functions?

3. *SECTION 7: FUNCTIONS AND GRAPHS looks at the effect of transformations on graphs of functions. This could be used to investigate more general properties of functions.*

SOLUTIONS

1. (i), (ii), (iii), (v) and (vi) are graphs of functions; (iv) and (vii) are not.

2. (i) and (ii) are the only graphs of one-one functions.

3. (i) Suppose that a and b are two elements in the domain of f for which $f(a) = f(b)$.

This means that $f(a) = 2a - 1$
$= f(b)$
$= 2b - 1.$

That is, $2a - 1 = 2b - 1$

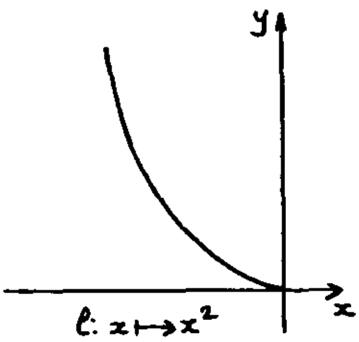
so $2a = 2b$

hence $a = b$.

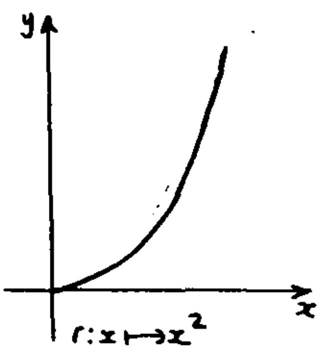
Therefore whenever $f(a) = f(b)$ it follows that $a = b$ and so f is a one-one function.

(ii) We need to find two numbers a and b in the domain of the cosine function such that $\cos a = \cos b$ but $a \neq b$. Two suitable values are $a = 0$ and $b = 2\pi$, since $\cos(0) = \cos(2\pi) = 1$. In fact any two numbers which are a multiple of 2π apart would do.

4. (i) (a)



(b)



(ii) (a) Suppose a and b belong to $\{x \in \mathbb{R} : x \leq 0\}$ and $\ell(a) = \ell(b)$.

This means that

$a^2 = b^2$.

So $a = b$ or $a = -b$.

If $a = b$ then we are finished; if $a = -b$ then if a and b are not both zero, they must have opposite signs. However this is not possible as both $a \leq 0$ and $b \leq 0$. Thus, whenever $\ell(a) = \ell(b)$ we have $a = b$ and hence ℓ is a one-one function.

5. (i) (a) $t(x) = 0$, so $4x - x^2 = 0$
and $x = 0$ or $x = 4$.

(b) $t(x) = 4$, so $4x - x^2 = 4$ or $x^2 - 4x + 4 = 0$.
This has solution $x = 2$.

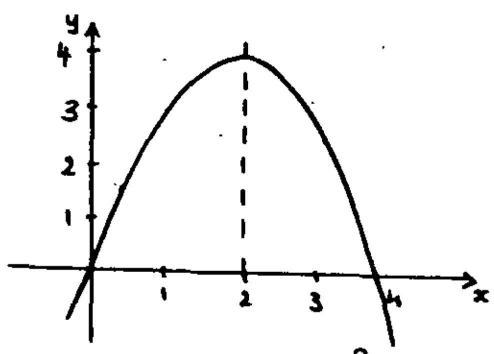
(c) $t(x) = 5$, so $x^2 - 4x + 5 = 0$. This has no solutions.

(ii) From Part (i) when $x = 0$, $t(0) = 0$ and when $x = 4$, $t(4) = 0$.
Hence t is not one-one.

(iii)
$$\begin{aligned} 4 - (x-2)^2 &= 4 - (x^2 - 4x + 4) \\ &= 4 - x^2 + 4x - 4 \\ &= 4x - x^2. \end{aligned}$$

(iv) From Part (iii), if $t(a) = t(b)$, then
$$4 - (a-2)^2 = 4 - (b-2)^2.$$
Thus $(a-2)^2 = (b-2)^2$
so either $a - 2 = b - 2$, in which case $a = b$,
or $a - 2 = -(b-2)$
$= 2 - b$ which gives $a = 4 - b$.

(v)



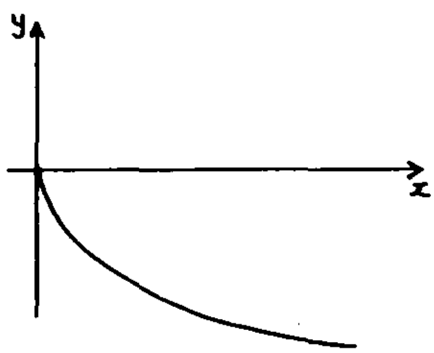
$t : x \longmapsto 4x - x^2$

(vi) The image set is $[0, 4]$

(vii) From the graph in (v) the two intervals are $[0, 2]$ and $[2, 4]$

6. $\ell^{-1} : \{x \in \mathbb{R} : x \geq 0\} \longrightarrow \{x \in \mathbb{R} : x \leq 0\}$

$\ell^{-1} : x \longmapsto -\sqrt{x}$



$\ell^{-1} : x \longmapsto -\sqrt{x}$

7. (a) $k^{-1}(4) = 2$ and $k^{-1}(7) = 3$.

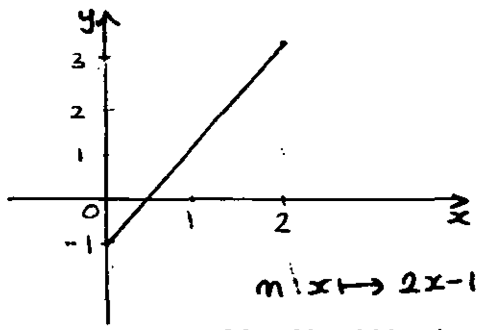
8. (i) Suppose a and b belong to [0, 2] and $m(a) = m(b)$.

Then $2a - 1 = 2b - 1$

hence $a = b$.

Thus m is one-one.

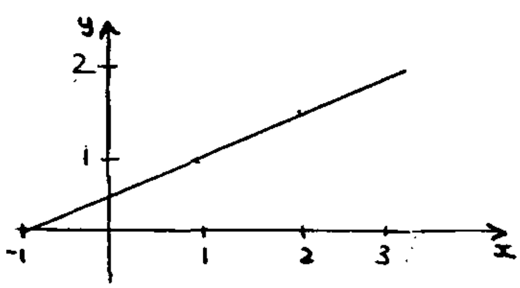
(ii)



(iii) The image set is [-1, 3].

(iv) $m^{-1}: [-1, 3] \rightarrow [0, 2]$.

$m^{-1}: x \mapsto \dfrac{x + 1}{2}$.



9. $v^{-1}: [0, 4] \rightarrow [2, 4]$.

Putting $y = 4x - x^2$

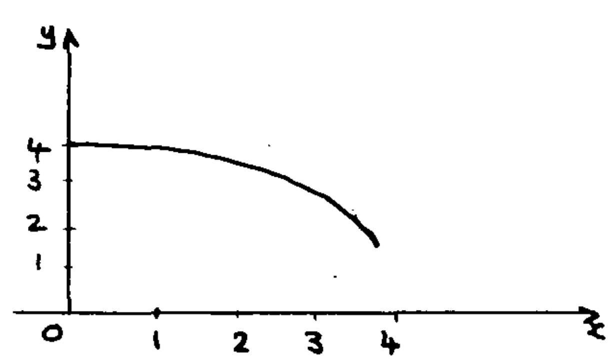
we obtain $y = 4 - (x - 2)^2$

which gives $x = 2 \pm \sqrt{4 - y}$.

v has domain [2, 4] so we must take

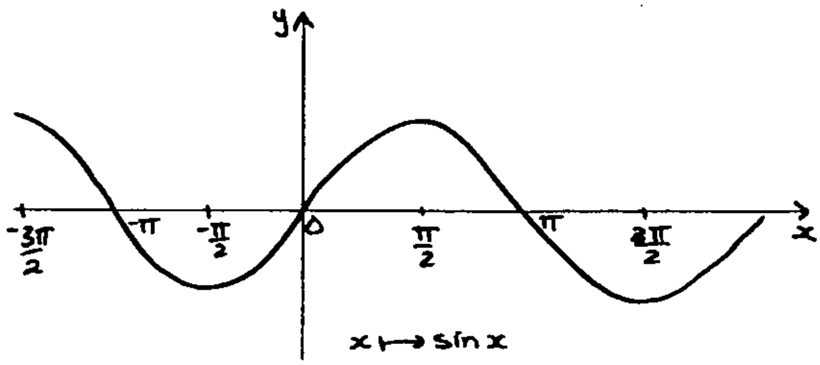
$x = 2 + \sqrt{4 - y}$.

Hence $v^{-1}: y \mapsto 2 + \sqrt{4 - y}$ or $v^{-1}: x \mapsto 2 + \sqrt{4 - x}$.



10. (i) $\sin 0 = \sin \pi = 0$ so the sine function is not one-one.

(ii)



From the graph three intervals are

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ although there are many other possibilities.

(iii) You should have found that the inverse sine of 1 is $\frac{\pi}{2}$ and the inverse sine of -1 is $-\frac{\pi}{2}$. So if the function $\sin^*$ is defined by

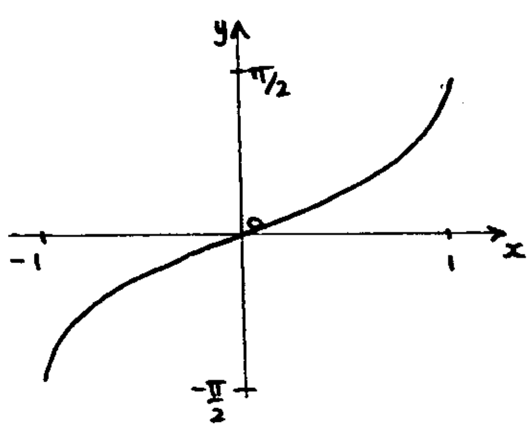
$\sin^*: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$.

$\sin^*: x \mapsto \sin x$

then $\sin^{*-1}: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{*-1}: x \mapsto \arcsin x$.

Its graph is illustrated below.



6. RATIONAL NUMBERS AND $\sqrt{2}$

PROGRAMME SUMMARY

This programme considers two views of numbers - first in a computational sense such as used in numerical calculations; second as numbers to measure with - lengths of lines, constructed geometrically. We consider the meaning of $\sqrt{2}$ from both points of view.

1 min

The programme begins with a short introduction about functions whose domain and codomain are both $\mathbb{R}$. For such functions it is necessary to have some understanding of the properties of numbers.

2 mins

There are two views of numbers - first as numbers to calculate with (calculations); second as numbers to measure with (geometry) and the same thing holds for functions. There is however a conflict between the two approaches which has its origins in greek mathematics. By resolving this conflict it is hoped to increase students understanding of the concept of number.

3 mins

Natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$) and rationals ($\mathbb{Q}$) may all be constructed geometrically.

2 mins

We can also arrive at $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ using the computational approach. Notice that all calculations on a calculator or computer are done using rational numbers in the form of decimals.

4 mins

The conflict arises over $\sqrt{2}$. This can be constructed exactly by means of Pythagoras' Theorem but we can only obtain an approximation by calculation. [The method used is the ten-subdivision method - see suggested Pre-programme work]. This suggests that $\sqrt{2}$ is not a rational number - so how can we calculate with it?

5 mins

We then prove that $\sqrt{2}$ is irrational using the standard proof by contradiction.

3 mins In fact there are many examples of irrational numbers. Such numbers may not be nice (in the sense that they can't be calculated exactly) but they are important - particularly when it comes to graphs of functions. If we just ignored the irrational numbers then there would be gaps in the graphs of functions. This leads to the idea of the Reals ($\mathbb{R}$) as a number line.

4 mins In fact $\sqrt{2}$ does have a meaning when looked at in a computational sense. Its value can be thought of either as a symbol whose square is 2, or as a length of a line which we can calculate as accurately as we please. This means that we can use approximations to perform calculations which involve irrational numbers - and the conflict between the two approaches is resolved.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) the natural number (1, 2, 3 ...) represented by $\mathbb{N}$

(ii) the integers (... -3, -2, -1, 0, 1, 2, 3, ...) represented by $\mathbb{Z}$

(iii) the rationals, represented by $\mathbb{Q}$ as fractions of the form $\frac{a}{b}$ where a and b are integers, $b \neq 0$

(iv) geometrical constructions

(a) of a line perpendicular to a given straight line through a given point - using ruler and compass.

(b) of parallel lines - using set square and ruler

(v) angle properties of a line intersecting parallel lines

(vi) similar triangles and the properties of ratios of sides

(vii) the idea of functions and the notation

$f: x \mapsto x^2, (x \in \mathbb{R})$

and the concept of domain and codomain

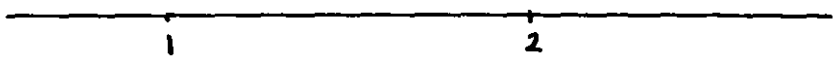
(viii) the idea of intervals on the Real line of the form [a, b]

(ix) use of a scientific calculator

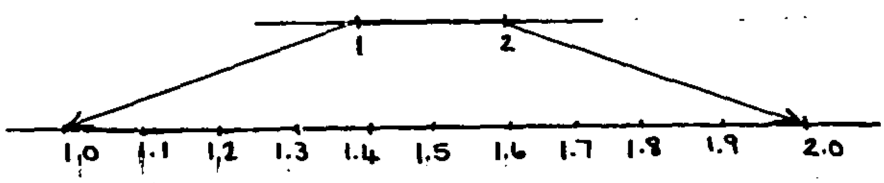
PRE-PROGRAMME WORK

1. (i) Calculate (a) $1^2 - 2$ (b) $2^2 - 2$ (c) $3^2 - 2$.

(ii) Between which pair of consecutive whole numbers does $\sqrt{2}$ lie?



So, $\sqrt{2}$ lies somewhere in the interval [1, 2]. We can of course locate it more accurately. One possible way is to divide the interval [1, 2] into ten intervals of equal length — [1.0, 1.1], [1.1, 1.2] ... [1.9, 2.0].



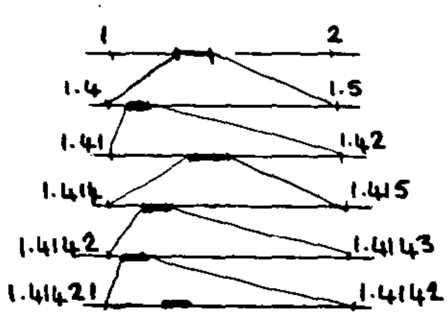
2. (i) Calculate $1.1^2 - 2$, $1.2^2 - 2$, $1.3^2 - 2$ and so on.

(ii) Hence find which of these smaller intervals contains $\sqrt{2}$.

3. Divide the interval you obtained as the solution to Question 2 (ii) into ten intervals of equal length. By a similar method to that used in Question 2, locate $\sqrt{2}$ in one of these smaller intervals.

This process can be repeated indefinitely — pinching $\sqrt{2}$ in smaller and smaller intervals as follows:

$\sqrt{2} \in [1, 2]$
$\sqrt{2} \in [1.4, 1.5]$
$\sqrt{2} \in [1.41, 1.42]$
$\sqrt{2} \in [1.414, 1.415]$
$\sqrt{2} \in [1.4142, 1.4143]$
$\sqrt{2} \in [1.41421, 1.41422]$.
etc ...



This method is called the <u>ten-subdivision method</u> of calculating $\sqrt{2}$.

<u>Note</u>: The Bisection method is similar to the ten-subdivision method. This time the interval [1, 2] is divided into two equal intervals. Then, by evaluating $1.0^2 - 2$, $1.5^2 - 2$ and $2.0^2 - 2$ we can locate $\sqrt{2}$ in the interval [1.0, 1.5]. The process is repeated to give the sequence:

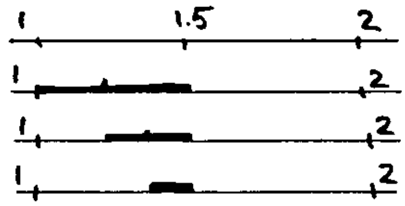
$\sqrt{2} \in [1, 2]$

$\sqrt{2} \in [1, 1.5]$

$\sqrt{2} \in [1.25, \; 1.5]$

$\sqrt{2} \in [1.375, \; 1.5]$

etc.



4. (i) Plot the graph of the function

$$f: \mathbb{R} \to \mathbb{R}$$
$$f: x \mapsto x^2$$

(ii) Use your graph to find (a) $f(1)$ (b) $f(-2)$ (c) $f(\frac{3}{2})$.

(iii) (a) Try to find $f(\sqrt{2})$ from the graph.

(b) Evaluate $f(\sqrt{2})$.

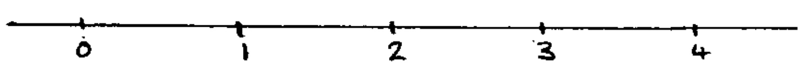
*5. (i) Show that the square of an even integer is always a multiple of four.

(ii) Show that the square of an odd integer is always odd.

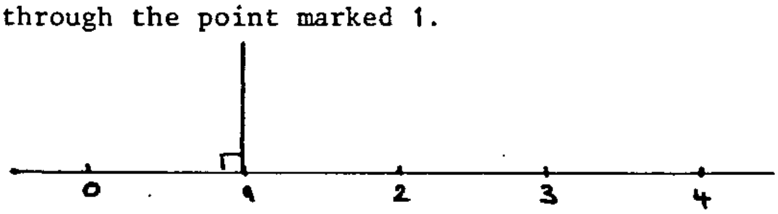
<u>WORK TO BE TACKLED BEFORE OR DURING THE PROGRAMME</u>

6. Using ruler and compasses only

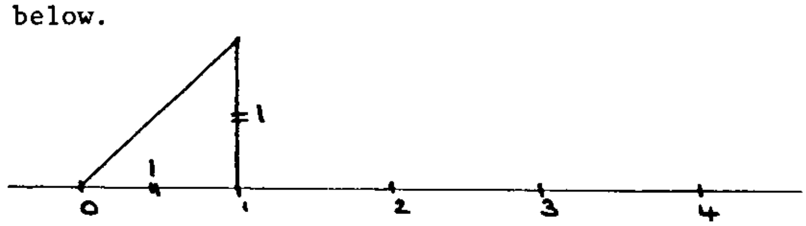
(i) Mark off four equal distances on a straight line and label your diagram as illustrated below.



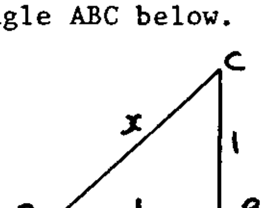
Construct a line at right angles to this straight line through the point marked 1.



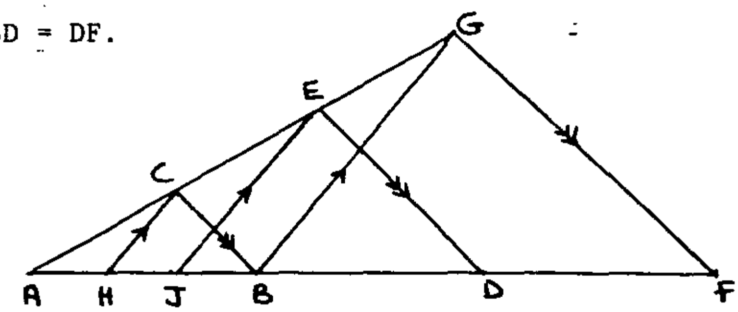
Hence construct the isosceles triangle illustrated below.



Use Pythagoras' Theorem to calculate the length of x in triangle ABC below.



7. In the diagram below
the lines BC, DE and FG are parallel
the lines HC, JE and BG are parallel,
and AB = BD = DF.



(i) (a) Show that triangle ABC and triangle AFG are similar
(b) Show that $AC = \frac{1}{3} AG$.

(ii) (a) Show that triangle AHC and triangle ABG are similar.
(b) Hence show that $AH = \frac{1}{3} AB$.

POST-PROGRAMME WORK

We do not suggest any specific post programme work as the programme itself provides a nice conclusion.. However, the proof that $\sqrt{2}$ is irrational and the nature of a proof by contradiction could promote an interesting discussion.

POSSIBLE EXTENSIONS

1. This section could form the basis of a historical discussion on different types of numbers. In particular, geometrical constructions, such as those used by the Greeks could be further investigated.

2. It could also lead to further work on the nature of irrational numbers and how we deal with them. For example, what is the meaning of $2^{\sqrt{2}}$? This could be looked at both from a purely computational view; and via the graph of the function

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

$$f: x \longmapsto 2^x.$$

3. We describe the ten-subdivision method of obtaining $\sqrt{2}$ to any desired degree of accuracy. This could be converted to a computer programme. The bisection method may also be investigated using a computer.

4. The ten-subdivision method is only one type of iteration process. You may wish to discuss other methods of finding $\sqrt{2}$. For example the iteration process based on

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(known as Newton's method).converges quite quickly.

NOTE *SECTION 1: SYMBOLS AND EQUATIONS discusses what is involved in an iteration process. This is followed up in SECTION 9: ITERATION AND CONVERGENCE where we examine whether the iteration process based on a given rearrangement of an equation converges or diverges.*

SOLUTIONS

1. (i) (a) $1^2 - 2 = -1$ change in sign

(b) $2^2 - 2 = 2$

(c) $3^2 - 2 = 7$

(ii) The change in sign shows that $\sqrt{2}$ lies between 1 and 2. [Note: This may also be illustrated by the graph of

$f: x \mapsto x^2 - 2 \ (x \in \mathbb{R})$.]

2. (i) $1.1^2 - 2 = -0.79$

$1.2^2 - 2 = -0.56$

$1.3^2 - 2 = -0.31$

$1.4^2 - 2 = -0.04$ change in sign

$1.5^2 - 2 = 0.25$

(ii) Because there is a change of sign $\sqrt{2} \in [1.4, 1.5]$.

3. Divide the interval [1.4, 1.5] into ten intervals of equal length.

$1.40^2 - 2 = -0.04$

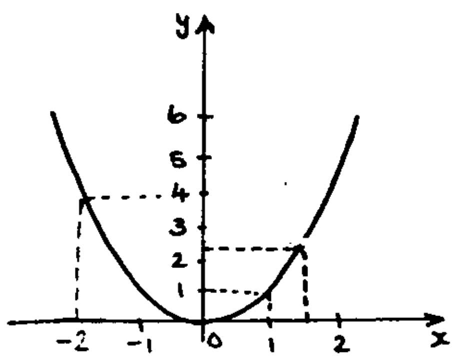
$1.41^2 - 2 = -0.0119$ change in sign

$1.42^2 - 2 = 0.0164$

$1.43^2 - 2 = 0.0449$

This suggests that $\sqrt{2} \in [1.41, 1.42]$.

4 (i)



$f: x \mapsto x^2$

(ii) From the graph

(a) $f(1) = 1$

$f(-2) = 4$

$f(\frac{3}{2}) \simeq 2.3$

(Note this is only approximate because of the scale used on the graph).

(ii) (a) It is not possible to locate $\sqrt{2}$ exactly on the x-axis. However, from Question 3 above we know that $\sqrt{2} \in [1.41, 1.42]$. This gives an approximate value for $f(\sqrt{2})$ as 1.9.

(b) $f(\sqrt{2}) = (\sqrt{2})^2 = 2$. So it <u>is</u> possible to calculate $f(\sqrt{2})$ exactly, although it is not possible to read the value exactly from the graph.

5. (i) An even integer must have the form 2n where n is an integer. Now $(2n)^2 = 4n^2$ which is a multiple of 4, so the square of an even number is always a multiple of 4.

(ii) An odd integer must have the form $2n + 1$ where n is an integer.

$$(2n + 1)^2 = 4n^2 + 4n + 1$$
$$= 4(n^2 + n) + 1.$$

Since $4(n^2 + n)$ must always be even, $4(n^2 + n) + 1$ must always be odd. Hence the square of an odd integer is always odd.

6. (a) (b) (c). Your diagrams should look like the illustrations - although they should be more accurate!

(d) Pythagoras' Theorem states that

$$AB^2 = AC^2 + BC^2.$$

Hence $x^2 = 1^2 + 1^2 = 2$

and $x = \sqrt{2}$. (As x is a length we must take the positive square root).

7. (i) $A\hat{B}C = A\hat{F}G$ (corresponding angles for the parallel lines BC and AG)

$\hat{A}$ is common to both triangle.

Hence $A\hat{C}B = A\hat{G}F$. (either by corresponding angles or from the angle sum of a triangle).

As corresponding angles are equal triangle ABC and triangle AFG are similar.

(ii) The property of ratio of corresponding sides of similar triangles shows that

$$\frac{AB}{AF} = \frac{AC}{AG} = \frac{BC}{FG}.$$

$AB = BD = DF$ so $AB = \frac{1}{3} AF$ and $\frac{AB}{AF} = \frac{1}{3}$.

Hence $\frac{AC}{AG} = \frac{1}{3}$ and $AC = \frac{1}{3} AG$.

(iii) (a) A similar argument to that used in Part (i) shows that triangle AHC and triangle ABG are similar.

(b) The ratios of corresponding sides are the same, so

$$\frac{AH}{AB} = \frac{AC}{AG} = \frac{HC}{BG}.$$

From Part (ii) (b) $\frac{AC}{AG} = \frac{1}{3}$, so $\frac{AH}{AB} = \frac{1}{3}$

and $AH = \frac{1}{3} AB$.

7. FUNCTIONS AND GRAPHS

PROGRAMME SUMMARY

The programme deals with the effects of the following transformations on graphs:

(i) translations in the x- and y-directions

(ii) scaling in the x- and y-directions

(iii) reflections in the x- and y-axes.

3 mins

We begin by looking at the graph of $f: x \mapsto \frac{x + 4}{x + 2}$ $(x \in \mathbb{R})$. This can be obtained from the graph of $f: x \mapsto \frac{1}{x}$ as follows:

(i) first a translation left by 2

(ii) followed by a scaling of 2 in the y-direction

(iii) followed by a translation up by 1.

2 mins

In fact, the graph of any hyperbolic function can be obtained in the same way. To cope with functions such as $f: x \mapsto 1 - \frac{2}{x + 2}$ we need to introduce the idea of reflection in the x-axis...

3 mins

Indeed we can generalize for any function:

Starting with the graph of $x \mapsto f(x)$:

(i) an x-translation through α to the left gives the graph of $x \mapsto f(x + \alpha)$

(ii) a y-translation through β, up gives the graph of $x \mapsto f(x) + \beta$

(iii) a y-scaling by a factor μ gives the graph of $x \mapsto \mu\ f(x)$

(iv) a reflection in the x-axis gives the graph of $x \mapsto f(x)$.

2 mins

Scaling in the x-direction seems rather odd, since, for example, stretching the graph out by 2 corresponds to a scale factor of $\frac{1}{2}$. Thus we show that an x-scaling by a factor λ gives the graph of $x \mapsto f(\frac{1}{\lambda}x)$.

1 min

We then examine the effect of reflecting in the y-axis.

This takes the graph of $x \mapsto f(x)$ to the graph of $x \mapsto f(-x)$.

| 4 mins | Some problems about the properties of functions are introduced. These should be worked through after the programme. They are included in the post-programme work.

| 8 mins | Finally we consider the effect of a reflection in the line $y = x$. Some practical examples of functions and their inverses illustrate what is involved and we look at the graph of $x \mapsto \frac{77}{100}x$ and its inverse in some detail. This demonstrates how to obtain the graph of $x \mapsto f^{-1}(x)$ from the graph of $x \mapsto f(x)$.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) manipulation of functions and their graphical representation

(ii) plotting graphs of the form

$$x \mapsto ax^2 + bx + c$$

$$x \mapsto \frac{ax + b}{cx + d}$$

and some knowledge of the graphs of

$x \mapsto \sin x$ and $x \mapsto 2^x$

(iii) the effects of translation (in the x- and y-directions)

scaling (in the x- and y-directions)

and reflection (in the x- and y-axis)

although these could be introduced via this section of work

(iv) the notation and manipulation of inverse functions and in particular the necessary conditions in order that a function has an inverse.

(NOTE SECTION 4: INVERSE FUNCTIONS covers this area of work.)

PRE-PROGRAMME WORK

[Note: All functions here have domain $\mathbb{R}$ unless otherwise specified]

1. (i) Write each of the following expressions in the form $(x + \alpha)^2$.

(a) $x^2 + 6x + 9$ (b) $x^2 - 2\pi x + \pi^2$ (c) $x^2 + \frac{2}{m}x + \frac{1}{m^2}$

(d) $x^2 \pm \frac{bx}{a} + \frac{b^2}{4a^2}$

(ii) Write each of the following expressions in the form $\lambda(x + \alpha)^2$

(a) $\frac{1}{2}x^2 + x + \frac{1}{2}$ (b) $2x - (1 + x^2)$ (c) $m^2x^2 + 2mx + 1$

(d) $ax^2 + bx + \frac{b^2}{4a}$

(iii) Write each of the following expressions in the form $(x + \alpha)^2 + \beta$

(a) $x^2 + 2x + 3$ (b) $x^2 - \frac{2}{3}x + \frac{2}{9}$ (c) $x^2 + 2mx + 1$

(d) $x^2 + bx + c$

(iv) Write each of the following expressions in the form $\lambda(x + \alpha)^2 + \beta$

(a) $4x^2 + 8x + 3$ (b) $2x - (2 + x^2)$ (c) $m^2x^2 + 2mx$

(d) $ax^2 + bx + c$.

2. Question 1 demonstrates that any quadratic of the form $ax^2 + bx + c$ can be written in the form

$$\lambda(x + \alpha)^2 + \beta;$$

which is called the completed-square form.

Which of the following quadratic functions are in completed-square form:

(i) $x \mapsto \frac{1}{4}x^2 - \frac{1}{2}x - \frac{11}{4}$ (ii) $x \mapsto (x + \frac{1}{2})^2 + 2$

(iii) $x \mapsto x^2 + x + \frac{1}{4}$ (iv) $x \mapsto (x + \frac{1}{2})^2 + 2x + 2$

(v) $x \mapsto \frac{1}{4}(x - 1)^2 - 3$ (vi) $x \mapsto (x + \frac{1}{2})^2$

(vii) $x \mapsto x^2 + 2$ (viii) $x \mapsto (2x + 1)^2 + 1$.

3. (i) Plot the graphs of each of the following functions

(a) $x \mapsto x^2$ (b) $x \mapsto x^2 + 2$ (c) $x \mapsto (x + \frac{1}{2})^2$

(d) $x \mapsto (x - 1)^2 - 1$ (e) $x \mapsto (x + \frac{1}{2})^2 - \frac{1}{4}$.

(ii) All the graphs in Part (i) are parabolas.
For each of these parabolas write down the coordinates of the vertex.

Parts (i) and (ii) of this Question, applied to the graph

$x \mapsto x^2$, demonstrate the following properties:

A A translation of α to the left changes the graph of $f(x)$ to the graph of $f(x + \alpha)$. (What happens when α is negative?)

B A translation of β up changes the graph of $f(x)$ to the graph of $f(x) + \beta$. (What happens if β is negative?)

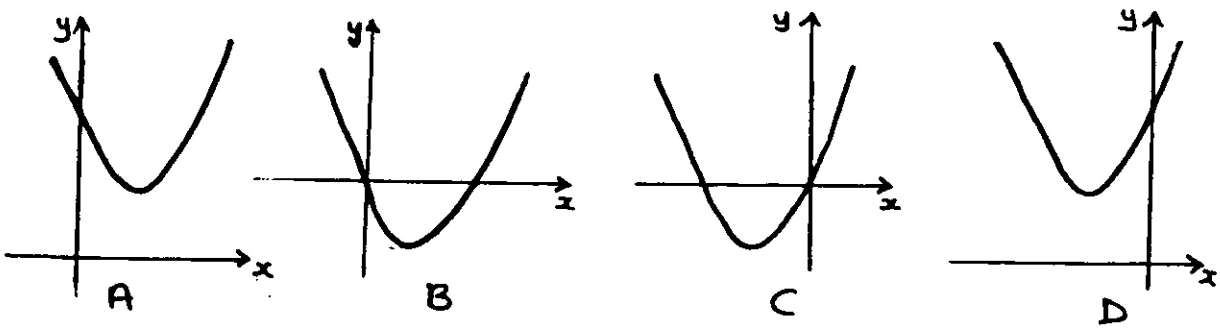
C A translation by α to the left followed by a translation of β up changes the graph of $f(x)$ to the graph of $f(x + \alpha) + \beta$. That is, the graph of $x \mapsto x^2$ is changed to $x \mapsto (x + \alpha)^2 + \beta$.

4. So putting the quadratic into completed square form provides a useful way of sketching the graph of a quadratic function.

Match each of the following functions to its graph

(i) $x \mapsto (x - 1)^2 + 1$ (ii) $x \mapsto (x + 1)^2 + 1$

(iii) $x \mapsto (x + 1)^2 - 1$ (iv) $x \mapsto x^2 - 2x$



5. (i) Plot the graphs of each of the following functions
(a) $x \mapsto 2x^2$ (b) $x \mapsto \frac{1}{2}x^2$ (c) $x \mapsto -x^2$.

6. Question 5 demonstrates that a scaling of λ in the y-direction changes the graph of $f(x)$ to the graph of $\lambda f(x)$. (What happens if λ is negative?) So, the graph of $x \mapsto \lambda(x + \alpha)^2 + \beta$ can be obtained from the graph of $f: x \mapsto x^2$ as follows.

A first a translation of α left.

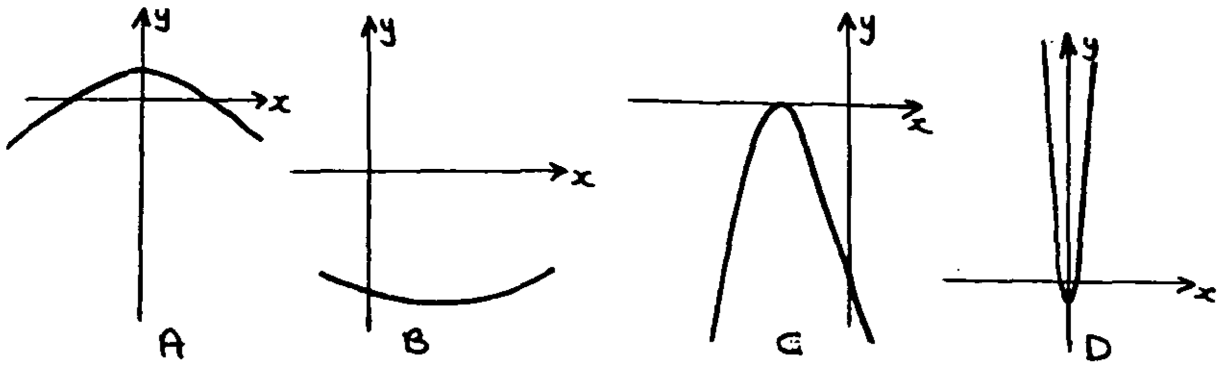
B followed by a scaling of λ in the y-direction.

C followed by a translation of β up.

Use this information to match each of the following functions to its graph

(i) $x \mapsto \frac{1}{4}(x - 1)^2 - 3$ (ii) $x \mapsto 5x^2 - 1$ (iii) $x \mapsto -3(x + 1)^2$

(iv) $x \mapsto 1 - \frac{1}{2}x^2$



7. Sketch the graphs of each of the following functions by putting the quadratic into completed-square form.
(i) $x \mapsto x^2 + 6x + 9$ (ii) $x \mapsto x^2 - \frac{2}{3}x + \frac{2}{9}$ (iii) $x \mapsto 4x^2 + 8x + 3$.

8. Write down the inverse of each of the following functions.

(i) $g: \mathbb{R}^+ \to \mathbb{R}^+$ $\quad$ $g: x \mapsto \frac{77}{100}x$

(ii) $h: \mathbb{R} \to \mathbb{R}^+$ $\quad$ $h: x \mapsto 2^x$

<u>POST-PROGRAMME WORK</u>

9. If the graph of $x \mapsto x^3$ is reflected in the y-axis and the result is then reflected in the x-axis the final graph coincides with the original one. What property of the function explains this phenomenon.

10. (i) If the graph of $x \mapsto \sin x$ (or $x \mapsto 2 \sin x$) is translated parallel to the x-axis sufficiently far we obtain the original graph. Why? How far do we have to translate?

(ii) The graph of $x \mapsto \sin x$ (or $x \mapsto 2 \sin x$) is unchanged by reflection in the y-axis followed by reflection on the x-axis. Why?

11. A y-scaling by a factor 2, and an x translation to the left of 1 unit have the same effect when applied to the graph of $x \mapsto 2^x$. Explain.

12. The function $x \mapsto \sin x$ does not have an inverse but the process of finding an angle given its sine is quite common in trigonometry. Resolve this contradiction.

POSSIBLE EXTENSIONS

1. The graph of $x \mapsto \frac{px + q}{rx + s}$ can be obtained from the graph of $x \mapsto \frac{1}{x}$ by expressing $\frac{px + q}{rx + s}$ in the form $\frac{\frac{q}{r} - \frac{ps}{r^2}}{x + \frac{s}{r}} + \frac{p}{r}$

   and applying the following transformations:

   (i) first a translation of $\frac{s}{r}$ to the left.

   (ii) followed by a scaling of $\frac{q}{r} - \frac{ps}{r^2}$ in the y-direction

   (iii) followed by a translation of $\frac{p}{r}$ up.

   You may wish either to examine this general result — or the more particular case of $x \mapsto \frac{x + a}{x + b}$.

2. This section could lead to further work on the properties of functions. The particular properties of $x \mapsto 2^x$ may be worth investigating.

3. It could also lead to further work on the geometrical effects of transformations applied to coordinates.

   (SEE SECTION 3. TRIGONOMETRIC FORMULAS).

SOLUTIONS

1. (i) (a) $(x + 3)^2$ (b) $(x - \pi)^2$ (c) $(x + \frac{1}{m})^2$ (d) $(x + \frac{b}{2a})^2$

   (ii) (a) $\frac{1}{2}(x + 1)^2$ (b) $-(x - 1)^2$ (c) $m^2(x + \frac{1}{m})^2$ (d) $a(x + \frac{b}{2a})^2$

   (iii) (a) $(x + 1)^2 + 2$ (b) $(x - \frac{1}{3})^2 + \frac{1}{9}$ (c) $(x + m)^2 + (1 - m^2)$

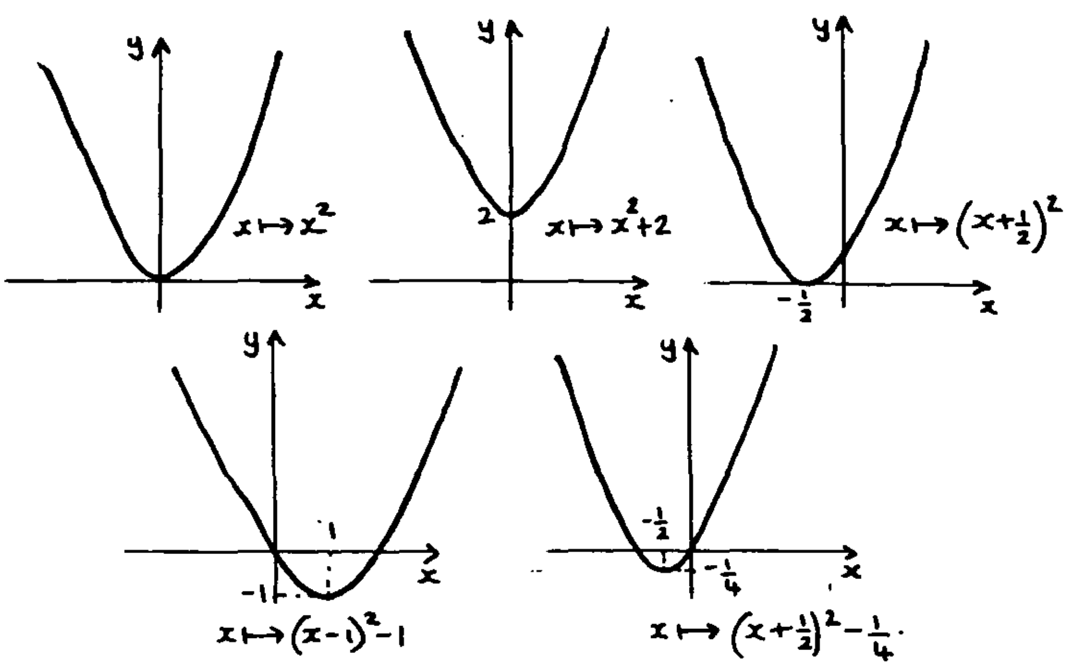
   (d) $(x + \frac{b}{2})^2 + (c - \frac{b^2}{4})$

   (iv) (a) $4(x + 1)^2 - 1$ (b) $-(x - 1)^2 - 1$ (c) $m^2(x + \frac{1}{m})^2 - 1$

   (d) $a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$.

2. Functions (ii), (v), (vi) and (vii) are in completed square form; the other functions are not.

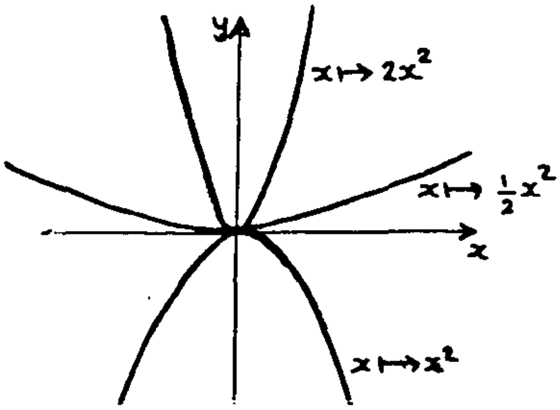
3. (i)



   (ii) The vertices of each of the graphs are marked on the diagram above. The vertices are:
   (a) (0, 0) (b) (0, 2) (c) $(-\frac{1}{2}, 0)$ (d) (1, -1) (e) $(-\frac{1}{2}, -\frac{1}{4})$.

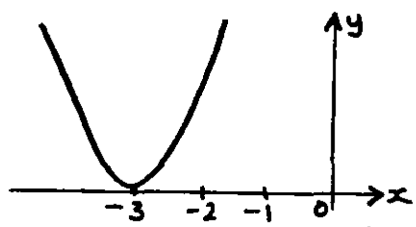
4. (i) —— A
   (ii) —— D
   (iii) —— C
   (iv) —— B.

5.

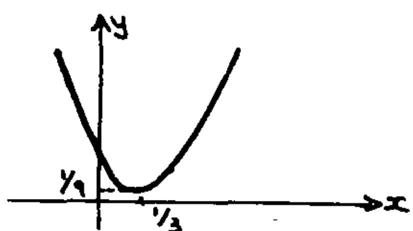


6. (i) —— B
(ii) —— D
(iii) —— C
(iv) —— A

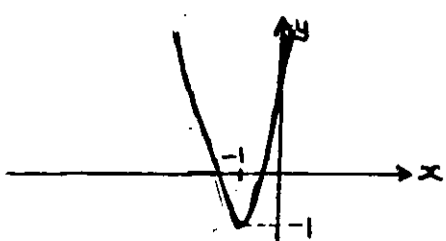
7. (i) $x \mapsto x^2 + 6x + 9 = x \mapsto (x + 3)^2$.



(ii) $x \mapsto x^2 - \frac{2}{3}x + \frac{2}{9} = x \mapsto (x - \frac{1}{3})^2 + \frac{1}{9}$.



(iii) $x \mapsto 4x^2 + 8x + 3 = x \mapsto 4\ (x + 1)^2 - 1$



8. (i) $g^{-1}: \mathbb{R} \rightarrow \mathbb{R}$
$g^{-1}: x \mapsto \frac{100}{77}x$

(ii) $h^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R}$
$h^{-1}: x \mapsto \log_2 x$

9. Consider the function f. Reflection in the y-axis gives the graph of f(-x). If the result is reflected in the x-axis we obtain the graph of -f(-x). For $x \mapsto x^3$ this coincides with the original graph because $-(-x)^3 = x^3$.

(<u>Note</u> A function, f, such that

$$f(x) = -(f(-(x))$$

is called an <u>odd function</u>.)

10. (i) An x translation through α changes the graph of $x \mapsto \sin x$ to the graph of $x \mapsto \sin(x + \alpha)$. But $\sin(x + 2\pi) = \sin x$, so a translation to the left of 2π leaves the graph unchanged. (The same is true of translations through 4π, 6π and so on — and of translations to the right by the same amounts).

    (ii) $-\sin(-x) = \sin(x)$.

11. A y-scaling by 2 changes the graph of $x \mapsto f(x)$ to the graph of $x \mapsto 2\, f(x)$. An x-translation by 1 unit to the left results in the graph of $x \mapsto f(x + 1)$. For the function $x \mapsto 2^x$ these two are equal since

$$2 \times 2^x = 2^{x+1}.$$

12. The function

$$\sin^*: [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to \mathbb{R}$$

$$\sin^*: x \mapsto \sin x$$

is one-one and so <u>does</u> have an inverse which is

$$\arcsin: [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

arcsin: $x \mapsto$ the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (inclusive) whose sine is x.

[Compare this with what happens on your calculator].

9 ITERATION AND CONVERGENCE

(Note This section follows on directly from the work in SECTION 1: SYMBOLS AND EQUATIONS).

PROGRAMME SUMMARY

The variations in behaviour of formula iterations may be explained and analysed using the idea of the scale factor.

2 mins

Two problems concerning the equations

$$x = \frac{5}{8}(x + 0.3)^2$$

and

$$x = \frac{1}{8}(1 - 20x^2)$$

are introduced.

We recall that the formula iteration based on

$$x_{n+1} = \frac{5}{8}(x_n + 0.3)^2$$

with starting value $x_1 = 0.5$, converges to 0.1.

5 mins

But why does this process converge? And why is it that we don't get exactly 0.1? The reasons are demonstrated pictorially using mapping diagrams where we investigate the effect of successive iterations near to the solution, It turns out that the idea of convergence can be related to the scale factor λ at X, which is defined as

$$\lambda = \frac{\mu}{\delta} = \frac{f(X + \delta) - f(X)}{\delta}, \qquad \text{(to first order)}$$

Here X is the value of x at the solution to $x = f(x)$. In fact the iteration process will always converge if λ is positive and less than 1 near the solution.

3 mins

This can also be demonstrated by looking at the graphs of $x \mapsto \frac{5}{8}(x + 0.3)^2$ and $x \mapsto x$. We illustrate the iteration process by means of a staircase diagram.

4 mins

Using a staircase diagram we show that the formula iteration

$$x_{n+1} = \frac{5}{8}(x_n + 0.3)^2$$

with a starting value $x_1 = 1$ diverges. Again this is related to the scale factor at the solution $X = 0.9$, which in this case is greater than one.

| 2 mins | The idea of convergence and divergence is generalised by looking at the slope of the function at the solution to $x = f(x)$.

(Note This part may need some expansion).

| 5 mins | We now turn to the function

$$x \mapsto \frac{1}{8}(1 - 20x^2)$$

which has a negative slope at the solution to $x = \frac{1}{8}(1 - 20x^2)$.

This time successive iterations are illustrated by means of a cobweb diagram. Again such a diagram predicts convergence or divergence for any iteration process based on $x = f(x)$ depending on the slope of the function f at the solution to $x = f(x)$.

| 3 mins | We summarize what's been done. A sketch of the graphs of $x \mapsto f(x)$ and $x \mapsto x$ predicts convergence (or divergence) of the iteration process based on $x = f(x)$. If the sketch indicates convergence then we can obtain a starting value from the diagram and so find a solution. If the sketch indicates divergence then the modulus of the scale factor at the solution is greater than one and we need to investigate alternative arrangements of the original equation.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) functions and mappings; domains, codomains and images

(ii) mapping diagrams

(iii) iteration processes as introduced in SECTION 1: SYMBOLS AND EQUATIONS.

(iv) the idea of the slope of a graph at the point X in terms of the tangent

(v) use of a scientific calculator.

PRE-PROGRAMME WORK

Students will benefit by working through SECTION 1: SYMBOLS AND EQUATIONS *before starting this Section.*

Suppose an iteration process is based on the formula

$$x_{n+1} = f(x_n)$$

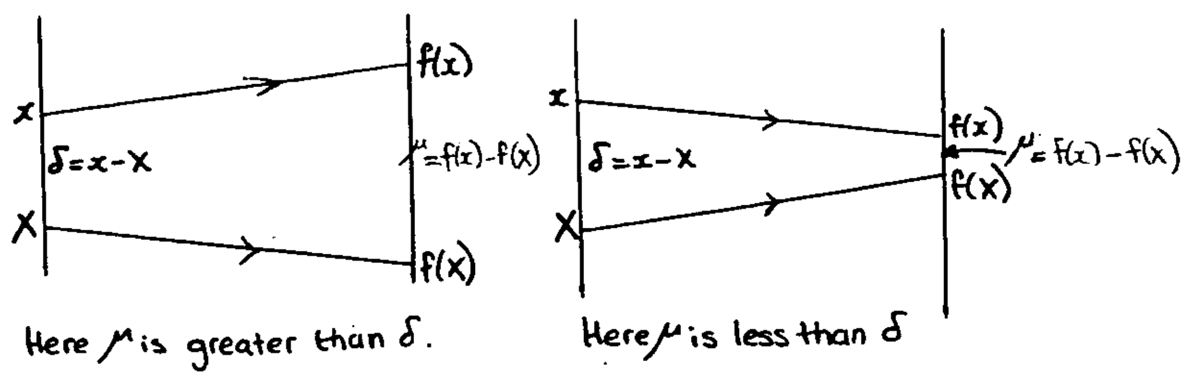
and the exact solution is X. That is, X satisfies $X = f(X)$ exactly.

A starting value x_1 is fed into the formula to give $x_2 = f(x_1)$. Now, for the iteration process to converge we want x_2 to be closer to X than x_1. Since $x_2 = f(x_1)$ and $X = f(X)$ this means that we want $f(x_1)$ to be closer to $f(X)$ than x_1 is to X. This suggests that we should investigate the effect of functions on errors and that's the purpose of the following work.

Suppose f is a function and x is an approximation to X with error $\delta = x - X$. Then $f(x)$ is an approximation to $f(X)$ with error μ, where

$$\begin{aligned}\mu &= f(x) - f(X)\\ &= f(X + \delta) - f(X).\end{aligned}$$

and μ is called the error propagated by f. This error may be larger or smaller than the original error as shown by the following mapping diagrams.



Example. f is the function $x \mapsto 4x^2 - 56x + 192 \quad (x \in \mathbb{R})$. In this case

$$\begin{aligned}\mu &= f(X + \delta) - f(X)\\ &= \{4(X + \delta)^2 - 56(X + \delta) + 192\} - \{4X^2 - 56X + 192\}\\ &= 4X^2 + 8X\delta + 4\delta^2 - 56X - 56\delta + 192 - 4X^2 + 56X - 192\\ &= 8X\delta - 56\delta + 4\delta^2\end{aligned}$$

If δ is small then we can ignore the term involving δ^2 to get

$$\mu = (8X - 56)\delta \qquad \text{to first order}$$

and the coefficient of δ, $(8X - 56)$, is called the scale factor λ.

Alternatively, λ can be defined as

$$\lambda = \frac{\mu}{\delta} = \frac{f(X + \delta) - f(X)}{\delta}$$

in which case in the example above

$$\lambda = \frac{\mu}{\delta} = \frac{8X\delta - 56\delta + 4\delta^2}{\delta} = 8X - 56 + 4\delta.$$

This time if δ is small we ignore the term involving δ to get $\lambda = 8X - 56$ as above.

Now λ can be evaluated for various values of X.

1. f is the function $x \mapsto 4x^2 - 56x + 192 \quad (x \in \mathbb{R})$.
   Evaluate the scale factor, λ, at each of the following values of X.
   (i) $X = 2$ (ii) $X = 0.1$ (iii) $X = 8$

2. (i) (a) What is the error μ in $f(x)$ as an approxiation to $f(X)$? (to first order) when f is the function $x \mapsto 3x^2 + 7x + 2 \quad (x \in \mathbb{R})$.
   (b) Hence write down the scale factor λ for f at X.
   (c) Evaluate λ for
   $X = 1$ and $X = -3$.
   (ii) (a) What is the error μ in $f(x)$ as an approximation to $f(X)$? (to first order) when f is the function $x \mapsto 14x - 3x^2 + 9 \quad (x \in \mathbb{R})$.
   (b) Hence write down the scale factor λ for f at X.
   (c) Evaluate λ for
   $X = 0.1$ and $X = 2$.

3. (i) For each of the functions
   (a) $f: x \mapsto 2x - x^2 + 3 \quad (x \in \mathbb{R})$
   (b) $g: x \mapsto x^2 - \frac{1}{4}x + 3 \quad (x \in \mathbb{R})$
   calculate the scale factor at $X = 2$.
   (ii) For each of the functions in part (i) find the image of
   (a) $x = 2$ (b) $x = 2.1$.
   and illustrate your answers using mapping diagrams.
   (iii) What does this suggest about the effect of the functions f and g on errors near $X = 2$?

*4. In this problem we give you a number of functions f and starting values x_1. You have to calculate successive terms in the iteration process based on $x_{n+1} = f(x_n)$ using your calculator, until you are confident that you understand the long term behaviour of the process.

Does it converge? If so, which solution of $x = f(x)$ is it approaching? Do any of the processes diverge?

	function f ($x \in \mathbb{R}$)	Solutions of $x = f(x)$	Starting value x_1	Number of iterations to be performed
(i)	$x \mapsto \frac{5}{8}(x + 0.3)^2$	0.1 and 0.9	(a) 0.5	12
			(b) 1.0	9
(ii)	$x \mapsto \frac{1}{8}(1 - 20x^2)$	-0.5 and 0.1	0.05	12
(iii)	$x \mapsto \frac{1}{4}x^2 + 0.3$	approx 0.33 and 3.7	3.0	7

*5. The scale factor of f at various points is going to be important in the discussion of why some of these iterations processes converge and some diverge.

(i) (a) Write down the scale factor λ of $x \mapsto \frac{5}{8}(x + 0.3)^2$ ($x \in \mathbb{R}$) at X.

(b) Evaluate λ for X = 0.1 and X = 0.9.

(ii) (a) Write down the scale factor λ of $x \mapsto \frac{1}{8}(1 - 20x^2)$ ($x \in \mathbb{R}$) at X.

(b) Evaluate λ for X = -0.5 and 0.1.

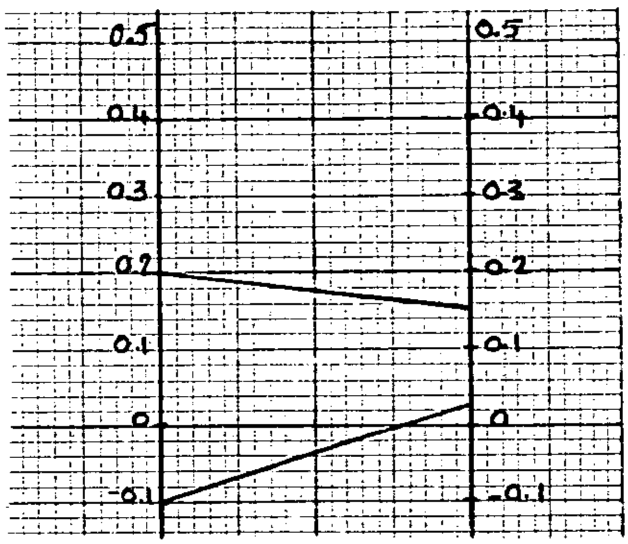
(iii) (a) Write down the scale factor λ of $x \mapsto \frac{1}{4}x^2 + 0.3$ ($x \in \mathbb{R}$) at X.

(b) Evaluate λ for X = 0.33 and X = 3.7.

*6. Complete the mapping diagram on the next page for the function

$$f: x \mapsto \frac{5}{8}(x + 0.3)^2 \quad (x \in \mathbb{R})$$

Use the points marked in the domain; calculate their images; mark the corresponding points in the codomain, and join them up.

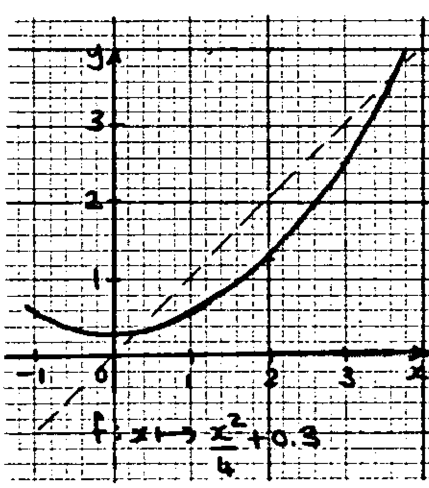


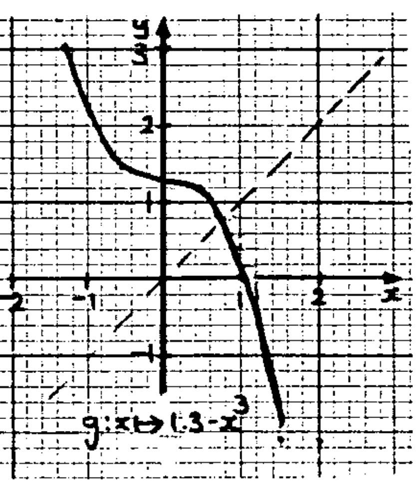
POST-PROGRAMME WORK

7. Here are the graphs of the functions

$$f: x \mapsto \frac{x^2}{4} + 0.3 \qquad (x \in \mathbb{R})$$

$$g: x \mapsto 1.3 - x^3 \qquad (x \in \mathbb{R})$$





(i) Draw on the first graph the staircase or cobweb diagram for the sequence starting at x = 2. On the second graph draw the diagram for the sequence starting at x = 0.6.

(ii) Use your diagram to predict whether the iterative process based on

$$x_{n+1} = 1.3 - x_n{}^3$$

with starting value $x_1 = 0.7$ will converge or diverge.

(iii) Check your prediction by using the formula to work out

x_2 x_3 x_4 x_5 and x_6.

8. (i) Use the Quadratic Formula to work out approximate solutions to $x^2 - x - 1 = 0$.

(ii) The function $f: x \mapsto 1 + \frac{1}{x}$ has scale factor $\lambda = -\frac{1}{X^2}$

and the function

$$g: x \mapsto \frac{1}{x - 1} \text{ has scale factor } \lambda = \frac{1}{(X - 1)^2}$$

Write down the scale factor of the function $h: x \mapsto x^2 - 1$.

(iii) In Section 1 we discussed solving the equation $x^2 - x - 1 = 0$ using the rearrangements

(a) $x_{n+1} = 1 + \frac{1}{x_n}$ (b) $r_{n+1} = \frac{1}{x_n - 1}$ (c) $x_{n+1} = x_n^2 - 1$.

with various starting values.

Use Parts (i) and (ii) to predict which of these rearrangements you would use to find the positive solution of $r^2 - r - 1 = 0$. Which would you use to calculate the negative solution?

9. This problem illustrates the dangers of using formula iteration to solve an equation when the chosen rearrangement has scale factor close to one near the solution.

The equation

$$8x^3 - 12x^2 + 6x - 1 = 0$$

has one solution in the interval [0, 1]. Calculate this solution to three places of decimals using the rearrangement

$$x = \frac{1}{6}(1 + 12x^2 - 8x^3)$$

with starting values

(i) 0.505 (ii) 0.501

POSSIBLE EXTENSIONS

1. This section could be used to introduce the concept of the derivative, or if students are already familiar with calculus, it presents a different way of looking at the derivative and this can be followed up by a comparison of the various approaches. *(SEE SECTION 11: THE DERIVED FUNCTION).*

2. In Exercise 8 we gave the scale factors for the functions $x \mapsto \frac{1}{x}$ and $x \mapsto \frac{1}{x - 1}$. You may wish to explore the derivation of the scale factors from first principles.

3. Students are now in a position to investigate the solution of any equation using iterative methods. They can formulate their own rearrangements to get the equation into the form

   $$x = f(x)$$

   and use the methods introduced here to predict whether such a rearrangement converges or diverges.

4. The work in 3 above could also be extended to cover the solution of equations to a desired degree of accuracy using a computer.

SOLUTIONS

1. $\lambda = 8X - 56$

(i) When $X = 2$, $\lambda = -40$

(ii) When $X = 0.1$, $\lambda = -55.2$

(iii) When $X = 8$, $\lambda = 8$.

2. (i) (a) $\mu = f(X + \delta) - f(X)$

$= \{3(X + \delta)^2 + 7(X + \delta) + 2\} - \{3X^2 + 7X + 2\}$

$= 6X\delta + 7\delta + 3\delta^2$

(b) To first order $\mu = (6X + 7)\delta$ and so $\lambda = 6X + 7$. Alternatively, $\lambda = \frac{\mu}{\delta} \simeq 6X + 7$ (ignoring the term involving δ).

(c) When $X = 0.1$, $\lambda = 7.6$

When $X = 2$, $\lambda = 19$.

(ii) (a) $\mu = f(X + \delta) - f(X)$

$= \{14(X + \delta) - 3(X + \delta)^2 + 9\} - \{14X - 3X^2 + 9\}$

$= -6X\delta + 14\delta - 3\delta^2$.

(b) To first order $\mu = (-6X + 14)\delta$ and so $\lambda = -6X + 14$ Alternatively, $\lambda = \frac{\mu}{\delta} \simeq -6X + 14$ (ignoring the term involving δ.

3. (i) (a) $\mu = f(X + \delta) - f(X)$

$= \{(2(X + \delta) - (X + \delta)^2 + 3\} - \{2X - X^2 + 3\}$

$= - 2X\delta + 2\delta - \delta^2$

$= (-2X + 2)\delta$ to first order.

Hence $\lambda = - 2X + 2$ and when $X = 2$, $\lambda = -2$

(b) $\mu = g(X + \delta) - g(X)$.

$= \{(X + \delta)^2 - \frac{1}{4}(X + \delta) + 3\} - \{X^2 - \frac{1}{4}X + 3\}$

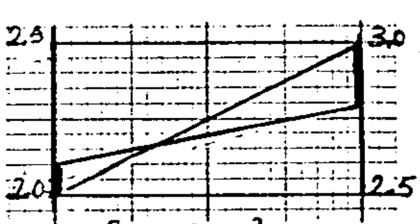
$= 2X\delta - \frac{1}{4}\delta + \delta^2$.

$= (2X - \frac{1}{4})\delta$ to first order.

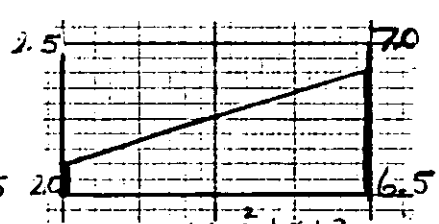
Hence $\lambda = 2X - \frac{1}{4}$ and when $X = 2$, $\lambda = 3.75$.

(ii) $f(2) = 3$ $g(2) = 6.5$

$f(2.1) \simeq 2.8$ $g(2.1) \simeq 6.9$



$f: x \mapsto 2x - x^2 + 3$



$g: x \mapsto x^2 - \frac{1}{4}x + 3$

(iii) In both cases the distance between f(2) and f(2.1) is larger than the distance between 2 and 2.1. This suggests that the effect of both functions f and g is to <u>increase</u> the errors.

Notice that when the scale factor is negative the lines in the mapping diagram cross over.

4. (i) (a) When $x_1 = 0.5$, the iteration process converges to 0.1.

(b) When $x_1 = 1$, the iteration process diverges.

(ii) The process converges to 0.1; terms alternate smaller and larger than 0.1.

(iii) The process converges steadily to 0.33.

5. (i) (a) $\lambda = \frac{5}{8}(2X + 0.6)$.

(b) When $X = 0.1$ $\lambda = 0.5$

When $X = 0.9$ $\lambda = 1.5$

(ii) (a) $\lambda = -5X$

(b) When $X = -0.5$ $\lambda = 2.5$.

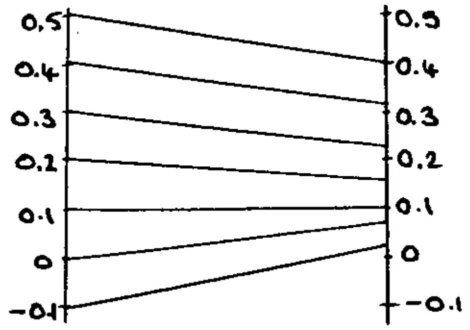
When $X = 0.1$ $\lambda = -0.5$.

(iii) (a) $\lambda = \frac{1}{2}X$

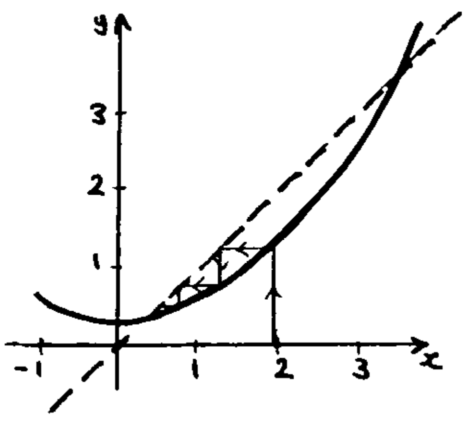
(b) When $X = 0.33$ $\lambda = 0.165$

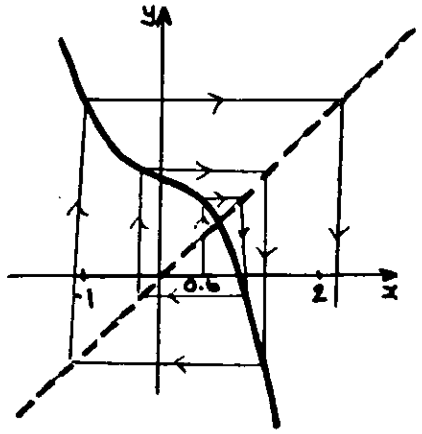
When $X = 3.7$ $\lambda = 1.85$.

6.



7. (i) Your diagrams should look like those below





(ii) The process will diverge.

(iii) $x_2 \simeq 1.084$ $\quad$ $x_5 \simeq -0.897$

$x_3 \simeq 0.026$ $\quad$ $x_6 \simeq 2.022$

$x_4 \simeq 1.23$

8. (i) $x \simeq 1.618$ and $x \simeq -0.618$

(ii) The scale factor for h is 2X.

(iii) For each of the rearrangements we need to calculate the scale factors at the approximate solutions.

$f: x \mapsto 1 + \frac{1}{x}$ has scale factor $\lambda = -\frac{1}{X^2}$

At X = 1.618 $\lambda \simeq -0.382$

At X = -0.618 $\lambda \simeq -35.431$.

At X = 1.618 $|\lambda|$ is less than 1, so iteration based on this rearrangement with starting value near to 1.618 will converge to the positive solution.

At X = -0.618 $|\lambda|$ is greater than 1, so, iteration based on this rearrangement with starting value near to -0.618 will diverge.

(b) $g: x \mapsto \frac{1}{x-1}$ has scale factor $\lambda = -\frac{1}{(X-1)^2}$.

At X = 1.618 $\lambda \simeq -2.618$

At X = -0.618 $\lambda \simeq -0.382$.

At X = -0.618 $|\lambda|$ is less than 1, so iteration based on this rearrangement with starting value near -0.618 will converge to the negative solution.

At X = 1.618 $|\lambda|$ is greater than 1, so iteration based on this rearrangement with starting value close to 1.618 will diverge.

(c) $h: x \mapsto x^2 - 1$ has scale factor $\lambda = 2X$.

At X = 1.618 $\lambda \simeq 3.336$.

At X = -0.618 $\lambda \simeq -1.336$.

In both cases $|\lambda|$ is greater than 1 so iteration based on this rearrangement will diverge for starting values near to both solutions.

9. If you used the starting value 0.505 you would conclude that the solution was 0.505 to three decimal places: if you used the starting value 0.501 you would conclude that the solution was 0.501 to three decimal places. In fact the solution is 0.5 exactly. This illustrates the danger of working without checking the scale factor, which in this case is 1 at $X = 0.5$. So the 'convergence' which appears to take place is unreliable. Check this by constructing the appropriate staircase diagram. You will find that the process appears to converge while still far from the solution.

10. $x \mapsto \frac{1}{x}$: AN AREA FOR REVISION

PROGRAMME SUMMARY

This programme deals with the link between $\log_e x$ and the corresponding area under the graph of y: $x \mapsto \frac{1}{x}$. Furthermore, the rule

$$\log ab = \log a + \log b$$

can be demonstrated by adding together the corresponding areas.

3 mins

The graph of f: $x \mapsto \frac{1}{x}$ ($x \in \mathbb{R}^+$) has the following property: if W(r) denotes the weight of the area under the graph from x = 1 to x = r, then

$$W(r) + W(s) = W(rs).$$

2 mins

In fact the areas themselves are connected and

$$A(r) + A(s) = A(rs).$$

This is related to the logarithm rule

$$\log r + \log s = \log rs.$$

We aim to show that A(r) = log(r) to a certain base.

8 mins

First we show numerically that $A(2) = \log_e 2$. This is done by repeatedly calculating under-estimates and over-estimates of the area between 1 and 2 by dividing it into rectangles. Each calculation gives an interval which includes the area we want. By dividing the area into more and more rectangles the interval becomes smaller and we can trap the exact area in a nest of intervals.

1 min

In fact $A(2) = \log_e(2)$.

7 mins

Next we prove that the property

$$A(r) + A(s) = A(rs).$$

This involves two results about scaling in the x- and y-directions. First, a scaling by a factor of s in either direction multiplies areas by s. Second, a scaling by s in the x-direction followed by a scaling of $\frac{1}{s}$ in the y-direction leaves the graph of f: $x \mapsto \frac{1}{x}$ ($x \in \mathbb{R}^+$) unchanged.

1 min

This demonstrates that A(r) is indeed $\log_e(2)$. Since there is a connection between $\log_e x$ and e^x, this suggests that there is a link between the function $x \mapsto \frac{1}{x}$ and $x \mapsto e^x$.

1 min	Finally we recap the main ideas used in the programme:

the properties involved in scaling functions and the idea of trapping a real number in a nest of intervals.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) the idea of logs and knowledge of the rule

$$\log ab = \log a + \log b.$$

Preferably students should know about $\log_e$ - although this could be introduced via the programme.

(ii) the idea of an interval on the real line of the form

$[a, b]$

(iii) inequalities of the form $a < b < c$

(iv) scaling graphs in the x- and y-directions. Students should have some idea of the fact that an x-scaling of λ transforms $f(x)$ to $f(\frac{1}{\lambda}x)$, and a y-scaling of μ transforms $f(x)$ to $\mu f(x)$ (although not necessarily in this form).

NOTE The effect of scaling on graphs is illustrated IN SECTION 7: FUNCTIONS AND GRAPHS

(v) use of a scientific calculator.

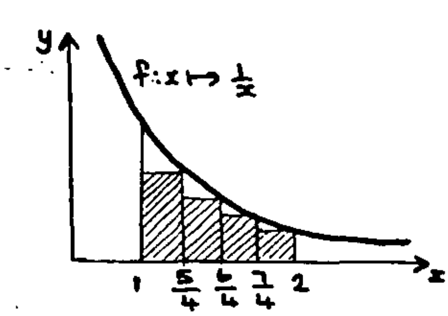
PRE-PROGRAMME WORK

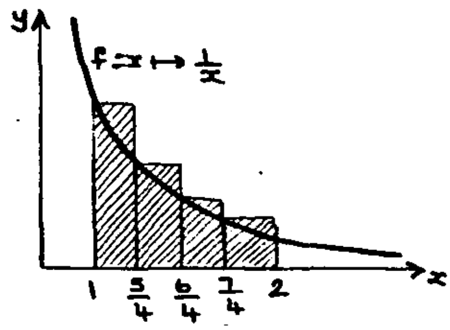
*1. (i) Complete the table below and so plot the graph of $f: x \mapsto \frac{1}{x}$.

x	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	6
$\frac{1}{x}$											

(ii) (a) $f(\frac{5}{4})$, (b) $f(\frac{6}{4})$, (c) $f(\frac{7}{4})$.

(iii) Use your answers to Parts (i) and (ii) to calculate the following shaded areas.





*2. This question involves logarithms to the base 10.

(i) Use your calculator (log-tables) to find

(a) $\log_{10}3$, (b) $\log_{10}6$, (c) $\log_{10}2$

(ii) Hence find (a) $\log_{10}18$, (b) $\log_{10}12$.

(iii) Now write down $\log_{10}(ab)$ in terms of $\log_{10} a$ and $\log_{10} b$.

*3. Your calculator may also have a key which gives logarithms to the base e. (e is just a number, its value is about 2.72), These logs are often called <u>Naperian</u> or <u>Natural logarithms</u> and are frequently denoted by Ln. - So the key on your calculator may be labelled Ln.

(i) Find $\log_e 2$ using your calculator. You should get a result in the interval [0.693, 0.694].

(ii) Use your calculator to find

(a) $\log_e 3$, (b) $\log_e 6$.

(iii) From parts (i) and (ii) calculate

(a) $\log_e 3 + \log_e 6$, (b) $\log_e 6 + \log_e 2$.

(iv) Use the inverse key on your calculator to check

that (a) $\log_e 3 + \log_e 6 = \log_e 18$ $\boxed{\text{INV}}\ \boxed{\text{Ln}}$

and (b) $\log_e 6 + \log_e 2 = \log_e 12$.

This demonstrates that the addition rule also works for

$\log_e$. That is $\log_e ab = \log_e a + \log_e b$.

In fact this rule work for logs to <u>any</u> base.

4. The shaded square below has area 1.

(i) The square is scaled in the x-direction by a factor of 2. Draw a diagram to represent the resulting shape. What is the are of the resulting rectangle?

(ii) The square is scaled in the y-direction by a factor of $\frac{1}{2}$. Draw a diagram to represent the resulting shape. What is its area?

(iii) The square is first scaled in the x-direction by a factor 2, then scaled in the y-direction by a factor $\frac{1}{2}$. Draw a diagram to represent the resulting square. What is its area?

(iv) The square is now scaled in the x-direction by a factor r, then in the y-direction by a factor s. Draw a diagram to represent the shape. What is its area?

<u>POST-PROGRAMME WORK</u>

No specific post programme work is suggested here although it may be helpful to go over the part on scaling at the end of the programme, relating this to the scaling of shapes as in the pre-work. Also, some explanation may be needed to show why a scaling of s in the x-direction followed by a scaling of $\frac{1}{s}$ in the y-direction leaves the actual graph of $x \mapsto \frac{1}{x}$ unchanged. Perhaps try it with some specific examples.

e.g. when x = 2, s = 3. etc.

POSSIBLE EXTENSIONS

1. This section could clearly lead into a discussion of integration.

   NOTE *SECTION 13: THE FUNDAMENTAL THEOREM OF CALCULUS investigates the link between the area under a curve f between a and b and the integral.*

   $$\int_a^b f(x)dx$$

2. Specifically, it could be used to show that

   $$\int_1^r \frac{1}{x}dx = \log_e r$$

   with a neat demonstration that $\log_e 1 = 0$. The programme draws attention to the fact that, when integrating, Naperian (Natural) logs should always be used rather than $\log_{10}$.

3. The section could also lead to further work on the properties of logarithms.

4. We mention that there is some link between the graph of $x \mapsto \frac{1}{x}$ and $x \mapsto e^x$. This could be followed up.

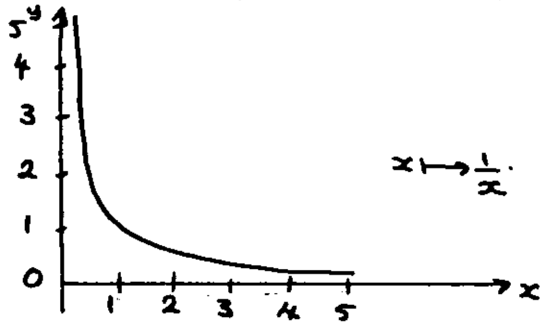
   Note *SECTION 15 WHY* e? *discusses this connection*

5. In the programme we suggested that $\log_e 2$ can be calculated as accurately as we please by dividing the area into narrower intervals. This could be converted into a computer program to calculate $\log_e 2$ and indeed $\log_e n$ for any $n \geq 1$.

SOLUTIONS

1. (i)

x	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5	6
$\frac{1}{x}$	5	3	2	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$



(ii) (a) $f(\frac{5}{4}) = \frac{4}{5}$, (b) $f(\frac{6}{4}) = \frac{4}{6}$, (c) $f(\frac{7}{4}) = \frac{4}{7}$.

(iii) (a) Area $= (\frac{1}{4} \times \frac{4}{5}) + (\frac{1}{4} \times \frac{4}{6}) + (\frac{1}{4} \times \frac{4}{7}) + (\frac{1}{4} \times \frac{4}{8}) = 0.634$

(b) Area $= (\frac{1}{4} \times 1) + (\frac{1}{4} \times \frac{4}{5}) + (\frac{1}{4} \times \frac{4}{6}) + (\frac{1}{4} \times \frac{4}{7}) = 0.760$

2. (i) (a) To 4 decimal places $\log_{10}3 = 0.4771$.

(b) $\log_{10}6 = 0.7782$

(c) $\log_{10}2 = 0.3010$.

(ii) (a) $18 = 3 \times 6$, so $\log_{10}18 = \log_{10}3 + \log_{10}6 = 1.2553$

(b) $21 = 6 \times 2$, so $\log_{10}12 = \log_{10}6 + \log_{10}2 = 1.0792$

(iii) $\log_{10}ab = \log_{10}a + \log_{10}b$.

3. (i) $\log_e 2 = 0.6931$ (to 4 decimal places)

(ii) (a) $\log_e 3 = 1.0986$; $\log_e 6 = 1.7918$

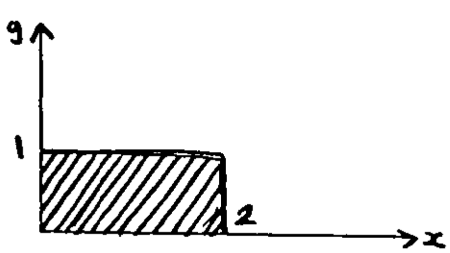
(iii) (a) $\log_e 3 + \log_e 6 = 2.8904$

(b) $\log_e 6 + \log_e 2 = 2.4849$

(iv) From Part (iii) (a) $2.8904 = \log_e 18$

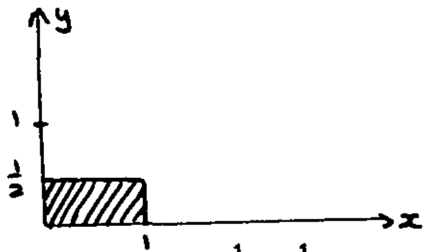
(b) $2.4849 = \log_e 12$.

4. (i)



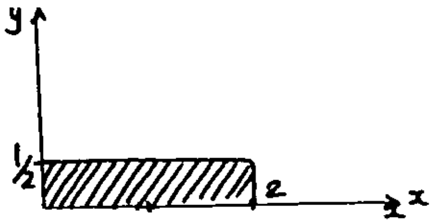
Area $= 2 \times 1 = 2$ sq units.

(ii)



Area $= 1 \times \frac{1}{2} = \frac{1}{2}$ units.

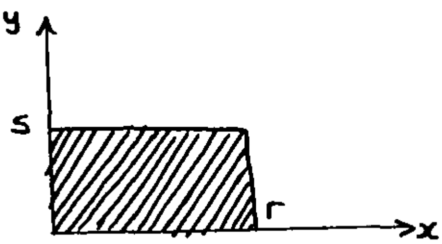
(iii)



Area $= 2 \times \frac{1}{2} = 1$ sq units.

So the resulting area is the same as the original.

(iv)



Area = rs sq units.

11. THE DERIVED FUNCTION

(NOTE *This section builds upon the work of scale factors in SECTION 9: ITERATION AND CONVERGENCE).*

PROGRAMME SUMMARY

The programme is in three main parts. First by looking at scale factors we lead up to the definition of derivative. Then, we look at a geometrical interpretation of the derivative. Finally we look at the derived function for which we use the notation f'.

2 mins	For some functions it's quite easy to find the scale factor

using the definition

$$\lambda = \frac{\mu}{\delta} = \frac{f(X + \delta) - f(X)}{\delta} \quad \text{(to first order).}$$

10 mins	But there are snags with $f: x \mapsto \sin x$. However, it is

possible to evaluate $\frac{\mu}{\delta}$ for X = 0.6 and various values of δ. And as δ gets smaller we get a sequence of values $\frac{\mu}{\delta}$ which converges as δ gets closer to zero. This gives a new definition for the scale factor as

$$\lim_{\delta \to 0} \frac{\mu}{\delta} = \lim_{\delta \to 0} \frac{f(X + \delta) - f(X)}{\delta}$$

This definition applies more generally and we say that the derivative of f at X is

$$\lim_{\delta \to 0} \frac{f(X + \delta) - f(X)}{\delta}.$$

1 min	We turn to the geometrical interpretation of $\frac{\mu}{\delta}$ as the slope

of the chord joining

$(x, f(X)$ and $((X + \delta), f(X + \delta))$.

As $\delta \to 0$ the slope of this chord approaches the tangent to the graph of f at the point X and this gives a different picture of the derivative at X as the tangent to the graph of f at the point X.

7 mins	This alternative approach allows us to build up the derived

function of $f: x \mapsto 0.7x^2 + 1$ geometrically. We look at the derived functions for various functions. For polynomials the derived functions has degree one less than the original function.

$\boxed{\text{1 min}}$ We then return to $x \mapsto \sin x$. Again the derived function is obtained diagramatically. It looks as though the derived function is $x \mapsto \cos x$.

$\boxed{\text{2 mins}}$ Finally we look at increasing and decreasing linear functions and constant functions. The geometrical approach turns out to be quite useful.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) the definition of scale factor as introduced in *SECTION 9: ITERATION AND CONVERGENCE.*

(ii) the idea of the degree of a polynomial and some of the characteristics of polynomial graphs

(iii) the graph of $x \mapsto \sin x$ and $x \mapsto \cos x$

(iv) the idea of increasing, decreasing linear functions and constant functions

(v) an intuitive idea of a limit

(vi) the slope of a straight line as $\frac{f(b) - f(a)}{b - a}$

(vii) use of a scientific calculator.

PRE-PROGRAMME WORK

Students will benefit by working through at least the first three exercises of SECTION 9: ITERATION AND CONVERGENCE before starting this section.

1. (i) Find the scale factor at X for the function

$$f: x \mapsto 0.7x^2 + 1 \qquad (x \in \mathbb{R}),$$

(ii) Hence find the scale factors at

(a) $X = 0.6$ (b) $X = 2.0$ (c) $X = 3.5$ (d) $X = -1.5$.

2. (i) Draw the graph of $g: x \mapsto 1.4x$.

(ii) Use your calculator to draw the graph of

$$f: x \mapsto 0.7x^2 + 1.$$

3. (i) Use your calculator to complete the table below.

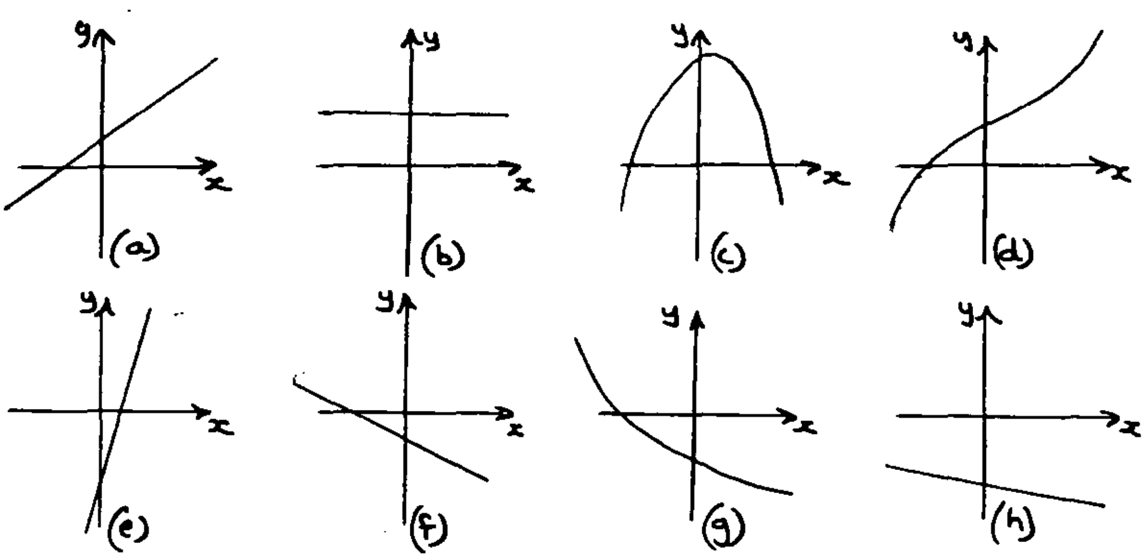
(Note θ is measured in radians, so make sure your calculator is in the appropriate mode).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$																	
$\cos\theta$																	

(ii) Use the table to plot the graphs of

(a) $\theta \mapsto \sin\theta$ (b) $\theta \mapsto \cos\theta$.

4. Decide whether each of the following graphs is a graph of an increasing linear function, a decreasing linear function or a constant function.



POST-PROGRAMME WORK

5. Let f be the function given by

$$f(x) = x^3 - 3x^2 + 2.$$

(i) Find the scale factor for f at X and calculate its value when X = 1.
(Use the method introduced in Section 9).

(ii) Use your calculator to calculate

$$\frac{f(1 + \delta) - f(1)}{\delta}$$

for (a) $\delta = -0.1$ (b) $\delta = -0.01$ (c) $\delta = -0.001$
(d) $\delta = 0.1$ (b) $\delta = 0.01$ (f) $\delta = 0.001$.

6. Use the definition

$$\text{derivative} = \lim_{\delta \to 0} \frac{f(X + \delta) - f(X)}{\delta}$$

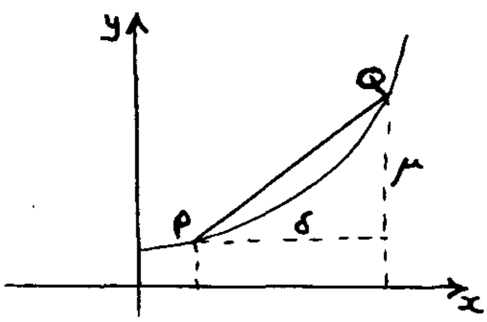
to find the derivatives of each of the following functions at X.

(i) $f: x \mapsto x$

(ii) $f: x \mapsto 9x^2 - 4x - 21$

(iii) $f: x \mapsto \frac{1}{x}$ $(x \neq 0)$.

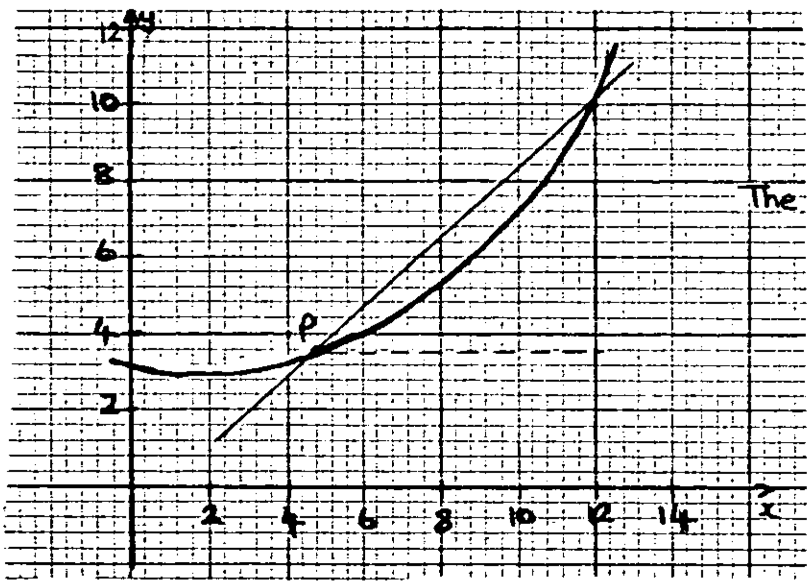
7. This question concerns the geometrical interpretation of the derivative as the slope of the tangent to the graph of a function.



The slope of the chord PQ is $\frac{\mu}{\delta}$.
What happens as δ gets smaller?

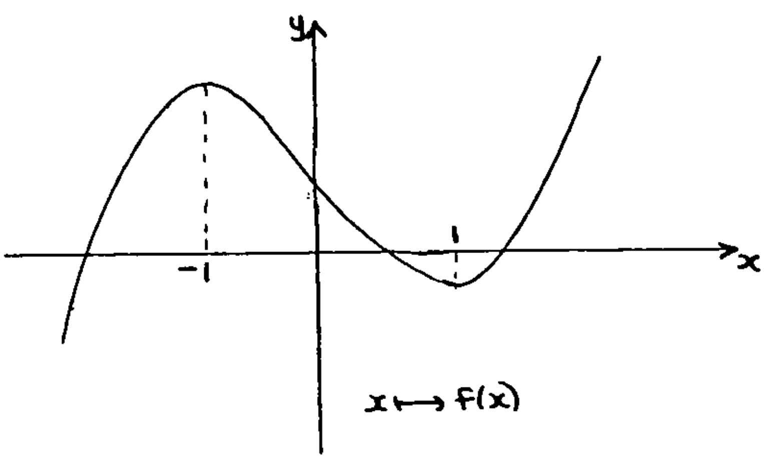
(i) On the diagram on the next page draw the chords through P corresponding to values of δ of your choice. Make sure that $|\delta|$ gets smaller and smaller. Try a few with δ negative, $\delta = -5$ for example.

(ii) What is happening to these lines?



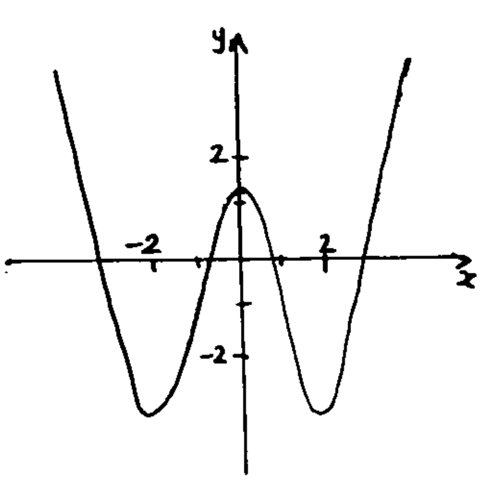
The chord drawn on the diagram corresponds to $\delta = 7$.

8. The graph of a function f is given below. Indicate where its slope is positive, negative and zero.

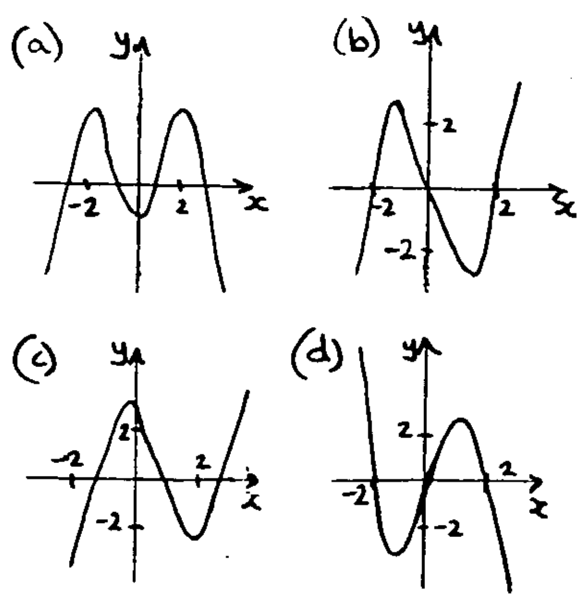


$x \mapsto f(x)$

9. The graph of a function g is given below. We have sketched four possibilities for the graph of the derived function. Which one is most likely to be correct?



$x \mapsto g(x)$



10. In this question we use the notation $\sin'(x)$ and $\cos'x$ to stand for the derived function of $\sin x$ and $\cos x$ respectively.

(i) (a) Write down an expression for $\sin'0$ using the limit definition of the derivative.

(b) Use your calculator to evaluate $\frac{\sin \delta}{\delta}$ for a range of positive and negative values of δ (measured in radians!)

(For example, try 0.1, 0.01, 0.001, 0.0001, -0.1, -0.01, -0.001, -0.0001.)

What can you conclude about $\lim_{\delta \to 0} \frac{\sin \delta}{\delta}$?

(c) What do you conclude is the value of $\sin'0$.?

(ii) Find $\cos'(0)$ using a similar method.

(iii) (a) Use the sine formula

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

to show that

$$\frac{\sin(0.8+\delta)-\sin(0.8)}{\delta} = \sin(0.8)\left(\frac{\cos\delta-1}{\delta}\right) + \cos(0.8)\left(\frac{\sin\delta}{\delta}\right).$$

(b) Hence explain why $\sin'(0.8) = \cos(0.8)$.

(c) Show that $\sin'X = \cos X$ at a general point X.

(iv) Use a similar argument and the cosine formula

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

to show that $\cos'X = -\sin X$ at a general point X.

POSSIBLE EXTENSIONS

1. This section could either be used as an introduction to differentiation or as an alternative approach to something already familiar. In both cases it opens the way to further work on differentiation (from first principles) - and also differentiation techniques.

2. In particular the derivatives of sin x and cos x might be further investigated — especially with reference to the graphs of these functions and the slopes at various points. The graphs of $\frac{\sin x}{x}$ and $\frac{\cos x - 1}{x}$ could be plotted using a calculator and this could help the understanding of the limits

$$\lim_{\delta \to 0} \frac{\sin \delta}{\delta} \text{ and } \lim_{\delta \to 0} \frac{\cos \delta - 1}{\delta} .$$

3. The graphical approach to the derived function leads to the principles involved in curve sketching.

(<u>*Note*</u> <u>*SECTION 12: THE BEHAVIOUR OF FUNCTIONS*</u> *looks at this aspect of curve sketching).*

SOLUTIONS.

1. (i) $\mu = f(X + \delta) - f(X)$

$= \{0.7(X + \delta)^2 + 1\} - \{0.7X^2 + 1\}$

$= 0.7X^2 + 1.4X\delta + 0.7\delta^2 - 0.7X^2 - 1$

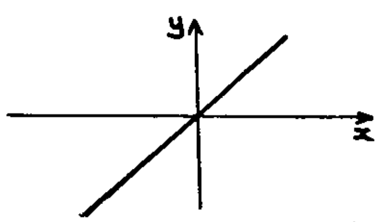
$= 1.4X\delta + 0.7\delta^2.$

$= 1.4X\delta$ (to first order)

Hence the scale factor is 1.4X.

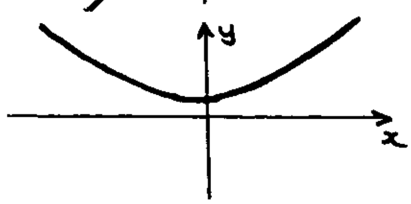
(ii) (a) 0.84 (b) 2.8 (c) 4.9 (d) -2.1

2. (i)



$g: x \mapsto 1.4x$

(ii)

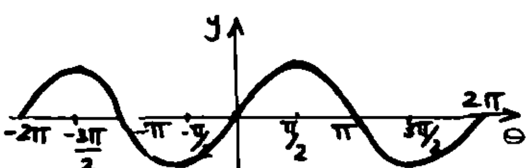


$f: x \mapsto 0.7x^2 + 1$

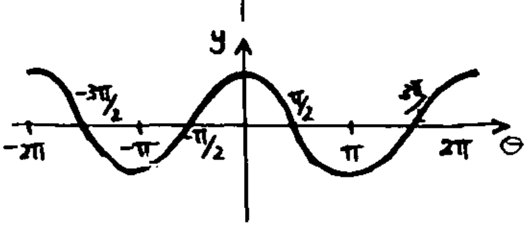
3. (i) To one decimal place:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin θ	0	0.5	0.7	0.9	1	0.9	0.7	0.5	1	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0
cos θ	1	0.9	0.7	0.5	0	-0,5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0	0.5	0.7	0.9	1

(ii)



(a) $\theta \mapsto \sin\theta$



(b) $\theta \mapsto \cos\theta$

4. (a) and (e) are graphs of increasing linear functions (b) is a graph of a constant function. (f) and (h) are graphs of decreasing linear functions.

Note (d) is an increasing function but it is not linear.
(g) is a decreasing function but it is not linear.

5. (i) $f(X + \delta) = (X + \delta)^3 - 3(X + \delta)^2 + 2$

$$= X^3 + 3X^2\delta + 3X\delta^2 + \delta^3 - 3X^2 - 6X\delta - \delta^2 + 2$$

$$= (X^3 - 3X^2 + 2) + (3X^2 - 6X)\delta + (3X - 1)\delta^2 + \delta^3$$

Hence $f(X + \delta) - f(X) = (3X^2 - 6X)\delta + (3X - 1)\delta^2 + \delta^3$

$= (3X^2 - 6X)\delta$ to first order.

Hence the scale factor λ is $3X^2 - 6X$ (the coefficient of δ)

When $X = 1$, $\lambda = -3$.

(ii) When $X = 1$, $f(X) = 0$. Hence

$$\frac{f(1 + \delta) - f(1)}{\delta} = \frac{f(1 + \delta)}{\delta}$$

δ	$\frac{f(1+\delta)}{\delta}$
-0.1	-2.99
-0.01	-2.9999
-0.001	-3.0

δ	$\frac{f(1+\delta)}{\delta}$
0.1	-2.99
0.01	-2.9999
0.001	-3.0

So, as δ gets smaller in magnitude $\frac{f(1 + \delta)}{\delta}$ gets closer to -3, the value of the scale factor at 1.

(Note the entries in the table are obtained using a calculator, this does not mean that $\frac{f(1 - 0.001)}{- 0.001}$ and $\frac{f(1 + 0.001)}{0.001}$ are exactly -3, only that the differences between their values and -3 is too small to be noticed by a calculator.)

6. (i) $f(X + \delta) - f(X) = (X + \delta) - X$

$= \delta$.

Thus $\frac{f(X + \delta) - f(X)}{\delta} = \frac{\delta}{\delta} = 1$.

As δ tends to zero $\frac{f(X + \delta) - f(X)}{\delta}$ does not change

so $f'(X) = \lim_{\delta \to 0} \frac{f(X + \delta) - f(X)}{\delta} = 1$.

(ii) $g(X + \delta) - g(X) = \{9(X + \delta)^2 - 4(X + \delta) - 21\} - \{9X^2 - 4X - 21\}$

$$= 9X^2 + 18X\delta + 9\delta^2 - 4X - 4\delta - 21 - 9X^2 + 4X + 21$$

$$= 18X\delta + 9\delta^2 - 4\delta.$$

So $\dfrac{g(X + \delta) - g(X)}{\delta} = 18X - 4 + 9\delta$

As δ tends to xero so does 9δ and so $18X - 4 + 9\delta$ gets closer and closer to $18X - 4$. Hence

$$g'(X) = \lim_{\delta \to 0} \frac{g(X + \delta) - g(X)}{\delta} = 18X - 4.$$

(iii) $h(X + \delta) - h(X) = \dfrac{1}{X + \delta} - \dfrac{1}{X}$

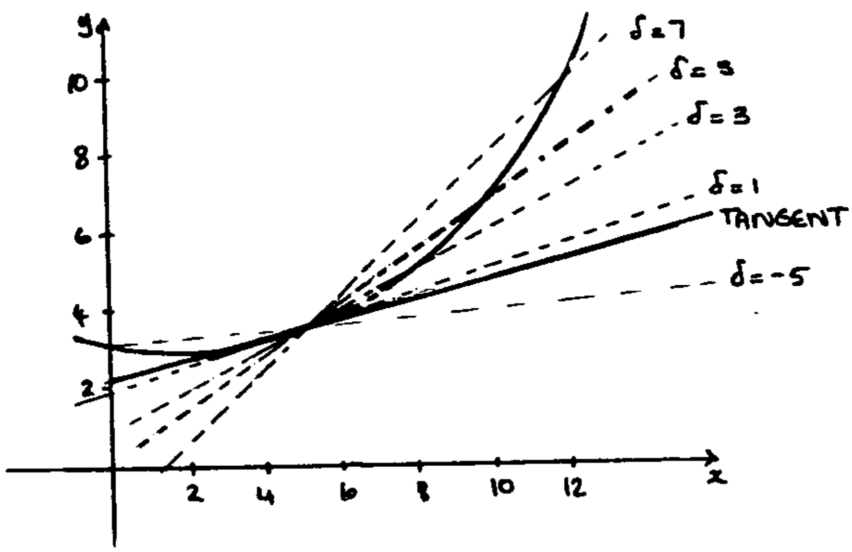
$$= \frac{X - (X + \delta)}{X(X + \delta)} = \frac{-\delta}{X(X + \delta)}.$$

Then $\dfrac{h(X + \delta) - h(X)}{\delta} = \dfrac{-1}{X(X + \delta)}$

As δ gets closer to zero, $X + \delta$ gets closer to X, so $\dfrac{-1}{X(X + \delta)}$ gets closer and closer to $\dfrac{-1}{X^2}$.

Hence $h'(X) = \lim_{\delta \to 0} \dfrac{h(X + \delta) - h(X)}{\delta} = \dfrac{-1}{X^2}$.

7. (i) Your diagram should look something like this.



(ii) The chords are getting closer and closer to the tangent to the curve at x = 5.

8. The slope is

positive for $x \in]-\infty, -1[$

zero at $x = -1$

negative for $x \in]-1, 1[$

zero at $x = 1$

positive for $x \in]1, \infty[$

9. The first thing to note is that g has zero slope at three points, $x = -2$, $x = 0$ and $x = 2$. So the derived function must have value

zero at these points. This eliminates (a) and (c). Also the slope of g is negative for $x < -2$ and $0 < x < 2$ and positive for $-2 < x < 0$ and $x > 2$, so the correct choice is (b).

10. (i) (a) $\sin'(0) = \lim_{\delta\to 0} \frac{\sin(0+\delta) - \sin 0}{\delta}$

$$= \lim_{\delta\to 0} \frac{\sin \delta}{\delta} \qquad \text{(as } \sin 0 = 0\text{).}$$

(b)

δ	$\frac{\sin \delta}{\delta}$
0.1	0.9983341
0.01	0.9999834
0.001	1
0.0001	1

δ	$\frac{\sin \delta}{\delta}$
-0.1	0.9983341
-0.01	0.9999834
-0.001	1
0.0001	1

The tables suggest that as δ tends to zero $\frac{\sin \delta}{\delta}$ gets closer and closer to 1.

That is, $\lim_{\delta\to 0} \frac{\sin \delta}{\delta} = 1$

(Note This is not a proof (that would be much more difficult.) But it is a useful demonstration).

(c) Hence $\sin' 0 = \lim_{\delta\to 0} \frac{\sin \delta}{\delta} = 1$.

(ii) By definition

$$\cos'(0) = \lim_{\delta\to 0} \frac{\cos(0+\delta) - \cos 0}{\delta}$$

$$= \lim_{\delta\to 0} \frac{\cos \delta - 1}{\delta} \qquad \text{(since } \cos 0 = 1\text{).}$$

This suggests that we should investigate the behaviour of $\frac{\cos \delta - 1}{\delta}$ for various values of δ.

δ	$\frac{\cos\delta - 1}{\delta}$
0.1	-0.0499583
0.01	-0.005
0.001	-0.0004999
0.0001	0

δ	$\frac{\cos\delta - 1}{\delta}$
-0.1	0.0499583
-0.01	0.005
-0.001	0.0004999
-0.0001	0

Again, the tables suggest that as δ tends to zero $\dfrac{\cos\delta - 1}{\delta}$ gets closer and closer to 0.

That is $\lim_{\delta\to 0} \dfrac{\cos\delta - 1}{\delta} = 0.$

Hence $\cos'(0) = \lim_{\delta\to 0} \dfrac{\cos\delta - 1}{\delta} = 0.$

(iii) (a) $\sin(0.8 + \delta) = \sin 0.8 \cos\delta + \cos 0.8 \sin\delta.$

Hence

$$\sin(0.8+\delta) - \sin 0.8 = \sin 0.8(\cos\delta - 1) + \cos 0.8 \sin\delta$$

and

$$\frac{\sin(0.8+\delta)-\sin 0.8}{\delta} = \sin 0.8\left(\frac{\cos\delta - 1}{\delta}\right) + \cos 0.8\left(\frac{\sin\delta}{\delta}\right)$$

(b) As $\lim_{\delta\to 0} \dfrac{\cos\delta - 1}{\delta} = 0$ and $\lim_{\delta\to 0} \dfrac{\sin\delta}{\delta} = 1$

$$\lim_{\delta\to 0} \sin 0.8\left(\frac{\cos\delta - 1}{\delta}\right) + \cos 0.8\left(\frac{\sin\delta}{\delta}\right) = \cos 0.8.$$

Hence $\sin'(0.8) = \lim_{\delta\to 0} \dfrac{\sin(0.8+\delta) - \sin 0.8}{\delta} = \cos 0.8.$

(c) Replacing 0.8 by X in Parts (a) and (b) gives

$$\sin' X = \lim_{\delta\to 0} \frac{\sin(X+\delta) - \sin X}{\delta}$$

$$= \lim_{\delta\to 0} \sin X\left(\frac{\cos\delta - 1}{\delta}\right) + \cos X\left(\frac{\sin\delta}{\delta}\right)$$

$$= \cos X.$$

(iv) By definition

$$\cos' X = \lim_{\delta\to 0} \frac{\cos(X+\delta) - \cos X}{\delta}$$

$$= \lim_{\delta\to 0} \frac{\cos X\cos\delta - \sin X\sin\delta - \cos X}{\delta}$$

$$= \lim_{\delta\to 0} \cos X\left(\frac{\cos\delta - 1}{\delta}\right) - \sin X\left(\frac{\sin\delta}{\delta}\right)$$

$$= -\sin X.$$

12. THE BEHAVIOUR OF FUNCTIONS

PROGRAMME SUMMARY

This programme show how the information that is obtained by analysing derivatives can be used to sketch the graph of a polynomial function.

5 mins

The graph of a quadratic function can be sketched using scalings and translations by completing the square, *(SEE SECTION 7: FUNCTION AND GRAPHS)* but this doesn't help with a function like $f(x) = 3x^4 + 2x^3$.

We could plot a few points and then 'fit' a curve but this isn't much good as several graphs provide a good fit. Indeed it is possible that even by plotting a lot of points we can make mistakes. For example what about the graph of

$$f(x) = \frac{10\,000\,x}{10\,000\,x - 1} \quad ?$$

4 mins

We look at the graph of a typical function f and its derived function f', and examine the relationship between the slope of the original function and the corresponding value of the derived function. This leads to the idea that a local minimum occurs at a point where the derived function is zero and changes sign from negative to positive. (A local maximum is also defined). This gives the first derivative test.

2 mins

The first derivative test is then applied to the function

$$f(x) = (x - 1)^2 + 2.$$

4 mins

It's not always easy to apply the first derivative test and we now look at the second derivative to find out what information that gives about the original function. This leads to the second derivative test: a local minimum occurs at a point where the first derivative is zero and the second derivative is positive. A local maximum is also defined.

The second derivative test is applied to the function

$$f(x) = 3x^4 + 2x^3$$

but it doesn't give a complete picture.

5 mins — In fact the second derivative provides more information about the graph of f. We introduce the idea of concave up and concave down and a point of inflection.

3 mins — Finally this is applied to the function

$$f(x) = 3x^4 + 2x^3$$

and we consequently sketch the graph.

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) plotting quadratic graphs

(ii) functional notation

(iii) differentiating polynomial functions and the terminology first derivative and second derivative

(iv) some idea of increasing and decreasing functions

(v) factorisation of algebraic expressions

(vi) interval notation of the form [a, b] and]a, b[.

Note *Sketching a quadratic by completing the square and using scalings and translations is mentioned in the programme. This is covered in SECTION 7: FUNCTIONS AND GRAPHS. However, the reference is only fleeting and it is not a necessary pre-requisite.*

PRE-PROGRAMME WORK

1. Find the first and second derivatives of each of the following functions

   (i) $f(x) = x^2$

   (ii) $g(x) = (x - 1)^2 + 2$

   (iii) $h(x) = x^5 - 4x^3 + 3x^2 - 2$

* (iv) $\ell(x) = 3x^4 + 2x^3$.

*2. Factorize (i) $3x^4 + 2x^3$

   (ii) $12x^3 + 6x^2$

   (iii) $36x^2 + 12x$.

3. Let $f(x) = 3 + 2x - x^2$.

   (i) (a) Plot the graph of f for x lying in interval [-1, 4]

   (b) From your graph determine where the slope of the graph is positive,negative and zero.

   (c) Which value of x corresponds to the maximum value of f(x)?

   (d) Which value of x corresponds to the minimum value of f(x)?

   (ii) (a) Differentiate f(x)

   (b) Determine where

   $f'(x) = 0$

   $f'(x) < 0$

   $f'(x) > 0$

   (iii) Compare your answers to Parts (i) and (ii). What do you notice?

4. Let $g(x) = x^2 - 2x + \frac{1}{2}$

   (i) (a) Plot the graph of g for x lying in the interval [-1, 4]

   (b) From your graph determine where the slope of the graph is positive, negative and zero.

   (c) Which value of x corresponds to the maximum value of g(x)?

   (d) Which value of x corresponds to the minimum value of g(x)?

   (ii) (a) Differentiate g(x)

   (b) Determine where

   $g'(x) = 0$

   $g'(x) < 0$

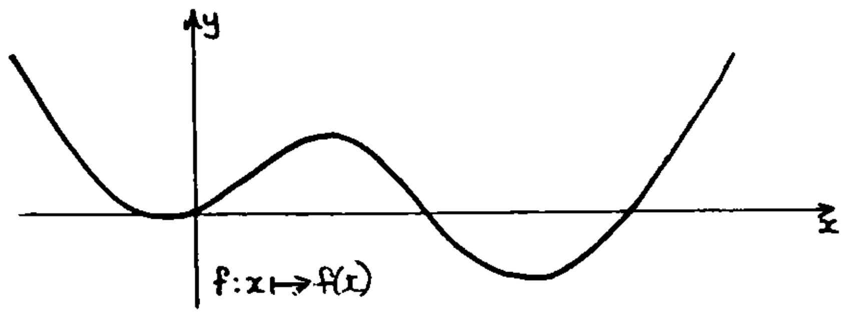
   $g'(x) > 0$.

Questions 3 and 4 demonstrate the following properties :

- if $f'(x) > 0$ for all x in some interval then f is increasing on that interval

- if $f'(x) < 0$ for all x in some interval then f is decreasing on that interval
- if $f'(c) = 0$ and $f'(x)$ changes from positive to negative as x increases through c then f has a maximum at c
- if $f'(c) = 0$ and $f'(x)$ changes from negative to positive as x increases through c then f has a minimum at c.

5. Use the properties listed above to mark on the graph below
   (a) the minimums of f(x)
   (b) the maximum of f(x).



This question demonstrates that, according to the properties listed above, f has two minima and one maximum.

And as for the maximum - there are other points where the function takes much higher values. Strictly speaking the properties above refer to a <u>local maximum</u> (or a <u>local minimum</u>) in the sense that of all the points nearby this is the one where the function takes its largest (or smallest) value. So all the rules for using derivatives will do is to find a <u>local extremum</u>. (Note the word extremum (plural extrema) is used for 'a maximum or minimum' when we don't want to specify which.)

<u>POST-PROGRAMME WORK</u>

6. The function $f(x) = x^3 - 6x^2 + 9x + 1$
   has local extrema where x = 1 and x = 3. Use the information in the table below to sketch the graph of f(x).

x	0		1			2			3	
$f(x) = x^3 - 6x^2 + 9x + 1$	1		5			3			1	
$f'(x) = 3x^2 - 12x + 9$	+	+	0	-	-	-	-	-	0	+
$f''(x) = 6x - 12$	-	-	-	-	-	0	+	+	+	+

7. Construct a similar table for $g(x) = 1 + 4x - \frac{x^3}{3}$ and so sketch the graph of g.

8. Construct a similar table for $h(x) = 2x^3 - 9x^2 + 12x - 4$ and so sketch the graph of h.

9. Sketch the graphs of each of the following functions:

(i) $x \mapsto 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$

(ii) $x \mapsto x^5 - \frac{5}{3}x^3 + 4$

(iii) $x \mapsto x^5 - 5x^4 + 5x^3$.

(iv) $x \mapsto \frac{x^5}{5} - \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{5}$

(v) $x \mapsto \frac{x^5}{5} - \frac{x^3}{3} + \frac{1}{5}$.

10. The second derivative test may seem attractive but it does not solve every problem.

The function given by $f(x) = x^4$ has one extremum.

(i) Find it.

(ii) Find the value of the second derivative of f at the local extremum. What help, if any, does it give you in classifying the local extremum?

(iii) Use the first derivative test to classify the local extremum.

POSSIBLE EXTENSIONS

1. This section has looked only at the techniques involved in sketching polynomial functions. In such cases it is not necessary to consider the behaviour of the function for large values of x. Nevertheless this additional technique is important when sketching other types of graphs. This could be added to the method introduced in the section and students could go on to sketching other types of graphs. In particular, they could explore the graph of $x \mapsto \frac{10\,000\,x}{10\,000\,x - 1}$.

2. We have concentrated on the derivative in terms of the slope of a graph. In many practical applications this has a special meaning, (speed, acceleration, rate of change). These practical applications could be introduced via this graphical approach.

SOLUTIONS.

1. (i) $f'(x) = 2x$; $f''(x) = 2$

(ii) $g'(x) = 2(x - 1)$; $f''(x) = 2.$

(iii) $h'(x) = 5x^4 - 12x^2 + 6x$; $h''(x) = 20x^3 - 24x + 6.$

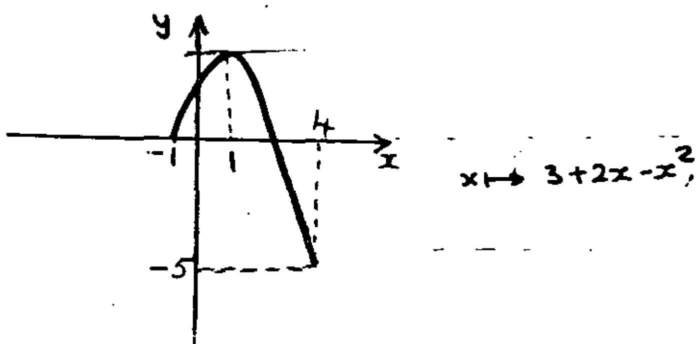
(iv) $\ell'(x) = 12x^3 + 6x^2$; $\ell''(x) = 36x^2 + 12x.$

2. (i) $3x^4 + 2x^3 = x^3(3x + 2)$

(ii) $12x^3 + 6x^2 = 6x^2(2x + 1).$

(iii) $36x^2 + 12x = 12x(3x + 1).$

3. (i) (a) Your graph should look something like this.



(b) The slope is

positive for $x \in]-1, 1[$

negative for $x \in]1, 4[$

zero at $x = 1.$

(c) $f(x)$ has its maximum vlaue at $x = 1.$

(d) On the interval $[-1, 4]$, $f(x)$ takes its minimum value at $x = 4$ when $f(x) = -5.$

(ii) (a) $f'(x) = 2 - 2x = 2(1 - x)$

(b) $f'(x) = 0$ when $x = 1$

$f'(x) > 0$ when $x < 1$

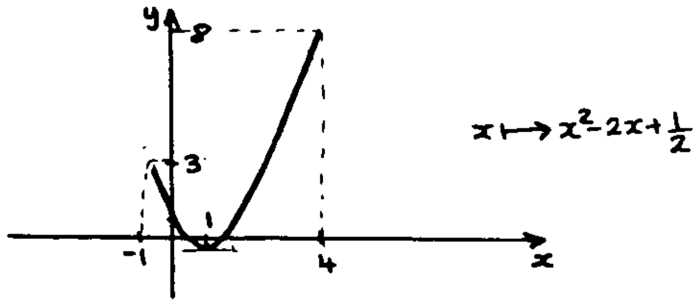
$f'(x) < 0$ when $x > 1$

(iii) When $f'(x) = 0$ $f(x)$ has its maximum value

When $f'(x) > 0$ the slope of $f(x)$ is positive and f is increasing

When $f'(x) < 0$ the slope of $f(x)$ is negative and f is decreasing.

4. (i) (a) Your graph should look something like this.



(b) The slope is

positive for $x \in]1, 4[$

negative for $x \in]-1, 1[$

zero at $x = 1$.

(c) On the interval $[-1, 4]$ $g(x)$ takes its maximum value at $x = 4$ when $g(x) = 8.5$.

(d) $g(x)$ has its minimum value at $x = 1$.

(ii) (a) $g'(x) = 2x - 2 = 2(x - 1)$

(b) $g'(x) = 0$ when $x = 1$

$g'(x) > 0$ when $x > 1$

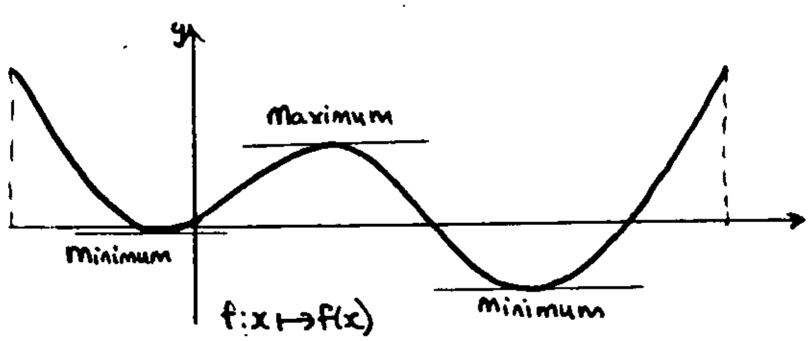
$g'(x) < 0$ when $x < 1$

(iii) When $g'(x) = 0$, $g(x)$ has its minimum value.

When $g'(x) > 0$ the slope is positive and g is increasing

When $g'(x) < 0$ the slope is negative and g is decreasing

5.



6.

x	0		1			2			3	
$f(x) = x^3 - 6x^2 + 9x + 1$	1 ↗	↗	5 →	↘	↘	3 ↘	↘	↘	1 →	↗
$f'(x) = 3x^2 - 12x + 9$	+	+	0	-	-	-	-	-	0	+
$F''(x) = 6x - 12$	-	-	-	-	-	0	+	+	+	+

The table show that f is increasing on $]-\infty, 1[$

has a local maximum at $x = 1$

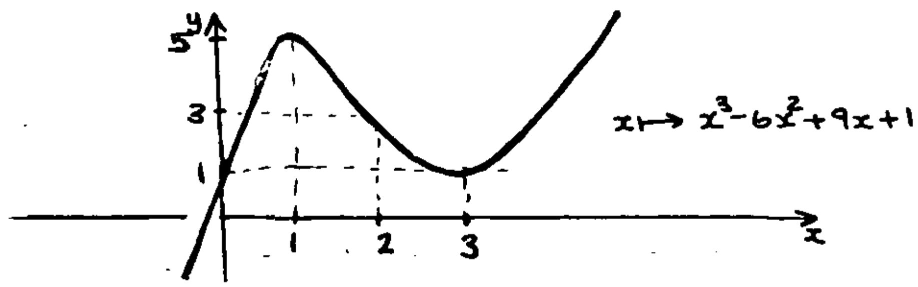
is decreasing on $]-1, 3[$

has a point of inflection at $x = 2$ where the graph changes from concave down to concave up.

has a local minimum at $x = 3$

is increasing on $]3, \infty[$

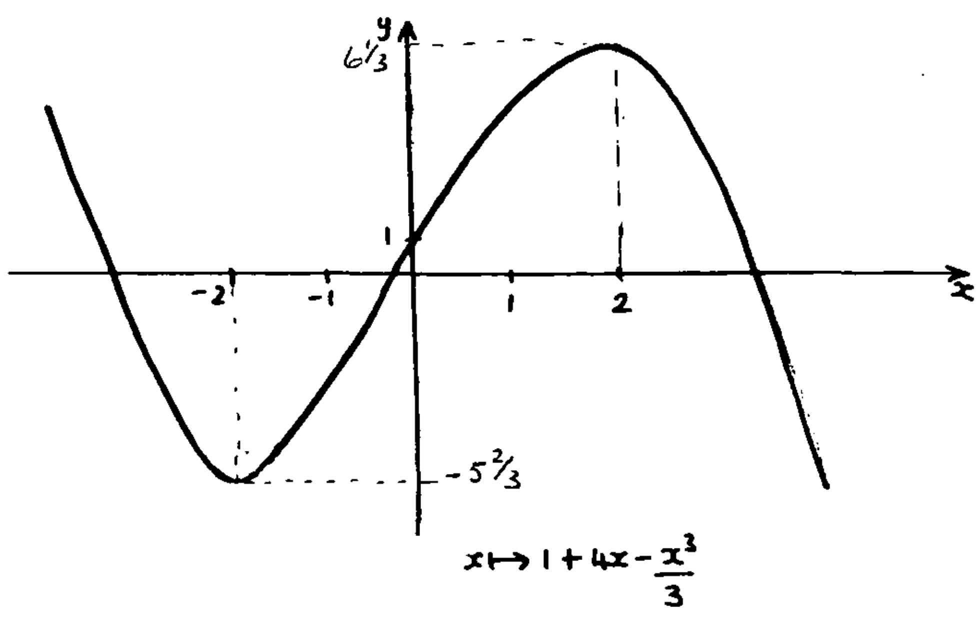
This gives the sketch



7. (<u>Note</u> it is always useful to consider the value of the function at $x = 0$.

x		-2			0			2	
$g(x) = 1 + 4x - \frac{x^3}{3}$	↘	$-5\frac{2}{3}$ →	↗	↗	1 ↗	↗	↗	$6\frac{1}{3}$ →	↘
$g'(x) = 4 - x^2$	-	0	+	+	+	+	+	0	-
$g''(x) = -2x$	+	+	+	+	0	-	-	-	-

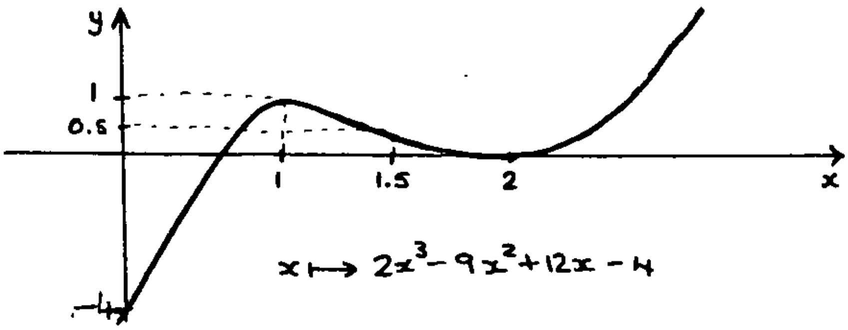
This gives the sketch below.



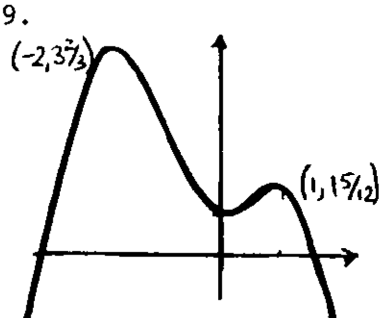
8.

x	0		1			$\frac{3}{2}$			2	
$h(x)=2x^3-9x^2+12x-4$	-4 $\nearrow$	$\nearrow$	1 $\rightarrow$	$\searrow$	$\searrow$	0.5 $\searrow$	$\searrow$	$\searrow$	0 $\rightarrow$	$\nearrow$
$h'(x) = 6x^2 - 18x + 12$ $= 6(x - 2)(x - 1)$	+	+	0	-	-	-	-	-	0	+
$h''(x) = 12x - 18$ $= 6(2x - 3)$	-	-	-	-	-	0	+	+	+	+

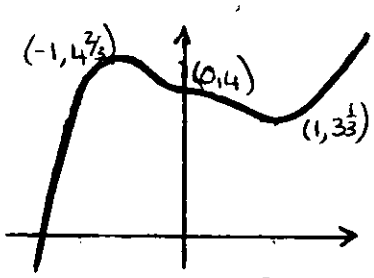
This gives the sketch below



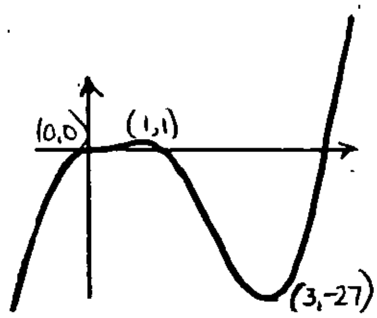
9.



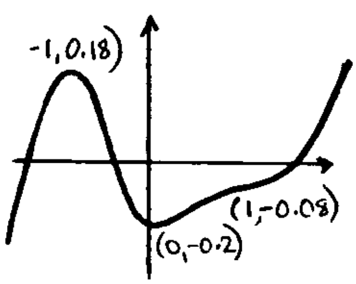
$x \mapsto 1 + x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$



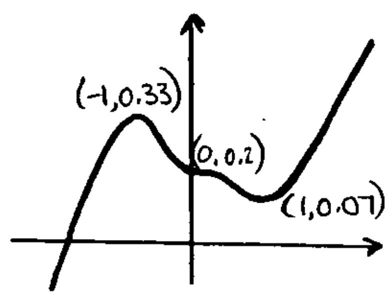
$x \mapsto x^5 - 5/3\,x^3 + 4$



$x \mapsto x^5 - 5x^4 + 5x^3$



$x \mapsto \frac{x^5}{5} - \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{5}$



$x \mapsto \frac{x^5}{5} - \frac{x^3}{3} + \frac{1}{5}$

10. (i) $f(x) = x^4$

$f'(x) = 4x^3 = 0$ at $x = 0$. So f has a local extremum at $x = 0$.

(ii) $f''(x) = 12x^2$

$= 0$ at $x = 0$.

This does not help us to classify the local extremum.

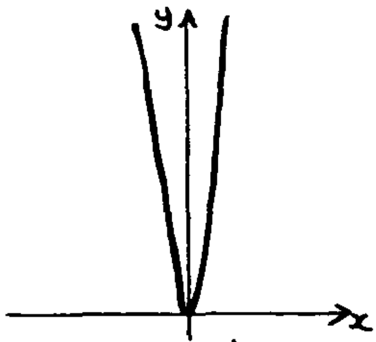
(iii) $f'(x) = 4x^3$

So $f'(x) > 0$ for $x > 0$

and $f'(x) < 0$ for $x < 0$

Hence using the first derivative test f is decreasing for $x < 0$ and increasing for $x > 0$ and hence has a local minimum at $x = 0$.

Here is the graph of $x \mapsto x^4$.



$x \mapsto x^4$

13. THE FUNDAMENTAL THEOREM OF CALCULUS

(NOTE: This programme builds upon the work on areas in SECTION 10: $x \mapsto \frac{1}{x}$: AN AREA FOR REVISION).

PROGRAMME SUMMARY

This programme demonstrates the links between the notions of differentiation and integration in the form of the Fundamental Theorem of Calculus, which states that if F is any primitive for the function f then

$$\int_a^b f(x)\,dx = f(b) - f(a).$$

$\boxed{\text{4 mins}}$ We begin by examining Question 4 which students should have attempted beforehand, and the concept of an area so far function is explored for a given function f. The area so far function, A, with starting point a is defined by

$A(x)$ = area under the curve $x \mapsto f(x)$ from a to x.

$\boxed{\text{3 mins}}$ By definition A(b) can be written as

$$\int_a^b f(x)\,dx$$

and this can be found by dividing the area between a and b into rectangles. This gives a sequence of over- and under-estimates for the area. *[Note: We refer here to SECTION 10: $x \mapsto \frac{1}{x}$: AN AREA FOR REVISION]*. More precisely we can define the area as

$$\lim_{i \to 0} \Sigma\, f(x_i)\, \delta x_i$$

but how can we evaluate this?

Sometimes it can be found by algebraic manipulation but only in particular cases. More generally, we need to look for another method.

5 mins

The area so far function can be defined precisely using the idea that

$$\text{Area} = \lim_{i\to 0} \Sigma\, f(x_i)\, \delta x_i .$$

This allows us to show that the area under f between x = a and x = b is

$A(b) - A(a)$.

2 mins

But surely A(x) depends on the starting value? - and indeed it does. There is a whole family of area so far functions. But the area between x = a and x = b is unaffected so in practice we only need to find one area so far function.

5 mins

We now investigate A(x) more closely, and by looking at the slope of A(x) and the corresponding value of f(x) we show that

$A'(x) = f(x)$.

Next, we explain how to use this observation to evaluate integrals. If A is an area so far function for f then

$$\int_a^b f(x)\, dx = A(b) - A(a).$$

So in order to evaluate the integral we should look for an area so far function.

4 mins

This theory is applied to

$$\int_{\pi/3}^{\pi/2} \cos x\, dx.$$

In fact we don't need to check that sin x is an area so far function. It is sufficient that sin x is a primitive for cos x — since all primitives differ by a constant. So, in order to evaluate any integral of the form

$$\int_a^b f(x)\, dx,$$

it is enough to find any primitive F for f, in which case

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PRE-REQUISITES

Before working through this section students should be familiar with the following:

(i) differentiation

(ii) integration as the process of finding a primitive - (that is - a function which differentiates to give the original function.)

(iii) the integral as a way of representing the area under a curve, including the idea that areas beneath the x-axis give a negative contribution to the integral

(iv) finding the area under a curve as introduced in *SECTION 10: $x \mapsto \frac{1}{x}$: AN AREA OF REVISION.*

(v) manipulation of expressions of the form

$$\sum_{i=1}^{n} a(b - (\tfrac{i}{n})^2)$$

(vi) subscript notation

(vii) finding the areas of rectangles and triangles

(viii) reading information from a graph

(ix) the slope of a straight line..

(x) the interpretation of the derivative as the tangent at a point and the definition of the derivative from first principles.

(xi) both functional and Leibnitz notation (that is $f'(x)$ and $\frac{d}{dx}$)

PRE-PROGRAMME WORK

Students will benefit by working through SECTION 10: $x \mapsto \frac{1}{x}$: AN AREA FOR REVISION *before starting this section.*

1. (i) Differentiate each of the following functions.

(a) $x \mapsto x^3 + 3$

(b) $x \mapsto 2x + 1$

(c) $x \mapsto \frac{x^3}{3} + x^2 + x - 1$

(d) $x \mapsto 2x^4 + 3x + 274$

(ii) Find primitives for each of the following functions

(a) $f(x) = 2$

(b) $g(x) = 8x^3 + 3$

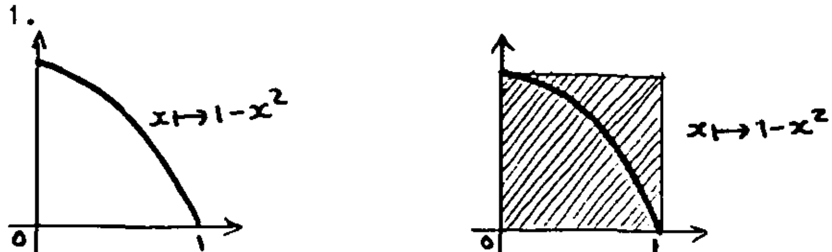
(c) $h(x) = 3x^2$

(d) $e(x) = x^2 + 2x + 1$

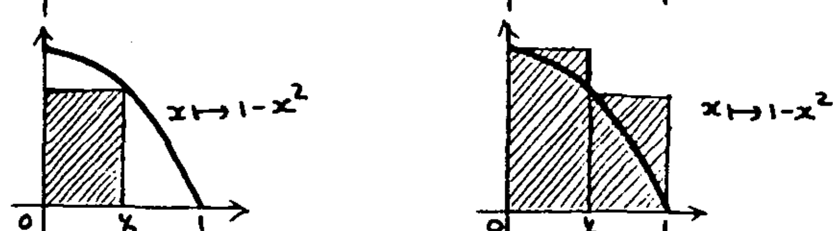
2. This question concerns the curve $f: x \mapsto 1 - x^2$.

(i) Calculate each of the shaded under-estimates and over-estimates for the area under the curve between $x = 0$ and $x = 1$.

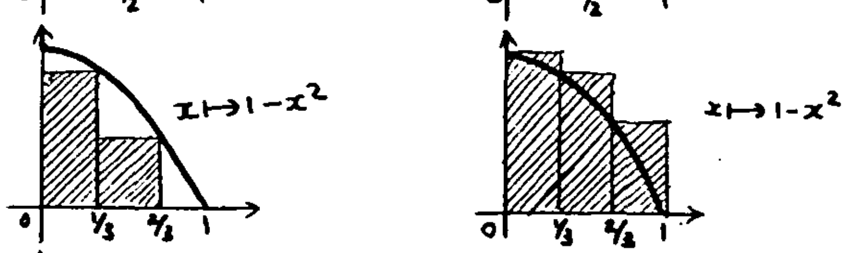
(a)



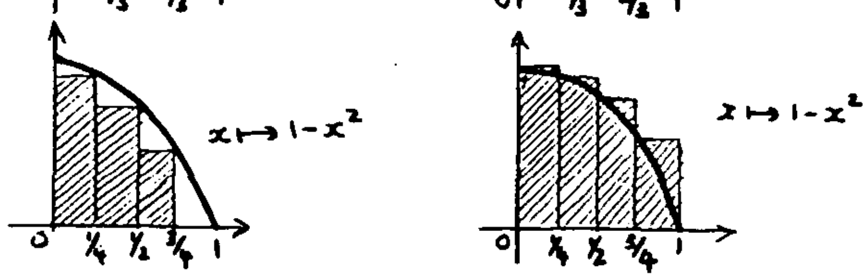
(b)



(c)



(d)



In each case the true area is pinched between the under-estimate and the over-estimate - but even with four rectangles it cannot be pinched very accurately.

(ii) Show that the under-estimate and over-estimate of the area under $f(x) = 1 - x^2$ between $x = 0$ and $x = 1$ obtained by dividing the area into n-rectangles are given by

$$\text{under-estimate} = \sum_{i=1}^{i=n} \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^2\right)$$

$$\text{over-estimate} = \frac{1}{n} + \sum_{i=1}^{n} \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^2\right)$$

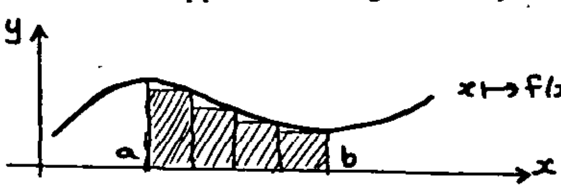
(iii)(a) Given that $\sum_{i=1}^{n} i^2 = \frac{1}{6} n (n + 1) (2n + 1)$

show that

the under-estimate is $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$

the over-estimate is $\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$

(b) What happens as n gets very large?



In general, the area under the function f between $x = a$ and $x = b$ can be obtained by dividing the interval [a, b] into a number of sub-intervals, n say. In each sub-interval we select a point - x_1 in the first sub-interval, x_2, in the second ... x_i in the ith. Then the required area is approximated by the sum of rectangles given by

$$\sum_{i=1}^{n} f(x_i)\ \delta x_i$$

where δx_i is the length of the ith interval. The exact area is denoted by

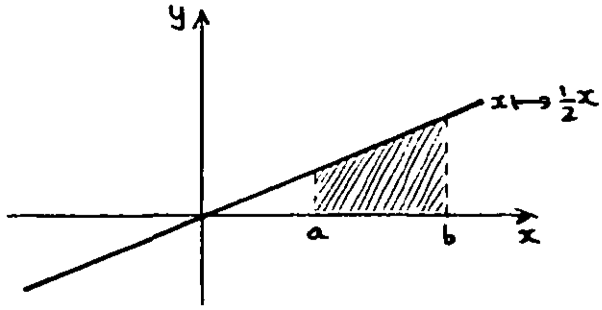
$$\int_a^b f(x)\ dx$$

and we say

$$\int_a^b f(x)\ dx = \lim_{i \to 0} \sum_{i=1}^{n} f(x_i)\ \delta x_i .$$

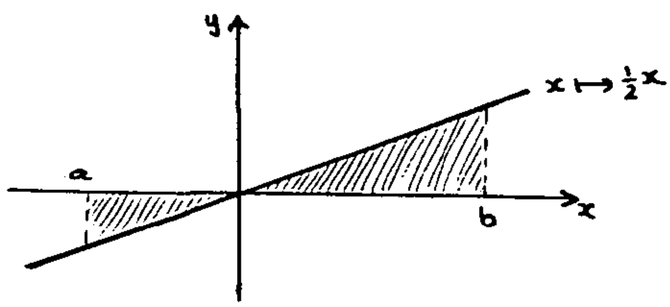
Thus $\int_a^b f(x)\,dx$ is the area under the function f between x = a and x = b

3. (i) $\int_a^b \frac{1}{2} x\,dx$ is the shaded area in the diagram below.



Write down an expression for this area by considering the areas of two triangles.

(ii) Use a similar method to find $\int_a^b \frac{1}{2} x\,dx$ in the diagram below.
(Remember areas under the x-axis give a <u>negative</u> contribution to the integral.)

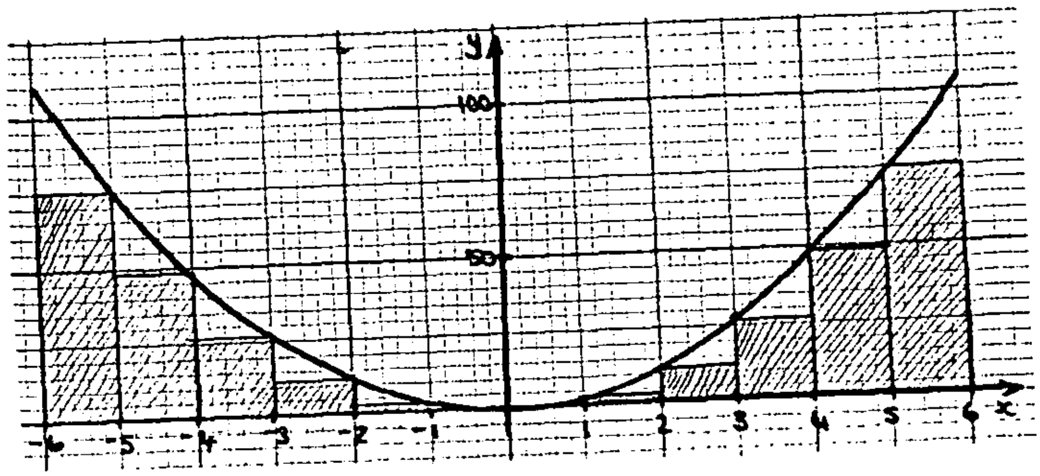


(iii) Let a = -1. Hence find $\int_{-1}^{b} \frac{1}{x}\,dx$.
You should obtain a function in terms of b. This function is called an <u>area so far function</u> because its value at a point is the area under the graph 'so far'. That is as far as the point in question. We can define an area so far function A for any function f and any starting point a as

A (x) = Area under the graph from a to x.

*4. (i) Plot the graph of g: $x \mapsto x^3 + 10$ for $x \in [-5, 6]$.
(ii) Find the derived function of the function g: $x \mapsto x^3 + 10$.

(iii)

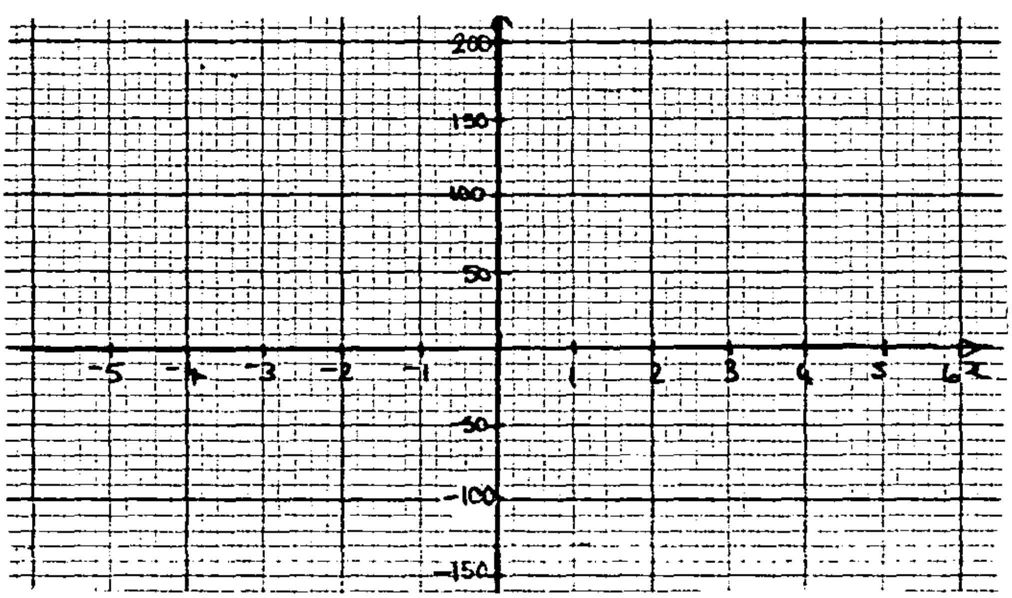


$g' : x \mapsto 3x^2$

Here is the graph of the derived function g'. Use the shaded areas to complete the table below to give approximate values of the area so far function A(x). Notice that areas to the left of x = 0 should be given a negative value.

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
A(x)													

(iv) Use the table to plot the graph of A(x) on the axes below.



POST-PROGRAMME WORK

5. Evaluate (i) $\int_{\pi/3}^{\pi/2} \cos x \, dx$

(ii) $\int_{0}^{\pi/2} \cos 3x \, dx$

(iii) $\int_{\pi}^{2\pi} \frac{1}{x^2}\, dx.$

POSSIBLE EXTENSIONS

1. This section could clearly lead to further work on finding integrals.

2. The programme demonstrates a very complicated concept which is often difficult to appreciate. To many students the Fundamental Theorem is just taken for granted. This programme could provoke discussion on why this is so.

   To this end students would probably benefit by going over the main points of the programme again - more slowly,

3. The proof of the Fundamental Theorem of Calculus is not usually required at A-level - but it could be tackled at this stage if wanted.

4. The interpretation of $\int_a^b f$ as an area could be used to explore rules such as

$$\int_a^a f = 0$$

$$\int_a^b f = -\int_b^a f$$

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

SOLUTIONS

1. (i) (a) $x \mapsto 3x^2$

(b) $x \mapsto 2$

(c) $x \mapsto x^2 + 2x + 1$

(d) $x \mapsto 8x^2 + 3$.

(ii) (a) $x \mapsto 2x + c$ (where c is some constant).

(b) $x \mapsto 2x^4 + 3x + c$

(c) $x \mapsto x^3 + c$

(d) $x \mapsto \frac{x^3}{3} + x^2 + x + c$.

Notice the connection between (i)(a) and (ii)(c); (i)(b) and (ii)(a); (i)(c) and (ii)(d); (i)(d) and (ii)(b).

2. (i) (a) Under-estimate = 0, Over-estimate = 1

(b) Under-estimate $= \frac{1}{2} \times \left(1 - \left(\frac{1}{2}\right)^2\right) = \frac{3}{8}$

Over-estimate $= \left\{\frac{1}{2} \times 1\right\} + \left\{\frac{1}{2} \times \left(1 - \left(\frac{1}{2}\right)^2\right)\right\} = \frac{7}{8}$

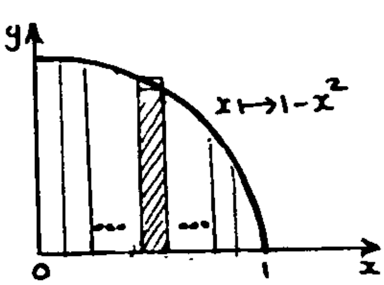
(c) Under-estimate $= \left(\frac{1}{3} \times \frac{8}{9}\right) + \left(\frac{1}{3} \times \frac{5}{9}\right) = \frac{13}{27}$

Over-estimate $= \left(\frac{1}{3} \times 1\right) + \left(\frac{1}{3} \times \frac{8}{9}\right) + \left(\frac{1}{3} \times \frac{5}{9}\right) = \frac{22}{27}$

(d) Under-estimate $= \left(\frac{1}{4} \times \frac{15}{16}\right) + \left(\frac{1}{4} \times \frac{12}{16}\right) \times \left(\frac{1}{4} \times \frac{7}{16}\right) = \frac{17}{32}$

Over-estimate $= \left(\frac{1}{4} \times 1\right) + \left(\frac{1}{4} \times \frac{15}{16}\right) + \left(\frac{1}{4} \times \frac{12}{16}\right) + \left(\frac{1}{4} \times \frac{7}{16}\right) = \frac{25}{32}$

(ii)



For n rectangles the points of sub-division are at $\frac{1}{n}, \frac{2}{n} \ldots \frac{n-1}{n}$, giving the sub-intervals $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}] \ldots [\frac{n-1}{n}, 1]$.

Hence each rectangle has base $\frac{1}{n}$.

For the under-estimate we take the rectangles formed by the heights of the graph at the right-hand end points of the sub-intervals

$$1 - \left(\frac{1}{n}\right)^2,\ 1 - \left(\frac{2}{n}\right)^2,\ \dots\ 1 - \left(\frac{n-1}{n}\right)^2,\ 1 - \left(\frac{n}{n}\right)^2 = 0.$$

Hence the under-estimate is

$$\frac{1}{n}\left(1 - \left(\frac{1}{n}\right)^2\right) + \frac{1}{n}\left(1 - \left(\frac{2}{n}\right)^2\right) + \dots + \frac{1}{n}\left(1 - \left(\frac{n-1}{n}\right)^2\right) + \frac{1}{n}\left(1 - \left(\frac{n}{n}\right)^2\right)$$

$$= \sum_{i=1}^{n} \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^2\right).$$

The over-estimate is given by the rectangles formed by the heights of the graph at the left-hand end points of the sub-intervals

$$1 - 0^2,\ 1\ \left(\frac{1}{n}\right)^2, \dots\ 1 - \left(\frac{n-1}{n}\right)^2.$$

Hence the under-estimate is

$$\frac{1}{n}(1 - 0^2) + \frac{1}{n}\left(1 - \left(\frac{1}{n}\right)^2\right) + \dots + \frac{1}{n}\left(1 - \left(\frac{n-1}{n}\right)^2\right)$$

$$= \frac{1}{n} + \sum_{i=1}^{n} \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^2\right).$$

(iii) (a) Under-estimate $= \left(\sum_{i=1}^{n} \frac{1}{n}\left(1 - \left(\frac{i}{n}\right)^2\right)\right)$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$= 1 - \frac{1}{n^3}\left\{\frac{1}{6}n\,(n+1)(2n+1)\right\}$$

$$= 1 - \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

$$= \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}.$$

$$\text{Over-estimate} = \frac{1}{n} + \sum_{i=1}^{n} \frac{1}{n}(1 - (\frac{i}{n})^2)$$

$$= \frac{1}{n} + (\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2})$$

$$= \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}.$$

(b) (b) As n becomes large both $\frac{1}{2n}$ and $\frac{1}{6n^2}$ tend to zero, so that the underestimate and the overestimate get closer and closer to $\frac{2}{3}$.

3. (i) Using the formula for the area of a triangle,

$$\text{shaded area} = \frac{1}{2}b\left(\frac{b}{2}\right) - \frac{1}{2}a\left(\frac{a}{2}\right)$$

$$= \frac{b^2}{4} - \frac{a^2}{4}.$$

(ii) Shaded area to the right of the y-axis is

$$\frac{1}{2}b\left(\frac{b}{2}\right) = \frac{b^2}{4}$$

Shaded area to the left of the y-axis is

$$\frac{1}{2}a\left(\frac{a}{2}\right)$$

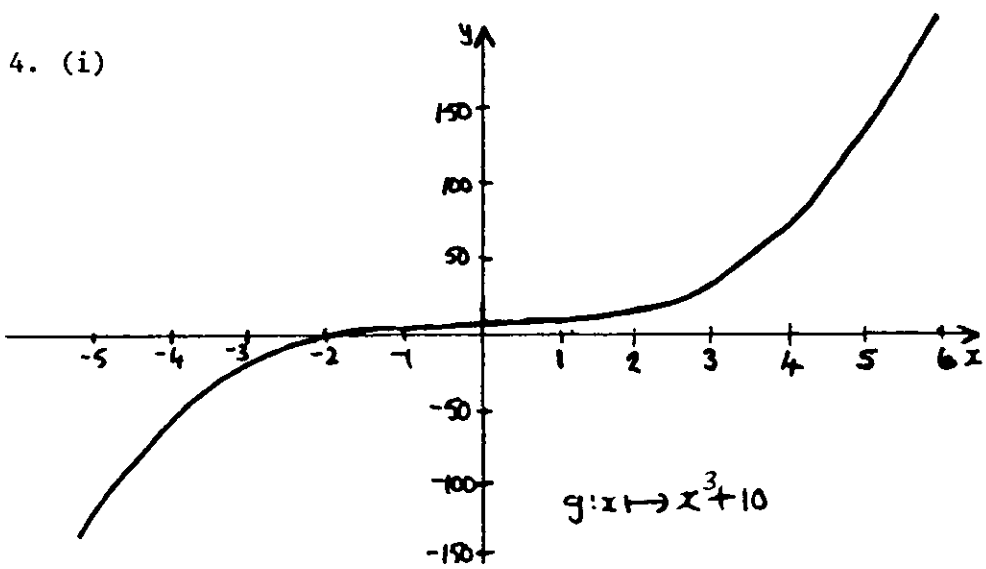
but this area is below the x-axis, so its contribution to the integral is negative.

Hence

$$\int_a^b \frac{1}{x}\, dx = \frac{b^2}{4} - \frac{a^2}{4}.$$

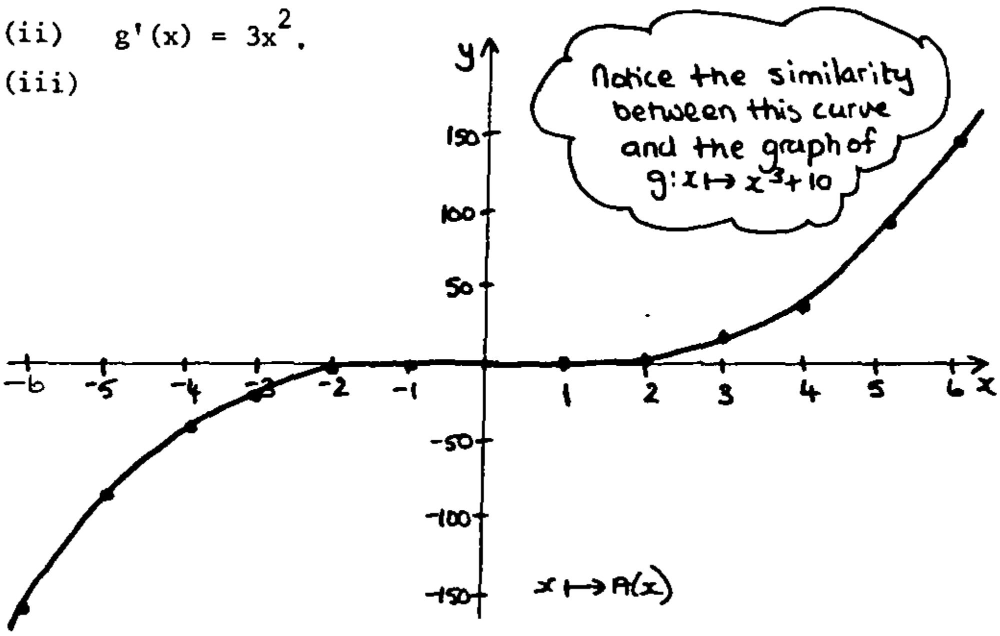
(iii) $\int_{-1}^{b} \frac{1}{x}\, dx = \frac{b^2}{4} - \frac{1}{4}.$ (from Part ii)

4. (i)



(ii) $g'(x) = 3x^2$.

(iii)



5. (i) Sin x is a primitive for cos x. Hence

$$\int_{\pi/3}^{\pi/2} \cos x \, dx = \sin \frac{\pi}{2} - \sin \frac{\pi}{3}$$

$$= 1 - \frac{\sqrt{3}}{2} = \frac{2 - \sqrt{3}}{2} .$$

(ii) $-\frac{1}{3}$ sin 3x is a primitive for cos x. Hence

$$\int_{0}^{\pi/2} \cos 3x \, dx = -\frac{1}{3} \sin \left(\frac{3\pi}{2}\right) - \frac{1}{3} \sin (0)$$

$$= \frac{1}{3} .$$

(iii) $-\frac{1}{x}$ is a primitive for $\frac{1}{x^2}$. Hence

$$\int_{\pi}^{2\pi} \frac{1}{x^2} \, dx = -\frac{1}{2\pi} - \left(-\frac{1}{\pi}\right)$$

$$= \frac{1}{2\pi} .$$

14. TAYLOR POLYNOMIALS

PROGRAMME SUMMARY

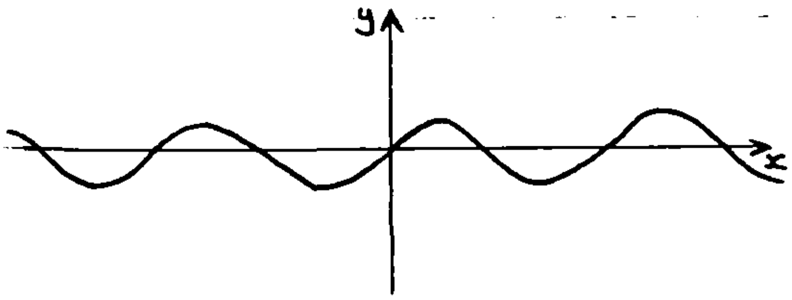
This programme compares the graphs of the Taylor polynomials of a function with the graph of the actual function. This demonstrates what we mean by an approximation. It also indicates that sometimes the approximation only holds for a particular range of values of x.

2 mins

Questions 3 and 4 which students should have tackled beforehand can be solved using the Taylor polynomials for sin x.

6 mins

But how can a function with a graph like



be approximated by a polynomial? Computer graphics allow us to demonstrate the graphical effect of adding the terms

$$x, \; -\frac{x^3}{3!}, \; \frac{x^5}{5!}, \; \frac{-x^7}{7!} \; .$$

Hence we show that the polynomial of degree 7 fits the graph of $x \mapsto \sin x$ to within one decimal place over one complete cycle.

2 mins

As the degree of the polynomial increases the graph becomes a better approximation to that of sin x, and the region of accuracy to one decimal place increases. For example, the polynomial of degree 17 gives one decimal place accuracy over two complete cycles.

1 min

We return to the method of finding the Taylor polynomials of sin x.

By assuming

$$\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

and repeatedly differentiating both sides we obtain

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \; .$$

$\boxed{\text{6 mins}}$ For the general function f, a similar process gives

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$$

This can be viewed as a recipe for f(x), with the ingredients 1, x, $\frac{x^2}{2!}$, $\frac{x^3}{3!}$ and so on to be taken in various quantities.

Computer graphics illustrate what happens as the quantities of the ingredients are varied. In particular we consider

$$f(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$\boxed{\text{2 mins}}$ In fact this is a Taylor polynomial for

sin x + cos x (among others),

$\boxed{\text{4 mins}}$ For some functions the Taylor polynomials behave exactly as we would like and get closer and closer to the actual function. But this is not always the case. For example the graphs of the Taylor polynomials of $\frac{1}{1 + x}$ only get close to the graph of $x \mapsto \frac{1}{1 + x}$ for x lying in [0, 1].

PRE-REQUISITES

Before working through this section of work students should be familiar with the following:

(i) differentiation of polynomials, sin x, cos x and expressions such as $\frac{1}{1 + x}$

(ii) the general form of a polynomial, and the degree of a polynomial

(iii) the graph of $x \mapsto \sin x$

(iv) the relationship between the degree of a polynomial (odd or even) and the behaviour for large x of the corresponding graph. In particular students should recognise the shapes of graphs of functions such as $x \mapsto k$, $x \mapsto kx$, $x \mapsto kx^2$, $x \mapsto kx^3$, for various values of k.

(v) the method of Integration by Parts

(vi) use of a scientific calculator

(vii) both functional and Leibnitz notation (that is f'(x) and $\frac{d}{dx}$).

PRE-PROGRAMME WORK

1. Write down the degree of each of the following polynomials

   (i) $x^2 + 3$

   (ii) $1 + x^4 - x^3 + 2x^2$

   (iii) $3 + x^5 - 2x^4 + 17x$

   (iv) $x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7$.

*2. The fourth Taylor polynomial for sin x about 0 is given by

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4.$$

   (i) Find (a) $P'(x)$ (b) $P''(x)$ (c) $P'''(x)$ (d) $P^{(iv)}(x)$
   This Taylor polynomial of degree four is such that its derivatives at 0 up to and including the fourth derivative are the same as the derivatives of sin x at x = 0.

   (ii) Find
   (a) $\frac{d}{dx}(\sin x)$ (b) $\frac{d^2}{dx^2}(\sin x)$ (c) $\frac{d^3}{dx^3}(\sin x)$

   (d) $\frac{d^4}{dx^4}(\sin x)$.

   (iii) Use Parts (i) and (ii) to write down the fourth Taylor polynomial for sin x about x = 0.

*3. The nth Taylor polynomial for f(x) about 0 is the polynomial with the same derivatives at 0 as f up to and including the nth derivative (provided of course that f can be differentiated repeatedy). This can be written as

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_nx^n \ldots$$

   (i) P(x) is such that P(x) = f(x) at x = 0. Use this to write down the coefficient a_0 in terms of f(0).

   (ii) Also P'(x) = f'(x) at x = 0. Differentiate P'(x) and hence obtain the coefficient a_1 in terms of f'(0).

   (iii) Continue this process to find

   (a) a_2 (b) a_3 (c) a_4 (d) a_n.

   (iv) Hence write down the nth Taylor polynomial for f(x) about x = 0.

*4. (i) Write down the fifth Taylor polynomial for sin x about x = 0.

   (ii) Write down the seventh Taylor polynomial for sin x about x = 0.

*5. (i) Complete the table below to give the values of the Taylor polynomials for sin x of degree 1, 3 and 5 about $x = 0$ when

	(a) $x = \frac{\pi}{6}$	(b) $x = 0.5$
x		
$x - \frac{x^3}{3!}$		
$x - \frac{x^3}{3!} + \frac{x^5}{5!}$		

(ii) Check your results by using a calculator to find $\sin \frac{\pi}{6}$ and sin 0.5. [Note - These angles are measured in radians!].

6. (i) Would you say that the integral

$$\int_{0.5}^{1} \frac{\sin x}{x}\, dx$$

can be evaluated using Integration by Parts?

(ii) By replacing sin x with the Taylor polynomial of degree 3 about $x = 0$ find an approximate value of

$$\int_{0.5}^{1} \frac{\sin x}{x}\, dx.$$

*7. Suppose $f: x \mapsto \frac{1}{1+x}$.

(i) Find (a) $\frac{d}{dx}\left(\frac{1}{1+x}\right)$ (b) $\frac{d^2}{dx^2}\left(\frac{1}{1+x}\right)$ (c) $\frac{d^3}{dx^3}\left(\frac{1}{1+x}\right)$.

(ii) Sketch the graph of f(x)

(iii) Use the Result of Question 3 to write down the 3rd Taylor polynomial for $x \mapsto \frac{1}{1+x}$ about $x = 0$.

(iv) Find the nth Taylor polynomial for $x \mapsto \frac{1}{1+x}$ about $x = 0$.

POST-PROGRAMME WORK

8. Find the 6th Taylor Polynomial for cos x about $x = 0$.

9. Find the 8th Taylor polynomial of the function

$x \mapsto \sin x + \cos x$

about $x = 0$.

10. (i) Find the kth Taylor polynomial for $x \mapsto (1 + x)^r$ about $x = 0$.

11. (i) Find the kth Taylor polynomial for $x \mapsto \frac{1}{1 - x}$ about $x = 0$.

(ii) Use the identity

$$\frac{1}{1-x^2} = \frac{1}{1 + x} + \frac{1}{1 - x}$$

to find the 10th Taylor polynomial for $x \mapsto \frac{1}{1-x^2}$ about $x = 0$.

POSSIBLE EXTENSIONS

1. In this section we have restricted ourselves to the discussion of Taylor polynomials. Taylor Series are very similar but carry on indefinitely.

2. We have been concerned only with Taylor polynomials about $x = 0$. The work could be extended to Taylor polynomial about any value of x.

3. Questions 7 and 10 touch on the question of adding Taylor polynomials. Similarly, Taylor polynomials (and Taylor Series) may be differentiated and integrated, or other values of x may be substituted. For example, the Taylor Series for cos 3x may be obtained by substituting $x = 3\theta$ in the Taylor Series for cos x.

4. *In* *SECTION 15: WHY e?* *we investigate the Taylor Series for e^x and $\log_e(1 + x)$.*

SOLUTIONS

1. A polynomial of the form

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

has degree n. That is, the degree of the polynomial is the highest power of x.

(i) Degree = 2

(ii) Degree = 4

(iii) Degree = 4

(iv) Degree = 7.

2. (i) (a) $P'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$

(b) $P''(x) = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2$

(c) $P'''(x) = 6a_3 + (4)(3)(2)a_4x$

(d) $P^{iv}(x) = 24a_4$

(ii) (a) $\frac{d}{dx}(\sin x) = \cos x)$

(b) $\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}(\cos x) = -\sin x$

(c) $\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx}(-\sin x) = -\cos x$

(d) $\frac{d^4}{dx^4}(\sin x) = \frac{d}{dx}(-\cos x) = \sin x.$

(iii) The coefficients a_0, a_1, a_2, a_3 and a_4 can be found as follows:

When $x = 0$, $P(0) = a_0 = \sin 0$.

Hence $a_0 = 0$.

When $x = 0$, $P'(0) = \sin'(0)$.

From above, $P'(0) = a_1$

and $\sin'(0) = \cos 0 = 1$..

Hence $a_1 = 1$.

When $x = 0$, $P''(0) = \sin''(0)$.

From above, $P''(0) = 2a_2$ and $\sin''(0) = -\sin 0 = 0$.

Hence $a_2 = 0$.

When $x = 0$, $P'''(0) = \sin'''(0)$.

From above $P'''(0) = 6a_3$ and $\sin'''(0) = -\cos x = -1$.

Hence $a_3 = -\frac{1}{6} = -\frac{1}{3!}$.

When $x = 0$, $P^{iv}(0) = \sin^{iv}(0)$.

From above $p^{IV}(0) = 24\ a_4$ and $\sin^{IV}(0) = \sin 0 = 0$.
Hence $a_4 = 0$
So $P(x) = x - \frac{x^3}{3!}$.

3. (i) $P(0) = f(0) = a_0$
Hence $a_0 = f(0)$.

(ii) $P'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$
$P'(0) = a_1 = f'(0)$.
Hence $a_1 = f'(0)$.

(iii) (a) $P''(x) = 2a_2 + (3)(2)a_3x + \ldots + n(n-1)a_nx^{n-2}$

$P''(0) = 2a_2 = f''(0)$.

Hence $a_2 = \frac{f''(0)}{2}$.

(b) $P'''(x) = (3)(2)a_3 + (4)(3)(2)a_4x + \ldots + n(n-1)(n-2)a_nx^{n-3}$
$P'''(0) = (3)(2)a_3 = 3!a_3 = f'''(0)$

Hence $a_3 = \frac{f'''(0)}{3!}$.

(c) $P^{IV}(x) = (4)(3)(2)a_4 + (5)(4)(3)(2)a_5x + \ldots$
$+ n(n-1)(n-2)(n-3)a_4x^{n-4}$

$P^{IV}(0) = (4)(3)(2)a_4 = 4!a_4 = f^{IV}(0)$

Hence $a_4 = \frac{f^{IV}(0)}{4!}$

(d) $P^n(0) = n(n-1)(n-2) \ldots . (3)(2)(1)a_n$
$= n!a_n = f^n(0)$.

Hence $a_n = \frac{f^n(0)}{n!}$.

(iv) $P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^n(0)}{n!}x^n$.

4. (i) From Question 3 the fifth Taylor polynomial is

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \frac{f^V(0)}{5!}x^5$.

Question 2 tells us that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$,
$f'''(0) = -1$, $f^{IV}(0) = 0$.

$f^V(x) = \frac{d}{dx}(\sin x) = \cos x$.

Hence $f^V(0) = \cos 0 = 1$ and

$P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

(ii) Extending the process, $f^{(6)}(0) = 0$ and $f^{(7)}(0) = -1$.
Hence the seventh polynomial is

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$.

5. (a) $x = \frac{\pi}{6}$ (b) $x = 0.5$

	(a)	(b)
x	0.5235988	0.5
$x - \frac{x^3}{3!}$	0.4996742	0.4791667
$x - \frac{x^3}{3!} + \frac{x^5}{5!}$	0.5000021	0.4794271

(ii) $\sin \frac{\pi}{6} = 0.5$. The Taylor polynomial of degree 5 gives this to five decimal places.
$\sin 0.5 = 0.4794255$. Again, the Taylor polynomial of degree 5 is accurate to five decimal places.

6. (i) Trying to integrate by parts we get either

$$\int_{0.5}^{1} \frac{\sin x}{x}dx = \Big[\log_e x \sin x\Big]_{0.5}^{1} - \int_{0.5}^{1} \log_e x \cos x \, dx.$$

or $$\int_{0.5}^{1} \frac{\sin x}{x}dx = \left[\frac{-\cos x}{x}\right]_{0.5}^{1} - \int_{0.5}^{1} \frac{\cos x}{x^2}dx$$

In both cases the resulting integral is more complicated than the original, which does not help!

(ii) Putting $\sin x = x - \frac{x^3}{3!}$,

$$\int_{0.5}^{1} \frac{\sin x}{x}dx = \int_{0.5}^{1} \frac{x - x^3/3!}{x}dx$$

$$= \int_{0.5}^{1} \left(1 - \frac{x^2}{6}\right)dx$$

$$= \left[x - \frac{x^3}{18}\right]_{0.5}^{1} \simeq 0.45.$$

7. (i) (a) $\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2} = -\frac{1}{(1+x)^2}.$

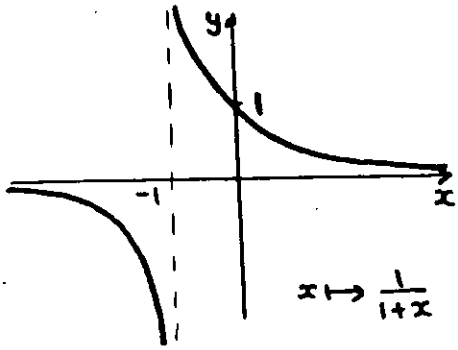
(b) $\frac{d^2}{dx^2}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(-(1+x)^{-2}) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}.$

(c) $\frac{d^3}{dx^3}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(2(1+x)^{-3}) = (-3)(2)(1+x)^{-4} = \frac{-3!}{(1+x)^4}.$

(ii) $\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{1+x^2}.$

This is never equal to zero so f(x) has no local extrema. For positive large values of x, f(x) tends to zero and is positive; for negative large values of x, f(x) tends to zero and is negative.

For x just greater than -1, f(x) is very large and positive; for x just less than -1, f(x) is very large and negative. At x = -1, f(x) is undefined. This gives the sketch below.



(iii) From Part (i) $f(0) = 1$

$$f'(0) = -1$$
$$f''(0) = 2$$
$$f'''(0) = -3!$$

Hence the Taylor polynomial of degree 3 is

$$f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$
$$= \quad 1 - x + x^2 - x^3.$$

(iv) Extending the process, the nth Taylor polynomial is

$$1 - x + x^2 - x^3 + x^4 - x^5 + \ldots + (-1)^n x^n.$$

[Note: This approximation only holds for $|x| < 1$].

8. The 6th Taylor polynomial for cos x is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

9. The Taylor polynomials for sin x + cos x can either be obtained from first principles as set out in Question 3, or they may be found by adding together the separate Taylor polynomials for sin x and cos x. Hence, the 8th Taylor polynomial is

$$1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!}.$$

10. The kth Taylor polynomial for $(1 + x)^r$ is

$$1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \ldots + \frac{r(r-1)\ldots(r-k+1)}{k!}x^k.$$

[Note: This only holds for $|x| < 1$].

11. (i) The kth Taylor polynomial for $\frac{1}{1 - x}$ is

$$1 + x + x^2 + \ldots + x^k.$$

[Note: This only holds for $|x| < 1$].

(ii) The Taylor polynomials for $\frac{1}{1 - x^2}$ can be found by adding together the polynomials for $\frac{1}{1 + x}$ and $\frac{1}{1 - x}$. Hence the 10th Taylor polynomial is given by adding

$$\frac{1}{1 - x} = 1 + x + x^2 + \ldots + x^{10}$$

and

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots + x^{10}.$$

to give $\frac{1}{1 - x^2} = 2 + 2x^2 + 2x^4 + 2x^6 + 2x^8 + 2x^{10}$.

[Note: Again, this only holds for $|x| < 1$].

15. WHY e?

(Note: This television programme makes explicit reference to SECTION 10: X $\mapsto$ 1/X: AN AREA FOR REVISION. It also uses the method of constructing the derived function which was introduced in SECTION 11: THE DERIVED FUNCTION.

PROGRAMME SUMMARY

What's so special about the number e? By examining the functions a^x and $\log_a x$ together with their derived functions we show the origins of e. This also demonstrates the significance of the functions e^x and $\log_e$, as we show that $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.

| 2 mins |

In the television programme from *SECTION 10: X $\mapsto$ 1/X: AN AREA FOR REVISION* we showed that

$$\int_1^x \frac{1}{x}dx = \log_e 2.$$

But what does e mean? — and where does it come from?

| 2 mins |

The Fundamental Theorem of Calculus *(SECTION 13)*. suggests that in order to find

$$\int_1^{1.5} \frac{1}{x}dx$$

we need to find a primitive F(x) such that $F'(x) = \frac{1}{x}$, in which case

$$\int_1^{1.5} \frac{1}{x}dx = F(1.5) - F(1).$$

But is $\log_e x$ a primitive for $\frac{1}{x}$?

| 2 mins |

We look at the characteristics of the graphs of 2^x, 3^x and a^x - and their inverses $\log_2 x$, $\log_3 x$ and $\log_a x$.

| 2 mins |

Now, if $f(x) = a^x$, then $f'(x) = a^x\lambda_a$ (See Question 3 in the Pre-programme work), and the graph of the derived function, f', is similar to the graph of f. The exact shape of f' depends on the value of a.

$\boxed{\text{3 mins}}$ Similarly, if $f(x) = \log_a x$, then $f'(x) = \frac{1}{x.\lambda_a}$ (See Question 4 in the Pre-programme work), and the graph of the derived function, f', is similar to the graph of $x \mapsto 1/x$. Again, the exact shape of f' depends upon the value of x.

$\boxed{\text{3 mins}}$ We look for the value of a such that λ_a is one and such that the derived function of a^x is <u>exactly</u> the same — a^x and the derived function of $\log_a x$ is <u>exactly</u> $1/x$.
This value of a is called e.

$\boxed{\text{2 mins}}$ Thus $\log_e x$ is a primitive for $\frac{1}{x}$ and so the Fundamental Theorem of Calculus tells us that

$$\int_a^b \frac{1}{x}dx = \log_e b - \log_e a.$$

Furthermore, if a = 1, then

$$\int_1^b \frac{1}{x}dx = \log_e b \text{ and so } \int_1^{1.5} \frac{1}{x}dx = \log_e 1.5.$$

$\boxed{\text{5 mins}}$ But how do we find $\log_e 1.5$? We could look it up in tables, or work it out by approximating the area with rectangles (as in Section 10). Alternatively, we could use Taylor polynomials. Unfortunately there are problems with the Taylor series for $\log_e x$ as $\log_e 0$ is not defined. However we can translate the function along by one to get

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$$

and this holds for $|x| < 1$.

$\boxed{\text{2 mins}}$ This allows us to find $\int_1^{1.5} \frac{1}{x}dx$ as accurately as we please.

<u>PRE-REQUISITES</u>

Before working through this section students should be familiar with the following:

(i) the definition of the derivative from first principles

(ii) the chain rule

(iii) construction of the derived function by looking at the tangent at each point (as introduced in *SECTION 11:* THE DERIVED FUNCTION

(iv) graphs of the form 2^x, $\log_2 x$, $1/x$

(v) the result

$$\int_1^2 \frac{1}{x}dx = \log_e 2$$

as derived in *SECTION 10: X ↦ 1/X: AN AREA FOR REVISION*
See Note below

(vi) algebraic manipulation of logs and indices

(vii) the Fundamental Theorem of Calculus (*as introduced in SECTION 13)*

(Note. This section provides an introduction to the derivatives of e^x and $\log_e x$. Familiarity with these functions is not required. It is sufficient that students know that there is a number e, and that its value is about 2.7. SECTION 10: X ↦ 1/X: AN AREA FOR REVISION provides an adequate background.)

PRE-PROGRAMME WORK

Students will benefit by working through SECTION 10: $X \mapsto 1/X$: AN AREA FOR REVISION *before starting this section.*

1. The derivative of a function f at X is defined as

$$f'(X) = \lim_{\delta \to 0} \frac{f(X+\delta) - f(X)}{\delta}.$$

Suppose $f(x) = 2^x$.

(i) Show that

$$f'(X) = 2^X \lim_{\delta \to 0} \left(\frac{2^\delta - 1}{\delta}\right)$$

(ii) (a) Use your calculator to find the value of $\left(\frac{2^\delta - 1}{\delta}\right)$ for $\delta = 0.1, 0.01, 0.001, 0.0001, -0.1, -0.001, -0.0001$.

(b) Hence write down $\lim_{\delta \to 0} \left(\frac{2^\delta - 1}{\delta}\right)$ to two decimal places.

(iii) Hence find approximate values for (a) $f'(2)$, (b) $f'(3)$.

2. (i) Use a similar argument to show that if $f(x) = 3^x$ then

$$f'(X) = 3^X\lambda$$

where λ is some number, and calculate the value of λ to two decimal places.

3. The calculation of the derivative of a^x (for any positive a) will be much the same. Show that if $f(x) = a^x$, then

$$f'(X) = a^X\lambda_a$$

for some number λ_a and find an expressions for λ_a.

4. $x \mapsto \log_a x$ is the inverse function of $x \mapsto a^x$.

(i) Draw a rough sketch of the graphs of a^x and $\log_a x$ for various values of a.

(ii) One way of saying that $x \mapsto \log_a x$ is the inverse of a^x is to say

$$a^{\log_a x} = x.$$

(a) Put $u = \log_a x$ and let $y = a^{\log_a x}$. Use the Chain Rule and the result of Question 3 to find $\frac{du}{dx}$ in terms of $\frac{dy}{dx}$.

(b) Now use the fact that $y = a^{\log_a x} = x$ to show that

$$\frac{du}{dx} = \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\lambda_a}.$$

(c) Use the results of Questions 1 and 2 to find

(a) $\frac{d}{dx}(\log_2 x)$ and (b) $\frac{d}{dx}(\log_3 x)$.

WORK TO BE TACKLED DURING OR AFTER THE PROGRAMME

5. Suppose $f(x) = \log_e (1+x)$

(i) Find

(a) $f'(x)$, (b) $f''(x)$ (c) $f'''(x)$ (d) $f^{IV}(x)$.

(ii) Hence write down the first four terms in the Taylor Series for $\log_e (1+x)$.

(iii) Guess the nth term in the series.

(iv) Use the Taylor Series to find $\log_e 1.5$ correct to three decimal places.

POST-PROGRAMME WORK

6. (i) Show that for any $a > 0$

$$a^x = e^{(\log_e a)x} .$$

(ii) Use this result to find the derivative of $x \mapsto a^x$ and so interpret the constant λ_a in the expression

$$\frac{d}{dx}a^x = \lambda_a a^x. \qquad \text{(See Question 3)}$$

7. (i) Calculate the values of $f'(0)$, $f''(0)$, $f'''(0)$ and $f^{IV}(0)$ for $f(x) = e^x$.

(ii) Hence write down the first five terms in the Taylor Series of e^x about $x = 0$.

(iii) Guess the nth term. Check your answer by differentiating.

(iv) Obtain a series for e by putting $x = 1$.

(v) Use the first 8 terms in your series to calculate the value of e correct to four decimal places.

8. (For those who have met Complex Numbers).

(i) Substitute $i\theta$ for x in the Taylor Series for e^x.

(ii) Separate your series into real and imaginary parts. Do you recognize the two series?

(iii) What happens when $\theta = \pi$.

POSSIBLE EXTENSIONS

1. This section introduces the derivatives of e^x and $\log_e x$ and does of course lead to further exercises involving differentiation and integration techniques on function involving these functions.
2. The graphs of functions involving e^x and $\log_e x$ could be explored using graph sketching techniques.

SOLUTIONS

1. (i) $$f'(X) = \lim_{\delta\to 0} \frac{f(X+\delta) - f(X)}{\delta}$$

$$= \lim_{\delta\to 0} \frac{2^{X+\delta} - 2^X}{\delta}$$

$$= \lim_{\delta\to 0} \frac{2^X 2^\delta - 2^X}{\delta}$$

$$= \lim_{\delta\to 0} 2^X\left(\frac{2^\delta - 1}{\delta}\right) = 2^X \lim_{\delta\to 0}\left(\frac{2^\delta - 1}{\delta}\right)$$

(ii) (a)

δ	0.1	0.01	0.001	0.0001	-0.1	-0.001	-0.001
$\left(\frac{2^\delta - 1}{\delta}\right)$	0.72	0.70	0.69	0.69	0.67	0.69	0.69

(b) Hence $\lim_{\delta\to 0}\left(\frac{2^\delta - 1}{\delta}\right) = 0.69$ (to two decimal places).

(iii) (a) $f'(2) \simeq (2^2)(0.69) \simeq 2.76$

(b) $f'(3) \simeq (2^3)(0.69) \simeq 5.52.$

2. A similar argument to Question 1 Part (i) gives

$$f'(X) = 3^X \lim_{\delta\to 0}\left(\frac{3^\delta - 1}{\delta}\right).$$

Hence $\lambda = \lim_{\delta\to 0}\left(\frac{3^\delta - 1}{\delta}\right)$.

δ	0.1	0.01	0.001	-0.1	-0.01	-0.001	-0.0001
$\frac{3^\delta - 1}{\delta}$	1.16	1.10	1.10	1.04	1.09	1.10	1.10

This table suggests that $\lim_{\delta\to 0}\left(\frac{3^\delta - 1}{\delta}\right) = 1.10$.

Hence $\lambda = 1.10$ (to two decimal places).

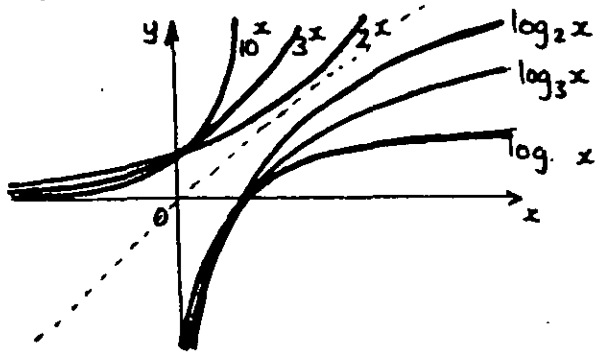
3. $$f'(X) = \lim_{\delta\to 0} \frac{f(X+\delta) - f(X)}{\delta}$$

$$= \lim_{\delta\to 0} \frac{a^{X+\delta} - a^X}{\delta}$$

$$= a^X \lim_{\delta\to 0}\left(\frac{a^\delta - 1}{\delta}\right).$$

Hence $\lambda_a = \lim_{\delta\to 0}\left(\frac{a^\delta - 1}{\delta}\right)$.

4. (i) The graph of $\log_a x$ can be obtained by reflecting the graph of a^x in the line $x \mapsto x$.



(ii) (a) $y = a^{\log_a x}$

Let $u = \log_a x$. We want to find $\frac{du}{dx}$.

Now, $y = a^u$ and

$$\frac{dy}{dx} = \frac{da}{du}\,\frac{du}{dx} \qquad \text{(using the Chain Rule)}$$

$$= a^u \lambda_a \frac{du}{dx} \qquad \text{(from Question 3).}$$

and $\frac{du}{dx} = \frac{1}{a^u \lambda_a}\,\frac{dy}{dx}$.

(b) Since $y = a^{\log_a x} = x$, $\frac{dy}{dx} = 1$.

Hence $\frac{du}{dx} = \frac{1}{a^u \lambda_a}$.

That is $\frac{d}{dx}(\log_a x) = \frac{1}{a^u \lambda_a}$.

But $a^u = a^{\log_a x} = x$.

So $\frac{d}{dx}(\log_a x) = \frac{1}{x\lambda_a}$.

(iii) (a) $\frac{d}{dx}(\log_2 x) = \frac{1}{a\lambda_2}$

From Question 1, $\lambda_2 \simeq 0.69$.

Hence $\frac{d}{dx}(\log_2 x) \simeq \frac{1}{0.69x}$.

(b) From Question 2, $\lambda_3 \simeq 1.10$.

Hence $\frac{d}{dx}(\log_3 x) = \frac{1}{x\lambda_3} \simeq \frac{1}{1.10x}$.

5. (i) (a) $f'(x) = \frac{d}{dx}(\log_e\ (1+x)) = \frac{1}{1+x}$.

(b) $f''(x) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}((1+x)^{-1}) = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$.

(c) $f'''(x) = \frac{d}{dx}(-(1+x)^{-2}) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$.

(d) $f^{iv}(x) = \frac{d}{dx}(2(1+x)^{-3}) = -6(1+x)^{-4} = -\frac{6}{(1+x)^4}$.

(ii) The Taylor Series for $f(x)$ is

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + \frac{f^n(0)x^n}{n!} + \dots$$

From Part (i) $f(0) = \log_e 1 = 0$

$f'(0) = 1$

$f''(0) = -1$

$f'''(0) = 2$

$f^{iv}(0) = -6$.

Hence the first four terms of the taylor series for $\log_e\ (1+x)$ are

$$\log_e\ (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

(Note This only holds for $|x| < 1$.

(iii) The nth term is $(-1)^{n+1}\frac{x^n}{n}$. This can be checked by differentiating.

(iv) $\log_e\ (1.5) = 0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} + \dots$

Taking the first eleven terms

$$\log_e\ (1.5) = 0.405 \quad \text{(to three decimal places).}$$

6. (i) Since $\log_e\ (a^x) = (\log_e a)x$

$$e^{(\log_e a)x} = e^{\log_e (a^x)} = a^x.$$

(ii) $\frac{d}{dx}(a^x) = \lambda_a a^x$ from Question 3.

Also $\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\log_e a)x}\right) = (\log_e a)e^{(\log_e a)x}$

$$= (\log_e a)\, a^x$$

Hence $\lambda_a a^x = (\log_e a)a^x$

and $\lambda_a = \log_e a$.

7. (i) If $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$ and so on.

When $x = 0$, $e^x = e^0 = 1$.

The Taylor series for $f(x)$ is

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots .$$

Hence the Taylor Series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots .$$

(ii) The nth term is $\frac{x^n}{n!}$

Differentiating n times, $\frac{d^n}{dx^n}\left(\frac{x^n}{n!}\right) = \frac{n(n-1)(n-2)\dots}{n!} = 1$

So, when $x = 0$, $\frac{d^n}{dx^n}\left(\frac{x^n}{n!}\right) = 1$.

Also $\frac{d^n}{dx^n}(e^x) = e^x$, and when $x = 0$, $e^x = 1$.

So $\frac{d^n}{dx^n}(e^x) = \frac{d^n}{dx^n}\left(\frac{x^n}{n!}\right)$ when $x = 0$, as required.

(iv) $e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

(v) $e \simeq 2.7183$

8. (i)

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= \cos\theta + i\sin\theta.$$

(ii) When $\theta = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0.$$

That is $e^{i\pi} = -1$.

This is a nice result as it combines the numbers e, π and i in one simple looking elegant result. What does it mean?

Television in the teaching of Sixth form Mathematics

Questionnaire

PART I

1. Name ______________________

2. School ______________________

3. A-level Syllabus used in your course ______________________

4. Have you used any of the Open University materials?

   Yes ☐ Please turn to Question 7

   No ☐

5. If you have not used any of the materials please indicate the reason.

   Inappropriate to A-level syllabus ☐

   Insufficient time ☐

   Difficult to organize ☐

   Unsuitable in other ways. (Please specify) ______________________

Continued at the top of next column

6. (a) Would you consider using any of the materials in the future?

   Yes ☐

   No ☐

   (b) If yes, which sections? ______________________

7. Which sections have you already used? (Please tick accordingly)

   1. ☐ 2. ☐ 3. ☐ 4. ☐ 6. ☐

   7. ☐ 9. ☐ 10. ☐ 11. ☐ 12. ☐

   13. ☐ 14. ☐ 15. ☐

8. Which sections do you plan to use but have not yet had an opportunity to do so?

   1. ☐ 2. ☐ 3. ☐ 4. ☐ 6. ☐

   7. ☐ 9. ☐ 10. ☐ 11. ☐ 12. ☐

   13. ☐ 14. ☐ 15. ☐

9. Please indicate your general comments about the suitability of the materials

____

The next part of this questionnaire comprises six identical sets of questions. It would be most helpful if you could complete one set of questions for each section you have worked with in the classroom. As I indicated, in my earlier letter I am particularly seeking comments on Sections 1 to 4, but I realize that these may not be relevant at the moment and of course, all your comments will be used to improve the materials. (If you have worked with more than six sections, and would like to forward more comments please let me know and I can send extra copies of the questions as required).

PART II Section Number ______ Title ______

1. How relevant was this work to your A-level syllabus?

Directly relevant, used as a teaching resource ☐

Not directly relevant, but supportive ☐

Not really relevant, but interesting ☐

Completely irrelevant. ☐

2. At which level (or levels) do you feel this work to be most useful?

Pre-A-level ☐

First year of A level Course ☐

Second year of A level Course ☐

3. If possible, please indicate how long you spent preparing for the teaching sessions which specifically involved this section of work?

______

______

______

4. How long did the students spend working on this section? ______

______

5. How did the students react to the work?

They enjoyed it ☐ Please tick as many as appropriate.

It helped with their understanding of the mathematical concepts ☐

It made a change ☐

It was useful ☐

It was too hard ☐

It was too easy ☐

It was boring ☐

Other (Please specify) ______

______

______

6. (a) Would you use these materials again?

Yes ☐ No ☐

(b) If no, please give the reason

______

______

______

7. The TV Programme

(a) Did the TV summary provide a reasonable idea of what was involved in the programme?

Yes ☐ No ☐

(b) If no, how could it be improved? ______

______

______

______

(c) How much of the programme did the students watch?

All of it ☐

Over half of it ☐ } Please give a rough indication of which parts

Less than half of it ☐

______

______

______

(d) How did the students watch the TV programme?

In one viewing ☐

In several viewings but on the same same occasion ☐

Several viewings on separate ocassions ☐

8. The Pre-Requisites

(a) Were the prerequisites realistic?

Yes ☐ No ☐

(b) If not, please indicate in which ways they were unsatisfactory

______

______

______

9. The Pre Programme Work

(a) Did the students work through any of of these exercises? If so, please indicate which ones ______

______

(b) Did you provide any additional material? If so please give a rough indication of what this covered?

______

______

______

(c) After having used the materials can you identify any areas where more/less pre programme work is needed? If so please indicate which topics require more attention ______

______

______

______

(d) In retrospect, how do you feel the pre programme work relates to the TV programme?

It is essential ☐

It aids understanding but is not essential ☐

It could be omitted ☐

Other (Please specify) ______

______

______

10. The Post Programme Work

(a) Did the students work through any of the post programme exercises. If so which ones? ______

______

______

Continued at the top of next column

(b) Did you provide any additional material? If so please give a rough indication of what this covered ______

______

______

______

______

(c) Can you now identify any areas where more/less post programme work is needed? If so please indicate which topics require more attention ______

______

______

______

(d) In retrospect how do you feel the post programme work relates to the TV programme?

It is essential ☐

It aids understanding but is not essential ☐

It could be omitted ☐

Other (Please specify) ______

______

______

11. Possible extensions

Can you identify any omissions here? If so please specify ______

______

______

______

______

12. Did you spot any errors? If so please indicate below

______

______

______

______

______

______

______

13. If you have any other general comments please indicate these below. (These could include suggested additions or deletions to the materials, a change in format, additional diagrams and so on.)

______

______

______

______

______

______