The effect of language in the process of mathematical education

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THE EFFECT OF LANGUAGE IN THE PROCESS OF MATHEMATICAL EDUCATION

by

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ABSTRACT

There continues to be much discussion as to why many children find mathematics difficult. Schools have attempted, over the past twenty years, to modify their mathematics curricula to make students feel that the subject is interesting and worthwhile. Changes have been made in content, teaching style and classroom organisation, but children continue to fail.

The role of oral and written language used in the mathematics classroom would seem to be critical when considering children's performance in this subject, and this will be discussed in the following chapters.

Assessing how difficult children will find a particular piece of mathematics text to read will also be discussed, as will the limitations of readability formulae to qualtify the degree of 'difficulty'.

Finally, some suggestions will be made as to how the presentation of mathematics text can be changed, so that children find it easier to read and understand.

It is hoped that the information provided in this study will enable teachers to think more about the importance of language in mathematics education and help improve the presentation and readability of written material used in the classroom.

It is not possible in a dissertation of this length to discuss all the language factors in the learning of mathematics, but the areas chosen are those that the author considers the most crucial for classroom teachers and for the writers of mathematical text to consider.

There is no doubt that there is room for much more research on this whole area of concern, and in Britain especially, on the establishment of criteria for evaluating written material in mathematical text.

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CHAPTER 1: THE ROLE OF LANGUAGE IN THE FORMATION OF MATHEMATICAL CONCEPTS

Language is an aspect of learning mathematics which is all too often overlooked. Language should be seen as an important aspect of learning, which can be used for the construction of thought.

One of the school's most important roles is to enhance the development of language, since language sophistication is very closely related to function in school. Language is the chief medium of instruction in schools, and its importance, therefore, must not be underestimated.

It most be noted, however, that language is not synonymous with communication. Animals are able to convey messages such as fear, hunger or pleasure. They communicate, but do not have a language. To communicate through language is to make use of sounds, symbols or gestures in a purposeful way, to convey meaning. The use of language also involves combining or changing sounds and gestures to produce a multitude of meanings.

Language is a central feature of classroom life. One of the major functions of language is in its use for learning, for trying to put new ideas into words, and for trying out one's thinking on other people. This does suggest, therefore, that language has an active use, rather than a passive one. Education is basically about the meanings pupils take away with them. Discussion in the classroom is important, so that individuals can make sense of what is presented to them. Whatever the teacher believes has been taught, we can be certain that individual children each take away something different. The importance of the pupils' own language in their learning process must be recognised. Paragraph 306 in "Mathematics Counts" (1982) says that "Language plays an essential part in the formulation and expression of mathematical ideas.... there is a need for more talking time... ideas and findings are passed on through language, and developed through discussion, for it is this discussion after the activity that finally sees the point home."

To what extent, then, is the development of mathematical concepts dependent on the language development of the child? When teaching any mathematical topic the teacher has to take decisions about the optimum time for introducing relevant specialised vocabulary and symbolism, but these decisions have to be made against a background of psychological uncertainty about the role of language in the acquisition of concepts. As teachers, how do we choose and use language in order to facilitate the acquisition of mathematical concepts?

Questions have been raised as to whether the growth in linguistic ability follows the development of concrete operational thought, or whether the development of adequate terminology is a prerequisite for cognitive growth. Piaget (1954) maintains that growth in linguistic ability follows the development of concrete operational thought rather than preceding it. However, Bruner (1966) maintains that the development of adequate terminology is essential for cognitive growth. It seems likely, however, that the acquisition of language is both a cause and an effect of cognitive development.

Bruner's view seems to be tied in with the American behaviourist tradition, where thought was treated simply as 'unsocialised speech', and hence thought without language was inconceivable. We therefore have two extremes, one the behaviourist view, where thought is felt to be verbal in nature, and the other, Piaget's, where thought is considered, essentially, to be spatial.

The spatial and verbal aspects of thought are basically complementary, but individuals may find one type of activity much more accessible than the other. There is psychological evidence to suggest that these two aspects of the learning of mathematics, spatial representation and language, may be linked to the activities of different halves of the brain; spatial processing generally being performed by the right hand side, while language functions are performed by the left.

It has been suggested by Wheatley and Wheatley (1979) that individuals may not be equally proficient in both types of activity, and they point out that many "low attainers" find a spatial approach "more accessible and more congenial."

Many children from deprived backgrounds enter school unable to articulate ideas because of language inadequacies, and this can restrict their learning of mathematics not only intially, but all the way through their school life. Choat (1974) suggests that a spatial approach may lead on to the introduction of appropriate language in a meaningful way, and may lead to a high degree of success where none may have occurred before. Wheatley (1977) uses Einstein as an example: "Einstein reported that his great discoveries came as flashes of images, not in words or symbols. It was often weeks before he could put the ideas into words and symbols... It should be noted that Einstein was unsuccessful in school: for example, he was not good at the left hemisphere tasks demanded in arithmetic."

Although this 'hemisphere theory' is speculative, and although some learners may favour one or other of these strategies, many will vary their approach depending on the context of the problem. Wheatley does feel, however, that "for all children spatial ('right hemisphere') development has been under-emphasized."

Wheatley also felt that the two aspects of mathematics represented by language and symbols on the one hand, and by spatial representation on the other, are entirely complementary in nature, and should each receive a reasonable share of attention in any mathematics curriculum. Some children may show a preference for one or the other, and in these cases the best approach might be to introduce a topic in the style which is most appropriate for the learner, but aim to use this to build up a capability in the less favoured aspect as well.

Piaget (1926) distinguished between 'egocentric' and 'socialised' talk. In egocentric talk, children do not worry about who they are speaking to, or whether they are being listened to. Vygotsky (1962) sees egocentric speech as the ability to think in verbal terms; he further felt that egocentric speech was a "transitional stage from vocal to inner speech, which characterised individual thought." As he put it, "speech for oneself". Piaget suggested that 'egocentric' speech begins to disappear at about the age of seven, although "this does not mean that from the age of seven or eight children can immediately understand each other."

The use of language is often very helpful to the mathematician. Austin and Howson (1979) noted the "sub-vocal movements of the tongue and lips" often observed when difficult material was being read. They also commented on the tendency of mathematicians to want to talk to a colleague in an apparently egocentric manner, in order to analyse a difficulty. As teachers, we often suppress children talking in the classroom, and by doing this we may be doing them a great disservice.

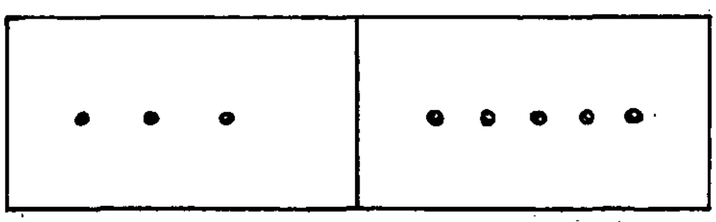
Sapir (1963) wrote "the feeling entertained by many that they can think, or even reason, without language, is an illusion... no sooner do we try to put an image into conscious relation with another than we find ourselves slipping into a silent flow of words." Although Sapir considered the use of language essential, there is much disagreement, and Skemp (1971) claimed to show the formation and use of low order concepts without the use of language, yet wrote that the emergence of higher order concepts would seem to be inextricably linked with language. Vygotsky (1962) performed some simple experiments with children under five years old, where he showed that they 'explain' the names of things by their qualities or attributes. He concluded from this experimental study of concept formation that the formation of concepts is dependent on linguistic ability. He wrote "the birth of a new concept is invariably foreshadowed by a more or less strained or extended use of old linguistic material; the concept does not attain to individual and independent life until it has found a distinctive linguistic embodiment." Piaget (1954) actually generalised his view of the relationship between thinking and language. He said, ".... language and thought are linked in a genetic circle, where each necessarily leans on the other in an independent formation and continuous reciprocal action. In the last analysis, both depend on intelligence itself, which antedates language and is independent of it."

Piaget and Vygotsky agree, therefore, that language does seem to have an essential part to play in the development of higher order concepts. They both provide evidence that the development of linguistic structure sometimes precedes the appreciation of the corresponding logical relationship. Piaget, especially, felt this and suggested in his work that children use subordinate clauses with 'because' or 'unless', for example, some time before they grasp the corresponding logical relationship. The statement 'grammar precedes logic' appears in the work of both Vygotsky and Piaget.

Choat (1974) noted the close interdependence of language and conceptual development. He said that: "Even if the learner interacts with the physical aspect of the learning situation, i.e., objects, the verbal element is necessary both as a means of communication and as an instrument of individual representation ..... in the acquisition of mathematical knowledge, a new concept brings a new word. Devoid of the conception, children will not understand; without the word they cannot as easily assimilate and accommodate the concept."

Many recognised tests of concept development, including those used by Piaget to test whether language precedes or succeeds concept development, unfortunately depend on children's understanding and use of language, and this is problematic. Piaget claimed that "thought precedes language", but he chose to use verbal interviews to judge whether concepts had been attained or not. Siegel (1978) pointed out this inconsistency in Piaget's position, and she attempted to evolve non-verbal tasks of concept development, some of which were related to Piagetian experiments.

In one task, she tried to train 3/4-year olds to consistently choose the larger of two sets of dots simultaneously presented to them, as in the diagram below, by rewarding them with a sweet each time they made the correct choice:



Siegel also trained a second group of children to select the smaller of the collections of dots presented to them.

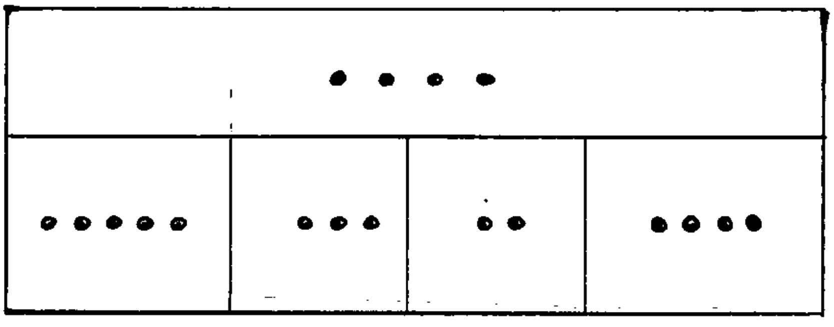
Each of the children were also asked to identify which of the two collections of dots was 'the big one' (or, for the second group of children, 'the little one'). The results were as follows:

CONCEPT AND LANGUAGE PERFORMANCE ON COMPARISON TASKS

GROUP	Pass language Pass concept	Fail language Pass concept	Pass language Fail concept	Fail language Fail concept	TOTAL
3-year olds	15	20	2	8	45
4-year olds	37	17	1	2	57

These results do tend to suggest that children's ability to learn the concept of comparison between smaller or larger groups of objects in the non-verbal task precedes the acquisition of the related language of little and big.

Siegel then repeated this experiment using the concept of equality (see diagram below):



In this case the children were trained to select the group of dots equal to the top one. They were also asked verbally to choose the set which was 'the same' as the top one. The results were as follows:

CONCEPT AND LANGUAGE PERFORMANCE ON EQUALITY TASKS

GROUP	Pass language Pass concept	Fail language Pass concept	Pass language Fail concept	Fail Language Fail concept	TOTAL
3-year olds	7	4	0	34	45
4-year olds	40	8	0	9	57

In this case it should be noted that there is a much closer relationship between language and concept success, and Dickson (1984) suggested that this was because both tasks rely on a 'matching' or 'counting' technique.

Siegel also asked some of the children to justify their choices verbally. The next table shows the proportion of those able to make the correct choice, but who were unable to give an appropriate verbal explanation for each of the tasks set.

CHILDREN PASSING CONCEPT ATTAINMENT TASK
BUT FAILING TO GIVE APPROPRIATE VERBAL JUSTIFICATION

GROUP (YEARS OLD)	COMPARISON TASK	EQUALITY TASK
3	69%	100%
4	32%	79%

This does suggest, for these experiments, that the children had the ability to attain the concept long before they could actually produce the associated language. We must bear in mind, though, that these results show little about whether the understanding of the language develops at the same time, or after, the attainment of the concept.

It is arguable, therefore, that language is a tool for the construction of thought. Language should be brought in to the mathematics lesson by the teacher, and active verbal involvement from children should be sought. Instead of making learners mimic someone else's rules, teachers should encourage discussion about the mathematical processes involved, and by doing this the learners would undoubtedly gain in understanding. Discussion in the classroom is an important aid to concept formation.

Paragraph 307 in 'Mathematics Counts' (1982) states that "All children need, as a first stage in the learning of mathematics, to develop their understanding of words and expressions by means of activities and discussion in the classroom." We must also not forget that discussion can help develop an understanding of the relationships which exist between the many different mathematical topics we look at in schools.

By discussing mathematical processes in the classroom we can encourage learners to reflect upon the mental procedures they are using, thereby making them meaningful both to themselves and others. Discussion can, therefore, improve the long term storage and retrieval of mathematical knowledge. It might also seem reasonable to suppose that requiring learners to verbalise a principle after they appear to understand it, might increase their degree of awareness of that particular abstraction and help fix it in their minds, but research findings are not consistent with this supposition.

It seems to be generally accepted that concept formation arises through verbal discussion. Stephens (1977) emphasised the need for a varied pattern of communication to be used in the classroom. Kysilka (1976) found that some mathematics teachers talked more than some social studies teachers; that they asked more convergent questions, made more directing and descriptive statements, and elicited and rejected fewer student responses. We must accept that awareness can only occur if children are allowed to participate in their learning, in a variety of contexts. The teacher should act as a guide.

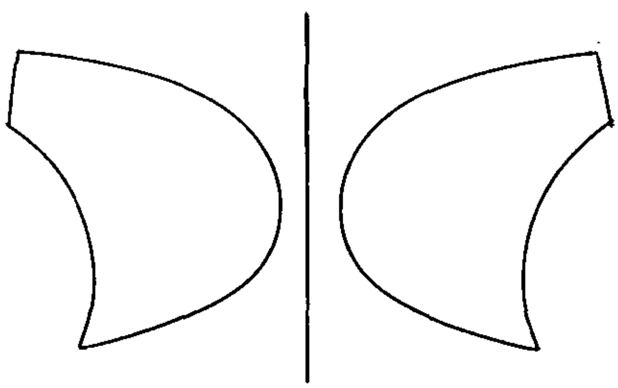
One of the aims of mathematical education must, therefore, be to encourage children to express their mathematical ideas verbally, with their peers, and their teacher. Work done in listening skills is also of importance. It was noted in the first APU secondary report on 15-year olds (1980b) that "....nearly all the testers commented that the ability of many pupils to express themselves clearly was the main stumbling block."

Hanley (1978) concluded that the best learning situations exist where language can be used freely as the interactive medium, and the best resource for this is the teacher. He expected teachers to introduce ideas, using words in an inexact way, and these can then be progressively refined until precision of thought is developed.

Children's understanding and use of language depends on how involved the children are in the situation in which the language is used, and how relevant this situation is to them. It is a dynamic process:

teachers must help children to express themselves more clearly and specifically. Larcombe (1985) felt that "It is easy to stifle the language and self-expression of learners by appearing to anticipate the language they should use rather than encouraging free expression and accepting whatever language they do use. Perhaps too much emphasis is placed upon 'correct mathematical language', and not enough on 'pupils' mathematical language'." Larcombe's remarks are very convincing. Many children are put off mathematics by the pedantry and narrow, complicated vocabulary used by their mathematics teachers. We must learn to compromise.

Nicholson (1980) believed that a 'negotiation of meaning' should take place. Children and their teachers should discuss various meanings of words, using less specialised terminology than that usually used in mathematics. An example he gave was:



symmetry
folding
reflection

Children will probably initially encounter a diagram like this when painting a shape and then folding it over. It would make sense at this stage, therefore, to call it a 'folding': this has meaning for them. Later on, when they meet the terms 'symmetry' and 'reflection', reference to the term 'folding' must be made when the meanings of these new words are being discussed. We must not underestimate the value of the everyday language used by children. Nicholson felt that "this will lead to the consolidation of the concept in relation to the word, and to the confirmation of the word as the name of the concept."

Annette Sweet (1972) stated that "as mathematics is a precise, unambiguous language, our first concern must be to teach the language, then the grammar will follow." She felt that there were specialised words that children should know, and that this should not prove difficult if "we remember how we learnt to speak." She went on that "no child's mother would attempt to give the verbal concept of 'chair' in one lesson: she would sit the child on one from a very early age, before the child had any idea of what it was. She will associate it in the child's mind with statements like "Get off that chair", or "Don't dirty the chair", ad infinitum. The child is repeatedly exposed to the idea, and it sinks in. The child is never sat on one and told 'that is a chair'." She went on to express the view that the concept of chair is far more difficult than many of the words we wish to introduce, and that if children grow up using the correct language, its grammar must be easier to learn.

There is no doubt that the learning of the mathematical language is time-consuming, and mathematics teachers are tempted to push children forward before they are ready, and they attain superficially high rates of progress. We must be more prepared to accept the children's language, or less precise words, while this process is going on. Merely teaching mathematics words to children will achieve little, and could well do harm if the children have little knowledge of the concept described. It has been suggested that children could be introduced, at an early age, to some of the specialised linguistic patterns used in mathematics, so while increasing children's mathematical vocabulary one could structure their learning of the syntax of mathematics. There could be a gradual progression from natural language to the formal language of mathematics.

It has often been pointed out, by Sweet and others, that mathematics itself is a formalised language, and it has even been suggested that it could be taught as such. Sweet (1972) said that "If we were to teach mathematics as we teach English, we would perhaps begin to have a few less negative attitudes to cope with. Children are not taught to read before they can speak, nor are they expected to write before they can read...." However, Austin (1979) felt that "mathematics was not a language, but an activity, and a treasure house of knowledge acquired over many centuries. The activity can be engaged in, and the results codified, using a variety of languages... the differences between the formalised language in which professionals present their mathematics and that in which they think, and talk about their work, would seem to be greater than those which normally exist between the written and spoken forms of a language." Mathematics should not be taught as a 'language'; it must develop through understanding, using a variety of styles, in the mathematics classroom.

We must also remember that the language used to talk about a particular mathematical concept may well be very different from that used in other contexts, and children have to learn these differences. There is often a mismatch between children's understanding of words and the teacher's. It can happen that a teacher tries to provide a familiar example for children, but in doing so triggers off a different meaning from the one intended.

Barnes (1969) drew attention to the consequences upon learning of a mismatch between the sort of language presented to the learner and the sort of language the pupil uses outside school. A danger here is that it is easy to assume that mathematical inactivity is due to conceptual problems in mathematics when in fact it is difficult for the pupil to gain access to the mathematics because of the language problem.

It appears, therefore, that language, especially oral language, plays a major part in the process of developing mathematical concepts. As mathematics teachers perhaps we should study the patterns of everyday speech, and by learning from these we may facilitate children's learning of mathematics.

CHAPTER 2: THE ROLE OF LANGUAGE WHEN PROBLEM SOLVING.

In most mathematics classrooms it is extremely difficult for teachers to listen to any one child for any length of time. Junior school teachers go to great lengths to hear children read so that individual problems can be indentified and help given. It would be an unusual mathematics classroom where a teacher gave individual attention to children attempting a problem, for any length of time, so that difficulties with the process of solution could be identified and help given. It is far more usual for short written tests to be given, but it could be argued that these can be a very poor indicator not only of children's ability but also as to where difficulties in the problem solving process are occurring. We need to know more accurately why errors have been made, so that children can be helped more precisely. Verbal questioning of individual children is necessary if we are to succeed in this.

Written responses to problems can only suggest the reasons why children make errors, but structured interviews must be conducted before consistent patterns of errors can be determined with any degree of certainty.

Watson (1980) noted that diagnostic interviews seem to have little place in most primary classrooms. He suggested several reasons for this: written tests are quick and easy to administer, teachers do not have the time to hold in-depth interviews in the classroom, and that interviews, in many cases, do not provide information that can be readily understood or used by the teacher.

Hollander (1978) believed that most studies she had reviewed only produced lists of broad, poorly defined error types, that did not indicate the causes of errors. It was suggested that teachers were unlikely to use a list of errors made by children if it did not readily suggest ways of helping them.

The teachers of young readers have been provided with a framework by Goodman (1965,1969) so that the reasons for errors made in the reading process can be more easily seen. If reasons why errors are made are known, then teachers can adjust their teaching to overcome any weaknesses.

Recent research from Australia has looked at the errors made by children during the problem-solving process, using a particular model for classifying errors made on verbal arithmetic problems. The classification of errors used was devised by Newman (1977). Its use is limited to problems involving only a single step to solution, but it is a useful guide for outlining the role of language in the problem solving process. The classification is useful for the teacher because it provides a clear framework for not only analysing errors made, but for questioning the pupils as well. The teacher can discover not only where, but why, a child made a mistake.

The classification is based on a model of how children go about solving problems. Newman made the assumption that associated with any given word problem are a number of hurdles that must be overcome if a correct solution is to be obtained. Failing at any one particular hurdle prevents people from going on to the next hurdle, and therefore from obtaining the correct solution unless they arrive at the 'correct' answer by faulty reasoning. Newman therefore defined a hierarchy of error causes for one-step mathematical problems. The model comprises a sequence of steps:

Reading

↓

Comprehension

↓

Transformation

↓

Process skills

↓

Encoding

Failures at different stages are shown as different errors.

Watson (1980) summarised the associated 'Criterion for Error Causes' as follows. The examples for each type of error are given by Clements (1980). It should also be noted that sub-categories under each main heading are also given.

CRITERION FOR ERROR CAUSES

1: READING ABILITY. Can the learner read the question?

Sub-categories: (i) Word recognition

(ii) Symbol recognition

EXAMPLE: A 12-year old pupil gave the answer '96 hours' for the question "What does fifty-six minus forty equal?". His response when asked how he obtained his answer was 'It says what does fifty-six minutes forty equal?'. It did not tell me what I had to do, so I added and got ninety-six. Now ninety-six is more than sixty, so the answer must be in hours." By misreading an important word in the question, he had been prevented from proceeding further.

2: COMPREHENSION. Can the learner understand the problem?

Sub-categories: (i) General understanding

(ii) Understanding of specific terms and symbols.

EXAMPLE: A 12-year old boy read the following question perfectly, but got an incorrect answer. "Sam goes to bed at 10 minutes to nine. John goes to bed 15 minutes later than Sam. What time does John go to bed?"

He gave an answer of '15' and explained: "it says John goes to bed fifteen minutes later, so the answer must be '15'. " He could read all the individual words in the question, but had not understood what was being asked of him. His progress towards a solution had been stopped because of comprehension problems.

3: TRANSFORMATION. Can the learner select the mathematical processes required to obtain a solution?

Sub-categories: None given.

EXAMPLE: A twelve-year old boy was shown a picture of twelve different children, and the question read: "Here are some children. I have 24 lollies and I want each child to have the same number of lollies. How many lollies will I give each child?" The pupil gave an answer of '144' and explained, "there are twelve children and twenty-four lollies; 12 into 24 goes 2, so we have two twelves; you multiply these two twelves: 12 times 12 is 144." When questioned further, it was obvious that not only could the pupil read the question but he also understood what he was being asked to do. He could not, however, transform the written problem into an appropriate ordering of mathematical procedures.

His progress towards a solution had been stopped because of a transformation problem.

4: PROCESS SKILLS. Can the learner perform the mathematical operations necessary for the task?

Sub-categories:

(i) Random response

(ii) Wrong operation

(iii) Faulty algorithm

(iv) Faulty computation

(v) No response.

(NOTE: These sub-categories are for arithmetical skills).

EXAMPLE: When asked the question "If you buy a bag of flour for £1.07 and pay the shopkeeper £2.00, how much change should you get?", a twelve-year old girl gave the answer £1.93. To get this answer she had used a faulty algorithm, and therefore the error made was due to a weakness in process skills.

5: ENCODING. Can the learner write the answer in an acceptable form?

Sub-categories: None given.

EXAMPLE: When asked the 'flour question' above, a twelve-year old boy wrote down the answer '93'. Although numerically he was correct, he was marked wrong, because he did not write his answer in the acceptable form of £0.93 or 93p. He had made an encoding error.

Newman pointed out that there are other types of error not included in this sequence of steps. They were listed as follows:

6: MOTIVATION. Many children just cannot be bothered to answer a question, although they could easily have done so if they had made an effort.

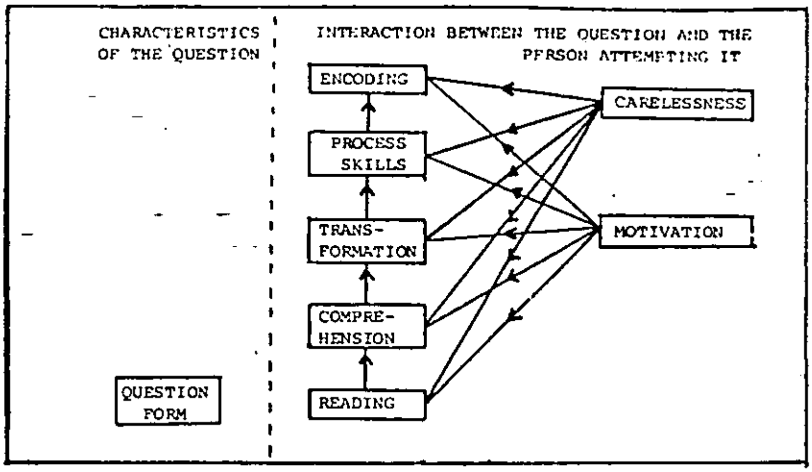
7: CARELESSNESS: Children often made a careless error at some stage of the solution, and although they can do all the steps in the solution this gives them an incorrect answer. Careless errors are unlikely to be repeated.

8: QUESTION FORM: The wording of a question causes children to make an error. The fault here lies with the writer of the question, not with the children who are attempting it.

EXAMPLE: When asked the 'lolly' question, already mentioned, a boy gave the answer as 'one'. Although this is not the answer expected, it should be regarded as being correct as the boy's reasoning of "I would give each one lolly and keep twelve for myself" is sound. His answer arose because of the badly worded question.

In diagram form we have:

THE NEWMAN HIERARCHY FOR ONE-STEP VERBAL MATHEMATICAL PROBLEMS

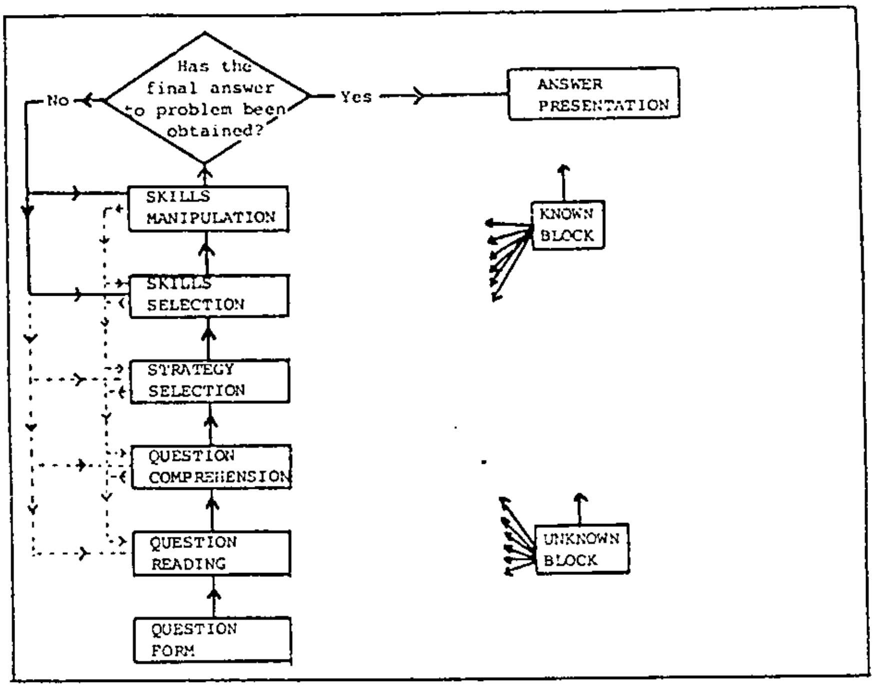


It should be noted that errors due to 'carelessness' and 'motivation' can occur at any of the first five stages, and this is reflected in the diagram.

Newman's hierarchy for one-step problems encouraged Casey (1978) into producing a more general hierarchy for use with many-step verbal mathematics problems. Casey modified and extended Newman's Hierarchy so that an analysis of errors could be made on many-step problems. Casey stressed that anyone attempting to solve a many-step problem has to identify and solve, in the correct order, a set of associated sub-problems. It is a cyclical process of solution, because the person often has to return to lower stages of the hierarchy when moving toward an overall solution.

In diagram form we have:

CASEY'S HIERARCHY FOR MANY-STEP VERBAL PROBLEMS IN MATHEMATICS



It should be noted that, unlike Newman, Casey actually included 'Question Form' in the hierarchy because, as Clements (1980) puts it, "this is the first point of interaction between the written task and the person attempting it." Newman's 'Transformation' category was changed to the two separate categories of 'Skills Selection' and 'Strategy Selection' to take account of the more complicated many-step problems now being looked at. Casey's 'Known Block' and 'Unknown Block' are the error categories outside the actual hierarchy. The 'Known Block' could include 'Motivation' and the 'Unknown Block', 'Carelessness'.

Newman investigated the errors made by Grade 6 (12-year old) pupils on a 40-item mathematics test containing numerical, spatial and logical questions. She gave this written test to 917 children in 31 classes in 19 schools in Melbourne, Australia. Within two weeks, four of the five lowest performing children in each of the 31 classes were interviewed:- 124 in all. The interviews were structured according to Newman's error classification list.

The interviewer would ask children to attempt a question they had originally got wrong. Once they had completed the question, whether right or wrong, the following questions were asked:

1: Please read the question to me. If you don't know a word, leave it out.

2: Tell me what the question is asking you to do.

3: Tell me how you are going to find the answer.

4: Show me what to do to get the answer. Tell me what you are doing as you work.

5: Now write down the answer to the question.

Note that each of these questions corresponds to a level in the Newman hierarchy.

The 124 low achievers interviewed had made 3002 errors on the original 40-item test, and over seventy per cent of these errors were repeated during the interview sessions. The task of each interviewer was to determine at which level the children first broke down in their attempt at solving the problem.

The results are shown in the following table.

NEWMAN'S CLASSIFICATION OF 3002 ERRORS MADE BY 124 SIXTH GRADE LOW ACHIEVERS (Melbourne, 1976).

ERROR CATEGORY	NUMBER OF ERRORS IN THIS CATEGORY	PERCENTAGE OF ERRORS IN THIS CATEGORY
Reading	390	13
Comprehension	665	22
Transformation	361	12
Process Skills	779	26
Encoding	72	2
Carelessness or Motivation	735	25
TOTAL	3002	100

Note that very special care had been taken in compiling the questions, and so no errors were attributed to question form.

From this table it can be seen that nearly 50% of the errors occurred before reaching the stage where process skills were needed. Only 16% of the 40 items on the test were considered the need the ability to transform, and 25% of the errors made on these 16 items first occurred at the transformation stage. (In other words, many questions were already in mathematical form, like $\begin{array}{r}624\\-312\\\hline\hline\end{array}$.)

Clements also interviewed 184 13-year old pupils, consisting of 92 low achievers and 92 average achievers, from 36 Australian schools. In all, 1981 errors were classified according to the Newman classification. The results are given in the following table.

CLASSIFICATIONS OF 1981 ERRORS MADE BY 92 LOW ACHIEVERS AND 92 AVERAGE ACHIEVERS (GRADE 7), 1977-1979.

	LOW ACHIEVERS (n=92)		AVERAGE ACHIEVERS (n=92)	
ERROR CATEGORY	No. of errors	% of errors	No. of errors	% of errors
Reading	117	8	18	3
Comprehension	225	16	32	6
Transformation	401	28	150	28
Process Skills	351	24	126	23
Encoding	37	3	12	2
Carelessness or Motivation	306	21	206	38
TOTAL	1437	100	544	100

The table is interesting, in that it is evident by looking at it that the low achievers are making nearly one quarter of their errors at the reading and comprehension levels, compared to 9% for the average achievers. So the data does seem to indicate that reading and comprehension difficulties play a significant role in children's low attainment in mathematics. It is also interesting to note that well over one third of the errors made by average achievers were due to carelessness or lack of motivation. Mathematics teachers should take note of this.

Although Casey developed his own method of classifying errors, and used a test which contained many-step problems only, he found, when testing 120 Grade 7 pupils of all abilities, that 45% of errors were made at or below the Newman 'Transformation' level.

Watson (1980) also conducted diagnostic interviews with third year pupils at a Melbourne primary school to determine why they had made errors on sixteen arithmetical tasks. Watson modified the Newman method of analysing errors so that the children's errors could be sensibly classified in terms of the sequence of steps they used when attempting to solve the problems. Watson felt that by finding where children were making their errors, he was able to devise more appropriate teaching procedures.

Watson used a 16-item test, given to thirty Grade 2 children, then fifteen children were interviewed, five of whom were gifted mathematically, and ten who had difficulty with mathematics. Watson found that with both groups, the large majority of initial errors were made at the stages of Reading and Comprehension. Watson makes the point that "The most interesting feature of the results, from the classroom teacher's point of view, is that it was possible, from the classification of errors for each of the children, to see precisely how they had approached the question, and to see where their strengths and weaknesses lay." Although Watson realised that class teachers did not have the time for diagnostic interviews for all children, he did suggest that for children experiencing persistent difficulties in mathematics this could be time well spent, so that difficulties could be identified and new lessons devised. Clements also felt that the Newman classification was a useful diagnostic strategy to follow by a teacher helping individuals. An individual's error pattern can illuminate why that individual makes mistakes on mathematical tasks.

In order to show how the error analysis procedure can provide useful information for teachers, we compare the results of two children taking exactly the same test, given by Clements:

> A fourteen-year old Grade 8 boy, John, made fourteen errors: two at the 'Reading' stage, four at the 'Comprehension' stage, three at the 'Process Skills' stage, and eight because of 'Carelessness'.
>
> Charles, the same age, also made fourteen errors: six at the 'Process Skills' stage and eight because of 'Carelessness'.

The Newman technique has clearly shown that although these two boys scored the same mark on the test, they needed different kinds of remedial help. It should be noted, however, that although the Newman technique is useful for gross diagnostic purposes, much deeper probing is necessary in order to discover more precisely how children think about a given area of mathematics.

Clements points out two warnings about the use of the Newman classification. Firstly, he says, "it should not be imagined that if two or more children have been identified as being especially prone to 'Transformation' errors, say, they need similar remedial treatment." As Marriott (1976) pointed out after asking 2826 pupils in Grades 5 to 8 in schools in Victoria, Australia, the question:

$$\begin{array}{r} 940 \\ -\ \underline{586} \\ \underline{} \end{array}$$

it resulted in 200 different answers being given.

Children with similar error profiles given by use of the Newman classification can make very different errors on the same problem.

The second warning given by Clements was that "the Newman hierarchy does not imply that a verbal arithmetic problem is necessarily more difficult than the corresponding arithmetic problem involving the direct application of the relevant process skills." When Grade 6 pupils were asked the questions:

(1): Write in the answer $1 - \frac{1}{4}$

and (2): A cake is cut into four equal parts, and Bill takes one of the parts, what fraction of the cake is left?

children seemed to find Question 2 easier than Question 1.

Clements suggested two reasons for this: the first that the imagery evoked by the cake problem helped pupils, and the second that some pupils just cannot cope with fraction sums when presented in numerical form.

The results of research contained in this chapter indicate that many of the errors made by children when solving mathematical problems are due to reading and comprehension difficulties. Because of this, children often choose a process at random in an attempt at finding a solution.. As Dickson (1984) points out, "reading and comprehension difficulties with words and symbols play a particularly crucial role in children's low attainment in mathematics."

CHAPTER 3: ORAL LANGUAGE IN THE MATHEMATICS CLASSROOM

In Chapter 1, it was clear that discussion in the classroom, whether between teacher and pupils or between different pupils, had an important role to play in concept formation. Bell, Costello and Kuchemann (1983) point out that "there is clearly some kind of connection between the way a teacher talks in class and the relationship between teacher and pupils". It has been noted that a deterioration in oral work can take place in mathematics classrooms when individualised schemes of work are used. Schoen (1976) states that " the educational quality of pupil-teacher interaction in the self-paced classroom is very poor, consisting mainly of procedural matters". There is no doubt that some teachers do let themselves become swamped with organisational matters in mixed ability situations but others manage to promote discussion, whether in groups or between individuals, far more easily than in the traditional classroom, where teacher led learning is the rule. As teachers we must be prepared to be flexible and as Stephens (1977) says, "provide a varied pattern of communication". Stephens also felt that the learning of mathematics requires the "negotiating of mathematical meaning for, and by, each student: the use of prepared programmes places the teacher in too inactive a role for him to exercise this negotiation, and tends to isolate children from one another". Stephens does point out a danger with individualised schemes here, but it could be argued that this is caused by the teacher, not the scheme.

Increasingly, as children get older, the oral side of the language of mathematics is neglected in the classroom and children are expected to obtain more of their information from texts or worksheets, thereby putting many children with reading difficulties at a disadvantage. Lovell (1971) felt there was a need for constant discussion between teacher and pupil and among pupils themselves in mathematics classrooms. Fey (1969) designed a procedure to analyse verbal interactions in the mathematics classroom. His results seemed to show a much greater verbal activity on the part of the teacher. He further found that 50% of the verbalisations made by teachers and pupils were statements or questions of fact, 25% were evaluations (made mainly by the teacher) and 25% were justifications and analytic processes.

Kysilka (1970) compared mathematics teachers with social studies teachers. He found that mathematics teachers talked more often and their pupils talked less frequently than those in social studies classrooms. The mathematics teachers asked more convergent questions and made more directing and describing statements, but interestingly also rejected fewer student responses than did social studies teachers.

Cooney (1970) paid attention to the way in which teachers' verbal behaviour helps students to learn and organise their knowledge.

He demonstrated how deduction, induction, classification and analysis of knowledge by the student could be helped by teacher verbal behaviour.

Practical evidence points, therefore, to the conclusion that in most mathematics classrooms the teacher does the majority of the talking and pupils respond as little as possible. It would seem safe to assume then that no matter what organisation is being used, a mixed ability individualized scheme, or a traditional teacher led approach, children do not use oral language as much as they could or should, to help facilitate a deeper understanding of mathematics. We need more group work and a willingness by the teacher to 'let go'. However, many teachers would be worried by this 'lack of discipline', as they would see it. Henry (1971) suggested that "there is a dialectic opposition between mobilising children's attitudes to a pitch of excitement and attempting to control them".

Arnold (1973) pointed out that children in a teacher directed class will use the teacher's language, however uncomprehendingly, whereas in a child-centred class unorthodox language will be used and will present the teacher with new opportunities and challenges. How many teachers would actually take advantage of these opportunities or rise to these challenges is difficult to say, but there is no doubt that many teachers would find the 'lack of structure' difficult to cope with.

Oral language must be encouraged in the mathematics classroom. The organisation used in the classroom makes little difference; it is the attitude of the teacher that is important. Less direction and a willingness to listen and accept non-exact child language is important if firm conceptial foundations are to be laid and children made to feel a part of their own education. It is important for children to feel that they can voice their thoughts or worries about a problem, without fear of being derided. If children feel they can make mistakes during a discussion, without penalty, they will be more encouraged to join in and take an active part in the problem solving process. Children's understanding of mathematics can only be helped by this.

CHAPTER 4: DIFFICULTIES CHILDREN ENCOUNTER WHEN DEALING WITH INDIVIDUAL WORDS IN MATHEMATICS

Shuard and Rothery (1984) make the point that "Mathematical text is more complex than ordinary English text, partly because mathematics uses a technical vocabulary which overlaps with the vocabulary of ordinary English." They distinguished three broad categories of words:

(i) Words which appear in mathematics and ordinary English texts, but have different meanings in these two contexts,

(ii) Words specific to mathematics, not occurring in everyday language,

and (iii) Words which appear in mathematics and ordinary English texts, which have the same or roughly the same meaning in both contexts.

Words in each of these categories cause children problems and we look at each category in turn.

There are some words which children are used to using in their everyday language that have a different meaning when used in mathematics. This causes children great difficulty because of the confusion which can arise when a word has a number of meanings and the meaning has to be inferred from the context.

Matthews (1980) considered the word 'difference' when investigating the problem of children's understanding of 'subtraction' words. She gave the following question to 81 six or seven year olds', from three London schools.

"I've got 7 fir-cones and you've got 9. What is the difference between the number you've got and the number I've got?"
Of the 81 children taking part, only 30 answered the question correctly. Of the remaining 51 children, 19 were considered to have got the question wrong because they misunderstood the meaning of the word 'difference'.

When the children were asked to explain the word 'difference' many interesting replies were given, including "if I were 9 and your were 7 then I would be older than you" and "if I had 9 sweets and you had 7 it wouldn't be fair". When another child was asked the question "what is the difference between 7 and 10", the answer given was "one is odd and the other is even". All of these examples demonstrate the confusion children have over understanding this particular word.

There are basically two types of words having different mathematical and ordinary English meanings. There are some words which have a mathematical meaning unrelated to their everyday usage, for example 'product' or 'difference'. There are other words where the mathematical meaning is similar to the meaning in ordinary English, but the mathematical word has a more specialised meaning. Examples include 'divide', 'average', 'remainder', 'reflection', 'gradient' and 'similar'. Looking at the word 'similar' in detail, we see that the two shapes below are similar if we use the everyday meaning of the word,

because their shapes could be considered to be nearly the same as one another - they are both right angled triangles. However, mathematically they are not similar, because corresponding angles are not equal. It is fairly easy to see how a pupil could become confused when reading the word 'similar' in a mathematics text book.

Austin and Houson (1979) felt that "a very precise geometric definition is required if we are to distinguish between two shapes to say whether or not they are mathematically similar". They felt that we must consider how students first meet words and form concepts to the point at which a technical term is introduced. Skemp (1971) felt that children learn to use a word through its verbal context rather than through a formal definition - children construct their own definition based on colloquial usage. But, many mathematical terms are common words used by mathematicians in a specialised way. Austin and Houson felt that there was a danger that children arriving at their own definition would arrive at the wrong one. Moreover "if their usage, say of the word 'similar', embraces that of the more limited mathematical one, then no apparent contradiction will occur when they receive information. It is only when they transmit it that inconsistencies may become apparent. Too frequently this inconsistency may pass unnoticed".

We therefore have the situation where children may understand some mathematics through using the English meaning without fully grasping the more specialised full mathematical meaning. Hence, their mathematical education will be limited, as the children may feel that they have grasped the content of the lesson, because they are only thinking in terms of the colloquial meaning.

Time should be spent in mathematics lessons, discussing the meaning of particular words children find confusing. Children should be encouraged to write down sentences giving different meanings of the same word - it is by bringing out different mathematical and ordinary English meanings that mathematical understanding may develop. Krulik (1980) gave some examples for the word 'prime':

> A prime number is a number only divisible by itself and one.
>
> Popular TV programmes are given prime time to attract large audiences.
>
> To make a water pump work, you have to prime it.

Children's understanding of a mathematical term may be influenced by which meaning they meet first, the mathematical or the English meaning. 'Parallel', for example, is far more likely to be met in its mathematical sense before being used in everyday English usage. Other words, like reflection, will most probably be encountered in ordinary English usage first and then the children will have to refine the meaning to deal with it in the mathematical sense.

It must also be remembered that there are different degrees of relatedness between words in their ordinary English and mathematical contexts. The more similarity in meaning that exists, the better able children will be to understand the mathematical meaning. Also, some mathematical words can have multiple mathematical meanings, for example, the word 'base'. We have a number base, the base of a triangle, bases in vector spaces and bases of logarithms. It was further pointed out by Shuard and Rothery (1984) that "several dual meaning words carry a derogatory connotation, and pupils may detect this nuance without realising that it is prejudicing them against the ideas. Fractions may be vulgar or improper, a mean is a rather 'underhand' average, a negative number feels less good than a positive number, and when numbers become irrational or imaginary, mathematics has apparently entered the realms of nonsense."

There are some words that are specific to mathematics and only have a mathematical meaning and examples of these are 'parallelogram', 'rhombus', 'hypotenuse' and 'coefficient'. These words are extremely unlikely to be used by children in their everyday speech or be used in the home. Their meanings must therefore be learnt either from the mathematics teacher or from mathematics texts, but they cause reading and understanding problems because they are so infrequently met. For example, they may be defined by the teacher and then not met again for six months. It is no wonder children cannot remember their meaning.

If children cannot remember the meaning of a word used in everyday language, they are encouraged to look up the word in a dictionary. However, when dealing with words only having a mathematical meaning this can be difficult as dictionary definitions leave a lot (mathematically) to be desired and very few school mathematics textbooks have an index. Even if a textbook gives the definition of a word, it will rarely be repeated, and as every teacher realises, constant repetition is often needed to reinforce the meaning of a word in children's minds. Once children forget the meaning of a mathematical word, it can be very difficult for them to find out what it means.

Wilmon (1971) found that about 500 mathematical words were introduced in texts designed for young learners. At this stage in their development, these children could only be expected to have a total reading vocabulary of about 4000 words. Once again it is not surprising that children find remembering not only the words, but also their mathematical definitions, difficult to manage.

It was recognised by Austin and Howson (1979) that in talking in the classroom, both teachers and pupils use informal language for a great deal of the time. Text books present things in a very formal way and it could be that children do not have the language experience to enable them to read the text accurately. They felt that a more explicit approach to mathematical vocabulary may well be needed in the classroom. Teacher's need to repeat definitions often and maybe even get pupils to make up their own dictionary of mathematical words, using definitions they can understand. However, as Ginsberg (1977) commented, "The artificial language of mathematics can present the child with considerable difficulties. 'Plus', 'congruent', 'minus' and the like are unfamiliar words that children frequently misunderstand. Defining the artificial word does not guarantee comprehension".

Another problem with mathematical words is that they often have Greek or Latin origins and most children nowadays do not have a knowledge of ancient languages, hence the roots of many words will be unknown. They will therefore be unable to associate them with words in their own vocabulary. Because of the Greek or Latin origins, many mathematical words have unfamiliar spelling patterns as well, which makes initial recognition a problem for pupils.

Comprehension of a whole passage in a mathematical text may depend entirely on the understanding of one or more key mathematical words. To overcome this problem many teachers and authors of textbooks have avoided the use of words like numerator and denominator and replaced them with short, more easily understood, descriptive phrases. It could be argued that for many children, especially those who will not take mathematics as a subject after leaving school, this is the correct approach. However, with pupils who may wish to take mathematics further in the future, many would agree with Shuard and Rothery (1984) that "omitting all technical words is a short term policy which makes text easier to read but it may bring long term disadvantages to the pupil". They go on to say that "Many technical terms have an essential place in mathematics, children cannot proceed without knowing them. Pupils will have even greater difficulty in learning words if they never meet them in their reading. So a practice which may seem to be a kindness, may in fact lessen pupils' experience of essential vocabulary, and so may work against the pupils' future comprehension of mathematics". In all textbooks and in the classroom, it is better to avoid the use of unimportant and unnecessary technical words like 'minuend' or 'subtrahend', for example. Where technical words have to be used, it must be remembered that children need help with understanding them.

There are also words that have the same meaning in ordinary English usage as they do in mathematics. Examples given by Shuard and Rothery (1984) are "cat, dog, because, it, taxi, shelves and climb". These are examples of ordinary English words which are used in mathematics textbooks. These words are frequently used by children and are the words more easily understood, between the technical words used in the text. In an investigation by Earp and Tanner (1980), where they looked at an American sixth grade textbook, they found when interviewing fifty pupils that common non- mathematical words were understood by 98% of the pupils, while only 50% understood the mathematical words.

One of the problems children have with words which have the same meaning in both mathematical and ordinary English contexts, is actually knowing that they do mean the same. Children sometimes don't understand the ordinary English meaning of a word in the first place and, even if they do, many feel that ordinary words take on a very special meaning when used in a mathematics text. We should realise that children should be given the opportunity to discuss even ordinary English vocabulary when it appears in mathematics text books.

The difficulty of an individual word can be changed using context clues. The word may be made easier to understand by putting clues to its meaning in the surrounding sentence or chapter. Earp (1971) found that mathematics texts often use simple words in such a way that they become more difficult to understand. Earp suggested that mathematical text offered less context clues than did an ordinary English passage, so that a word appearing in a mathematical text may be more difficult to read and understand than the same word appearing in an ordinary English passage. In mathematics, exact comprehension is necessary for successful completion of work, and often the lack of context clues makes exact comprehension impossible. Otterburn and Nicholson (1976) studied children's understanding of words commonly used in CSE mathematics courses. They took a sample of 300 pupils from about the middle 50% of the ability range. Each of the children was given a list of thirty-six such words and were asked to answer four questions on each word:

(1) Do you understand it? Yes/No

(2) What is its symbol ?

(3) Draw a diagram to show its use.

(4) Describe it in words.

For the word 'minus', say, a child may have answered question 3 with

and question 4 with " Take away, subtract. 7 Minus 4 equals 3".

The way children responded was classified in one of three ways:

(i) Correct - showing they knew what the word meant.

(ii) Blank - if they hadn't clearly demonstrated understanding (even if they had written 'yes' in answer to question 1.)

and (iii) Confused - if their answer appeared muddled.

Some of the results, expressed as percentages, mostly to the nearest whole number, are shown in the following table:

Word	Correct	Blank	Confused
Multiply	99.7	0.3	0
Remainder	92	8	1
Rectangle	88	4	8
Parallel	77	19	3
Reflection	45	51	4
Square Root	40	44	16
Rotation	37	60	3
Parallelogram	37	41	22
Factor	32	62	6
Square Number	65	24	10
Prime Number	52	34	13
Union	26	65	9
Mapping	16	81	3
Rhombus	31	47	22
Product	21	59	20
Multiple	20	45	34
Integer	15	76	9

It seems clear from this piece of research that a problem exists here. There is a mismatch between mathematical words teachers and examiners expect children to understand and those that they do understand. (It should be noted that the word 'similar' that has already been looked at in this chapter, was understood by only 19% of the children).

Nicholson (1977) followed up this particular piece of research with two further investigations. In the first of these the words were given in a sentence,i.e. they were set in a particular context, rather than being given as an individual word. For example "Give one example of a multiple of 8".

The results, for the same words already listed, are given as percentages in the table below. It should be noted that the children tested were a totally different group than those used in the first investigation.

Word	Correct	Blank	Confused
Multiply	99.5	0	0.5
Remainder	98	0	2
Rectangle	64.5	4.5	31.5
Parallel	86	5	9
Reflection	86.5	0.5	13
Square Root	81.5	3	15
Rotation	34.5	6	59.5
Parallelogram	65.5	5.5	29
Factor	91	5	4
Square Number	75	4	21
Prime Number	68.5	9.5	21.5
Union	55	5	40
Mapping	41	32.5	26.5
Rhombus	32.5	24	44
Product	20.5	4.5	75
Multiple	11.5	2	86.5
Integer	9.5	18.5	72

In the second of the follow up investigations, Nicholson asked children to fill in a missing word. For example, the numbers 2,3,5, 7,11,13,17,19 are all examples of ______________ numbers. Children's responses were noted using a similar classification as before. The results as percentages, are shown in the table below, using two columns of results - 'Acceptable' and 'Confused'. There were 46 children tested in all.

Word	Acceptable	Confused
Multiply	98	2
Remainder	74	26
Rectangle	93	7
Parallel	91	9
Square Root	54	46
Parallelogram	48	52
Factor	50	50
Square Number	30	70
Prime Number	39	61
Rhombus	37	63
Product	17	83
Multiple	28	72

Although these three investigations are not comparable, it can be seen that many children have difficulties with these commonly used mathematical words. As Nicholson commented "Pupils who enter for CSE Mathematics have significant difficulties in understanding some of the mathematical terms in common use. It is important that teachers should recognise the extent of their difficulties and work continuously for better understanding".

It should be noted that children improved on some words on the second investigation. The words 'Remainder', 'Reflection', and 'Square Root' are examples of this. It could be that context clues actually helped children better understand these words. There is no doubt that some words (such as 'reflection') are made much easier to understand using context clues. We must remember, however, that different scores on words could easily have occurred due to differences in mathematical and general background of the different groups of children interviewed.

Nicholson does suggest then, that, broadly speaking, the middle 50% of the whole ability range are seriously held back by lack of vocabulary, although many could most probably understand the mathematical principles involved. These are the children who will be involved in taking CSE's. Earp and Tanner (1980) did find that by using context clues, they could significantly improve children's ability to define individual words.

Very few mathematics teachers have been concerned with actually teaching children to read mathematical English. We could borrow some of the techniques used by the teachers of young readers and apply them to teaching children how to read mathematics text. Certainly primary teachers, who already use flashcards, personal dictionaries and comprehension exercises in teaching children how to read ordinary English, are well placed to transfer these methods to the teaching of mathematical English.

There are certain activities that we could use to further reinforce the mathematical meaning of words in children's minds, and some are given as examples below:

1. Matching words and descriptions of words:

   Match each word with its description.

Polygon	A six sided shape.
Pentagon	A many sided shape.
Hexagon	A five sided shape.
Quadrilateral	A four sided shape.

2. Unscrambling words that are already defined.

   3 sided shape; LGNRTIAE.
   Cut into two equal pieces; ISCBET
   Put in table form; ETAATLUB

3. Selecting the correct definition.

   A square is (a) A four sided shape.
   (b) A shape with four right angles.
   (c) A quadrilateral with equal sides.
   (d) A quadrilateral with all sides and all angles equal in size.

4. Putting ordinary English words together to give phrases which have a mathematical meaning.

Children are asked to define, in ordinary English terms words like pie, root, square, chart, prime, number etc. They are then asked to write down the definition of phrases like

pie chart

prime number

square root etc.

which all have specific mathematical meanings. In this way, teachers can teach children in the mathematics classroom, to read mathematics more fluently and with greater comprehension.

Of course, the difficulty children experience with individual words is only one part of the problem associated with mathematical text.

Many problems arise, for example, from the linguistic structure used and the way diagrams and text are presented together on a page. How readable children find any mathematical text will be dealt with in Chapter 6.

CHAPTER 5: DIFFICULTIES CHILDREN ENCOUNTER WHEN DEALING WITH MATHEMATICAL SYMBOLS

When children learn to read they learn to recognise individual letters and to know the sounds they, or a combination of them, make. There is a definite relationship between letters and sounds and therefore children can attempt to read and vocalise unfamiliar words they meet in a text. Unfortunately, when faced with mathematical symbols there is no relationship between symbols and sounds for children to rely on. Many mathematical symbols, especially those for young children, are pictorial, and children are expected to grasp the reasoning behind the symbols used, so that they can interpret the text..

Shuard and Rothery (1984) point out that the code used in writing mathematical symbols is almost entirely conventional. There is no reason to do with the operation of multiplication or the sound of the word 'multiply' that makes the symbol x better to represent multiplication than, say, the symbol T. Also much of the symbolism used in mathematics has meaning which varies with the way the symbols are spatially presented to the reader. For example:

(4,2)	coordinates or ordered pairs of numbers
42	forty-two, or 4 10's and 2U's
4^2	four squared
$\frac{4}{2}$	four divided by two, and so on.

Because the symbolism of mathematics is not based on a code of sounds, children have to learn the meaning of each symbol or groups of symbols as they meet them, and as in the problem with words having only a mathematical meaning, if they don't meet them often enough for reinforcement to occur, then they will not remember their meaning. When children are actually in the process of learning a new symbol they have to link together the symbol, the concept it represents and a definition that they understand to explain the symbol. This is a difficult exercise for many children and explains partly the reason why children have problems with mathematical symbolism. We must remember that many mathematical symbols can be read in different ways and this adds a further area of confusion for many children.

4 x 8, for example, can be read as times, multiply, 4 lots of 8, four eights and so on.

8 ÷ 2 can be read as 'how many 2's in 8' and 'divide' 8 into 2 equal parts. The first of these is a grouping exercise and the second a sharing one. To fully grasp the concept of division, a child must link these two ideas together - how many children see two entirely different operations occurring here that, by coincidence, happen to give exactly the same answer? There are also a number of different symbols attached to the concept of division, $8 \div 2$, $\frac{8}{2}$, 8/2 and $2\overline{)8}$ all meaning the same thing, and children need to appreciate this. In fact they have to appreciate that each of the symbols used for +, –, x and ÷ have a number of meanings, depending on the context.

Preston (1978) analysed the mathematical terminology used in primary texts and workcards. In eight different schemes, he identified 18 ways in which the operation of addition is presented, including:

The sum of 6 + 4 is
Add 6 and 4
6 add 4
The total of 6 and 4 is
4 greater than 6

6 plus 4

6 + 4

6 and 4 equal, and so on.

Each scheme he looked at, on average, used seven of the alternative forms, and one text, supposedly for the less able, used fourteen of these alternative forms in the space of two pages. Surely, when language development is going on, so many different forms presented in a short time to children can only further increase confusion.

There are symbols that are not specific to the technical vocabulary of mathematics, and children may confuse their mathematics meaning with their ordinary English meaning. Trivett (1978) points out that 'x' may mean 'multiply' or a kiss, or an incorrect answer and that some punctuation marks such as '!' or '-' or '...' take on very specific mathematical meanings.

Another problem that children have to contend with is that mathematcal symbols are often combined with others, and different positioning of symbols conveys different meanings. These different meanings attached to different positionings of symbols are normally determined by convention, and once again children have to learn this as they go along. For example, 71 and 17 are different numbers although they use the same mathematical symbols, albeit in a different order. The symbols '71' match the order of speech 'seventy one' but the symbols '17' do not match the order of the spoken word 'seventeen', thus introducing scope for further confusion.

We therefore have many irregularities that children meet and have to deal with. A further example could be children measuring a room with a metre rule. When doing so they will write down a measurement of 4 metres as 4m. When they come on to deal with fractions, at a later stage, they can be forgiven for thinking that four thirds are written as 4⅓ and not as $\frac{4}{3}$, they are confusing the principles involved here. When meeting algebraic multiplication for the first time they have to learn that, if $x = 7$, then $5x$ does not mean the same as 57. The system of coding used in each case is very different - we have tens and units as opposed to multiplication.

Booth (1982) gives an example of an interview with an above average 4th year pupil in a secondary school. The pupil was asked to explain the meaning of the y in '5y'.

" Pupil: y could be a number, it could be a 4, making 54. Or could be 5 to the power of 4, making 20.

Interviewer: Do you think it could be either?

Pupil: It could be either - you can't really say.

Interviewer: So y could be any number? (pupil nods). Suppose I made it 23. What would you write down then?

Pupil: Five hundred and twenty three? But I dunno - it doesn't seem very promising. Wait it could be 28, 5 plus 23... yes,... There again, it could be 5 to the power 23. (Writes 5 + 23, 5^{23}).

Not surprisingly, Booth recommends that the multiplication sign in $5 \times y$ be left in until such time that a pupil recognises, consistently, that 5y is the product of 5 and y.

Many pupils fail to understand the symbolism used in algebraic manipulation, and unfortunately, mathematical development is very much dependent on being able to cope easily with this. Much testing of algebraic competence has been done by the APU (1981) in its Secondary Survey Report and by the CSMS team.

One group of items on the APU survey gave children a letter representing a number and they were asked to write down numbers that were 'one bigger than' or 'three less than' or 'twice that number' represented by the letter. There was a facility level of only 45% when 14/15 year olds attempted this group, and over 20% of children made the mistake of representing 'twice x' as x^2.

When asked to simplify a + a + a, 17% gave the answer a^3 and for a x a x b, 11% gave the answer 2ab. In answer to the question, if $y = d^3$, find y if d = 3, there was only a facility level of 38% and 19% of pupils gave the answer to be 9. There is obviously a great deal of confusion in children's minds as to what the index notation means, many thinking it stands for the multiplication of the letter by the index. There seems no doubt at all that many pupils, even the most able, have not understood fully the symbolism in algebra.

In what order mathematical symbols should be read is another source of confusion for children. Reading ordinary English, we read each line from left to right and each page from top to bottom. Once again, this convention does not always transfer to the reading of mathematical text. If we look at the addition of fractions, Krulik (1980) gives the example:

$$\frac{3}{4} + \frac{1}{4} = 1$$

which has to be read in the order given by these arrows:

$$\downarrow\frac{3}{4}\nearrow + \nearrow\frac{1}{4}\downarrow\nearrow = \rightarrow 1$$

Krulik suggests that children should be encouraged to do such arrow diagrams as a part of the process of learning to read symbolic expressions, and we should note this.

Many children become very confused when faced with symbolism that looks different, but actually means the same for example, $8 \div 4$ and $4\overline{)8}$ both ask you to divide 4 into 8, but the symbols are reversed in order. Many children just divide the smaller number into the larger on all occasions, to get over this apparent contradiction in meaning.

Kieran (1979) investigated arithmetical methods used by a small number of American 12 to 14 year olds. All the children evaluated the expressions given by working, in prose convention, from left to right, completely disregarding the symbolic conventions for 'order of operations' that they had been taught in their mathematics lessons. For example, 8 + 3 x 2 was evaluated as 22. This question was followed up with a request from the interviewer for the children to insert brackets at an appropriate place and two of the children placed them around the first two numbers (5 + 2) x 3, to further reinforce their belief that the left to right order was the correct way to work out the answer.

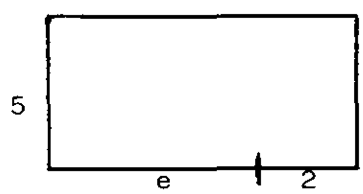
In the APU Secondary survey (1981), when asked the question 'Which of the following is NOT equal to any of the other three

A	$a - b + c$
B	$(a - b) + c$
C	$a - (b + c)$
D	$a + (c - b)$

There was a facility level amongst 15 to 16 year olds of only 30%, and interestingly 47% of pupils chose the expression where the order of the letters was different, thereby ignoring the brackets altogether.

Many children will derive their own conventions to avoid the use of brackets altogether, as they find the use of brackets to be an unnecessary complication. Booth (1982) asked the question:

Which of the following can you write for the area of this rectangle? Tick everyone you think is correct.



$5 \times e + 2$
$5 \times (e + 2)$
$10e$
$5 \times e2$
$5\ (e + 2)$
$e + 2 \times 5$
None correct.

This item was given to 991 pupils aged from 13 to 16, from the full ability range. The results for this item are given in the table below.

	YEAR GROUP			
	13	14	15	16
Correct answer	8%	10%	20%	34%
5 x e2 included	37%	31%	19%	16%
Brackets and non-brackets equivalent	32%	37%	46%	38%
Brackets excluded	25%	27%	16%	15%

From this table it can be seen that a significant number of children in each year group considered expressions with and without brackets to be equivalent, indicating that they have regarded brackets to be irrelevant. A high percentage also excluded all answers with brackets from their answers. Children from the average and above average ability groups were further questioned and it appeared that although they were familiar with the bracket notation they considered the use of brackets to be largely optional, because of three main misconceptions, which were:

(i) Operations are performed from left to right, in the order written.

(ii) The context (if one is given) determines the order in which the operations are to be performed.

(iii) The same answer will be obtained no matter what order is used.

As teachers, we obviously need to be aware of these and must amend our teaching methods to show how wrong they are.

Austin and Howson (1979) made the point that "when one moves on to algebra, then complex situations arise. In other words, that apart from the normal level of abstraction involved with mathematical symbolism, algebra takes this abstraction a stage further in difficulty. The structuring of brackets is but the first of several complications not present in ordinary language". They further pointed out that the algebraic language "can be handled autonomously, independently of the meaning of content" and they quoted Thom (1973) who asserted that "algebra is rich in syntax but weak in meaning". The strength of algebra, they feel, is that it "extends the number of important operations we can perform without thinking about them".

Many children also experience great difficulty in understanding number symbolisation and this obviously leads to difficulty with reading numbers.

The difficulty children find in translating numbers to words and vice-versa may occur due to the different rules, already noted, for the verbalisation of numbers and their symbolisation. The number 45321, for example, is read as 45 thousand, three hundred and 21, and read from left to right, whereas symbolically it is worked out from right to left as 1 unit, 2 tens, 3 hundreds, 5 thousands and 4 ten thousands. Brown (1981), as a part of the CSMS Project, found that children are generally able to recognise on sight numbers under a thousand. With numbers above that size at least 20% do not recognise them and have no systematic method of pronouncing the numbers. However, the proportion who cannot apply the symbolic, as opposed to the verbal, system of place value to writing and evaluating numbers seems likely to be considerably greater than this, and it probably reaches 50% or more of the secondary population. Brown also found that many children become confused with decimal fractions. Many do not realise the significance of the decimal point (40% at 11 years old to 25% at 15 years old). Typical mistakes are to consider 0.63 to be bigger than 0.7, as 63 is bigger than 7, and 3.02 to be bigger than 3.3 as 302 is bigger than 33.

When children first meet mathematical symbolism they are often working in the concrete mode, using say blocks or counters, and are dealing with the operation of addition. With addition the order of the symbols used may not seem to matter, for if four counters and five counters are put together to make nine counters in all, the position of the counter may not be important and the child writes down $4 + 5 = 9$ or $5 + 4 = 9$. When they move on to subtraction and say 5 blocks are taken away from 7 blocks, children may feel that $5 - 7 = 2$ is just as acceptable as $7 - 5 = 2$. As Rothery (1984) points out, "the reason that they are not allowed to write 5 - 7 when they are allowed to write 5 + 7 and 7 + 5 is beyond their experience at the time".

Brown (1981) further points out the serious consequences for the future, that the limited interpretation of arithmetical operations caused by the concrete setting children get initially can have. In response to the question 'Divide by 20 the number 16', many children considered that there was no answer, since 16 things cannot be shared between 20 people. The following table gives the percentage of children who considered there was no answer to 16 ÷ 20.

Age (Years)	12	13	14	15
% considering no answer	51	47	43	23

Further to this, many children faced with problems like $4 + \boxed{} = 7$ or $\boxed{} + 4 = 7$, put 11 into the box, to complete the number sentence. Work done by Kieran (1981) may point to the reasons why children do this. Because children often start their work on addition practically, they initially regard the = sign as an actual physical signal to carry out a combination of two sets of counters or blocks. This means that a sentence like $\boxed{} + 4 = 7$ does not make sense to them, so they rewrite it as $4 + 7 = \boxed{}$, to make 11 the correct answer to put in the box.

Austin and Howson (1979) point out the conflict between mathematical symbolism and natural language when dealing with the = sign. "In the teaching of primary school mathematics the introduction of such symbolism as 8 - 2 = 6 can cause major difficulties. Why write '2 from 8' in that particular way, and why does the (=) sign which in 6 + 2 = 8 meant 'makes' now mean 'leaves'?" The conflict they note will frequently occur. It is obvious then, that teachers need to do oral work with children to explain the meaning of equivalence attached to the '=' sign. Through discussion, children can be helped to move away from the concrete mode, towards the abstract mode.

It would seem then that many children do not fully understand the abstract symbolism used in mathematics, especially that used in algebra. The teacher has a vital role to play in ensuring that children understand the symbolism used and how symbols are combined, so that mathematical text can be correctly interpreted.

There are a number of skills that children need if they are to decode symbols easily. They need to decode symbols which straightforwardly replace a word or phrase, for example

$$12 < 14 \qquad \text{or} \qquad 7 - 3 = 4.$$

They need to be able to decode symbols which have no obvious verbal equivalent, for example,

$$(5 + 7) \times 4 = 48.$$

They need to be able to decode different spatial arrangements of symbols, for example;

$$4^5, \quad 45, \quad 5^4, \quad \frac{4}{5}, \quad \frac{5}{4}, \quad \frac{x}{5}^4, \quad 5x.$$

They also need to be able to decode symbols which do not conform to the English prose convention of reading from 'left to right', for example

$$3 - (2 + \frac{5}{8})^2.$$

We can use a number of strategies to help children become more proficient at reading the symbolism prevalent in mathematics texts.

We can use a 'matching' technique, and one suggested by Earle (1976), provides an example of this. We have three sets of cards, one set has the mathematical symbol on, the next its equivalent in words, and the third an example of mathematical expressions which might contain the symbols, and we ask the children to match these up. For example:

SET 1	SET 2	SET 3
$=$	equal	5/five
$<$	is less than	7/9
$\neq$	is not equal to	3 x 2 / 3 + 2
$\geqslant$	is bigger than or equal to	5 / 3
X	multiplied by	4?3 = 12
$\leqslant$	is less than or equal to	14/16

We could also ask children, during oral work, to translate symbols into words and words into symbols, and once again the teacher's role is extremely important here. Through groupwork and discussion in the classroom, children's understanding of the intricacies of mathematical symbolism can be improved.

CHAPTER 6: THE READABILITY OF MATHEMATICS TEXT

In chapter four we noted the difficulty children have with individual words when reading mathematical text. This is not the only concern, however, when deciding how difficult children will find a particular piece of text. We are interested in 'readability' - some passages are easier for pupils to read and understand than others.

We have already discussed how the difficulty of a particular word may be softened by its context. The surrounding sentence, or whole passage, may give valuable clues to the meaning of a word or words. A number of 'readability' formulae were developed in Britain and in other areas of the world, following on from a definition of readability given by Dale and Chall (1948), who said that "In the broadest sense, readability is the sum total of all the elements within a given piece of printed material that affects the success which a group of readers have with it. The success is the extent to which they understand it, read it at optimum speed and find it interesting".

The readability formulae produced, typically involved taking a number of sample pages from a text and counting the average word length and the average sentence length and this information was then used to estimate the age of the children for whom the text was suitable. This is all very well for extensive passages of prose, but mathematics text books are far more complicated because of the many special features included, such as symbols, formulae, diagrams, tables and graphs. In the light of this, readability formulae may seem to be rather unsuitable at gauging the age group for which a particular mathematical text is suitable.

Kane (1970) pointed out that he felt it was impossible to apply standard readability formulae to mathematics texts. Kane (1967) also introduced the terms Ordinary English (OE) and Mathematical English (ME) to stress the different nature of written mathematics. He said that "ME is a hybrid language. It is composed of OE co-mingled with various brands of highly stylised formal symbol systems. The mix of these two kinds of language varies greatly from elementary school texts to books written for graduate students". Kane also felt that ME and OE were so dissimilar that they require different skills and knowledge on the part of readers to achieve appropriate levels of reading comprehension. Kane also gave detailed reasons why readability formulae for Ordinary English Prose were inappropriate for use with mathematics texts. He felt that ME is essentially different from OE for four main reasons:

(i) letter, word and syntactical redundancies differ
(ii) in contrast to OE, in ME the names of mathematical objects usually have a single denotation.
(iii) adjectives are far more important in OE than in ME
and(iv) the grammar and syntax of ME are less flexible than in OE.

Unlike texts written for most other subjects, mathematical texts differ in that even on a single page there exist a variety of types of writing and a variety of purposes of writing. Shuard and Rothery (1984) gave a broad classification of the purposes of current mathematics textbooks, although they themselves admit that they have ignored some minor items.

They summarise the mathematical goals for which most texts aim, as the "acquisition of

concepts
principles
skills
and problem solving strategies."

Within these goals they felt that a particular passage may be intended to:

"1: teach concepts, principles, skills and problem solving strategies;

2: give practice in the use of concepts, principles, skills and problem solving strategies;

3: provide revision of 1 and 2 above;

4: test the acquisition of concepts, principles, skills and problem solving strategies."

In addition to these, they felt that texts should also attempt to:

5: develop mathematical language, to broaden children's mathematical vocabulary and their skill in the presentation of mathematics in a written form."

They felt that by providing these lists, authors and teachers would find them useful in helping them define the intended purpose of a particular passage of text.

Rothery and Shuard also give a classification of categories of writing found in mathematical text books, because "mathematical authors use several different types of text in order to carry out their intended purpose, and each type has its own characteristics." They are as follows:

EXPOSITION of concepts and methods, including explanations of vocabulary, notation and rules. Summaries are included here.

INSTRUCTIONS to the reader to write, or draw, or do.

EXAMPLES AND EXERCISES for the reader to work on; usually they are routine problems that have to be solved.

PERIPHERAL WRITING: introductory remarks, writing about the exposition, giving clues, etc.

SIGNALS: headings, letters, boxes and logos, for example.

Shuard and Rothery felt that these five categories provide us with a crude system that we can use to analyse a particular piece of text. A reader will respond in a different way when faced with a particular category of writing.

The exposition does not have to be acted on immediately by the reader; it is often for later use. It is to be read, and, hopefully, retained. It may be felt that the reading of mathematical exposition is a passive activity, like the reading of a piece of English prose. When reading a piece of mathematical exposition, however, children often need to work out the steps in an argument for themselves, and it is therefore essential for them to use pencil and paper as the reading continues. Exposition can, therefore, be presented as exercises to force children into actively involving themselves in the reading.

Instructions ask the reader to carry out any tasks that are described. Interestingly, because authors have not really thought about the instructions they have given, some instructions <u>cannot</u> be carried out, although they look like real instructions. Pupils have to learn quickly to recognise which instructions they should take literally, and those that cannot feasibly be carried out. An example is: "Draw a rod 28cm long and cut it into:

(a) 4 equal pieces
(b) 7 equal pieces
(c) 2 equal pieces

Record the length of the pieces in each case."

In no way is the author asking children to actually physically cut anything up.

Examples and exercises involve learners in the solution of problems: learners must be able to carry out a number of steps, starting with finding out what it is the problem is asking them to do, then finding out how to do it. The final stage is actually finding a solution and then, hopefully, checking its validity.

Peripheral writing is not intended to be read in such an intense way as an exposition. It is designed to help the reader, but does not contain information of vital importance. It can be scanned and its contents noted for immediate use, but not necessarily remembered. Examples of peripheral writing are introductory passages which provide links with ideas already met, or to give advance warning of what is to come in the particular section, so that children can fit what follows into some form of framework.

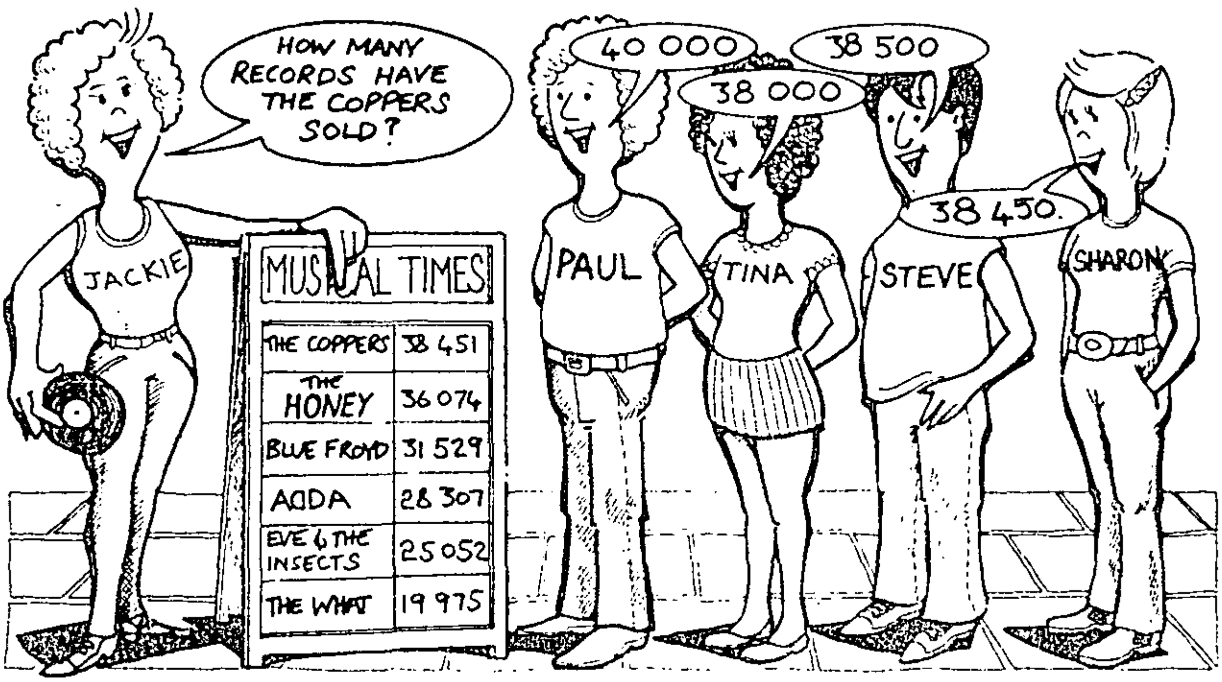
Signals are used in such a way as to help guide readers through the text, or to identify, highlight or make clear different passages. They are an extremely important part of a mathematics text, because mathematical writing is often complex and page layouts not straightforward. Bold type, italics or underlining can be used to provide signals.

Children are also expected to be able to distinguish between two particular types of question. Some questions are simply set as exercises, and the expected response is for children to write down an answer. For example, "What is the sum of 3 and 14?" Other questions can be used as signals to give an idea of the direction the text is moving in. These are rhetorical questions, and an example is "Can you think of the names of some two-dimensional shapes?" The author does not expect a long list of names of polygons.

Although the use of these categories to analyse a particular piece of text is rather crude, it does have the advantage of concentrating attention on the purpose of the separate parts of the text, and may well lead authors to think of ways of improving their presentation and hence the readability of the text.

The following example is of two separate pages from a CSE text book, that I have analysed using Shuard and Rothery's classification system. It is interesting to note that although the sentences used could be read by most 14/15-year old children, many would not understand the pages without teacher guidance, mainly because of the way in which the work is presented.

Significant figures



Each of Jackie's friends 'rounded off' the number.
But their answers were all the right size.

38 451

$\boxed{4}0\,000$

Paul looked at the '2nd figure'.
He corrected 38 451 to **1 significant figure**.

38 451

$\boxed{38}000$

Tina looked at the '3rd figure'.
She corrected 38 451 to **2 significant figures**.

Memo

Rounding off

If the number you look at is:
less than 5 – forget it
5 or more – add 1 to the figure in front of it

A

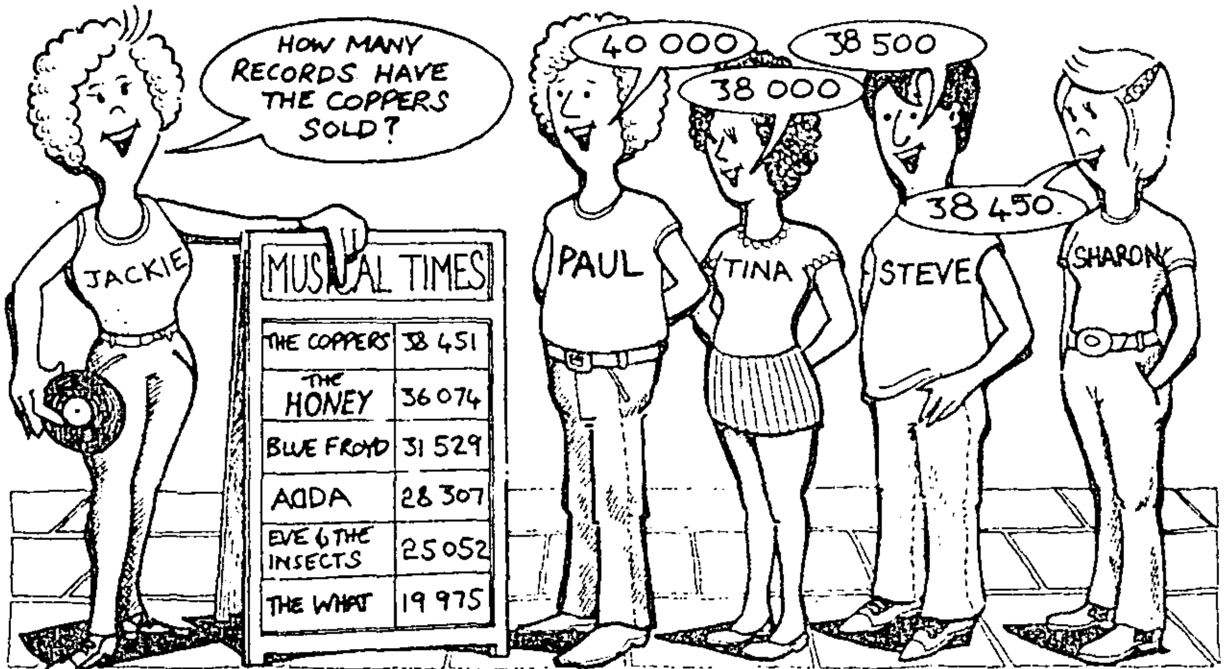
1. How many significant figures did Steve and Sharon correct the number to?
   Which figures did they look at?

s.f. means significant figures

2. Correct the other record sales figures to:
   (a) 1 s.f. (b) 2 s.f. (c) 3 s.f. (d) 4 s.f.

SIGNAL

Significant figures



Each of Jackie's friends 'rounded off' the number. PERIPHERAL WRITING
But their answers were all the right size.

EXPOSITION

38 451 WORKED EXAMPLE — Paul looked at the '2nd figure'. He corrected 38 451 to **1 significant figure**.

$\boxed{4}$0 000 SIGNAL

38 451 WORKED EXAMPLE — Tina looked at the '3rd figure'. She corrected 38 451 to **2 significant figures**.

$\boxed{38}$000 SIGNAL

Memo SIGNAL

Rounding off

If the number you look at is:
less than 5 – forget it
5 or more – add 1 to the figure in front of it

EXPOSITION

A SIGNAL

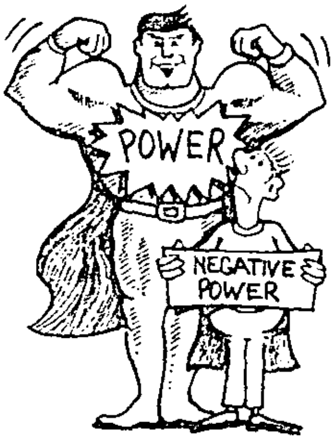
1. How many significant figures did Steve and Sharon correct the number to? Which figures did they look at?

2. Correct the other record sales figures to:
   (a) 1 s.f. (b) 2 s.f. (c) 3 s.f. (d) 4 s.f.

EXERCISES

PERIPHERAL WRITING

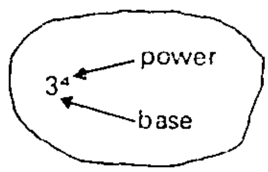
s.f. means significant figures



Negative powers

You should have done: More directed numbers (pages 17–19).

In arithmetic
$3 \times 3 \times 3 \times 3$
is 3^4
We say '3 to the power 4'.



A Write these out in full:
1. 3^5 2. 2^3 3. 7^6 4. 8^2 5. 4^4

Memo

> To divide numbers with the same base we subtract the powers:
> $4^5 \div 4^3 = 4^{5-3} = 4^2$

B Use this short method to do these:
1. $2^6 \div 2^2$ 2. $7^9 \div 7^6$ 3. $5^3 \div 5^3$ 4. $10^2 \div 10^7$

Your answer to **B** 4 should have been:

$2 - 7 = {}^-5$

$$10^2 \div 10^7 = 10^{2-7} = 10^{-5} \longleftarrow \text{negative power}$$

But what is 10^{-5}? We can work it out.

rewriting division

$$10^2 \div 10^7 = \frac{10^2}{10^7}$$

rewriting numbers and cancelling

$$= \frac{\cancel{10}^1 \times \cancel{10}^1}{\cancel{10}_1 \times \cancel{10}_1 \times 10 \times 10 \times 10 \times 10 \times 10}$$

$$= \frac{1}{10 \times 10 \times 10 \times 10 \times 10}$$

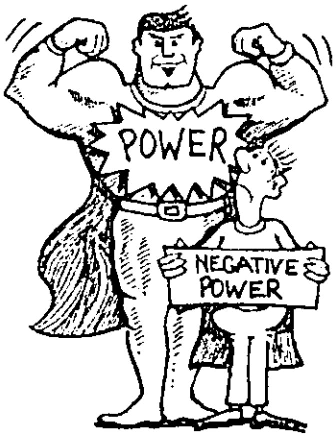
using powers

$$= \frac{1}{10^5}$$

Since both answers must be the same:

$$10^{-5} = \frac{1}{10^5}$$

SIGNAL



Negative powers

PERIPHERAL WRITING

You should have done: More directed numbers (pages 17–19).

EXPOSITION

In arithmetic
$3 \times 3 \times 3 \times 3$
is 3^4
We say '3 to the power 4'.

3^4 ← power
← base

SIGNAL

SIGNAL → **A** Write these out in full: EXERCISES
1. 3^5 2. 2^3 3. 7^6 4. 8^2 5. 4^4

SIGNAL

SIGNAL

To divide numbers with the same base we subtract the powers:
$4^5 \div 4^3 = 4^{5-3} = 4^2$

EXPOSITION

B Use this short method to do these: EXERCISES
1. $2^6 \div 2^2$ 2. $7^9 \div 7^6$ 3. $5^3 \div 5^3$ 4. $10^2 \div 10^7$

PERIPHERAL WRITING
$2 - 7 = {}^-5$

Your answer to **B** 4 should have been:
$10^2 \div 10^7 = 10^{2-7} = 10^{-5}$ ← negative power

WORKED EXAMPLE

RHETORICAL QUESTION → But what is 10^{-5}? We can work it out.

rewriting division

rewriting numbers and cancelling

PERIPHERAL WRITING

using powers

$$10^2 \div 10^7 = \frac{10^2}{10^7}$$
$$= \frac{\cancel{10}^1 \times \cancel{10}^1}{\cancel{10}_1 \times \cancel{10}_1 \times 10 \times 10 \times 10 \times 10 \times 10}$$
$$= \frac{1}{10 \times 10 \times 10 \times 10 \times 10}$$
$$= \frac{1}{10^5}$$

WORKED EXAMPLE

Since both answers must be the same:
$$10^{-5} = \frac{1}{10^5}$$

EXPOSITION

As already discussed, word length is considered as a factor in some readability formulae, where long words are considered to be difficult and short words regarded as easy. This can be explained by the fact that short words will most probably be more familiar to children and, also, longer words are more difficult to say and are therefore less easily recognised. Word length is a crude method of assessing the difficulty of vocabulary, for there are lots of examples of unusual short words, and there exist some familiar long ones.

It has been suggested, therefore, that a better measure of the difficulty of words is their familiarity, with reference to the vocabulary children are used to using in the classroom. Lists of familiar words have been made, and some readability formulae make use of these, but these formulae do have their limitations. One limitation is the time factor: as time goes on, the vocabulary in use changes, and so the lists of familiar words have to be changed. Also lists of familiar words are influenced by the nature of a particular nation's vocabulary and word use. As most lists are American in origin, this is problematic when using them with British texts, as the two language forms are far from identical.

There are, therefore, a number of ways of assessing the difficulty of the words used in a text. Four of these are suggested by Shuard and Rothery:

i) whether the word is familiar to the reader;

ii) whether the reader has met the ME word before, and whether it is confused with any other OE word;

iii) whether the word is on its own, or in context;

and iv) whether the text helps the reader to understand the word, i.e., does the book contain a glossary, for example.

To fully grasp the meaning of a particular piece of text, it is not enough to just understand individual words; the difficulty of the syntax used must also be taken into account. As well as taking word length into account, readability formulae often take sentence length into account. In other words, the longer the sentence the harder it is to read. Once again we find that this measure may not be very reliable. It has been found that sentences of the same length vary greatly in difficulty, and that some long ones are easier to understand than some shorter ones. Children would find sentences like the ones they themselves would write much easier to read and understand than those often found in text books. Familiarisation is an important aspect here.

A useful guide for assessing the complexity of a particular piece of text has been provided by Botel, Hawkins and Granowsky (1973). They listed types of sentence which cause syntactical difficulties, and gave a numerical score for each one, to give an idea of its level of complexity. The scores for each sentence were summed to produce a score for the overall complexity of a particular passage. Although the use of this sort of measure presents enormous practical problems, it can provide authors and teachers with an assessment of how complex a particular text is.

We must remember that both the vocabulary used and the syntax are highly significant factors in deciding how difficult children will find a particular text. Linville (1976) found this to be the case when investigating children doing word problems. Sentences containing complex syntax caused children as many reading difficulties as those containing easy syntax and difficult words. Linville developed four arithmetic word-problem tests, each consisting of the same problems but varying in difficulty of syntax and vocabulary. These were:

i) easy syntax and easy vocabulary;

ii) easy syntax, difficult vocabulary;

iii) difficult syntax, easy vocabulary;

iv) difficult synatax, difficult vocabulary.

These four tests were given to 408 fourth grade pupils in twelve American schools. The conclusions drawn from the results were that both syntactic structure and vocabulary level, with the latter more crucial, are important variables in solving verbal arithmetic problems. Another finding of this study was that pupils of higher general ability and/or higher reading ability gained significantly higher scores on the arithmetic problems than pupils of lower ability.

In other words, we cannot accurately measure how complex a passage of mathematical text is by looking at the length of words or sentences in that text. We do, however, have to note the importance of syntax and vocabulary when making an assessment of the complexity, and hence the readability, of a text.

We have used the idea of readability so far to mean 'how easy it is for children to understand what they are reading.' Unfortunately, most readability formulae have been produced to assess the difficulty of ordinary English prose. We must therefore remember that most mathematics texts contain little continuous prose. Although looking at particular readability formulae is outside the scope of this dissertation, looking at the criticisms of them for use on mathematics texts is an area worthy of investigation.

Readability formulae necessarily only use variables that can be easily quantified. As we have just seen, the syntax of a sentence greatly influences its readability, but the only measure used in any of the formulae reflecting syntax is that of sentence length. As we have already seen, this is not a reliable indicator. The formulae also fail to take into account the legibility of the text. Factors such as line length, page layout and typeface(s) used can have a great influence on the readability of a text. As mathematics texts have graphs, tables, diagrams, etc., this idea of layout is of great importance to the overall level of readability.

Different readability formulae may also give very different estimates of the readability of a particular passage and, more worryingly, the results do not differ consistently, and of course different parts or chapters of a book may give very different results, changing, say, with the content covered. On top of all this, readability formulae also ignore 'reader enjoyment'. Many children manage to read, understand and enjoy stories in books with readability well above their supposed capabilities. We need to be aware that few mathematics texts stimulate children enough for them to <u>want</u> to read them. Taylor (1953) pointed out that readability formulae fail to reflect the effect the book has upon the children reading it. Children's initial reaction to a book will almost certainly not concern its mathematical features; it will concern things like attractiveness of design, quality of illustrations, the clarity of the type face used, and so on. This initial reaction will very much affect the way in which the reader will feel about dealing with the rest of the book . We must remember, also, the defensiveness of those children who feel that they always fail with mathematics. Low ability 14- to 16-year olds can object to nicely presented, fragmented materials because they too closely resemble books for young children.

We can see, then, that readability tests designed for use on OE texts have little application to texts using ME. Readability formulae developed for use with school mathematics text books are almost unknown in Britain, but two such formulae have been developed in the USA. Hater (1969) made a distinction between 'word tokens' and 'mathematics tokens.' A mathematics token was taken to be a basic piece of mathematical symbolism. For example, 4 x 3 has three mathematical tokens: 4, x and 3; x^3 has two: x and ^{3}. Word tokens are ordinary words. Hater modified the Cloze procedure, where words of a text are replaced by blanks and readers are asked to supply the missing words, so that it could be used with mathematical text. The Cloze procedure therefore measures the reader's reaction to the text. The easier readers find the passage to understand, the more missing words they will be able to fill in, and vice-versa.

The procedure Hater adopted was to replace every fifth token with a blank. So that readers could distinguish between different types of token, mathematics tokens were replaced by short lines and word tokens by long lines. Hater found that this procedure measured the difficulty children have with passages of mathematical English in a way which correlated well with measurements obtained using comprehension tests.

Kane, Byre and Hater (1974) correlated a large number of the features of mathematics text books with the reading ease of these texts, and developed two readability formulae. To use these formulae lengthy lists of mathematical words and symbols are necessary, and these are given by Kane et al (1974). The formula regarded as the most useful is given here:

Predicted Readability = -0.15A + 0.10B - 0.42C - 0.17D + 35.52
where A, B, C and D are variables.

To use this formula to assess the readability of a book, samples of a text are taken containing 400 tokens, and their mean readability score found. At least ten of these individual scores are needed to assess the overall readability of the book.

The values of the variables are found as follows:

A - the number of words not on the Dale list of common non-mathematical words and also not on the list 'Mathematics Words known to 80% of children'.

B - the number of changes of word to mathematical token and vice-versa, in the passage chosen.

C - the number of terms not on the list 'Mathematical words known to 80% of children' and the number of different symbols not on the list 'Symbols known to 90% of children'.

D - the number of question marks in the passage chosen.

This formula does not give a 'reading age', but it does allow us to rank text books in order of readability difficulty' - the larger the score, the easier the text is to read.

Because the value of the variables is obtained from lists prepared from American textbooks, we would not be able to use this test without changing the lists to take account of differences of language and mathematical content found in Britain. The test is also, it must be said, far too tedious for the classroom teacher to administer. However, we must not lose sight of the fact that a readability test has been developed that takes account of some of the features peculiar to mathematics texts.

It must be stressed at the end of this chapter that readability formulae all have their limitations, and can only give some idea as to the degree of difficulty children will have understanding a particular text. The results from readability tests should be seen only as one of the criteria used to judge the difficulty of a text, along with the personal judgement and experience of teachers and other specialists involved in using the text. However, they do have a place, and those involved in mathematics education in Britain should take more account of their usefulness and help initiate further research into their development. It is only by establishing criteria for evaluating mathematical writing that we can begin to make some form of quantitative judgement about the level of difficulty of individual texts.

CHAPTER 7: IMPROVING THE READABILITY OF MATHEMATICS TEXT

In the preceding chapters we have identified and discussed some aspects of the language difficulties children experience when reading mathematical text. There is no doubt that there is room for substantial improvment in the way mathematical text is presented and hence, in its effectiveness. In this chapter we will look at ways mathematical authors and teachers can improve the presentation and readability of their work.

Before identifying areas where improvements can be made, it is worth remembering that the reading of mathematical text cannot be divorced from the teaching methods used in the classroom. Teachers, by interacting with the children in the right way, and at the right time, can bring out the best in any piece of mathematical writing. We must not lose sight of this - the teacher, many would feel, must be viewed as the most important resource in the classroom.

Austin & Howson (1979) point out that children's initial reactions to a text are extrememly important, as this will affect the way they deal with it in the future. These initial reactions will most probably have nothing to do with its mathematical features but with the layout and attractiveness of individual pages. This must be noted, but it must also be remembered that improving layout by, say, leaving more spaces or adding extra colour can add significantly to the price. It is no good having an extremely readable text that schools cannot afford to buy! A compromise must be reached between readability and cost.

How then do we make texts more readable? The following suggestions arise from work already covered in this dissertation.

When new technical words are introduced, we must be aware that many children will not only have initial understanding difficulties, but they will also forget definitions they have actually understood and learnt. Clear, simply-worded definitions of all new words and symbols must be given to aid initial understanding, and a list of all technical words and symbols used in that text, along with their definitions, must be compiled in a glossary. Children can therefore more easily look up the meanings of terms they forget.

Any new words or symbols introduced should be used as often as possible in the text and in as many different contexts as possible, to reinforce meaning. At the end of each chapter, reviews or tests on these meanings should be included. Peripheral remarks, redefining important words at regular intervals, could also be used. It must be remembered that many of the technical terms used in mathematics are not met elsewhere, and reinforcement can therefore only occur in the text or in the lesson.

Having indicated that simple definitions are important, it must be realised that problems do exist with this. It is possible, when actually trying to simplify language, to become long-winded, boring and ambiguous. We can actually end up with prose having a higher reading age than the original, more precise, definition. Care is needed here.

When introducing new technical words, we must ask ourselves whether that word is actually needed by the children. If a more familiar one can be used, with no loss of meaning, then it should be.

Too many new words or symbols should also not be introduced in too short a space of time. Children cannot cope with too much new vocabulary all at once. As an example of this, a page from an 'O' level text book is reproduced below. The amount of new vocabulary introduced in one page is astonishing, and would be extremely difficult for the learner to cope with. No concrete examples are given in the text, and this further compounds the problem.

Binary operations

Many sets which are met in mathematics have rules laid down in order to combine the elements in a set. Addition, subtraction, multiplication and division are four rules used for combining elements from sets of numbers like the positive integers, the rational numbers and the real numbers, except, of course, division by zero is not allowed. In sets we have met the operations of union and intersection. Such rules of combination are called **binary operations.** Let us suppose that we have a set A and a rule for combining ordered pairs of the elements of the set A. If the rule of combination produces for each pair an element which also belongs to A, then the rule is called a **binary operation on** the set A.

The associative property

Definition A binary operation $*$ on the set A is associative if and only if, for all elements $a, b, c, \ldots \in A$.

$$a * (b * c) = (a * b) * c.$$

The commutative property

Definition A binary operation $*$ on the set A is commutative if and only if, for all elements $a, b, \ldots \in A$.

$$a * b = b * a.$$

The distributive property

The binary operation ∇ is distributive over the binary operation $*$ if and only if, for all elements $a, b, c, \ldots \in A$,

$$a \nabla (b * c) = (a \nabla b) * (a \nabla c).$$

Under addition any real number remains unchanged when zero is added and under multiplication any real number remains unchanged when multiplied by 1.

Identity element

Definition An element e of a set A is called an identity element for the binary operation $*$ if and only if

$$a * e = e * a = a,$$

for every element $a \ldots \in A$.

Inverse elements

If $a, a' \in A$, whose identity element under the binary operation $*$ is e, and $a * a' = a' * a = e$, then a' is called the inverse of a under $*$ for the set A.

Symbols, too, should only be used when absolutely necessary. When introducing a new symbol we have to be sure that children understand the concept behind it, using remedial work if necessary. The symbol must not be used in too wide a sense too soon - initial practice with simple, straightforward examples, is essential.

Careful attention has to be paid to words that have different Mathematical English and Ordinary English meanings. The specialised mathematical meaning of a commonly used OE word, like 'similar' say, needs extra special emphasis, and frequent revision.

The text produced must be as succinct as possible. All OE words that are not necessary for complete understanding should be omitted. It is a good practice to put new pieces of information on new lines and to make the question being asked absolutely clear.

For example, the question " A man is told that he has won £3000 on the football pools and decides to split the money equally between his five children. How much money does each of his children get?"

Is better written as: A man wins £3000.
He divides it equally between his 5 children.
How much does each get?

The information given in the second draft of the question is much more clearly presented, and the question being asked is obvious.

In general, it is wise to avoid the use of rhetorical questions like ' Do you remember where we met this before? ' These can cause great confusion. Children often do not know what to answer or, indeed, whether they should give an answer at all!

When setting problems, it is as well to avoid the use of sentences containing information presented initially. For example, the question:

" for $x = 3.71$, find the value of $3x^2$ ",

is better written as,

"find the value of $3x^2$, if $x = 3.71$".

In mathematics, many different types of graphs, tables, diagrams, plans or other types of illustration are used to amplify or complement written text. Children need as much help in interpreting these as they do with words and symbols.

Shuard and Rothery (1984) identified three levels of importance of illustration in children's mathematical text:

1: Decoration

2: Related, but non-essential

and 3: Essential.

Decorative illustrations contain no exposition. They have aesthetic appeal and are used to fill in spaces in the text, to create interest or to generally set the scene.

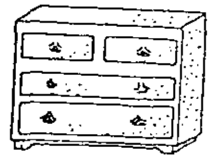
Related, but non-essential illustrations are used to emphasise the text and can be used to reinforce work that has already been met. It can help children visualise concepts, as an aid in developing language skills.

Essential illustrations cannot easily be produced in written form - they convey information that, at best, would be unwieldy and difficult to understand, written in words. It is necessary for learning, and is an essential part of the text. Without it, full understanding would not take place.

In many texts children find it extremely difficult to distinguish between the relative importance of different illustrations. They cannot decide what is essential for them to read and what they can just glance at. Children can therefore easily overlook essential illustrations and hence not make sense of the work. If it is essential for a diagram to be read for children to completely understand a piece of text, then this must be made apparent in some way, maybe by using the written part of the text, or by the use of signals. Likewise, it must be made apparent to children which of the illustrations can just be glanced at or enjoyed, without worrying too much about any meaning that may be attached to them.

If the text uses colour, then maybe different coloured boxes could be drawn around each diagram. Each colour used would relate to a different level of importance, in a consistent way. A simpler method would be to have a small number of symbols, each denoting a different level of importance, one of which would be placed on each diagram used.
When positioning diagrams on the page, the way children normally read text, from left to right and from the top of the page to the bottom, should be remembered. Diagrams should be placed on a page in such a way that they do not disturb the regular eye-movement. They must be positioned so that it is obvious which part of the text they relate to, instead of, as so often happens, being placed to make the maximum use of available space. Drawing arrow diagrams showing how the children's eyes have to move to read a particular page is a very illuminating exercise. The path that often has to be followed can be very complicated, thus indicating that the text itself can be confusing to read. The use of arrow diagrams can make authors and publishers think more about the way they present work - page layout can, there is no doubt, aid understanding. As an example, a page of text analysed using arrow diagrams, is included below. It is reproduced from Shuard and Rothery (1984). It should be noted that the text required some complicated eye movements for it to be read.

HIRE PURCHASE (HP) - 2



What do you pay extra when you buy on hire purchase?

This is how you work it out.

Deposit	= £100
HP payments 104 x £5	= £520
(2 years (2 x 52 weeks)	£620
Cash price	£560
Extra cost on HP	£ 60

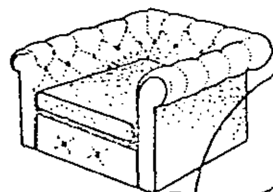
£100 deposit
£5 per week for 2 years

Now work out extra cost on hire purchase as a percentage of the cash price.

$\frac{60}{560} \times 100 = 10.7\%$

Work out the cost on hire purchase of buying these goods.

Work out the extra cost as a percentage of the cash price, to one decimal point.

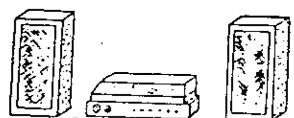


£375

HP terms
£100 down payment
£3 a week over 2 years

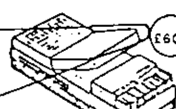
£96

HP terms
£28 down payment
£1.50 weekly for a year.



£250

HP terms
£56 down payment
£2.20 weekly over 2 years

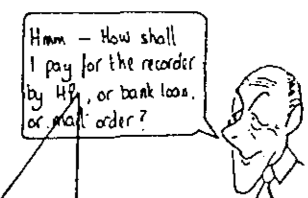


£60

HP terms
Deposit 15%
Then pay 60p a week over 2 years.

Bank loan £60 for 1 year at 11½% interest per year.

Mail order catalogue £3.50 a week over 20 weeks



Work out the cost of the three ways of buying.

Which is the cheapest?

Having commented on the importance of page layout, it is interesting to note that Smith and Watkins (1972), reporting on research carried out by the Typography Unit at Reading University, said that changing the relative positions of text and illustrations made very little difference to the comprehension of a passage, but that the presence of the illustrations themselves caused a significant increase in learning. This is all very well, but they miss the point that badly laid out, and difficult to follow pages, can quickly turn children off attempting any further work. The motivational quality of text is an important consideration here.

Apart from the relative positioning of text and illustrations, the illustrations themselves need some attention. They should be as uncluttered and as easy to read as possible. Graphs should have linear scales that are easily used by children. For example, if the scale of a graph drawn on metric graph paper goes up in jumps of 15, it would not be easy for many children to accurately place a number like 7 or 27.

In any one text, the presentation of any illustrative material must be consistent. For example, if shading or colour is used then it should, where possible, represent similar ideas throughout the text, otherwise children may misinterpret its meaning. Using shading or colour consistently also means that children will learn to recognise what it represents in that text.

When discussing the readability of mathematics text, Austin and Howson (1979) said that "It's not only the choice of words, sentence lengths, etc., that will be significant factors, but also other features such as content, style, format, organisation, illustrations, humour..." Many would agree with this, and therefore argue that how pleasing a page is to look at is an important factor in determining how motivated children will be when reading the text. Many mathematics texts have cluttered pages and, although the content covered may be straightforward, children may well immediately, just by glancing at the page, be put off.

The use of colour and different type faces should also be considered. Text in lower case is more legible and therefore easier to read than text in upper case. Also, text in bolder type is more easily read than that in italics. Appropriate use of colour can help children read the page more easily, and also make the text more attractive. It has been found that the positive effect of colour is more pronounced with younger children or children of lower ability. Smith and Watkins (1972) gave evidence to suggest that the addition of colour to illustrations sometimes led to a better understanding of the mathematics involved.

Many feel that for a text to be successful it must involve learners in an interactive way. Learning should not be a passive activity, and the more involved learners are, the more they will get from the text. Open University mathematics course units are an example of this. The text needs to draw the learner in, to encourage the use of pen and paper to work through problems, and to encourage the learner to ask questions without feeling under any sort of threat. We must note, though, that texts which set out to involve the learner in discovery must contain in-built checks, so that the learners or their teachers can discover whether understanding of the work covered has actually taken place.

There are many suggestions made in this chapter about the ways in which mathematics text can be changed so that readability and understanding are improved. It would obviously be very difficult for authors and their publishers to follow all the guidelines laid down, but they do provide a useful focus when writing material to be used in the mathematics classroom. It must be remembered, however, that no matter how careful we are when writing mathematical text, children will always have problems with understanding the written word. This was pointed out in paragraph 312 of 'Mathematics Counts' (1982), which said, "Even although children may, without difficulty, be able to read what is written in a mathematics textbook, they may well find great difficulty in learning an unfamiliar piece of mathematics from the written word. This is likely to be the case, however careful has been the choice of the language which is used."

It is up to the teacher to help children deal with the text as much as possible. Teachers need to regularly discuss text with children, to reinforce new words and symbols and to revise those previously met. They need to directly question children to ensure that they are getting the correct meaning from the text, and teachers need to identify areas of weakness in the text and reinforce them. Discussion of formats used and strategies to get 'unstuck' is also needed. We must remember that it is a responsibility of the teacher of mathematics, as it is of any teacher, to improve the reading ability of the learner. As paragraph 311 of 'Mathematics Counts' puts it, "Reading skills in mathematics should be built up alongside other reading skills, so that children can understand the explanations and instructions which occur in the mathematics books they use. If the skills of reading mathematics are not developed, many children will evolve their own strategies for avoiding such reading.

The power of the teacher in aiding children to understand, interpret and use the mathematical text they read must not be under-estimated or under-used.

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