Games of strategy in the teaching of mathematics

PUBLISHER

Loughborough University of Technology

REPOSITORY RECORD

Maddern, Stephen. 2021. “Games of Strategy in the Teaching of Mathematics”. Loughborough University. https://doi.org/10.26174/thesis.lboro.14805987.v1.

GAMES OF STRATEGY

IN THE TEACHING

OF

MATHEMATICS.

by

STEPHEN MADDERN B.Sc.

A Master's Dissertation submitted in partial fulfilment for the award of the degree of M.Sc. in Mathematical Education of the Loughborough University of Technology, January 1986.

Supervisor: R.P.KNOTT, B.Sc., M.Sc., Ph.D.

DEDICATION

To my son Peter, who received very little of his Daddy's attention during the first fifteen months of his life.

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INTRODUCTION

Throughout my career as a mathematics teacher, I have always believed that pupils must be allowed to consider the problems in mathematics. It was probably my PGCE tutor that laid the foundation to this belief, and 'consider the problem' has been a favourite catch phrase of mine ever since, only being beaten by my absolute favourite term in mathematical education, 'teacher lust'. Teacher lust, which is mentioned many times in the following work, is the desire by a teacher to tell all, a dreadful 'illness'! Allowing pupils to consider problems will involve them thinking, talking and writing about mathematics at their own level and within their own time. These ideas may seem rather obvious, however, if we look at the traditional approach to mathematics teaching, then we observe a situation which certainly has not allowed these events to occur. Pupils have often been given problem and solution hand in hand and therefore have not been considering the problems themselves. They have often been taught in such a narrow way, rote learning of routines, methods and techniques, that they have failed to understand their mathematics and hence have not developed the ability to apply their mathematics to new situations.

Since 1982, when the Cockcroft report was published, mathematics teachers have been examining what is happening in the mathematics classroom in perhaps more detail, and certainly in greater numbers than previously. It is my belief that mathematics teaching has to base its forward progress on an open ended, enquiry based learning style if we are to overcome the traditional problems and move forward to achieve a breadth of knowledge and depth of understanding in our future pupils. We have to beware of teacher lust; we must allow children to develop in the subject more naturally and lead them to an understanding of the structure and working of mathematics. This involves a complete change of attitude and approach by mathematics teachers. We have to adopt a teaching style which integrates the six aspects of good mathematics teaching, as outlined in Cockcroft paragraph 243, together with other aspects, such as the use of the calculator and computer, in a natural way.

It is with this type of approach that this dissertation deals, and in particular it discusses the use of games within such an approach. The discussion of a modern teaching approach is supported by a teaching package of games of strategy. The development of the discussion moves from the basic, but important, question of why and therefore how we should be teaching mathematics, onto a particular look at the problem solving aspect and finally onto the use of games of strategy within this. There is also a report on the games package with a few examples of pupils' work and a discussion of how the games may be, and have been, used in the classroom. I have also included a little of the mathematical background and analysis of the games for the interested reader.

ACKNOWLEDGEMENTS

I would like to express my gratitude to all the people who have helped me with this dissertation. Particular thanks must be expressed to my supervisor, Dr.R.P.Knott, for his comments and guidance, and the mathematics staff and pupils of Haywood School, Sherwood, for their help in testing the games. I should also mention those friends and colleagues whose support and encouragement have lightened the burden of the work and kept me going in times of difficulty. Finally, my greatest thanks must go to my wife Avrina, who painstakingly and critically proof read the work, and son Peter for their sacrifice, patience and encouragement throughout the preparation period of this dissertation and indeed the full duration of the course.

NOTE OF AUTHENTICITY

I declare that this dissertation consists entirely of my own work and has been prepared for the previously mentioned degree.

Stephen Maddern

January 1986

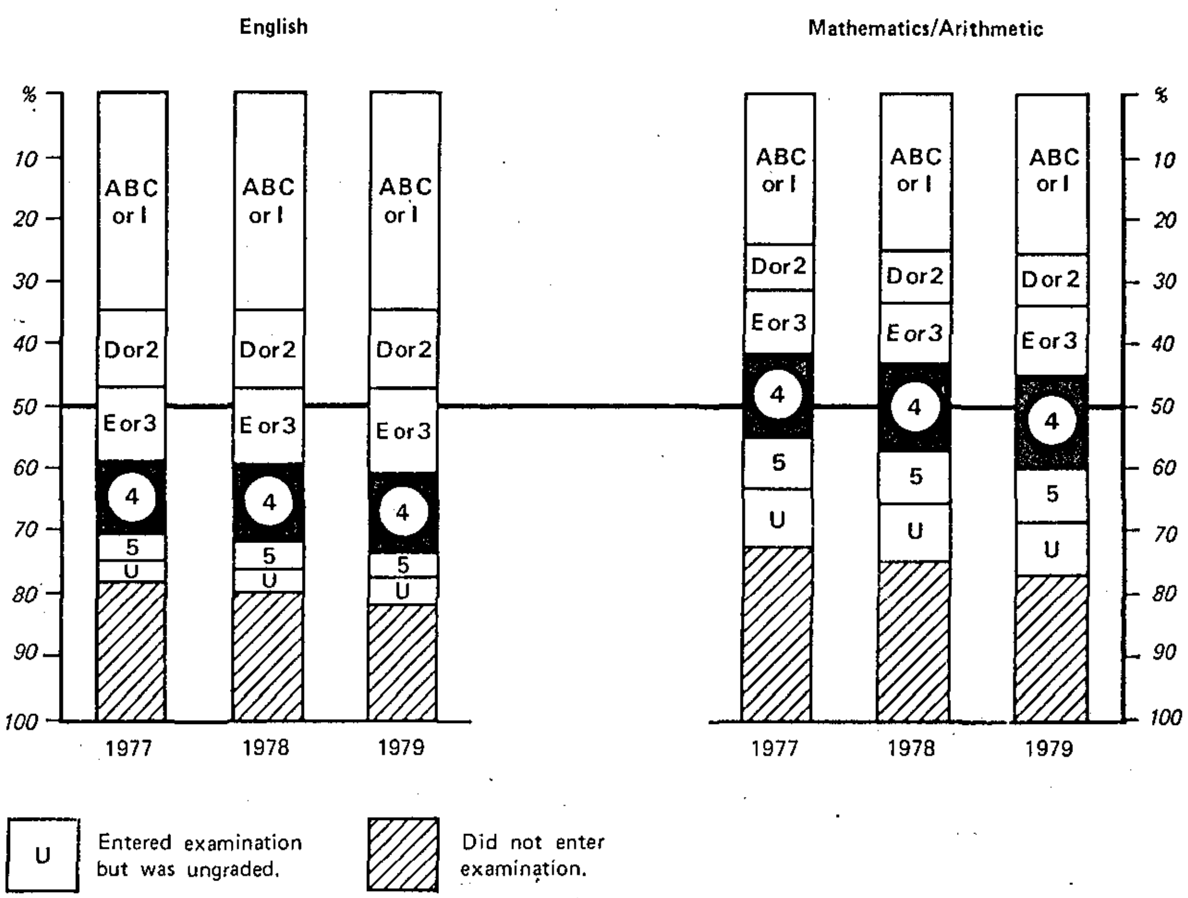
A Modern Approach to Mathematics Teaching.

A modern approach to the teaching of any subject within the curriculum must take account of a multitude of factors. However, I do feel that mathematics itself has one particular factor to take account of which perhaps does not exist in many other subjects, that of a traditional negative attitude towards it. 'Of all the subjects taught at school, maths provokes the strongest emotions of distaste, anxiety and feelings of incompetence' (Buxton 1981), so why ? We need to be able to answer this question, as mathematics teachers, if we are to try to overcome this problem.

Perhaps it is the impression of impracticality that this practical subject has given when taught in the traditional manner. The problems tackled have often been so far removed from real situations that one could not blame anyone for failing to see the relevance of them. It is only when the mathematical models become further advanced that account of reality may be considered, a stage which unfortunately the majority never experience, through either a lack of motivation or ability. It is also within the examination system that I feel this negative attitude flourishes. The situation nationally is indeed grave, as so correctly pointed out by the Cockcroft Committee in their report Mathematics Counts (Cockcroft 1982). 'It is the case that the positions of the grade boundaries in English are very different from those in mathematics and it is therefore to be expected that there will be many pupils who will achieve significantly better grades in English than they will achieve in mathematics' (Cockcroft 1982, paragraph 197). A child who is slightly above average ability, say the

45th percentile, in both English and mathematics may achieve the apparently uncomparable grades of CSE 2 and 4 respectively. Is it any wonder therefore, that many think that they are better at various subjects than they are at mathematics. I in fact, was once asked by the Head of Fifth Year in my own school why the estimated grades in my subject were generally so much lower than in all other subjects. Many teachers, including those of mathematics, are unaware of the facts with regard to these grade boundaries.

Proportions of pupils awarded O-Level grades A to E and CSE grades 1 to 5 in English and Mathematics/Arithmetic



Is it then surprising that research carried out at Bath University (Mathematics in Employment 16-18) and complementary research at the Shell Centre for Mathematical Education, at Nottingham University, showed that many pupils had a negative approach towards the subject. It has also been shown that there is a tendency among pupils of all ages to believe mathematics to be useful but not necessarily interesting or enjoyable. Whilst accepting that there is only evidence of slight correlation between attitude and achievement, it cannot be a healthy situation for the subject.

Another concern which I have is the impression of a 'binary state' of solutions that the subject has tradionally given, that of right and wrong. How many times has a mathematics teacher at a parent's evening had to listen to comments such as 'I know that they are not very good', 'I always think that you can either do it or not in maths', 'I could never do maths' and 'It's all changed since my day'. So many people have the idea that mathematics is difficult and you can either do it or not, and that most cannot. If this is the parents' attitude then what hope have we, as mathematics teachers, in trying to instil positive attitudes within their children ? So what do we need to take account of in our modern approach to mathematics teaching, if we are to avoid the traditional pitfalls and start to develop some sort of positive approach to the subject and broad understanding of it ? Why are we teaching the subject and what do we hope to achieve ? What justification have we got for our long-standing existence on the school curriculum ? In these times of ongoing curriculum change we, as mathematics teachers, should be both willing, and able, to offer such a justification.

'The aims and objectives of mathematics teaching have changed over the course of time. This has happened in response to the changing needs of society and to a deepening understanding of what can be learned and retained so as to influence a pupil's capabilities after he has left school' (School's Council 1977). It is because of these changes in need and understanding, both of which I agree have occurred, that we must adapt our approach and be continually questioning ourselves as to what we are doing and why ? The main objective of teaching must be the preparation of the individual for the real world, and the teaching of mathematics should form an integral part of this overall plan. Some may argue that this has always been the main objective and it has been successfully achieved; I would argue differently. Clearly the statement is rather ambiguous since it may be considered in a subjective manner as to what this preparation should be, and for what ? I hope that my interpretation and views on this statement will become clear as my discussion continues. Various authors would of course give different objectives and hence reasons why mathematics should be taught to all children in schools. I would say that such reasons cannot be given in a singular form and there have to be many, which are by no means disjoint. I would claim that the reasons may be broadly given as the following four:

a) Utility

b) Social

c) Cultural

d) Personal

These together form an integrated plan to support the education system as a whole, as mentioned above, and therefore each is essential and necessitates consideration. However, it is the way in which mathematics is taught which determines whether or not it contributes within these factors, a point again made by the Cockcroft Report.

Clearly the utility of the subject has always been recognised and indeed quoted as a justification for the teaching of it, and I would argue that perhaps it has been over-emphasised. I would state here that the interpretation of the term utility in mathematics may be extremely varied from one teacher to another. One has to agree that mathematics is encountered in many forms in everyday life both at home and at work, but the subject should not merely be directed at teaching specific utility topics; this would give an undesirable narrowness to the subject and would not therefore justify the existence of mathematics on the curriculum. This however, has been the case with many courses offered to less able pupils in schools in the past. Within the section of utility I would include reasons such as the need for mathematics across the curriculum, although it is often the case that people fail to apply their mathematics in other disciplines since they have not learnt the required mathematics in context. This does not necessarily mean that an increase in contextual teaching is required but does, I believe, indicate a lack of understanding of the underlying concepts and related structures, which has therefore to be corrected. It would be impossible to teach mathematics within all possible contexts since the range would be so vast and the human memory unable to cope with this inefficient method. I believe that it is an understanding of mathematics which has to be taught, so that the ideas may be applied to new problems when encountered. There are of course similar contextual problems with mathematics in everyday life and work. When this is considered on the school curriculum, then often the contextual aspect of the problem is not relevant to the pupils' experience and therefore the mathematics is rather meaningless. I would therefore suggest that such contextual mathematics be left until it is encountered naturally, probably at a later stage in life. This would of course leave the mathematics time, in schools, available to offer the pupils experiences of such a type that they may understand and tackle these areas of mathematics when necessary. This would be my understanding of utility in mathematics and I would therefore claim that it is both a valid reason to teach the subject and a factor for consideration in the approach to mathematics teaching.

An important part of the education of an individual is their own personal and social development. I have already stated the case for mathematics to support the education system as a whole, and here, in this section, perhaps mathematics has not always fulfilled its responsibility in the past. Working together in a group, taking responsibility for equipment so that it is there for the next member of the group, and discussion work are only a few examples which I would include as the social reasons to teach mathematics. Personally, I think that a lot may be done in this area with statistical topics. Such topics would give pupils the opportunities and experiences to think about society in general and to investigate, interpret and understand decision making in human affairs. The Schools Council's statistics package is typical of how mathematics can be used to support this aim and shows statistics in the true sense, rather than a mere set of mathematical techniques. These areas of education have not always got to be left to the staff of the Humanities Department, after all, mathematics teachers are members of society, aren't they ?

Education, and indeed mankind in general, has a responsibilty to pass on the culture of the race and this cannot be completely transmitted without considering man and mathematics. Mathematics has always been, and I think always will be, a language for communication. It is probably the most commonly known language in the world. Its powers of reduction and universal understanding are tremendous, it provides a means of communication which is powerful, concise and unambiguous and this in itself is a good reason to teach mathematics. However, the communication aspect perhaps has not been fully developed in the past. Pupils need to talk and write about mathematics in their own words as much as possible, and at every opportunity, if this is to be fully emphasised. Pupils ought to be aware of the significant mathematical developments of the past, the power of the language and the beauty of the occurrence of mathematics in the world. They all generally accept these things but usually without realising it. How often has a child in their spare time constructed a mathematical pattern, often based on symmetry, to occupy themselves, or measured their height in cm. to communicate their growth ? Also 'we should not overlook the pleasure that comes from an argument', (Marjoram 1974). Many arguments have been resolved with some simple logic or indeed mathematical comparisons of the various proposals. This again demonstrates mathematics as a means of communication and also the overlap of the cultural and social reasons to teach mathematics. If a child, within their own ability, encounters mathematics without a full realisation of the subject, then an opportunity has been missed and the child, in my view, has been deprived.

The fourth reason that I have given here for the teaching of mathematics is one of being personal to the child. Again this cannot be considered on its own and I have already mentioned various activities which would lead to personal development and others which give personal pleasure. 'We wish to develop in all students, attitudes and appreciations which lead to curiosity, initiative and confidence and to interests in various facets of mathematics' (Scopes 1973). 'The general attraction of puzzles and patterns suggest that the capacity to appreciate mathematics as an art to be enjoyed is present in people' (Schools Council 1977). Such mathematics that occurs in games and puzzles is popular 'because people enjoy trying to solve them' (Marjoram 1974), and I therefore believe that this type of mathematics has a major role to play in mathematics teaching. The subject should also offer further areas of study to those of all abilities who wish to continue. This all leads to personal enjoyment and development which therefore has to be included as a reason for teaching mathematics.

I have briefly stated my main reasons for the existence of mathematics within education today and also outlined my general educational ideals which will form the basis of my modern approach. I have not made explicit the interactions between each for they are many and complex but their existence should be implicit from my brief discussion.

I would claim that the mathematics classroom should offer the possibilities for all of these reasons to be fulfilled, and hence any modern approach to mathematics teaching should take account of these and allow this fulfilment. It is my view that the six aspects of good mathematics teaching, as outlined in Cockcroft paragraph 243, would encompass everything that I have discussed and summarise the approach which I have in mind. Clearly from my discussion, I believe mathematics to have a much broader role within education than it has traditionally taken.

Cockcroft 243:

> Mathematics teaching at all levels should include opportunities for:
>
> * exposition by the teacher;
> * discussion between teacher and pupils and between pupils themselves;
> * appropriate practical work;
> * consolidation and practice of fundamental skills and routines;
> * problem solving, including the application of mathematics to everyday situations;
> * investigational work.
>
> (Cockcroft 1982)

Matthews, in a discussion document entitled 'Why teach mathematics to most children?', states 'the clear objective of the teacher of young children is to provide experiences so that they can be helped forward in their development', a view that I would totally agree with provided young meant all children up to the age of sixteen, the maximum compulsory education age, and this would lead me to see his further statement that 'it is far easier to dish out the theorems to the willing' as both incorrect and irrelevant and also to disagree with him when he says 'those who have the ability will look after themselves'. (Matthews 1976).

Children need to learn to analyse, synthesise and construct problems, and how to select methods of enquiry. They need to know when to look for help, how to get it and how to present solutions and arguments. 'All of this can be summarised as developing an attitude of independence within the overall framework of the social context in which they find themselves.' (Scopes 1973). 'Such a contribution to thinking for themselves will fit the children for later life' (Matthews 1976). Education has a job to develop individuals who have the fundamental skills of investigation, problem solving and discussion. Mathematics can clearly, and must, play a large role in this development and therefore our approach must enable us to do so. I would therefore define my modern approach to mathematics teaching as one which allows this development. Indeed any subject within a discipline based curriculum which cannot justify its existence and approach as being that of producing and developing these skills should not, I believe, be on the modern day curriculum.

It is the problem solving aspect that I believe must be the foundation to a modern approach to mathematics teaching. It allows all the aspects of Cockcroft's paragraph 243 to be included and integrated into the mathematics curriculum with relative ease, offering varying levels of study and success to all in the same sense as a piece of creative or investigative work in say English or History. It also allows all pupils to study real problems which are relevant to them and within their own realisation. The pupil sets out and works to achieve success at their own level.

It is, however, no good 'doing' problem solving, then investigations etc.; it has to be an integrated attitude and approach of the teacher. This I believe is also true of the use of calculators and computers in mathematics teaching. What is needed is an overall realistic, enquiry based and integrated approach, using all the aspects which I have outlined. This I see as a much more natural approach to mathematics teaching than some which have been used in the past. Teachers must be willing to allow pupils to 'CONSIDER THE PROBLEM' and avoid 'TEACHER LUST' at all costs. Teacher lust is the desire to tell all; I believe this to be a traditionally 'deadly disease' within mathematics teaching, deadly because it kills positive attitudes, interest and understanding.

I would therefore set the following broad aims of mathematics teaching, with perhaps the first five being not that much different from say those of CDT, Schools Council History, Physics or many other subjects.

1. To enable every child to experience success.
2. To enable every child to experience enjoyment.
3. To develop an investigative and enquiring mind.
4. To develop problem solving techniques.
5. To develop acceptable social and personal skills.
6. To make all pupils numerate.
7. To develop the full mathematical potential in each child.
8. To develop an understanding and ability to see the

relevance of mathematics in everyday life.

9. To develop an ability to apply mathematical knowledge to new situations.

A modern approach to mathematics teaching has to achieve these aims and therefore I believe that we must offer courses which are:

PRACTICAL

RELEVANT

SUITABLE

and INTERESTING.

Only then will we have a modern approach to mathematics, one which is positive and enthusiastic, and that will generate these same qualities within the attitude and approach of the pupils. They always take our lead in these aspects.

Cockcroft, paragraph 242, states 'We are aware that there are some teachers who would wish us to indicate a definite style for the teaching of mathematics, but we do not believe that this is either desirable or possible.' (Cockcroft 1982). I would agree with this statement whole heartedly and would therefore conclude this first chapter by pointing out that the approach that I have outlined, as my modern approach, is basically a broad ideology and not a single teaching approach. Particular approaches must depend upon the individual and the situation with which the individual is working. However, the general approach or teaching style should be common within all teachers, not only those of mathematics, if we hope to develop individuals who are ready to join the real world and think for themselves in a positive way. This common general approach must, I believe, follow the guidelines of the modern approach to mathematics teaching that I have put forward in the preceding discussion of this chapter.

The Problem Solving Aspect.

The modern approach, as I have outlined in the previous chapter, is not based on content in any way. I would argue here that the actual content in a school mathematics course is not at all crucial when considered against the skills, structures, strategies and attitudes which are to be developed, and hence content does not influence a general approach. My argument is of course fully supported by the fact that there are such a large number of different external examination syllabi. If content were so important, then there simply would not exist such a choice. It is the methods and experiences that I consider important in a good approach to mathematics teaching since it is through these that ones aims are achieved.

Pupils have to be involved in their learning if they are to build an understanding and confidence in mathematics. It is my belief that such an understanding and confidence can only be developed in children by emphasising the problem solving and investigative aspects within the teaching of the subject. Here I am reminded of an old Chinese Proverb:

I HEAR, I FORGET.
I SEE, I REMEMBER.
I DO, I UNDERSTAND.

Meyer states that 'Many good students could not solve problems' (Meyer 1983), a statement which is fully suported both by research carried out at Bath and Nottingham University and evidence from industry, these all being mentioned in the first chapter. In the past, and in fact in many schools today, it would be correct to say 'the classroom instruction we provide for our children puts too much emphasis on action and not enough on thought' (Charles 1984).We want children to do mathematics and be involved in it but routine practice alone prevents the involvement and makes the doing become isolated from understanding.

Having now stated my claim for the emphasis to be placed upon these aspects, we need to look at what I mean by problem solving, why I think that this is so important and how we should go about integrating such aspects into mathematics teaching.

What is problem solving then, when considered in the context of mathematics teaching ? The Cockcroft Report, paragraph 249, offers the following comment:

> 'The ability to solve problems is at the heart of mathematics. Mathematics is only 'useful' to the extent to which it can be applied to a particular situation and it is the ability to apply mathematics to a variety of situations to which we give the name 'problem solving'.' (Cockcroft 1982, paragraph 249)

It was probably Polya in the 1950's who revived the problem solving aspect with his 'heuristics', 'the study of the methods and rules of discovery and invention.' (Polya 1957). Polya wrote ' to have a problem means to search consciously for some data appropriate to attain some clearly conceived but not immediately attainable aim. To solve a problem means to find such an action.' (Polya 1957).

It is, however, more recently that mathematical education discussion has turned towards this aspect in schools. Many may argue that problem solving has always existed and taken place in the classroom and it is here that problem solving perhaps therefore needs to be further defined. Often what is meant by the term problem solving is the solution to a set of problems which can be solved using a 'Tree Search' method and these are often restricted to transition problems. Tree search methods are when the solver has to follow along the 'tree' ensuring that the correct branch is taken at each stage, this is in fact, in my view, not problem solving but simple practice of routine. Meyer states that 'solving problems is an art, not an exact science' (Meyer 1983), a view with which I totally agree since this emphasises the creative aspect of mathematical problem solving rather than the routine.

I believe that it is the heuristic strategies that are to be related to problem solving and not the tree search methods. Heuristic strategies are the general, broad and non-specific strategies. Tree search methods may be memorised without understanding, they therefore have an obvious lack of structure and are inefficient since only a finite number of such mental trees may be memorised. If the catalogue of trees fail to solve a problem, then the solver is stuck since there is no alternative. With heuristic strategies, knowledge of a small number may well cover almost an infinity of problems since they are both broad and general. 'Heuristic guidelines are almost universally applicable. It does, however, require time and thought to see just which guideline is useful and what the application is. Studying and applying heuristics has another, much greater, benefit, however. It is impossible to apply a heuristic strategy to a situation which you don't understand. The very use of these strategies forces understanding of the underlying mathematical framework.' (Meyer 1983). It is exactly these points that suggest that heuristic strategies are better able to deal with new and unfamiliar situations and hence will more easily lead to a thorough understanding of the subject. I therefore suggest that those who claim that mathematical problem solving has always played a full role in the classroom are unaware of the full breadth of problem solving themselves.

There is of course, however, a place for the routine tree search methods in mathematics and a good problem solver probably uses them regularly for the more routine parts of their heuristic based solution. Success in problem solving clearly comes from a combination of both, but it is the tree search method that has traditionally been used, to the neglect of the heuristic. I believe that it is this latter approach that we now have to emphasise to mathematical educators at all levels. Only then have we any hope that this type of approach to solving problems will be adopted by our pupils.

Pupils have to be able to tackle both problems within their own experience and those which are new to them and hence they require a breadth of knowledge and depth of understanding in the subject. Meyer suggests that 'The major barrier to effective problem solving is psychological rather than intellectual. Many students are so accustomed to having someone else solve their problems, that they simply do not try to solve them on their own' (Meyer 1983), this is exactly my point with regard to 'teacher lust' and letting the children 'consider the problems'. However, if we are to help them to solve their own problems then we need to concentrate our teaching on the understanding and use of heuristic problem solving strategies and give them plenty of opportunities to apply these and discuss how the application of different strategies can lead to a solution. This, I believe, is an absolutely essential feature of the problem solving aspect of mathematics teaching, the understanding that there always exists more than one method of solution. This leads to a situation where pupils then realise that they should think about strategy selection rather than jumping into the so called 'correct routine'. The emphasis must be placed on the process rather than on the solution alone.

'At each stage of the mathematics course the teacher needs to help pupils to understand how to apply the concepts and skills which are being learned and how to make use of them to solve problems' (Cockcroft 1982, paragraph 249). Often it is the initial translation of problems to mathematical terms which causes problems, a fact which is often too little appreciated by those in mathematics education. This may well be a direct result of the use of tree search methods when the pupil had to pick the exact correct tree, whereas with heuristic methods, there may be many general strategies which will lead to a solution. This does not, however, take away the need for a major emphasis to be placed upon strategy selection, quite the opposite, since there may be both efficient and inefficient methods and pupils will only discover this through experience and practice. Such strategy selection and problem translation is important for all pupils and 'for many pupils this will require a great deal of discussion and oral work before even simple problems can be tackled in written form.', (Cockcroft 1982, paragraph 249). 'The quality of pupils' mathematical thinking as well as their ability to express themselves are considerably enhanced by discussion' (Mathematics 5-16, DES 1985).

It must now be to these specific heuristic strategies that our discussion turns. Various authors give slightly different lists of such strategies and all agree that there are many, but basically, there are about half a dozen which cover the majority of problems which will be encountered. I would like to suggest the following seven which seem to summarise the various sets of suggestions:

* Guess and check.
* Work backwards.
* Look for a pattern.
* Use logical reasoning.
* Find a suitable form of representation.
* Find and solve a sub-problem.
* Try simple cases.

Each is relatively self explanatory and it would not be suitable to give examples for each strategy since by definition these strategies are broad and any one problem may be solved in many different ways.

Polya states 'A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But, if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking' (Polya 1957), and I would see this statement being echoed some 25 years later when Cockcroft suggests that many mathematics lessons are about nothing, an indication of the lack of incidental and real learning in the subject. This of course led on to his now famous paragraph 243, where he lays down his six aspects of good mathematics teaching, problem solving, investigations and discussion work all being included.

I have tried to outline what I mean by problem solving and with it, my justification for emphasising this particular aspect. It is, however, the inclusion and integration of this into the school mathematics curriculum that concerns me most. All the ideas that I have discussed in this work have been in the past, and probably will be in the future, open to subjective interpretation and it is for this reason that this dissertation and discussion exists; I wanted to try to offer a more definite and precise interpretation.

So how can we include problem solving into a mathematics programme ? I believe that we have to integrate this in a natural way and not isolate each aspect for separate and individual treatment. This view is now being supported by the latest DES document on mathematics teaching, Mathematics 5-16. When this document refers to paragraph 243 of Cockcroft, it states that 'The various approaches considered in this chapter need not be isolated from each other but may all be part of the same activity' (Mathematics 5-16, DES 1985). It is my view that these aspects are at their strongest if they are simply a part of a teacher's usual approach rather than being dependent upon a particular piece or type of work. Whilst suggesting that the individual strategies have to each be given a thorough treatment, they should then be taken on board as general strategies to be used at all times, not just when it comes around to doing problem solving. I am a great believer in an integrated attitude on the part of the teacher, we have to be willing to pose problems at all times, asking children to investigate, discuss and explain the why and how of their views. I am sure that this can be learned by a teacher over a period of time, given help; however for others this is a natural and more normal approach.

I suppose, by emphasising the problem solving aspect in mathematics teaching, we have to emphasise certain teacher behaviour which is necessary. I would list, perhaps, the following:

* Teacher enthusiasm.
* Recognition and reinforcement of willingness and perseverance.
* Rewarding of risk takers.
* Encouragement of hunches and guesses.
* Acceptance of unusual and unexpected solutions.
* Emphasis on thoroughness and persistence rather than speed.
* Emphasis on method and strategy selection.
* Allowance of personalised problems either by allowing the pupils to bring in their own problems or adapting the problems to fit the pupils' interests, names, situations etc.
* Acting as a provocateur, questioner and evaluator and

never giving definite statements, answers or judgements.

All authors agree that it is the classroom climate that has to be right if good problem solving experiences are to take place within a classroom. There are many factors which contribute to the correct classroom climate, most important of all being teacher attitude and action. There is a need for a positive and enthusiastic climate with the absence of teacher lust and the presence of the above listed teacher behaviours.

Many would argue, and I would agree, that if all teachers are to be encouraged to use a problem solving approach then there is a desperate need for good material to support the work in the classroom. In the same way that pupils have to consider the problems, then teachers have to try this approach and to do this, they initially need to be offered material which does in fact emphasise the need for such an approach. It is often not appropriate for the traditional teacher to try to introduce this emphasis with the materials and ideas that they may have used for many years. However, with practice and experience, a teacher may then turn this approach back onto such work, often with surprising results. This view of the need for material development has been held in the USA for a long time and a lot of new material has been gradually developed, mainly along the lines, and in support of, Polya's heuristic approach. Much of this material is now becoming available on this side of the Atlantic and there is also similar material development in this country itself. One of the leading groups in this type of material development is the Shell

Centre for Mathematical Education (SCME) who in conjunction with the JMB (Joint Matriculation Board) have to date, produced two modules of material, with more to follow. The JMB are now introducing the problem solving aspect of mathematics into their examinations. The first two SCME modules are:

1. Problems and Patterns in Number.
2. The Language and Functions of Graphs.

In the first module, which my department used throughout the trial period, the SCME have developed a package to introduce and build on the heuristic ideal using a more sequenced approach, to help with the introduction of these ideas. This package uses the following strategies:

* Try some easy cases.
* Find some form of suitable representation.
* Try to spot a pattern.
* Form a rule.
* Test the rule.
* Adapt, adopt or scrap the rule.
* Express the rule verbally and algebraically if possible.

Pupils are encouraged to follow through the sequence of strategies in their solution process. This gives a natural insight into a number of heuristic strategies and also when each is appropriate, this allows the selection problem to be taken on board very easily.

Pupils who have followed this approach certainly seem to have a much broader understanding, appreciation and confidence in their mathematics when tackling new problems than those who have not. The new GCSE examinations are of course also looking for the inclusion of the problem solving aspect of mathematics teaching within their assessment.

It is with this type of material development that this dissertation is based. I do feel that teachers have to view this type of material, see it in action, try it, discuss it and involve themselves in this style of mathematics teaching if it is to become generally and more widely accepted as a positive and successful method. However, such material is open to abuse, in the wrong hands, hence the necessity of a full discussion prior to offering the teaching package in this work.

The Use of Games in Mathematics Teaching.

In the first chapter of this dissertation, I have outlined my views on a modern approach to the teaching of mathematics. There is not necessarily a new approach to mathematics implied by the word modern, but it is the approach which I think should be being adopted by a mathematics department, if it does not already exist. I have also discussed, in the second chapter, the particular aspect of problem solving within this approach to mathematics teaching. I now turn to the use of games and in particular games of strategy, the role that they play in mathematics teaching today, and their link with a problem solving approach.

There has been much recent research and discussion in the world of mathematical education based on practical work, problem solving, investigations and applicable mathematics. I feel, and hope to argue, that the use of games of strategy supports all of this work and allows the integrated approach to the teaching of these aspects, which I have previously mentioned. Games have a huge advantage in themselves, in the sense that they have a high motivation factor and a natural interest for the children. This does of course make a major step towards overcoming the traditional problems of negative attitude and the 'can or cannot do' syndrome. However, these problems will certainly not be overcome simply because you present the pupils with games to play. The games can merely emphasise the fact that some can do and some cannot if they are not presented and used in the correct manner, and the motivation factor will soon vanish and die if this is the case. I believe that the correct approach, as I have outlined in the first two chapters, must be adopted if this work is going to contribute in a positive way. Everyone can play a game at their own level. Games, I feel, have their own inherent individualised levels of study, both within the practical and theoretical aspects.

A major aspect of problem solving is the selection and use of a suitable strategy. Although various strategies may be taught and demonstrated, as outlined in chapter two, I do feel that it is important for pupils to initially realise the existence of strategy and experience this themselves through the use of such ideas. I again feel that games allow this and the true beauty, in my opinion, of games of strategy is that they can be used to demonstrate strategy, its existence and use in mathematics, with particular relevance to their use in problem solving. Then, when a problem solving approach has been taught and grasped, the problem solving strategies may be turned back onto the games to investigate them further. We therefore start with a game that may simply be played, then intuitively analysed, hence strategy understood, then by way of mathematics we may move on to a more formal analysis and hence the development of the mathematical strategies for that game.

Looking at games and asking pupils to think about strategies, makes pupils think about and discuss the methods that they are employing. I believe that this is a major battle in the elimination of pupils' misconceptions in mathematics. A great deal of research by such groups as the CSMS (Concepts in Secondary Mathematics and Science), SESM (Strategies and Errors in Secondary Mathematics) and SCME (Shell Centre for Mathematical Education) teams has looked at pupil misconceptions and these often stem from erroneous strategies. However, such strategies and misconceptions can often be cleared up by pupils discussing and arguing about their methods, a natural aspect of games playing.

Cockcroft, paragraph 240, points out that 'A Review of Research in Mathematical Education' (Bell et al 1983) drew a clear distinction between three elements in mathematics teaching:

a) facts and skills;
b) conceptual structures;
c) general strategies and appreciation.

Facts are the items of information which are essentially unconnected or arbitrary. Skills include the established procedures which it is possible to carry out by use of a routine. 'They need not only to be understood and embedded in the conceptual structures but also to be brought up to a level of immediate recall or fluency of performance by regular practice.' (Cockcroft 1982, paragraph 240). It is the conceptual structures, the inter-connected bodies of knowledge, which form the basis of that held in the long term memory. 'They underpin the performance of skills and their presence is shown by the ability to remedy a memory failure or to adapt a procedure to a new situation.' (Cockcroft 1982, paragraph 240). 'General strategies are procedures which guide the choice of which skills to use or what knowledge to draw upon at each stage in the course of solving a problem or carrying out an investigation. They enable a problem to be approached with confidence and with the expectation that a solution will be possible. With them is associated appreciation which involves awareness of the nature of mathematics and attitude towards it.' (Cockcroft 1982, paragraph 240). These of course, relate to the heuristic strategies, as discussed in the previous chapter. I have included Cockcroft's full description of this aspect of mathematics teaching since I believe that it is so important and basically fundamental in the discussion and approach taken throughout the whole of the work in this dissertation, and emphasises the points made in chapter two. Pupils must have a full breadth of knowledge and depth of understanding in the subject. Faced with a new situation or problem, they must be able to ask themselves the question of what mathematics they know that they may apply to this new situation and how they are to apply it in their search for a solution.

To give a simple example of what is meant here, we could consider the problem of looking at the area of a rectangle 12cm. by 5cm. To solve this problem we break it down into the following:

Fact:	that 12 means one ten and two units.
Skill:	to use the established procedure to multiply the two numbers.
Stucture:	of area, the understanding and meaning.
Strategy:	to select the correct process of multiplication for the area calculation.

Each element has to be understood if a successful solution to the problem is to be achieved.

'Research shows that these three elements; facts and skills, conceptual structures and general strategies and appreciation; involve distinct aspects of teaching and require separate attention. It follows that effective teaching must pay attention to all three.' (Cockcroft 1982, paragraph 241). Whilst certainly agreeing with this statement, I would, however, be concerned as to the interpretation put on it by some. The separate attention to each aspect must certainly exist but again, in my view, it would be totally wrong if they were each considered in isolation and as discrete elements. As I have already argued in the previous chapters, the six aspects of good mathematics teaching, as outlined in Cockcroft paragraph 243, designed to ensure that all three of the above elements are covered, must be considered in an integrated fashion. It is intended by Cockcroft, I feel, that discussion should take place on all three elements during the solution to a problem, perhaps even emphasising the step like procedure through the problem using the various elements together with the heuristic and tree search aspects. Only then will an awareness and appreciation of the nature of each of the three separate elements be gained.

It is, I feel, the third element that has been traditionally overlooked, and one way of offering this element a separate treatment and emphasis is through games of strategy and problem solving, although this involves the other elements at the same time, as I believe is desirable. It is perhaps the neglect of this area in the past that has led to the 'can or cannot do and most cannot' syndrome since if this element of understanding and process selection has not been taught and emphasised, then it is extremely difficult to attempt problems and hence impossible to obtain solutions to them.

It is now to games in particular that I turn my concentration and discussion. There are a variety of types of games and many so called mathematical games are merely puzzles. Such puzzles make their own valuable contribution to mathematics teaching. Although my present discussion does not involve such puzzles, my general approach, as outlined in the first chapter, would obviously include these. The use of such puzzles should be implicit from my discussion of the use of games of strategy. Some authors give deep and detailed definitions as to exactly what a game is, although my discussion is not really involved with either game theory or theoretical games. I am more concerned with practical games which may be played and discussed in a mathematics classroom, and with the aims of studying strategy and its particular relevance to mathematics.

A game, to give a dictionary definition, is 'any sport, a pastime, a contest for amusement, a trial of strength, skill or chance, an exercise or play for stakes' and the corresponding definition for strategy is 'the art of conducting military or naval operations'. By making a few deletions from each definition, then we arrive at what I would find an acceptable working definition for a game of strategy 'a pastime, a contest for amusement, a trial of skill in the art of conducting operations and chance.' I find this to be an excellent and exciting definition for something that is going to be encountered by all in the mathematics classroom. It is certainly a sound start in the battle against traditional attitudes, but can we now support this definition ? I believe that we have to !

At this stage, I would like to give my classification of games. In my view, there are broadly speaking five main types of game.

1. Games which use facts or skills.
2. Games which develop particular concepts.
3. Games to develop mathematical thinking.
4. Games of pure strategy.
5. Games of strategy and chance.

These groupings are by no means either mutually exclusive or exhaustive and I am sure that other, seemingly different, classifications exist.

Games which use facts and skills would include the 'Hat Games', from the 'Maths Games in the Classroom' series, as developed at The Mathematics Education Centre of Sheffield Polytechnic by the Eigen Publications team. These are a series of games played with a set of cards which are numbered from 1 to 9. The cards are shuffled and one is picked at random, this process is repeated as many times as necessary to produce the required number of random digits. Pupils then have to use the digits to produce a number, say, between certain limits or one which is higher or lower than that of the other players. These games may be developed to cover a variety of themes including the understanding of decimals, fractions and various symbols such as < and > etc.

Games which develop particular concepts would include the old favourite to introduce coordinates, that of Battleships. This particular game, however, has a major problem with it, in the fact that it numbers the axes in between the grid points rather than on the grid points themselves. This is a general misconception with coordinates and graph work, and one which I would claim is therefore a teacher caused misconception, as I feel many are. These games may be very simple and take only a few minutes to play. They may well be made up by the teacher, almost on the spot, to demonstrate the concept under consideration. A game can be made out of almost anything and this always gives added interest to the lessons and pupil learning.

This conceptual game idea is probably an area which could be developed a great deal more in the mathematics classroom. Most traditional mathematics texts have of course ignored this, as would be expected, but the opportunities have often been missed, I feel, in more recent work. The SMP lettered books, although giving a few games in their interludes, some of which I have included in my package, see Sprouts and The L Shaped Game, generally did not include many conceptual games to develop or reinforce various concepts. This aspect has been included much more in the new SMP 11-16 material and is very evident in what I would describe as the very practical Modular Mathematics course produced by Heinemann. This latter course uses the conceptual game to reinforce various concepts at every possible opportunity and this not only works well, with regard to the particular concepts, but also produces extra motivation and understanding for the follow up work.

Games to develop mathematical thinking would perhaps include the commercially available Score Four and Mastermind. With these games, the players may not realise that they are using any mathematics or indeed strategies but, I believe, they are required to think as mathematicians if they are to play the game successfully. I have included some games of this type in the package simply to make pupils think in this way. The pupil strategies in this type of game would involve logical or mathematical thinking rather than those developed through any form of analysis of the game.

Noughts and Crosses would come under the next section, those of games of pure strategy. I would also include Chess in this section. It is this type of game that may be, and often is, analysed fully by mathematical methods, although some games have not been, due to their complexity. Many people may argue that these games are not worth playing once the strategy is known by the players and may further argue that for the more complex games of pure strategy, the good players are simply those who have read a great deal about the game and committed this to memory. This is certainly not true of Chess as we see the masters of the game still playing for various titles, although I am sure that they do read a great deal about the game and its strategies. They have by no means lost their interest for the game. This is perhaps also true of the gamesters, puzzlers and mathematicians who continue to investigate and analyse the game of Noughts and Crosses and other such similar games.

Finally, the section which I would claim is the most interesting of them all, games of strategy which also involve chance. These are the games that will entice you back time and time again for further encounters. I would include such games as Bridge under this heading. It is the element of chance that makes you seek further strategies in perhaps trying to tame this sometimes vicious chance element. It is also perhaps the excitement of the unknown that thrills in these games. Games of pure chance are of course also very popular with both children and adults alike, we need only think of the popularity of games such as Snap and the seemingly addictive Bingo to confirm this. I have always enjoyed studying and analysing games of pure strategy but for me strategy games incorporating a chance element have always offered something more. I am not sure whether this is a cause or effect of my general interest and enthusiasm for the study of probability.

We now need to address ourselves to the question of which games are suitable for the mathematics classroom. I would argue that they all have their place and may be used at some suitable stage. Clearly, games from the first two sections have an obvious and regular use and may be used to introduce the particular facts, concepts and skills. However, these games may sometimes be used at a later stage as higher order games.

Games to develop mathematical thinking can always be played and may be returned to at any time. They can be played at the level suited to the individual players. The playing of such games would probably not be linked to any particular content in a mathematics course. However, the games are related to the skills and attitudes discussed in the previous chapter and hence should be used on many occasions.

Games of pure strategy are ideal to demonstrate the existence of strategy and to allow pupils to develop their own, either informally or formally. Again it could be argued that once these strategies are known then they have no further use in the classroom. Well, I have given Chess as an example of this type of game and stated that this is not true in this particular case, however, it may be true for some games of this type. I would not, however, use Chess in the classroom to investigate strategies due to its obvious complexity, but it certainly could be used as a game to develop mathematical thinking.

Games of strategy with an element of chance have a similar use to the two types of games that I have just mentioned. It is through the use of these three types of game that I wish to emphasise strategy and approach, which is not only important in playing games but also in solving problems in mathematics, other areas of the school curriculum and indeed life itself. The development of strategy may simply be intuitive or more formal, through some mathematical analysis. One gains a much greater success, I feel, in the solutions to problems if a strategic approach is used and if this strategy is known from the outset of the problem solving task. I also believe that this must be demonstrated to the pupils if they are to become aware of this fact. It was with this in mind that I adopted the current topic of discussion as the theme for my research and study, as outlined in this dissertation.

I certainly feel that the secrets of good and useful games for the mathematics classroom are openended-ness and returnability as implied in the above discussion. This is perhaps true of all good classroom material, games or otherwise. This allows a game to be used time after time. It may well be used as a game to introduce or reinforce a particular concept or skill, it may then be used to demonstrate the existence of strategy and allow the pupils to informally develop their own. Once the problem solving skills have been suitably developed, then the game may be used once again, this time as a problem to be solved, the problem being to investigate the strategy. Again, this investigation may also be carried out at different levels. However, each time that a game is revisited, I would envisage a new approach or level of study being adopted. It must not simply be re-offered if pupils are to play it merely as a game with no further mathematical discussion or discovery taking place.

When thinking of this idea of levels of study within a game and the concept of returnability, I am reminded of the game of Frustration. This is a commercially available game which is in fact the age old game of Ludo, but with a mechanical dice thrower. This game had been given to my younger sister when she was about five years old. The game was very useful to her because she was at the stage of learning to count and the amusement of the game led to her achieving considerable reinforcement of the idea of number and counting. At the same time my younger brother, who was about twelve, would also play the game, he was investigating the best use of a six. Was it best to use it to start another counter off around the board or was it best to use it to hurry along the counter that he already had on the board ? This was his development of an intuitive strategy by use of a practical approach through the playing of the game many times. At the time, I was an 'A' level double mathematics student, and having often to look after the younger members of the family, I became quite interested in this game as to whether the mechanical dice thrower was in fact random. I carried out a statistical experiment on this together with a significance test to investigate the problem; in fact the thrower was not at all random and the number six was much more likely to show on the dice than any other. I think that this was probably the start of my interest and fascination for games.

There are of course problems with the use and introduction of games into the classroom. The major problem is probably that of justification. Justification has to take place at several levels, the first being self justification. A teacher can only perform well if they are convinced that what they are doing is justified within their overall aims and goals for mathematics teaching. There is also the problem of justification to the pupils themselves and perhaps to the rather traditional Head of Department who is keeping a watchful eye on their new games playing probationer teacher. There is also the visiting parent or the roaming Headteacher to whom you may have to justify yourself. However, this justification is easy if you have the belief that what you are doing is allowing you to achieve your desired aims. This of course returns to the arguments put forward with regard to approach in the first chapter.

Games, like any other aspect in teaching will only work well in the classroom if supported by the necessary preparation and follow up work. It is no good simply using a game as a fill in lesson or an end of term treat, this is also true of any investigation or problem solving task. It is from this type of use that pupils may get the impression that this is simply a time filler and not a planned piece of work. This leads to pupil comments such as 'When are we going to do some work ?', when the teacher introduces further work of this type. It is precisely this type of pupil comment that necessitates a justification to them, although this need really only be a thoroughly planned approach linking up with previous or future related work. Pupils will then not even think about questioning the approach, seeking a justification or making such comments.

The introduction of any new method into the classroom needs time for acceptance by the children and for the member of staff to feel at ease, the methods, ideas and general approach that I have outlined here are by no means exceptions. An innovative member of a department has to tread carefully and take things slowly if they are trying to encourage other staff to adopt and try new ideas and methods with the long term aim of accepting the modern approach discussed in chapter one.

A Chemistry teaching colleague of mine was not long ago telling me that he intended to try an alternative approach to his practical lessons. He had always previously told the pupils what the experiment was trying to show and then how they were to do it. His new approach was to talk to the pupils and discuss what they were trying to investigate and then give them an envelope which contained a number of pieces of card, each showing a single instruction as to how to tackle the experiment. The pupils' task was therefore to firstly work out how to do the experiment and then carry it out. He was, however, most disappointed with the way the lesson went. The pupils became totally involved in solving the 'puzzle' about getting the instuctions in the correct order and very little practical work was done. The teacher felt that the lesson was noisy, chaotic and generally a failure because the pupils had not achieved what he had hoped and wanted. However, this situation was bound to happen I feel, the novelty of something completely different had got hold of the children, but this was good in a sense as they were obviously totally involved in the first problem that they had been set, that of solving the 'puzzle'. Clearly they were keen and interested in this new approach.

The amount of chemistry that they will have discussed during this particular lesson was probably a lot more than they would have with the more traditional approach, as previously used. When pupils then have a further session, and tackle the actual practical experiment, they will achieve a much greater understanding of what they are doing and will probably work with the continued enthusiasm that had previously been developed. Teachers must not be put off from introducing new approaches into the classroom simply because pupil response is not as we would have expected, after all, it is often a desired change in pupil response, attitude and understanding that leads us to want to introduce new methods, approaches and ideas.

In the games teaching package, given in the Appendices, I have tried to put together a variety of games of strategy without too much overlap of general game similarity, although I have included some to emphasise this point and to allow a particular theme to be followed through, if this is desired by the teacher. I have also, as previously stated, included some games simply to develop mathematical thinking where perhaps the use of a strategy is not obvious but logical thought, based on a strategic approach, is. Such games of this type from the package include Pico Centro and The Black Box Game. The sheets are designed possibly more as teacher sheets with suggested diversions, questions and extensions. However, they may be used as pupil sheets if this is preferred. I would, however, re-emphasise at this stage the essential aspect of avoiding teacher lust, as defined in chapter one. On the sheets I have tried to provoke, prompt and question, not prescribe. This is of course the essential aspect of the whole approach that I have outlined throughout these first three chapters.

The package has not merely been put together for the sake of completing this dissertation but as a practical package to support work in the mathematics classroom. Teaching material under development has to be fully tested in the classroom and good material is successful classroom material which works for both pupils and teacher alike. The final teaching package is the result of my initial ideas, a lifetime interest in games playing, an interest in improving mathematical education in schools, intermediate material and many hours of extensive classroom trials together with much valued discussion, criticism and support from the teachers involved in the trials.

Although I have used these games, and encouraged others to do so, over many years, the trials themselves, like most with this type of material, took place within a relatively short period of time, a single term. This naturally leads to a rather false situation and one which I have openly criticised in my discussion so far. We found that the strategies of earlier studied games were often influencing later work when the ideas were inappropriate. This may not have been a bad thing however, since pupils were clearly building up their armoury of problem solving approaches and their strategic awareness. They also usually managed to eliminate unsuitable approaches, through investigation. This did, however, cause concern for children if they had just studied a game with an absolute definite best strategy, see for instance the game Century, and then were asked, perhaps the following lesson, to look at a game with a more vague or informal strategy such as Pig.

As I have previously discussed, the use of games should be integrated into a general approach by the teacher and this would overcome the problems mentioned above, which have been caused by too much of the same all at once. This I believe is true of any aspect of mathematics teaching, whether games, investigations, problem solving or the traditional teacher exposition and practice of routines, which have dominated the mathematics classrooms of the past.

I certainly feel that games, in particular games of strategy, have a place in mathematics teaching, particularly in support of the development of general heuristic strategies, and it is with this final statement that I now move to the next chapter which looks at the package of games both practically and in more detail.

The Classroom Trials and Games in Detail.

The classroom trials, as mentioned at the end of the last chapter, were rather false and not really carried out under ideal and natural conditions. It would have taken a whole academic year for such 'natural' trials to take place and this would have had to have been preceded by considerable planning for the integration of the games into the courses in any department which was involved in the trials. It would be rather naive to think that such a situation could ever occur within a short period of time, and hence, the testing conditions for the material in the games package were considered to be both normal and sufficient, for such trials, to obtain an insight into the appropriateness and success of the material.

The major planning that is required to fully incorporate this material into a mathematics department's courses is now in fact taking place within my own school, although this is not considered as a part of this dissertation. During the trials, I was able to collect some pupils' work on the majority of the games in the package and some of this has been used in the following discussion of the material, trials and mathematical background of the games.

The discussion in this chapter looks at the games within the package, a little of the mathematical background and strategies of the games, and their use in the classroom. No game is given a complete analysis since such a treatment of any one of the twenty five would probably satisfy the full requirements of a piece of work such as this, and clearly this was not the object. The most important aspect to me, throughout this whole project, has been to express what I feel is both a correct and necessary approach to mathematics teaching together with a justification for this approach and the development of a teaching package based on Games of Strategy as a support for this.

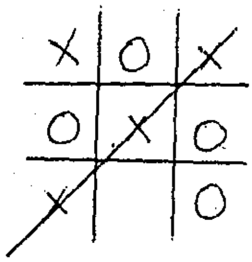
I believe that this type of games material was perhaps not available before and that it was therefore a necessary development. It is therefore the material in the Appendices together with the discussion in the first three chapters which is the more important and not the actual 'solutions' and commentaries in this section. However, the discussion of ideas on how the material may be used in the classroom may be helpful to some, while others may prefer to think of how to use it within their own style. There is no reason why a teacher should not tackle an investigation, problem or game without knowing the solution themselves, why shouldn't the pupils realise that we, as mathematics teachers, do not know everthing and that there is always something to learn irrespective of how much you already know. Pupils and teachers may consider the problems together and it is really with regret that I must plead guilty to teacher lust as I offer some of the strategies, solutions and ideas in this chapter, for without such a section, I feel that a study such as this would be incomplete. Therefore having stated that it is perhaps the other chapters which are the most important, I do feel that this is a valuable and necessary one.

The sheets should be seen as starting points for the problems and investigations and not as activities which have to be completely solved at one study, or as being the complete study of the particular game. Variations on the games may be preferred, by staff and/or pupils, to the actual game that I have suggested. Many of the games may be played in a large number of other forms including the misere version.

The complete list of games, in the order that they appear in the package, is given overleaf.

THE GAMES

1. Noughts and Crosses
2. 15 To Win
3. Fleet
4. Dots and Boxes
5. Century
6. Subgame
7. Multigame
8. Pig
9. Hunt the Hurkle
10. The Keyboard Game
11. Fox and Geese
12. Sprouts
13. The Tower of Hanoi
14. Sunflower
15. Pico Centro
16. Cover Up
17. The 'L' Shaped Game
18. Colours
19. Designer
20. The Black Box Game
21. Nim
22. Kayles
23. Nine Men's Morris
24. Hex
25. Wari



NOUGHTS AND CROSSES

You have probably all played this game before but play it again several times against your partner, taking turns to go first.

Try to write down some rules for trying to win, this is called a STRATEGY.

If you had a choice would you go first or second ?

Why ?

Can you write anything else about this game ?

NOUGHTS AND CROSSES

When pupils encounter this game, they are generally well aware that the result is usually a draw if the two players know how to play the game well. This probably comes from them having played the game from a very young age. Whilst having their own informally developed strategies for this game, pupils are probably unaware of the vastness of a complete strategy for the game, and will usually give a single strategy as to how best to play the game, this normally being a first player strategy. This will probably not allow for any symmetry in the game; a typical pupil's strategy being something like 'you always start in the top left hand corner'. A complete analysis is given by Berkelamp, Conway and Guy in their book 'Winning Ways' Volume 2 (1982). Assuming the labelling of the square in the form of a magic square

6	1	8
7	5	3
2	9	4

then the complete analysis is as follows:

Bold numbers represent forced moves;

! denotes a move that is better than some others;

? denotes a move that is worse than some others;

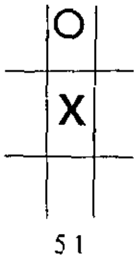
X denotes a win for Cross;

O denotes a win for Noughts;

⊗ denotes a tied game;

~ denotes an arbitrary move; and

v. is a cross reference to another column in the analysis.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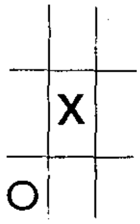
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Starting in the Centre.

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     ┌2!  8   6              X
     │3!  7   4! or 8!       X
     │4!      v.512
     │6!  4   2! or 7!       X
     │7!      v.513
51?  │8!      v.516
     │            ┌3  7  6  4  ⊗
     │            │4? 6        O
     │      ┌2!   │6  4  ~     ⊗
     │      │     │7  3  ~     ⊗
     │      │     └8  ~        ⊗
     └9?    │3?  2!            X
            │4! v.5192
            │6!  8  2  7  3  ⊗
            │7? v.5193
            └8! v.5196
```

```
     ┌1  9  4   6  7  3  ⊗
     │3  v.521
     │4  6  7   3         ⊗
     │6  v.524
     │         ┌1   9  4  6  ⊗
     │         │4   6  ~     ⊗
     │7  3     │6?  4        O
52?  │         │8   ~        ⊗
     │         └9   1  ~     ⊗
     │   ┌1?  3 or 4         X
     │   │3?  v. 5281
     │8  │4!  9   1  ~       ⊗
     │   │6!  v. 5284
     │   │7?  6              X
     │   └9?  v. 5287
     └9  v.527
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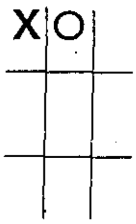
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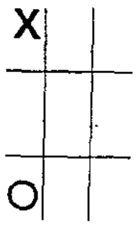
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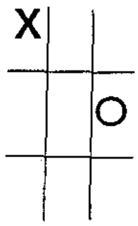
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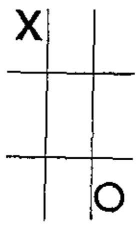
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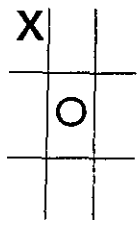
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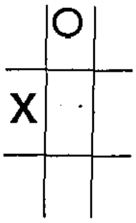
Starting in a Corner.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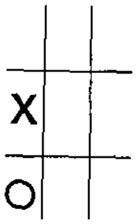
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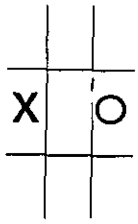
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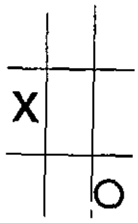
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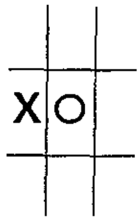

75

Starting on a Side.

```
61?
  2!  7 4! or 5!  X
  3?  2?4! or 5!  X
      4?7!  X
      5!9  ⊗
      7?4!  X
      8?5! or 7!  X
      9?5  X
  4?  5 9 2 8  ⊗
  5!  v.516
  7!  2 5!  X
  8?  5!!  ⊗
  9?  2~  ⊗
      3?2! or 4!  X
      4~  ⊗
      5?2!  X
      7?4!  X
      8?2! or 4!  X

62?
  1!  8 5  X
  3   1?v.6132
      4!9 1! or 5!  ⊗
      5?8  X
      7?v.6198
      8?5  X
      9?4  X
  4!  5 8  X
  5   v.524
  7?  1?v.6172
      3 4! or 8!  ⊗
      4!9 8!  O
      5 8 1 9  ⊗
      8?5  X
      9!4 5  O
  8!  1 4!  X
  9   1 v.6192
      3?1! or 5!  X
      4?1!  X
      5 8 1  ⊗
      7?1! or 5!  X
      8?5  X

63?
  1?8 4 5  O
  2!7 5  X
  4 v.614
  5! v.512
  72  1?8  O
      4581  ⊗
      5481  ⊗
      81~  ⊗
      9?8!  O
  8! v.612
  9   1?v.6193
      2?v.6293
      4?812!  X
      5 7281  ⊗
      7?v.6139
      8?v.6239

64?
  1?832!  O
  2! v.628
  3   1?v.6134
      2 v.6234
      5~  ⊗
      7~  ⊗
      819
      9~  ⊗
  7? v.641
  8! v.628
  9 v.643

65!
  18  v.6275
  27  v.6185
  3   1 v.6135
      2?v.6235
      4 v.6435
      7?v.6195
      8 v.6295
      9 v.6395
  4   1!9283  ⊗
      2?8  X

71?
  2?  v.631
  3?596! or 8!  O
  4   v.639
  5!  v.513
  6!  v.617
  8   v.613
  9   2   3584  ⊗
          4?8  O
          5 3~  ⊗
          6 3! or 5!  ⊗
          8 3!,4! or 5!  ⊗
      3?  2!  X
      4   2?6  O
          3562  ⊗
          5382  ⊗
          62~  ⊗
          85! or 2!  ⊗
      5?  2!  X
      6?8 23!  X
          32!  X
          45  X
          54  X
      8?625  X

72
  1?  3 v.7194
      4!95  O
      5 8643  ⊗
      6?5!  X
      8?5  X
      9?v.7196
  3?  584  O
  4   v.629
  5   v.527
  6?  v.627
  8   v.643
  9   1 v.7192
      4?5!  X
      5 8~  ⊗
      8?5  X

73
  1 v.719
  2 v.637
  4 v.619
  5 v.519

74?
  1?  2!95  O
      3?865  X
      5?6  X
      6?5  X
  2!  v.621
  3?  v.723
  5   v.521
  6?  v.641
  8   v.623
  9?  v.721

75
  1   28643  ⊗
      3?6!  X
      4?6  X
      6 4  ⊗
  2   6 v.6275
  3?  196! or 8!  O
      284  O
  4   v.653
```

This strategy allows for symmetry and each player being sensible enough to:

a) complete a line of his kind if he can;

b) prevent his opponent from doing so on his next move.

Clearly, for each 1st player strategy there is a 2nd player strategy, and hence two expert players should always draw the game. Many argue, because of this, that there is little point in experienced players continuing with the game, but others, including the above named authors and Martin Gardner, perhaps the most prolific author within this area of mathematics, would disagree and continue to look for more concise and efficient strategies for the game.

Pupils generally were quite able to express their strategies for this game in both verbal and written form, most giving examples to illustrate their ideas. The majority of pupils in their written strategies indicated a preference to take the centre square and therefore usually wanted to go first. Obviously some pupils gave a much more detailed account of their strategy, with most looking to set up more than one winning line simultaneously. The game of Noughts and Crosses is an ideal one to use in the classroom to demonstrate the existence of strategy. This is particularly true since the pupils usually already know that there is a method to playing this game and therefore the discussion can centre around the importance of using such a method, rather than actually finding one in this case. Pupils can be offered the opportunity to suggest and discuss their own strategies and may be asked to explain it to others verbally or give it in a written form.

One of the examples overleaf, given by an able eleven year old, shows a good understanding of the symmetry aspect of any strategy for this game, although such an understanding is not particularly common. One older and more thoughtful pupil, realising the difficulty in trying to win, gives his strategy to ensure that he does not lose.

Noughts and crosses

My strategy will give You a better chance of winning

Here are the steps to take

 put you X on the middle Square

 Next put 'X' on midde Right Square

Next put 'x' on top Right so your board Should be like this:

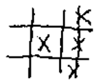
But what if someone put's their 'O' on middle right or Top Right.

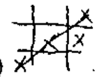
Well.. you will have to make a rotating effect like :

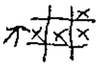
 or   and so on.

then there is no way of stopping this move

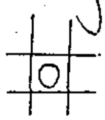
So you have 3 choices

 ← there

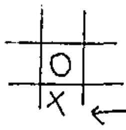
there ↗ 

and there ↑ 

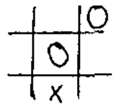
Noughts + Crosses



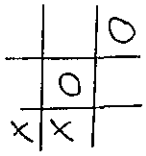
My way of winning is by going first in the middle



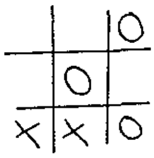
The next move is Somewhere like this.



Then would move Somewhere in a corner.

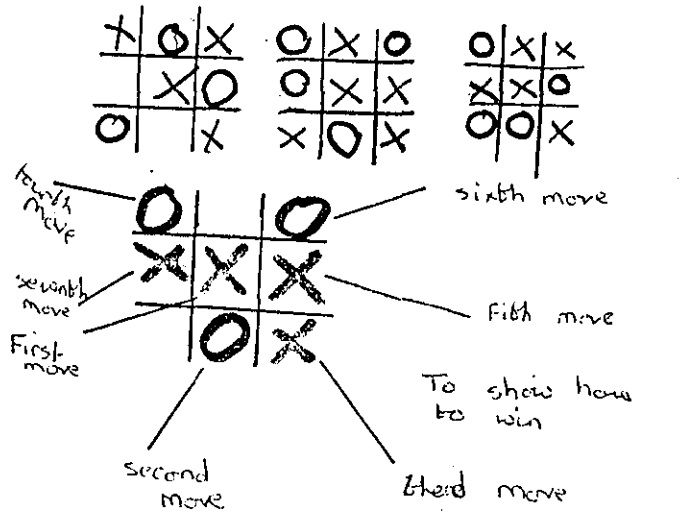


The next move would make is by Stopped the line.



Then I would Stop there line.

Making me win



I would rather go first because I could go in the middle and win Like the diagram and go in any direction

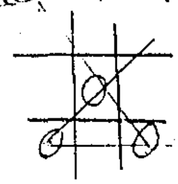
If you look at my diagram you can see that the opinent could win straight after me

Rules for how to win

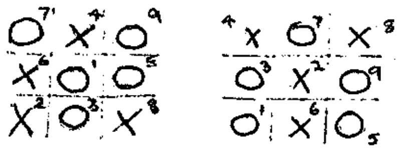
1 If you Start 1st put it in the middle.

2. If you start 2nd and the middle is already taken put it in one of the corners.

3. Try to do this:-



Because you can get it in 3 different ways



Noughts + Crosses

The strategy is that if you go 2nd and someone goes in the middle, then you go diagonaly, then block in turn to each move, this blocks all his moves so no-one can win. as long as you do this you won't lose..

15 TO WIN

9+3+6=?

1+4+8=?

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$

Try this game using the cards in your envelope.

Spread the cards out so that all the numbers are showing.

You and your partner now take turns to choose a card from the set. You must leave your cards in front of you with the numbers showing so that your opponent can see them. The idea is that you try to get a set of any 3 which add up to 15, the first to do this WINS!!

Play the game several times.

You may notice something familiar about this game.

Write down any ideas that you have about this game.

3+1+4=?

5+3+7=?

15 TO WIN

This game is an isomorphism of noughts and crosses. If we consider the possible winning triples then we obtain the following eight sets:

(6,1,8)	(7,5,3)
(2,9,4)	(6,7,2)
(1,5,9)	(8,3,4)
(6,5,4)	(8,5,2)

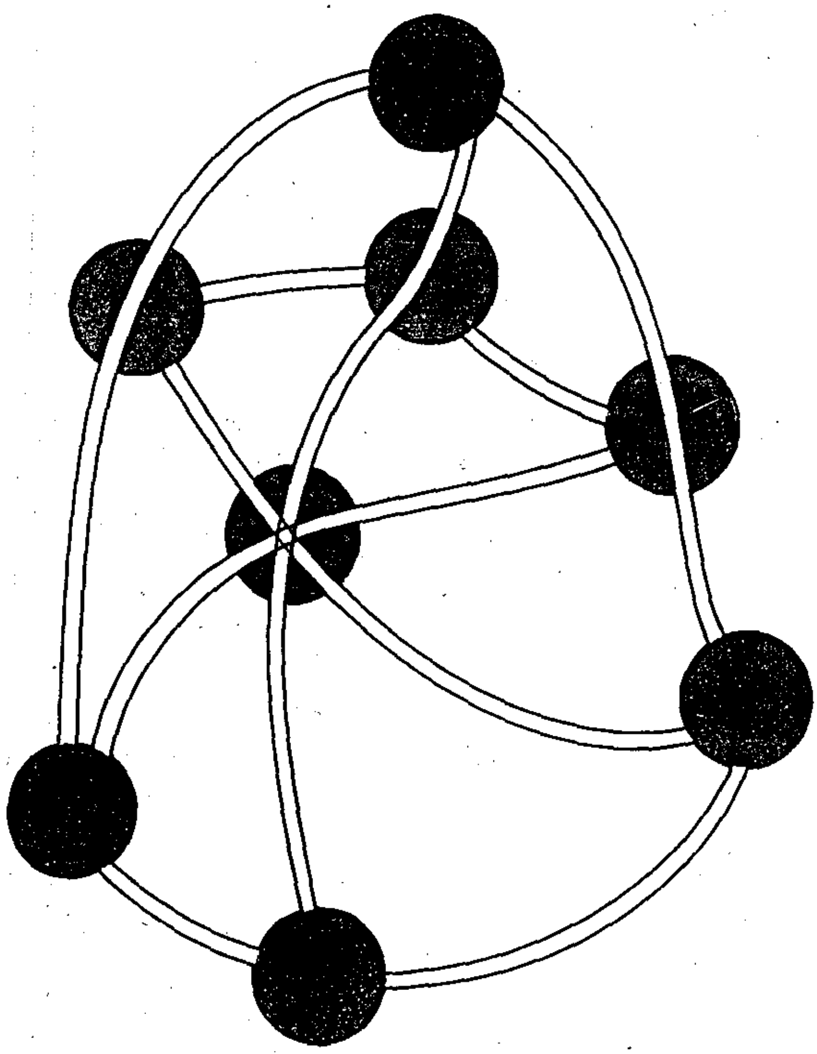
These sets may then be displayed within a magic square as:

6	1	8
7	5	3
2	9	4

The strategy for this game is therefore also isomorphic to that of Noughts and Crosses. If we consider the one-to-one correspondence between the numbers and the squares that they occupy within the magic square, then we have the strategy for this game, since it is identical to the one outlined for Noughts and Crosses.

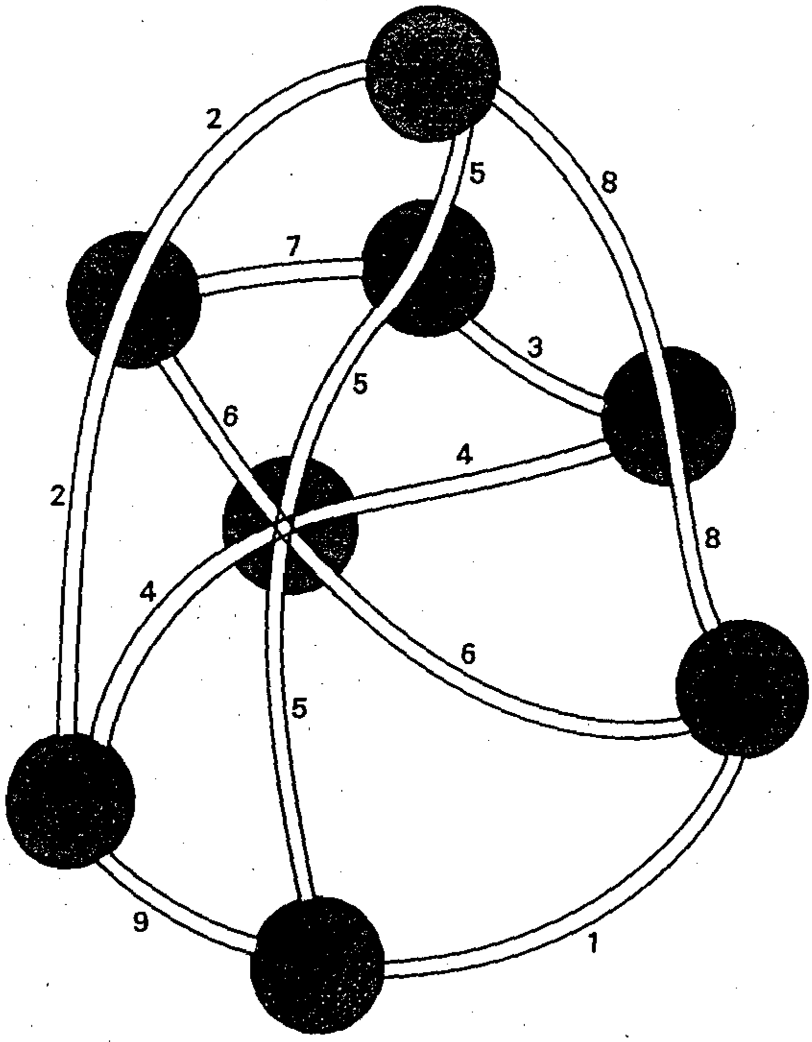
There are in fact other games which are also isomorphic to the game of Nought and Crosses. To briefly mention two: Jam, a game invented by the Dutch psychologist John. A.. Michon; and Hot, a game invented by the Canadian mathematician Leo Moser.

Jam involves two players playing on the following map and trying to jam any one of the 8 towns by colouring in all three roads that enter or pass through that town.



Map for the game of Jam

The isomorphism is easier seen if the road numbers are placed on the map as follows:



Map for the game of Jam

Again we have eight winning combinations i.e. towns and the three related road numbers, which always sum to fifteen. The map for Jam is in fact topologically equivalent to the dual of the Noughts and Crosses network.

Hot is a word game which has been copied by many, simply by using different sets of words. You are given the following list of words:

HOT

HEAR

TIED

FORM

WASP

BRIM

TANK

SHIP

WOES

Players take turns to select the words and the winner is the first to hold a set of three which contain a common letter. These words, like the digits in 15 To Win, may be arranged in a magic square.

HOT	FORM	WOES
TANK	HEAR	WASP
TIED	BRIM	SHIP

Again there are eight winning sets of three as would be expected since this is a condition of the isomorphism.

Generally, pupils did not see the relationship between 15 To Win and Noughts and Crosses although the magic square link was noticed by some low ability fourth year pupils who had recently studied the topic. They did, however, quickly pick up the idea that they had not only to get a score of 15 themselves, but also prevent their opponent from doing so, an identical strategy to that used in Noughts and Crosses. Some pupils, however, found the idea that you have to let your opponent see your numbers at all times rather strange, since this seems to go against all the games that they have ever played in the past. This in fact led one group to play a totally different game, one of total chance and similar in a way to Bingo. They simply picked the cards blindly and at random, and they had to again achieve a total of 15. This is certainly not a game of strategy but it does demonstrate an earlier point about how easy it is to make up simple concept or skills games, the skill here being basic addition.

All of the games belonging to the group which is isomorphic to Noughts and Crosses may be used when magic squares are being studied since they offer a very valid contribution to the work in this area of mathematics.

Ways to win 861
195
456
276
483
753
492
852

These are all I think.

15 To win

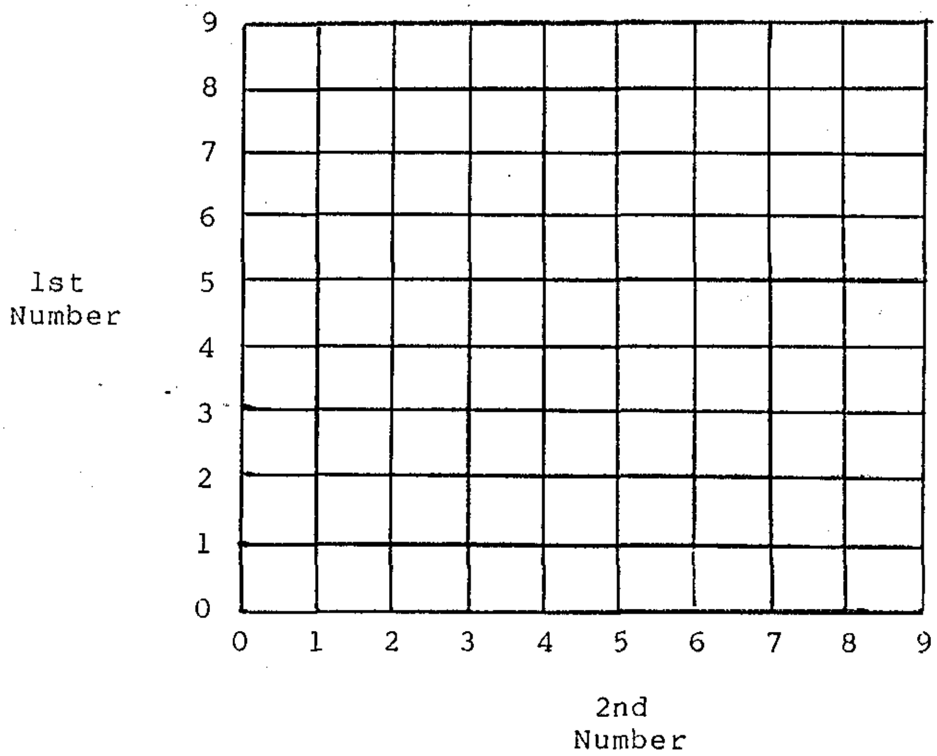
You have to stop the other peqson from getting 15 befer you. to win you have to look at what your parten has got and try and stop them from getting the number they want to win.

I would rather go first because You can get the numbers You want. You have more chance of wining.

FLEET

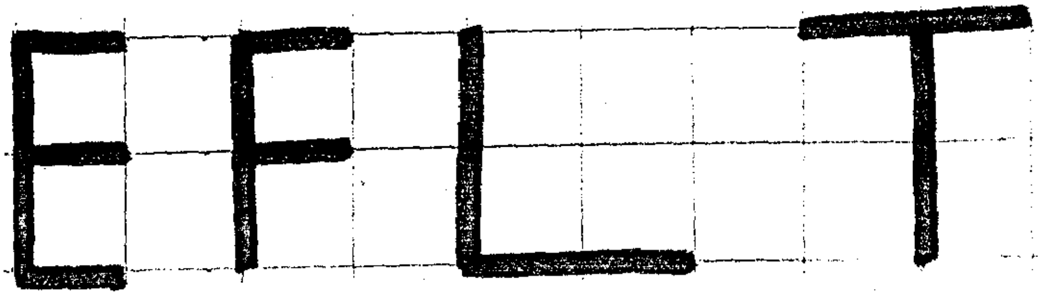
This is a game which is very similar to BATTLESHIPS, but there is one major difference which I will leave you to discover for yourself.

Fleet is played on a grid like this:



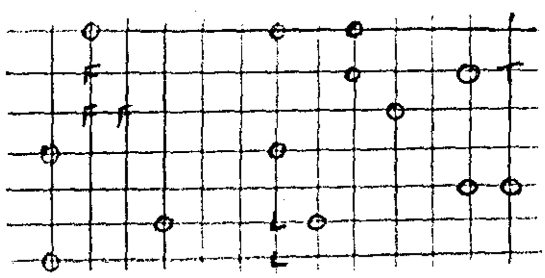
You have to hide your FLEET and try to find the fleet of your opponent.

Write the word FLEET on your grid using letters like these:



You then take turns to guess a position using coordinates. You will need 2 grids, one to hide your own fleet and one to help you find the opposition fleet.
When you have made a guess put an o on the point if you have not hit and put the letter that you have hit on the point if you have been successful.

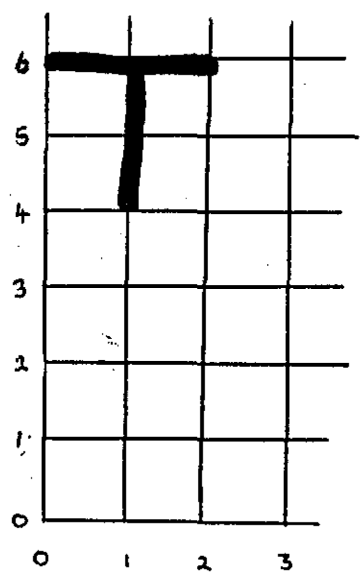
Eg



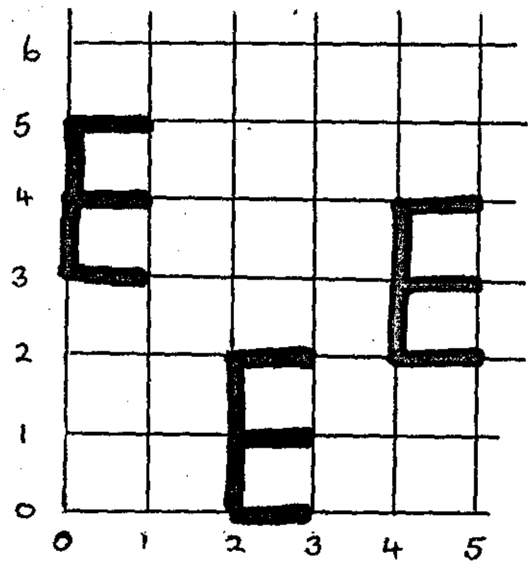
The WINNER is the first to destroy the opposition FLEET.

FLEET

Many may argue that this is not in fact a game of strategy but totally a conceptual one. However, I would disagree with this, although I would agree that its conceptual contribution may be greater than its strategic. There is a low level search strategy involved in this game and one which is extremely useful with problem solving in mathematics, guess and check. When pupils have made a correct guess then another form of thinking is initiated, that of searching around the area for the remaining related points. Such a search will lead to interesting discussion between children, if they are working together, as to the validity or necessity of the choice of further points. The idea of spotting a pattern and testing it is linked with physical restrictions, i.e. if a T point is found to be at the point (0,6), can another point of it be at (0,5) or can we say for sure that we know the exact point of the T that we have found.



The same ideas apply to all the letters, eg. if an E point is found somewhere in the centre of the grid then there is a 1 in 6 chance of realising which point of the E that it is. However, if this discovered point lies on the vertical axis then the assumption chance is immediately halved to 1 in 3 and the chance drops even further if the discovered point lies on the horizontal axis.



These ideas lead to a defence strategy to prevent the opposition from being able to make such elimination assumptions.

This type of game also emphasises, as do many of the guess and check type, the need to record both the guess and feedback, so that the information gained may be used efficiently at a later stage. Again this is an important aspect of mathematical problem solving.

Some may say that this game is no different from the traditional one of Battleships, which is regularly used in the classroom to introduce coordinates. However, the game of Battleships usually refers to squares by labelling the axes in between the grid points, a general misconception by many pupils in coordinate and graph work. This game does not set up such a misconception and therefore is perhaps one of the most simple but important in the collection.

The game may also be extended to include work on vectors by stating that each player must supply their next guess by giving a vector displacement from their last, and the starting point is assumed to be the origin. The game could, perhaps, also be used with work on directed number.

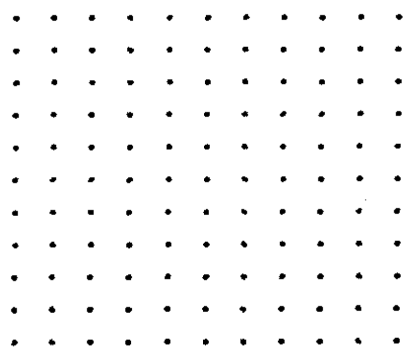
This game was used very little during the trials, mainly because the game of Battleships had previously been used by many teachers with most children for which this game would be suited. However, the general point about the labelling of the axes was discussed and accepted and the game may therefore be used in place of, or in conjunction with, Battleships in the future.

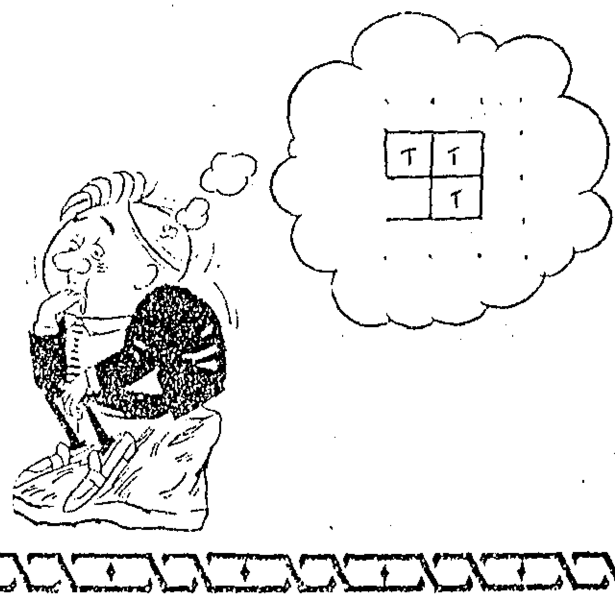
DOTS AND BOXES

This is a very popular game which you may have played before.
It is a game for 2 players and you take turns to join up any 2 dots next to each other, either with a vertical or horizontal line. The idea is to complete more boxes than your partner. If you complete a box then you put your initial in it. The winner is the player with most boxes when the playing sheet is full.

Play the game on a sheet which will give 10 boxes by 10 boxes. Notice how many dots there are in each row and column, can you explain this ?

Can you write down any ideas on how to play the game well ?



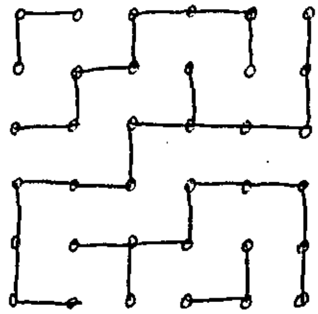


DOTS AND BOXES

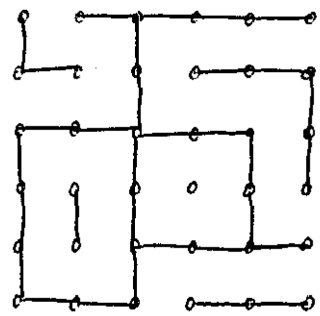
This is a traditional and well known game played by many generations with great enjoyment and competition. Some may be unconvinced that this is in fact a game of strategy and would state that it is little more than a game of luck, I would argue otherwise.

The initial ideas on strategy here may be thought about along the lines of who is going to win once the state of full partitioning has been reached, this is the state of the game when all possible moves will give away a chain of boxes. Assuming that the difference in the players scores at this stage is minimal, then the winning player depends on whether there is an odd or even number of chains.

If there is an even number of chains, then we can consider them in rank order of length and then paired off from the smallest. This implies that the player who opens up the first chain is the winner since they will choose to give away the smaller chain within each pair and accept the larger, thus doing no worse than a draw within each pair. If the number of chains is odd, then it is the person who takes the first chain who is the winner, since once they have done so then they are left with the paired state of partitioning. This strategy would imply that a player should try to get control of the game by juggling the number of chains to be odd or even, as they wish, once the partitioned state is being approached. Two examples of fully partitioned games using smaller starting grids are given overleaf:

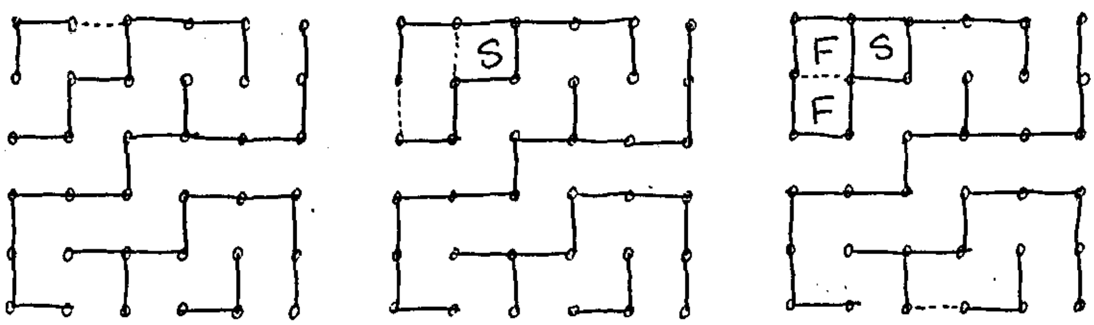


4 Chains



5 Chains

Let us now consider the double dealing strategy. This is where a player turns down the last two squares of any long chain, of length greater than three, in such a way that the opposition is forced to take the last two boxes and then open a new chain which will again be taken with the exception of the last two. Provided that there are sufficient long chains, then the first player to open a chain will lose, since the other player will then adopt the double dealing strategy. The full strategy here must therefore be to ensure that there are long chains and then to get control of the game by forcing the opposition to open the first chain when the partitioned state is reached. This is a higher level strategy than the first since it will always win against it.



The example given, shows the first player opening the shortest chain in an even partitioning of the game and therefore expecting to win by the first strategy. However, the second player is a double dealer and takes only one box to leave two for the first player, who has no choice but to accept them and open up the next shortest chain where the second player will once again double deal. This will continue and the game will be won by the second player with a large margin in the scores.

There is, however, still the problem of getting control. This is again related to keeping an eye on the number of chains when the state of full partitioning is being approached. It can be shown that:

NUMBER OF STARTING DOTS

\+ NUMBER OF LONG CHAINS

= TOTAL NUMBER OF TURNS + 1

$$D + C = T + 1$$

$$T = D + C - 1$$

This implies that each player will try to create either an odd or even number of long chains, those of length three or more, according to the number of starting dots and whether they are playing the odd or even moves.

This game was not played a great deal during the trials and when it was, the size of the game was probably too large since it took a long time to play. This prevented games being played repeatedly and it is really only when this occurs that any strategy development takes place. However, pupils were aware of the ideas that the early part of the game should be spent setting up chains and when the fully partitioned state was reached, then they should give away the shortest ones first. None, however, thought about trying to get initial control of the partitioned game or came up with the double dealing strategy. Many often started the game by trying to keep out of trouble by playing around the edge so that their opponent could not complete any boxes early on. One small group then went further and divided the game down the middle.

This game can also be linked, as it is on the sheet itself, with a small investigation looking at the number of spaces between a given number of dots.

I think that first you should try and make all of the corners then when its your go and you cannot draw a line without your partner making a square ~~so~~ you should look for a place ~~to~~ to draw a line where your partner cannot get too many squares. You should then win But you can't win if your partner has the same idea.

First try to do, all of the outside lines then the other player has a harder time wining

It is best to draw lines which will not give your partner to many squares.

first of all join the dots round the edge Second, when all the lines surrond the outside start doing the lines in side so only 2 lines are need to win boxes When all the spare dots have gone and if it is your go look where you opoinant has not go many boxes toget.

Try a make your partner gain the fewest boxes because then its there go and they will probably have to give you the most boxes.

CENTURY

Have a go at this game!

You can choose any number from 1 to 10 to add to the running total of the game. Starting at zero, you and your partner take turns adding to the total and the one who gets the CENTURY (Total=100) WINS!!!

Play the game several times to see who can score the most centuries.

Discuss with your partner any ideas you have on this game and try to write about them in your own words.

Now, you and your partner use your ideas to play against another pair.
Did you win ?

100

CENTURY

This game is typical of a family of 'add on and take away' games that exist. In this particular game, with the target number being 100 and the add on numbers available being 1 to 10, then it can be seen with relative ease, by a mathematician, that the number 89 is in fact a strategic number in the sense that the player who first reaches this number should win. This then has the effect of reducing the game to first to 89 and by similar reasoning, the game may be further reduced to 78 and hence a series of strategic numbers are set up, these being:

100
89
78
67
56
45
34
23
12
1

This does of course reduce the game to such a state that it is the first player who always wins, provided that their first choice is 1 and further choices simply take the game total to the next strategic number, this will lead to them winning on the 19th move every time. However, the experienced player playing against the novice need not give the strategy away by playing strictly to it throughout the whole game and may still win by adopting the strategic numbers only at a late stage in the game.

In general, all such games may be analysed into a set of strategic numbers, which depend only on the starting number and the range of numbers available for adding on. If the game is played with a target number of A and numbers must be chosen from 1 to N, then the strategic numbers will be:

$$A-(N+1)$$
$$A-2(N+1)$$
$$A-3(N+1)$$
$$\vdots$$
$$A-k(N+1)$$

Where k=Integer value of A/(N+1)

This game will always be won by the first player unless A divides exactly by N+1 in which case the game will be won by the second player.

$$\text{i.e.} \quad A=k(N+1)$$

Example:

In the game 'First to 65' and choosing numbers from 1 to 5, then

$$k=\text{Int.}(65/6)=10$$

and hence the first strategic number is

$$65-(10X6)=5$$

and the full set of strategic numbers is:

65

59

53

47

41

35

29

23

17

11

5

The misere game, i.e. the first person to reach 100 loses can be analysed in exactly the same way, the stategic set being:

99

88

77

66

55

44

33

22

11

0

This has a general solution:

$$A-1$$

$$A-1-(N+1)$$

$$:$$

$$:$$

$$:$$

$$A-1-k(N+1)$$

Where as before:

$$k=Int.(A/(N+1))$$

This implies that the winner is again the first player, provided that the first strategic number is not 0,

$$i.e. \quad A=k(N+1)+1$$

as it is here in the misere game of century, in which case the second player is the winner.

A player is only in a safe position in any such game if they leave a stategic number, if they are in an unsafe position they can only win if the opposing player makes a mistake and their best strategy is therefore to try to make the game last as long as possible, hence giving the opposing player more time to make the required mistake. There is, however, no way of extending this type of game against an expert strategic player.

Another game in the same family, but of the take away type, is the one which requires players to remove stones from a pile where the number in the pile is known and the players are only allowed to take a number within a given range.

Example:

Two players take turns to remove either 1,2 or 3 stones from a pile of 25 and the winner is the person who removes the last one. The winner in this particular game is the first player, who will win on the 13th move provided that they use the strategic numbers:

24

20

16

12

8

4

The general solution being:

$$k(N+1)$$
$$(k-1)(N+1)$$
$$:$$
$$:$$
$$:$$
$$(N+1)$$

Where $k = \text{Int.}(A/(N+1))$

The winner will always be the first player unless

$$A = k(N+1)$$

in which case the second player will win. This is in fact a similar position to the misere game of century.

The misere game of this may also be played and generates the strategic number set:

25

21

17

13

9

5

1

This shows that the second player wins the game, however, this is only because

$$A=k(N+1)+1$$

If this condition is not true then the first player will always win.

This, in my opinion, is an excellent game and proved to be very popular in the classroom since it could be used at various levels; from a number game with young pupils, who may only discover the importance of the number 89, through to using it with older and more able pupils who could be asked to tackle the problem of strategy development including a general algebraic solution, as given above.

Many pupils realised the significance of the number 89 but failed to look further until asked to play the new game of first to 89. However, others, of various ages and abilities, realised the existence of the full set of strategic numbers and some were able to write these numbers down together with a reason as to why they were the important ones and therefore a strategy as to how one should play the game of Century.

I noticed if you start with 1 and then if the other person does, say 8, then you do 3 so the total is always 11. In other words try and get the first digit 1 less than the second digit. You can't lose.

89 is a speshal number Becows The over player has to go 1 or more so The over man can win

The person who gets to 89 knows they have won because the next person has got to get 10 or less and they need 11 to get to 100 e.g.

R = ~~89~~ 6 = 89   L + 6 = 89
L = 10 = 99   R + 1 = 90
R = 1 = 100   L + 10 = 100.

The ~~pero~~ person who goes first has won because they allways get to a certain ~~note~~ number 89, 78, 67, 56 45, 34, 23, 12, 1

I have won three times because I have got to 89 first.
The reson for this is because their is only 11 number left and the person who ~~you're~~ playing against can ~~only~~ only choose from 1 to ten so what ever numbers they pick you can have the remainders.

Here are the numbers that you have to say if you are starting with 1 = 1
12
23
34
45
56
67
78
89.

89 is a stratigic number. Ayone who gets to 89 first is the winner of the game because you cant go higher than ten and you cant go lower than one.

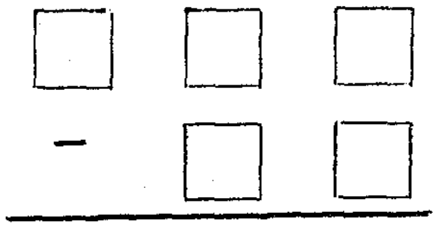
If the person who goes first keeps to the stratigic number they will win.
The stratigic numbers are the numbers that are eleven below the number you are aiming for.

These are the stratigic numbers
100.
89
78
67
56
45
34
23
12
1

SUBGAME

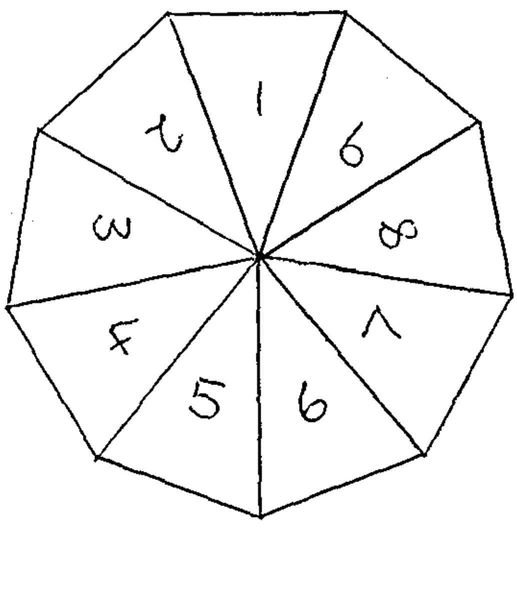
You may have played SUBGAME against the computer, if you haven't, then don't worry, here are the RULES.

You are given a subtraction sum like this:



You are then given five random numbers between 1 and 9, one at a time. (These are called digits.) Each time that you are given a digit, you have to decide which box to put it in, it cannot be moved after you have placed it in a particular box. The idea is that you have to get the highest answer that you can. The winner is the person who gets an answer which is bigger than anyone else's. You can either play this game against the computer or your friends.

Now that you all know the rules, play the game in pairs with the help of a 9 sided spinner like this one:



Using the Subgame grid and the spinner (to produce the numbers) see who can get the highest answer for the subtraction sum.
When you have played the game quite a lot write down in your own words the method that you used to try to win. A method like this is called your STRATEGY.

How would you change your strategy if you were trying to lose the game, in other words, you were trying to get a lower answer for your subtraction sum than your friends ?

SUBGAME

Subgame is a basic game of strategy developed on the idea of a subtraction sum. However, once again it has a significant conceptual contribution to make. The game is relatively simple and there is a clear strategy which pupils tend to pick up quickly and find little difficulty in expressing.

$$\begin{array}{cccc} & \boxed{a} & \boxed{b} & \boxed{c} \\ - & & \boxed{d} & \boxed{e} \\ \hline & \boxed{} & \boxed{} & \boxed{} \\ \hline \end{array}$$

The ideas behind this game are similar to that in Hat Games as discussed in the last chapter. The pupils have to use the random digits to produce either a high or low answer. The game, apart from offering conceptual help with subtraction, also allows valuable discussion on place value. The strategy used will, to a certain extent, depend upon personality, with pupils perhaps showing themselves as either a 'safe' or 'gambling' type of player. The strategy for this game may be expressed in a tabular form and pupils could be encouraged to use this form of representation for their solution. The strategy used by the computer in this game is given on the following page:

NUMBER	HIGHEST	LOWEST
1	DECBA	ABCED
2	DECBA	ABCED
3	EDCBA	CBAED
4	EDCBA	CBAED
5	EDCBA	CBAED
6	CBAED	EDCBA
7	CBAED	EDCBA
8	ABCED	DECBA
9	ABCED	DECBA

The strategy table above, shows the order of box selection for each digit. It is, I feel, the placement order of the middle digits that show whether a pupil is a gambling or a safe type of player. Clearly the strategy for the lowest game is simply the reverse of that for the highest.

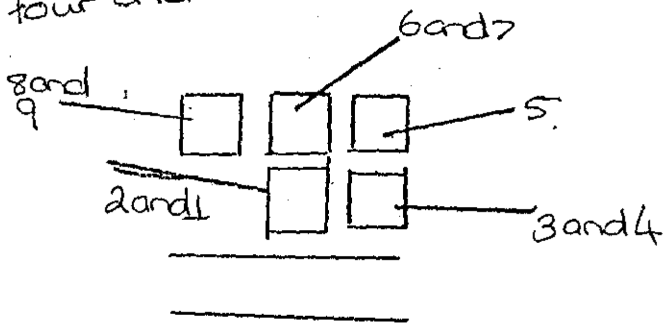
I have emphasised throughout this dissertation the importance of an integrated approach to mathematics teaching and this game, in my opinion, is a good example of what I mean. It offers basic number work, an investigation into the game strategy, a piece of practical geometry work in the construction of the spinner and, of course, the generally practical and enthusiasm building game. If this game is used correctly, then it offers a great deal of both introductory and follow up related work and therefore it is a game which may be used in an integrated form.

I have mixed feelings as to whether this game is in fact best played with or without the computer. The computer offers a colourful screen but may actually be guilty of teacher lust in the sense that pupils may be told the strategy by copying that used by the computer. Playing the game against friends offers the extra aspect of making the spinner to produce the random digits and the copying aspect does not exist to the same extent. I think that the computer is an excellent introduction to the game and then the game may be continued away from the computer. It is always useful if software can be used in this manner since most classrooms will only have the use of one computer at any time and, in my view, this situation is not likely to change in the near future.

This game, like the next, Multigame, was not used a great deal in the trials. This was due to the fact that the game is used to quite an extent normally by the teachers involved. However, when it was used the pupils found very little problem in expressing their strategy either orally or on paper. This was about as far as the game was used in the trials and pupils were not asked to develop a strategy table, like the one given on the previous page.

We think the method of this game is to put the highest numbers on the top line and the lowest numbers on the bottom line.

In the subgame if we get an eight or nin.. we
Put it in the first bot on the top, a six
and a seven goes in the middle box on the
top and the five in the last bot on top.
on the first box below we put a two or
one and the last box below has numbers
four and three.



If you get a high number you should put that in one of the top squares and if you get a low number you should try and put it in one of the bottom squares On the highest numbers say if you get 8 you should try and put them to the left.

If the number is four or less it should go on the bottom row. if the number is five or more it should go on the top row.

eg.

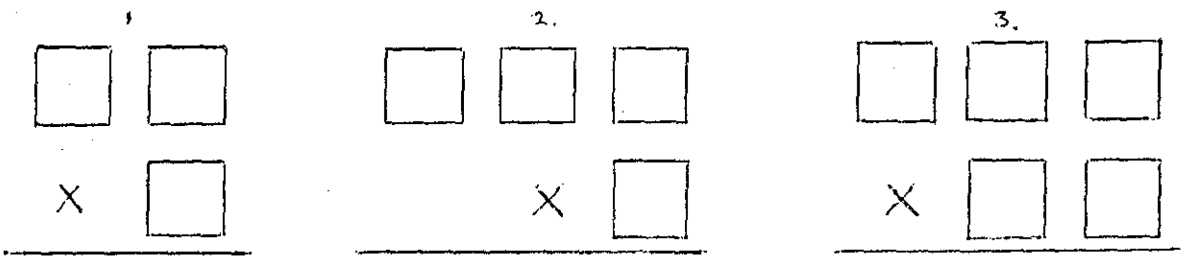
$$\begin{array}{r} \boxed{9}\ \boxed{7}\ \boxed{5} \\ -\quad \boxed{1}\ \boxed{3} \\ \hline 9\ \ 6\ \ 2 \end{array}$$

MULTIGAME

MULTIGAME is an exciting game about multiplying that you can play against the computer or your friends.

You are given some random numbers between 1 and 9 and you have to make them multiply together to give an answer bigger than that of your opponent.

If you play this game against the computer, then there are three levels of difficulty:



However, there are some more rules that you need to know before you can play this game. You only get one number at a time and you have to decide which box to put this into before you get the next one. Once you have decided on a box then it has to stay there. The same numbers may come up more than once !!!

Play the game against either the computer or your friends, you will of course have to work out a way to get your random numbers for the game. You may like to start with level 1.

Play the game lots of times.

Now that you have played the game on level 1 several times, can you write down your ways of trying to beat your friends ?

If all the boxes were empty and you got a 1 where would you put it ?

How about a 2 ?

What about the other digits up to 9 ?

What if you got one of these numbers and that box was full, which box would you use instead ?

Put your ideas into a table like this one:

Number	1st choice	2nd choice	3rd choice
1			
2			
3			
4			
5			
6			
7			
8			
9			

This table shows your STRATEGY or plan of attack.

Can you outline a strategy for playing the game on level 2 ?

What about level 3 ?

Can you support your strategies mathematically ?

MULTIGAME

Multigame is a natural follow on from Subgame. A similar game could be played with the operation of addition but it would be rather trivial and the game using division would, I feel, be over complicated with the problem of dealing with a remainder. Therefore, I feel, the only real extension game is the one using the multiplication operation. The game's strategy can again be expressed in a tabular form, as it was for Subgame. The strategy, while again seeming rather obvious for the first, and maybe the second, level, is certainly not as obvious as one may first think for the most difficult level. It is my view, that this, like Subgame, really is an open ended game since it may be played with very young children learning to multiply through to very able and older pupils trying to investigate a best strategy, with the use of an algebraic method. Such an investigation may be started along the lines of considering the expression

$$(100a + 10b + c)(10e + d)$$

$$\begin{array}{ccc} \boxed{a} & \boxed{b} & \boxed{c} \\ \times & \boxed{e} & \boxed{d} \\ \hline \\ \hline \end{array}$$

(Notice that the d and e boxes are in a different order here than they were for Subgame. The reason for this was ease of programming the three levels.)

Similar investigations may be carried out for the lower levels of the game to justify the intuitive strategies that pupils develop. It was in fact a totally intuitive approach which was used during the trials with no consideration given to the more advanced aspects of the game, this again being mainly due to a lack of time. The strategy tables that I used when writing the software for this game are as follows:

LEVEL 1

NUMBER	HIGHEST	LOWEST
1	CBD	DBC
2	CBD	DBC
3	CBD	DBC
4	BCD	BCD
5	BCD	BCD
6	BCD	BCD
7	DBC	CBD
8	DBC	CBD
9	DBC	CBD

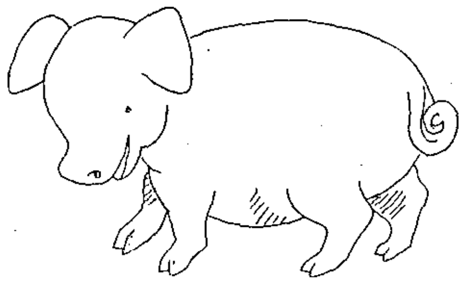
LEVEL 2

NUMBER	HIGHEST	LOWEST
1	CBAD	DABC
2	CBAD	DABC
3	BCAD	ADBC
4	BCAD	ADBC
5	BCAD	BCAD
6	ADBC	BCAD
7	ADBC	BCAD
8	DABC	CBAD
9	DABC	CBAD

LEVEL 3

NUMBER	HIGHEST	LOWEST
1	CBDAE	EADBC
2	CBDAE	EADBC
3	BCDAE	AEDBC
4	BCDAE	AEDBC
5	DBCAE	DBCAE
6	AEDBC	BCDAE
7	AEDBC	BCDAE
8	EADBC	CBDAE
9	EADBC	CBDAE

Although this game was not used much during the trials, when it has been used in the past, it has generated a great deal of mathematics beyond the ideas of multiplication and development of a game strategy. Various ideas on probability, the generation of random numbers and calculator work may be included as well as the more obvious skills of problem solving, investigation and discussion. I have not included any pupil work from this game since the majority of the work was clearly based on discussion and the written part was not significantly different from that of Subgame.



PIG

A dice game for two or more players.

The aim of the game is to be the first player to score over 100 .
You will need 2 dice and a piece of paper to score on.

When it is your turn, roll both dice. If you score a six with either dice (or both) your score is zero for that turn and you must pass the dice on to the next player. If neither of your two numbers is a six then add the two numbers together. You must now make a decision, either you record this total as your score for this turn and pass the dice on to the next player or you roll the two dice again. On this second throw the same rules apply, if neither number is a six then the two numbers are added to your score for the turn so far. However, if either number (or both) is a six then your score for the turn is immediately set to zero and your turn is over. So long as you do not roll a six you may continue rolling the two dice and adding to your score for the turn. The winner is the first player to reach a total score of 100.

Play the game a few times against your friends either in pairs or small groups.

Keep your score carefully.

Eg.

SCORES ON DICE	SCORE FOR TURN	TOTAL FOR GAME
(4+2),(1+5),(4+3),(6+4)	0	0
(3+5)	8	8
(4+4),(1+3)	12	20
(6+5)	0	20
(4+1),(2+2),(3+4),(5+2)	23	43

ETC...

Can you suggest any ways to improve your chances of winning ? This is called a STRATEGY. Does your strategy make sure that you never lose ? Write down your ideas on the game.

PIG

This is probably one of my most favourite games of all that I have included in this selection, for two reasons. The first must be because of its popularity within the classroom, to the extent that it became a major knock out event on the annual school camp, and secondly because it can be played with intuitive strategies by almost all children and these intuitive strategies are so close to the formal mathematical strategies which may be developed.

The development of any strategy for this game, in my view, must either be based on deciding an optimal number of throws or an optimal declaring score.

When looking for a 'no. of throws' strategy we need to take account of the expected scores.

For 1 throw.

We score 0 with probability $\frac{11}{36}$

If S is the expected non zero score, then we score S with probability $\frac{25}{36}$

Hence Expected Score $=\frac{25}{36}\times S+\frac{11}{36}\times 0$

$$=\frac{25S}{36}$$

For 2 throws.

$$\text{Expected score} = \left(\frac{25}{36}\right)^{2} 2S$$

For N throws.

$$\text{Expected score} = \left(\frac{25}{36}\right)^{N} NS$$

We can now maximise this function by normal methods:

$$E = \left(\frac{25}{36}\right)^{N} NS$$

$$\log E = \log \left(\frac{25}{36}\right)^{N} NS$$

$$= \log \left(\frac{25}{36}\right)^{N} + \log N + \log S$$

$$= N\log \frac{25}{36} + \log N + \log S$$

$$\frac{d\log E}{dN} = \log \frac{25}{36} + \frac{1}{N}$$

Stationary values occur when $\frac{d\log E}{dN} = 0$

$$\log \frac{25}{36} + \frac{1}{N} = 0$$

$$\frac{1}{N} = -\log \frac{25}{36}$$

$$\frac{1}{N} = \log \frac{36}{25}$$

$$N = \frac{1}{\log 1.44}$$

$$N = 2.74 \text{ (3SF)}$$

Therefore the optimum number of throws for the game of Pig is 3. This gives an expected score of:

$$\left(\frac{25}{36}\right)^{3} \times 3 \times S$$

However, since S is the expected non zero score

$$S = 2\times\frac{1}{25} + 3\times\frac{2}{25} + 4\times\frac{3}{25} + 5\times\frac{4}{25} + 6\times\frac{5}{25} + 7\times\frac{4}{25} + 8\times\frac{3}{25} + 9\times\frac{2}{25} + 10\times\frac{1}{25}$$

$$= \frac{1}{25}(2+6+12+20+30+28+24+18+10)$$

$$= \frac{150}{25}$$

$$= 6$$

This gives an expected score for 3 throws of 6.03 (3SF) and hence the game will be expected to last for approximately 17 throws each. The table given on the following page shows the expected score using the 'no. of throws' strategy for 0 to 10 throws.

No. of throws	Expected score
0	0
1	4.17
2	5.78
3	6.03
4	5.58
5	4.85
6	4.04
7	3.27
8	2.60
9	2.03
10	1.57

(All expected scores are given to 3sf)

A question must be raised at this stage, about how low these expected values are, and this may lead one to think that perhaps a player may be better served by a 'fixed declaring score' strategy. However, on examination of such a strategy, we have again to take account of the probability of actually scoring, since this effects the expected score with any such strategy.

If we let SN be the strategy of declaring as soon as a score of N is reached.

Then we have:

$p(scoring:S0)=1$

Minimum expected score $=0 \times 1=0$

$p(scoring:S1)=\frac{25}{36}=0.694$

Minimum expected score $=1 \times 0.694=0.694$

$p(scoring:S2)=\frac{25}{36}=0.694$

Minimum expected score $=2 \times 0.694=1.39$

$$p(scoring:S3)=p(>3 \text{ on the 1st throw}) + p(2 \text{ on the 1st throw}) \times p(scoring:S1)$$

$$=\frac{24}{36}+\frac{1}{36}\times\frac{25}{36}=0.686$$

Minimum expected score $=3 \times 0.686=2.06$

If we continue in this manner, then a recursive relationship is developed.

If we let $p(:K)$ mean $p(scoring:SK)$

and $p(N)$ mean $P(\text{scoring the number } N)$

then we have the following relationship:

$$p(:K) = p(10)Xp(:K-10) + p(9)Xp(:K-9) + \dots\dots\dots\dots + p(3)Xp(:K-3) + p(2)Xp(:K-2)$$

$$= \frac{1}{36}p(:K-10) + \frac{2}{36}p(:K-9) + \dots\dots\dots\dots + \frac{2}{36}p(:K-3) + \frac{1}{36}p(:K-2)$$

$$= \sum_{i=1}^{5} \frac{ip(:K-11+i)}{36} + \sum_{i=2}^{5} \frac{(i-1)p(:K-i)}{36}$$

This may be solved by a computer to give a maximum expected value of 5.11 which is the minimum expected score relating to the strategy of declaring as soon as a score of 17 is reached.. However, we can see that this strategy is not as good as the one of always having 3 throws. It may, however, be the case that a mixed strategy will be more appropriate than either separately, this certainly seemed to be favoured by the pupils when they played the game.

Usually, the pupils suggested the fixed number of throws strategy, with about two or three throws, but sometimes suggesting that you stop after only two if you have a particularly high score. The fixed score strategy, when suggested, was linked with scores of between 10 and 15. Most pupils were aware that they had to weigh up the chances of increasing their score against that of moving to a score of nought and used this idea as the basis of their intuitive strategy.

This game, like most in the package, may be linked up with various preparation and follow up work, mainly with ideas on probability and chance. Variations on this game may be introduced by changing the number of dice. The question may then be asked whether a change in strategy is required and if so then in what way.

I think that the dice usually
lands on a 3 or a 4 it usually
with both dies add up to 7
It is best to stick after 2 throws
because then is there hasnt been a six
there will probably be a 6

I think it is best to ~~get~~ go one if it is low try again but I wooldn't ~~g~~ do more than 4 gos becaose the six is boun to pop up in the end. I think that the best numbers to get are double 5 5 and 4 quite the high ones but if it is low it doesnt matter because it is better to be on the safe side than to get a six so it doesnt matter if your score is low but I think the game is good and very interesting because you don't know if to have another go that is why I like it very much

The best idea is to stick on between 12 and 15.

The best way to win at pig is to keep low numbers of throws about up to 3 throws or the number 13 is quite good.

Matthew won because he did one at a time and I gambled

Ryan won because he gamble paid off

I think that a Six has a chance in 1 in 3 to come up.

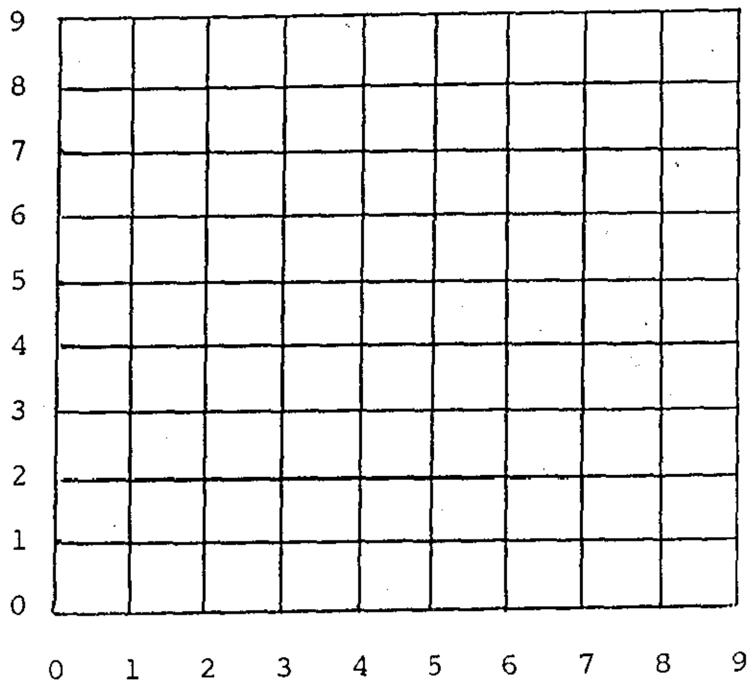
I always stick when I have got more than ten More often than not you get a high Score quite quickly.

I Could improve my chances of winning when I get to about 3 different numbers without getting a six thats how I could improve my chances of winning.

The best way to win at Pig's is until you get a reasonable Set of Numbers and then give up.

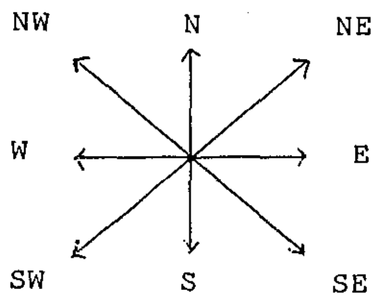
HUNT THE HURKLE

THIS IS A GAME FOR 2 PLAYERS.



The HURKLE is a strange creature and it hides in a grid and it's position is one of the grid points. These positions are given as a pair of coordinates.
Eg (5,3) meaning across to 5 then up to 3 .

One player is to be the Hurkle and the other is the hunter. The Hurkle decides where it is going to hide and the hunter has to make guesses to find the position of the Hurkle. After each guess the Hurkle gives the hunter a clue by saying the direction that the hunter has to travel to find him. If the direction is N,S,W OR E then this is the exact direction whereas the other 4 directions are only rough directions.



Guess how many goes you would need to catch the Hurkle if you were the hunter ?

What about if the numbers went up to 99 each way ?
What if they went up to 999 ?
When you play the game, write down your guesses and also the direction that you have to follow, this will help you to remember what guesses you have already made. Here are the details of a game that Mark and Sue have already started:

Sue is the hunter.
Mark is the Hurkle.

Sue	Mark
(8,2)	SW
(2,1)	NE

ETC...

Play the game several times and keep a note of how many goes each game took.

Game	1	2	3	4	5	6	7	8	9	10
No. of goes										

What is the average number of goes ?

How close was your guess ?

Write down anything that you have found out or worked out about this game.

Would you like to change your guess about how many goes you would need if the numbers went up to

a) 99 ?
b) 999 ?

What if the grid was not square, perhaps 0-24 across and 0-1999 up, now how many goes would you need ?

Is there any way to work this out ?

HAPPY HUNTING!!!

HUNT THE HURKLE

This is another of my long standing favourites from the collection. Again, this can be played like many others, with either a conceptual or strategic emphasis. Pupils may develop intuitive ideas of the game based on a binary search method but not actually be aware of this. The thing that impresses most children, and indeed teachers who have not met or considered the problem previously, is how quickly they can find the hidden point, despite their expectation of a high number of guesses.

Pupils should be asked to develop the game to a level which is suited to their ability. This may well be simply guessing the point within a given number of goes or it may be a full mathematical analysis of an NXM search area.

This game is in fact really a game involving two independent one dimensional searches. The game 'Higher or Lower' could well be played by younger children, or indeed by all as an introduction, to develop the same style of strategic thinking. The game would go something like this:

HOWARD: I have a number between 1 and 100, see if you can guess it ?

DOREEN: 72

HOWARD: It's lower than that !

DOREEN: 30

HOWARD: Now it's higher.

ETC.

An optimal strategy here is the binary search method of choosing a value in the middle of the range of possible answers. This method gives the most efficient reduction of the search area and with a search area of 100x100 the following reduction is achieved:

No.of guesses used.	Maximum range.
0	100
1	50
2	25
3	12
4	6
5	3
6	1
7	Correct

Since $2^6=64$ and $2^7=128$, then 6 goes are needed to reduce the range to just one possible answer and hence 7 goes are required to guess the position. We may, of course, be lucky and find the Hurkle in less guesses than this.

If this analysis is now applied to two dimensions, we simply have a twin guess, as an ordered pair e.g.(10,80), and a twin clue e.g.NW. This means that using the first coordinate with the second part of the clue i.e. 10 with W, then we have a one dimensional search and likewise for the other combination. This means that at most 7 guesses are again required. This would also be true for a three dimensional search provided that some triple clue could be given e.g. SW increasing altitude, and in fact this idea could be extended to N dimensions, provided that meaningful clues could be given.

We can therefore state that for N dimensions the maximum number of goes required using this strategy is n, where *n* is the lowest integer such that $2^n > d$, where d is the maximum of the N dimensions.

Example:

A rectangular search area 2000x50

d=2000

hence n=11

because this is the lowest integer such that

$$2^{11} > 2000$$

since $2^{10}=1024$ and $2^{11}=2048$

This game is now in fact availalble on computer and is known as Pirates. It has been developed by the ITMA collaboration and is published by Longmans. Pirates also allows further conceptual ideas on vectors and bearings to be used. The game is an excellent one to simply develop mathematical thinking from using the clues to decide upon future guesses. This idea is particularly emphasised by pirates when the clues are given simply as hot, warm and cold. This, I feel, is probably the most difficult version of the Pirates game.

When the game is played in the classroom, pupils quickly realise that their initial estimates for the required number of guesses are wildly high, and that their strategy will find the Hurkle in a very low number of guesses. However, they find great difficulty in using the ideas which they have developed to try to make any accurate predictions, rather than guesses, about games of other sizes, despite being quite clear on their own intuitive strategy. The usual suggestion is that if it takes 4 guesses for a 10 by 10 grid then it will take 40 guesses for a 100 by 100 grid.

This particular game could be used to develop the Shell Centre strategies of 'try simple cases', 'find a suitable representation', 'look for a pattern' etc. when investigating the number of guesses required for various size search areas.

If your were playing
this game it would
take you 4 goes
at the most because
each time we cut the
boxes in half and
guess in the middle
of them each time.

1. (5,5) SE

2. (7,2) W

3. (6,2) ✓

1. (5,5) NW

2. (2,7) NW

3. (1,8) N

4. (1,9) ✓

Only need about 3 or 4 goes.

1) (5,5) NE
2) (7,7) SE
3) (8,6) Hurkle

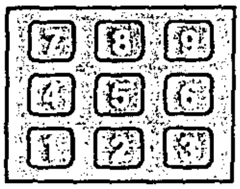
34
If the numbers went up to 99 each way I would think I would need 340 guesses.

The quickest way to find the Hurkle is to start in the middle of the grid. and then follow there instructions. you only need about 4 goes. to guess where the Hurkle is. when you are told your direction you go to the middle of that and so on.

THE KEYBOARD GAME

THIS IS A CALCULATOR GAME !

In this game you only use the numbers from 1 to 9 .



Start with 31 on the display.

The first player picks any number from 1 to 9 .
They then subtract this from the display.

The second player now picks a number but their number must be next to the key that the first player used.
The second player then subtracts their number.

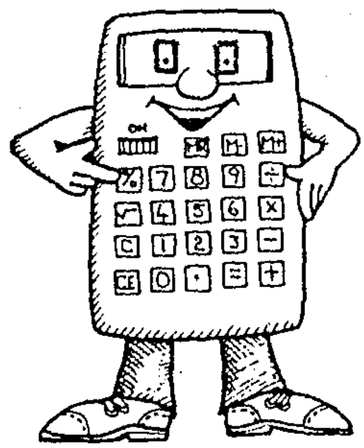
Then the first player has to go again but using the same rule about only being able to use a key which is next to the one that the other player used.

The game keeps going like this and the first player to end up with a minus number is the LOSER !!!

Can you find any ways of making sure that you win ?
If so write them down and try to explain why they help you to win.

Is this CHEATING ?

What if you start the game with a different number ?



THE KEYBOARD GAME

I first met this game when I was challenged by a fifth year boy who was in the bottom set for mathematics. Needless to say I lost several times when I first played him, and so did every other person, whether they were staff or pupil. After this, I did not reconsider the game again until I put this package together. Obviously, this is a relatively new game, and one of many of its type which have been suggested since the calculator has become common place within the mathematics classroom. This particular game may be used to develop both calculator use and various ideas on subtraction. However, I do feel that its main benefit, and the reason for which I have included it, is that it develops the idea of mathematical thinking in the sense that pupils have to build up their strategy from experience of playing the game.

As with some of the other games, see Century and later Kayles and Nim, there are both safe and unsafe positions to leave. The main difference with this game is that there are certain strategic numbers but they are also linked to the number that was last played.

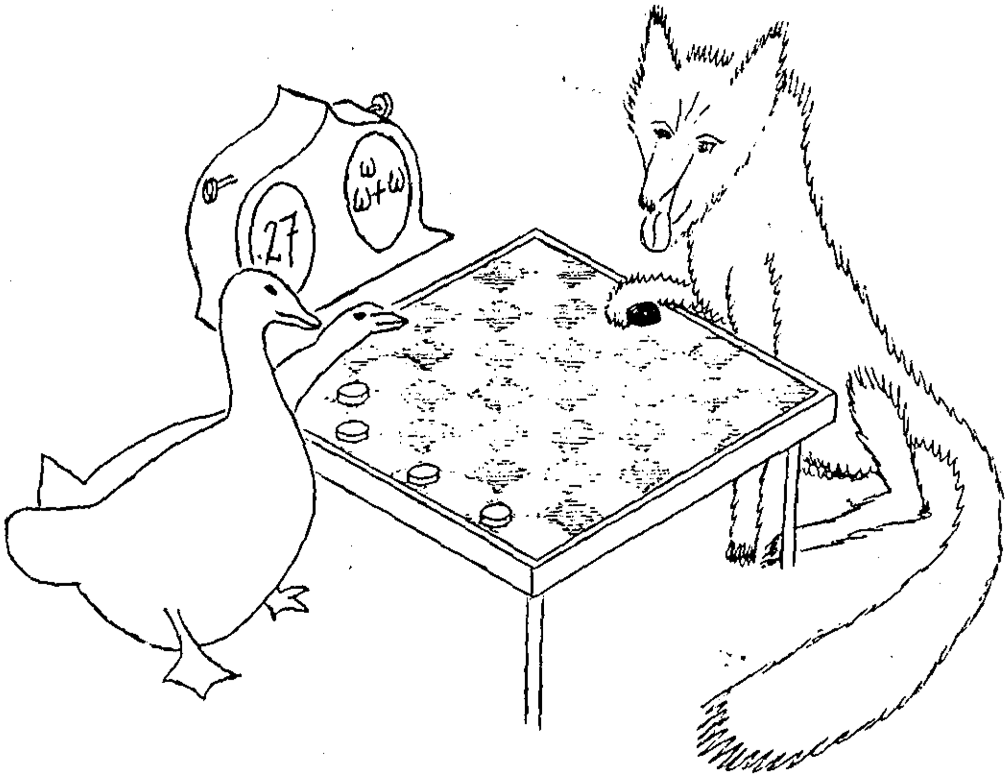
Example:

The first player should not leave any of the numbers 23, 25 or 26.
If we look at the number 23 then clearly the first player has played the number 8 and the second player will immediately reply with 9, leaving a total of 14.

This then leaves the first player with a choice of the numbers 5, 6 or 8, and whichever number they play, the second player can immediately reduce the score to zero, thus winning the game since the first player has no choice but to make the score negative. The other numbers given in this example may be given a similar treatment.

Children could play this game and develop seemingly winning strategies but were often unsure of what they were doing and therefore could not explain their methods that well. Many, however, quickly adopted the idea, which is often useful with games of strategy, of if in doubt then do as little as possible. This would lead them to subtract either the number one or the lowest number that was available in the given position.

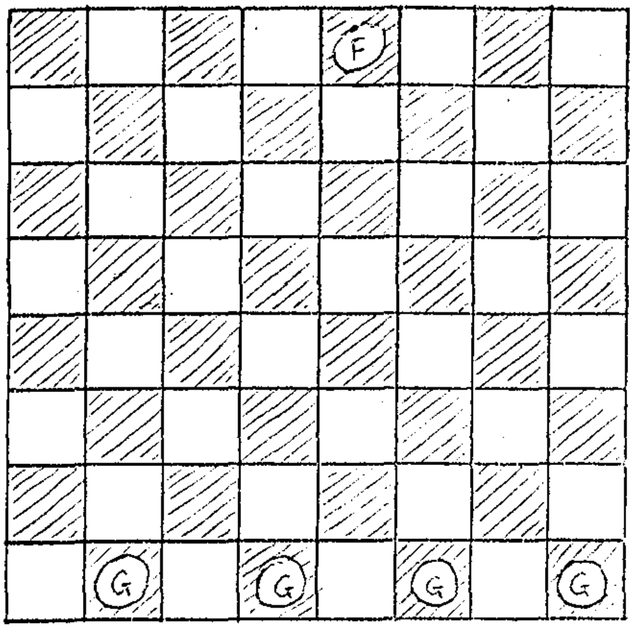
FOX AND GEESE



This is a game for 2 players.

It is played on an ordinary chess board using just the black squares. The Fox has just 1 counter F and there are 4 Geese counters G .

Play starts like this:



Sometimes we let the fox start on any black square on the back row. The Geese can only move diagonally forwards but the Fox can move both forwards and backwards. There is no jumping or taking. The Geese must try to trap the Fox and the Fox tries to break through the Geese and therefore get free.

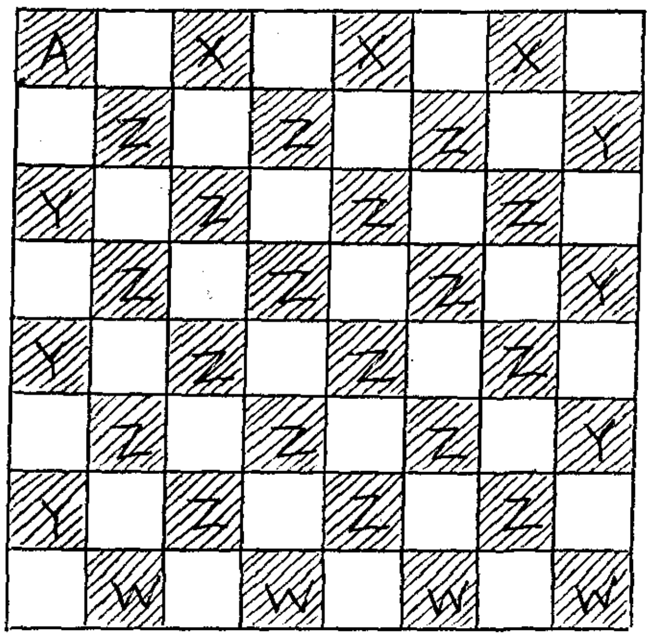
Let the fox go first and take turns to move.
Play the game several times changing over between being Fox and Geese.

Can you find any winning strategies ?
Write down your ideas on this game.

FOX AND GEESE

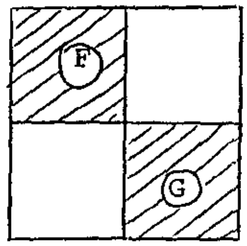
This is a game of strategy which again has a heavy bias towards one particular player, the Geese. This would probably appear quite obvious to pupils because they would see the game as a four against one game.

We can think of the board as being regionalised as follows:

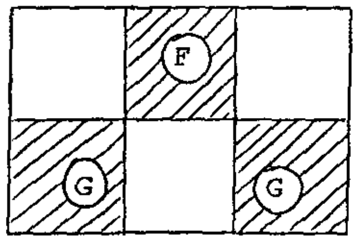


These regions are areas where the trapping pattern for the Geese are common within each.

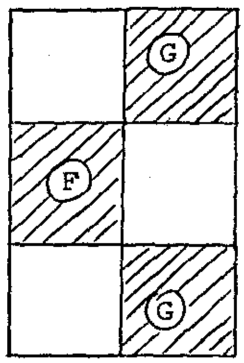
Region A:



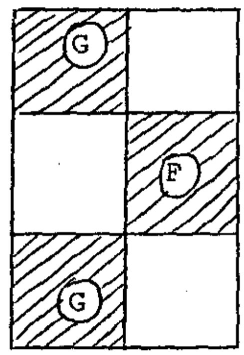
Region X:



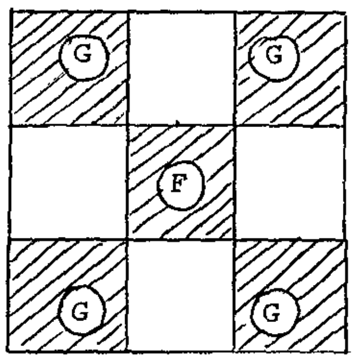
Region Y:



or



Region Z:



Region W: A WIN for the Fox

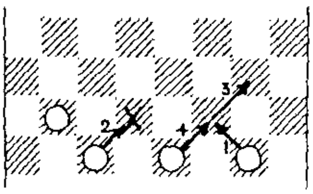
This is typical of many games of encirclement where there are certain areas for which the chance of being trapped, or trapping, is higher than for others. Clearly the

Fox has his best chance of surviving if play takes place within the centre of the board, however, surviving is not good enough for the Fox since if he does not escape at some stage then he is doomed to being trapped, eventually. Although the Fox has a better chance of survival by playing within the centre of the board, his best chance of creating a 'gap' in the Geese, and therefore escaping to freedom, is probably around the outside. The Fox's strategy must therefore be to entice the Geese towards the centre and then make a break for the outside when he has managed to create a 'gap'. This cunning of the Fox should be treated with calm by the Geese since their own best strategy is much more powerful than that of the Fox and they should play their strategy to ensure that the 'gaps', necessary for the Fox's escape, do not appear.

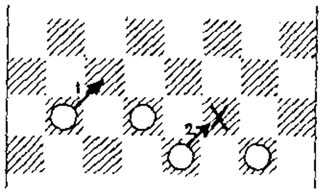
The majority of experienced players will play the Geese with a strategy to recreate a full row whenever possible, this sets up the starting position but with a smaller number of rows to play the game with, hence increasing their advantage, since there is less time for the Fox to find the 'gap'. However, the unique minimal winning strategy does not use this idea at all and in fact does not ever use a square which is in the extreme left or right columns, and hence, never recreates a complete row after breaking it with the first move.

This unique minimal winning strategy is named as such because it gives a minimal number of strategic moves for the Geese to follow and the whole strategy may be defined by just 5 such moves. The strategy is based on the fact that the Geese will find themselves in any one of just 5 unique positions and thus responds to each in a set way.

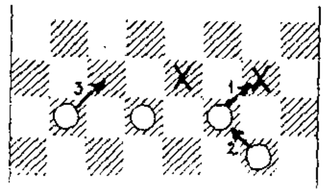
Other strategies cannot be defined by such a small number of moves and hence the Geese have more moves to memorise thus giving a greater chance to the Fox than would otherwise exist. The five moves which define this strategy are given below.



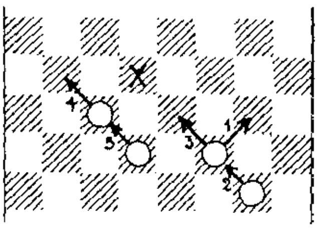
Position *A* 2*B*/1234*A*



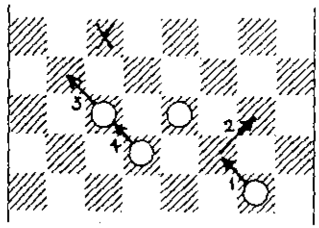
Position *B* 2*C*/12*D*



Position *C* 12*A*/3*D*



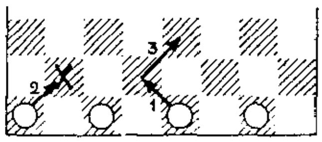
Position *D* 12345*A*/3*E*



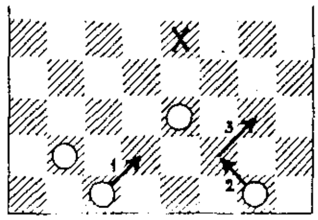
Position *E* 1234*A*/3412*A*

The O's indicate the positions of the Geese, and the X's indicate particularly critical points for the Fox. When the Fox is in one of these positions, then the Geese are in danger. For each move in the strategy, the Geese should play the moves given before the solidus(/) if they are not in danger, and the move given after it, if they are. The moves also show which new position the Geese will find themselves in, the new sequence for that position is then followed. These sequences may always be played since they guarantee that the Fox will not be in the way. The Geese can, of course, move into position A on their very first move.

A variation to the game is to let the Fox start from any position on the board, this introduces further problems for each strategy but does not necessitate a major change. The minimal strategy needs only the following two additions to its 5 moves.



Position *F* 2*A*/123*G*



Position *G* 123*B*/231*B*

These extra two moves are only necessary if the Geese are in danger before their first move, and cannot move immediately to position A.

This particular game was not used at all in the trials, although I have used it in my own classroom in the past. Initially the inexperienced pupil player, playing the Geese, tends to charge after the Fox and try to trap it, this usually creates the necessary gap for the Fox to escape. However, after a while, the strategy changes to the idea that the original line should be recreated at every possible chance. This is a highly successful strategy for the Geese and children are quite happy with it, although the Fox still manages to escape on the odd occasion.

This game has been included as a game to develop mathematical thinking whilst still allowing pupils to intuitively develop attack and defence strategies. This may be done without having to apply any mathematical analyses, but they are, throughout the game, developing the idea of strategy and understanding the importance of this in tackling problems, particularly in mathematics but generally also.

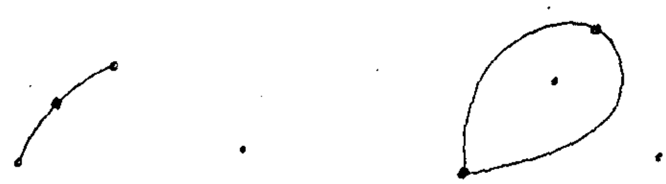
SPROUTS

Sprouts is a mathematical game which was invented by Professor J.H.Conway and M.S.Paterson in 1967 and is now quite popular.

It is a game for two players.

Mark 3 dots anywhere on a sheet of paper.

Each player now in turn draws a line joining a dot either to itself or to another dot and places a new dot on this line. Here are some possible starts:



No line may cross either itself or any other line and no dot may have more than 3 lines leaving it. The last player who is able to go is the winner.

Play the game several times with the person next to you. Sprouts may not be as easy as you first think!

Think about some of the following questions:

a) How many goes did it take to get a winner ?
b) How quick could you win if you were an expert playing against someone who had never played the game before ?
c) What would happen if you changed the number of starting dots ?

Try to work a few things out about this game and write about the ideas that you have. Can you suggest any winning strategies ?

SPROUTS

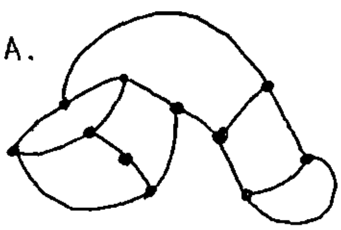
Sprouts is one of the few games that I have included which emphasises a particular area of modern mathematics, that of topology. This game is a recent invention by John Conway and Michael Paterson and was thought of only in 1967.

Every game of three dot sprouts must end in one of three topological equivalent states shown in the following table:

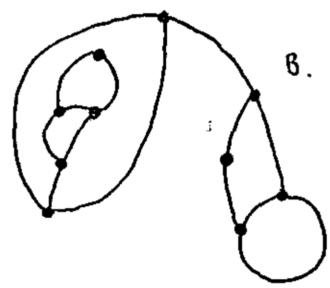
TYPE	NUMBER OF ARCS	NUMBER OF NODES	NUMBER OF 3-NODES	NUMBER OF 2-NODES	NUMBER OF MOVES
A	16	11	10	1	8
B	14	10	8	2	7
C	12	9	6	3	6

Typical examples of these three finishes are given below:

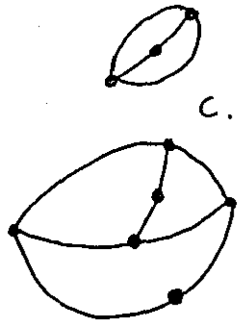
A.



B.



C.



The number of moves in any game of Sprouts is simply determined by the number of starting dots and all games do in fact come to an end, despite the fact that at first thought many people think otherwise. If m is the number of goes in a game of n dot sprouts then

$$2n \leqslant m \leqslant 3n-1$$

and we have already seen that for n=3

$$6 \leqslant m \leqslant 8.$$

The proof of these limits are relatively straight forward and could be an interesting exercise for sixth form students, whereas, the game itself may be played by young children in the secondary age group.

When pupils play Sprouts, they quickly build up ideas about the pattern produced in the network. The actual number of goes in a game of Sprouts is in fact related to the number of isolated points on the network. An isolated point is a 2-node which does not have a neighbouring 2-node to which an arc may be drawn.

PROOF OF THE ABOVE LIMITS.

The game of n dot sprouts begins with n 0-nodes. Each move adds 2 arcs and 1 node. Therefore after m moves there are 2m arcs and m+n nodes. The last node added must always be a 2-node and none of the other m+n-1 nodes can be of order greater than 3. Therefore the node sum of the network is less than or equal to $2+3(m+n-1)$. The node sum counts each arc twice, once at each end, and is therefore $2X2m$.

$$\begin{aligned} \text{So} \quad 2X2m &\leqslant 2+3(m+n-1) \\ 4m &\leqslant 2+3m+3n-3 \\ m &\leqslant 3n-1 \end{aligned}$$

This gives the required upper limit.

If we now consider the shortest possible game of n dot Sprouts, then this only occurs when the number of isolated points is a maximum. The number of isolated points cannot exceed the initial number of dots, therefore the minimum value for the node sum is $2n+3m$.

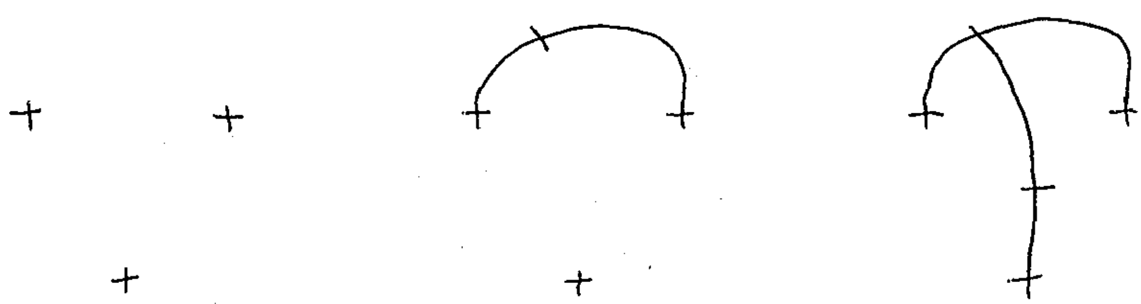
$$\begin{aligned} \text{So} \quad 2X2m &\geqslant 2n+3m \\ 4m &\geqslant 2n+3m \\ m &\geqslant 2n \end{aligned}$$

This gives the required lower limit.

Initially, the game of sprouts may seem relatively straight forward, but this is certainly not the case as the number of dots increases. The game has not been fully analysed beyond the six dot game and the four dot misere version. Conway claims that the eight dot game is even beyond analysis with a present day computer! The analysis of the lower order games show that the first player should win the 3, 4 and 5 dot versions and the second player the 1, 2 and 6 dot versions.

Although no strategy for perfect play has been formulated for Sprouts, one can often see towards the end of a game, how to draw closed curves that will divide the plane into regions in such a way as to lead to a win. This in fact is a stage the pupils can reach after playing the game several times. In all games of Sprouts, the number of goes usually comes down to m or m+1, with all other values being very unlikely. In the three dot game the majority of games will end in exactly 6 or 7 moves, hence the players are playing for one of these two states, according to whether they went first or second.

Conway gave a follow up 'game' to Sprouts, known as Brussel Sprouts, which in fact is not really a game. The 'game' starts with a number of crosses and each player makes a move by joining two free neighbouring arms of a cross or crosses. They then put a mark across their arc to produce a new cross with two free arms.



This is a trivial game since a game of n cross Brussel Sprouts always ends in exactly 5n-2 moves and therefore the first player wins if the game starts with an odd number of crosses and the second player wins otherwise. This again could be an interesting problem for the sixth form student.

Sprouts proved to be a very popular and exciting game in the classroom during the trials. Pupils' initial games often took the maximum 8 goes since their networks tended to be simple, with all points ending as 3-nodes with the exception of the last one to be created. This led pupils to believe that it was simply a second player game. However, this hypothesis was quickly proved to be incorrect, usually by luck rather than judgement. This, of course, gave them added motivation to look into the game further. Various groups realised that the winner was something to do with the number of trapped, or isolated, nodes, and many tried to cause these by producing loops which started and finished at the same point. This process can only be carried out on the n starting points since all newly created points are immediately 2-nodes. This was in fact the basis for the proof of the minimum number of moves in a game of Sprouts. However, very few sorted out a clear strategy, in the short time available, which they could effectively use or commit to paper. I am sure that if extra time had been given to the game, then intuitive strategies would have become more clear in the pupils' minds and hence definite.

There is an intersting point which occurs in this game with regard to Euler's relationship for networks. The usual idea that

$$N+R=A+2$$

does not apply to many of the completed Sprouts networks. In fact the more generalised form of the relationship

$$N+R=A+1+B$$

where B is the number of branches, has to be applied. Again this forms an interesting discussion point and investigation as an extension exercise to this game.

we could get up to ten dots but no more I found that because of this it was best to go second.

The dots mainly go up to nine or ten.

Try to box-in the Spots which only have two lines coming out.

Try to reduce the goes which each Player can have.

If the first person makes a Circle it is possible always for the Second person to win.

Odd number first person wins even Second person wins.

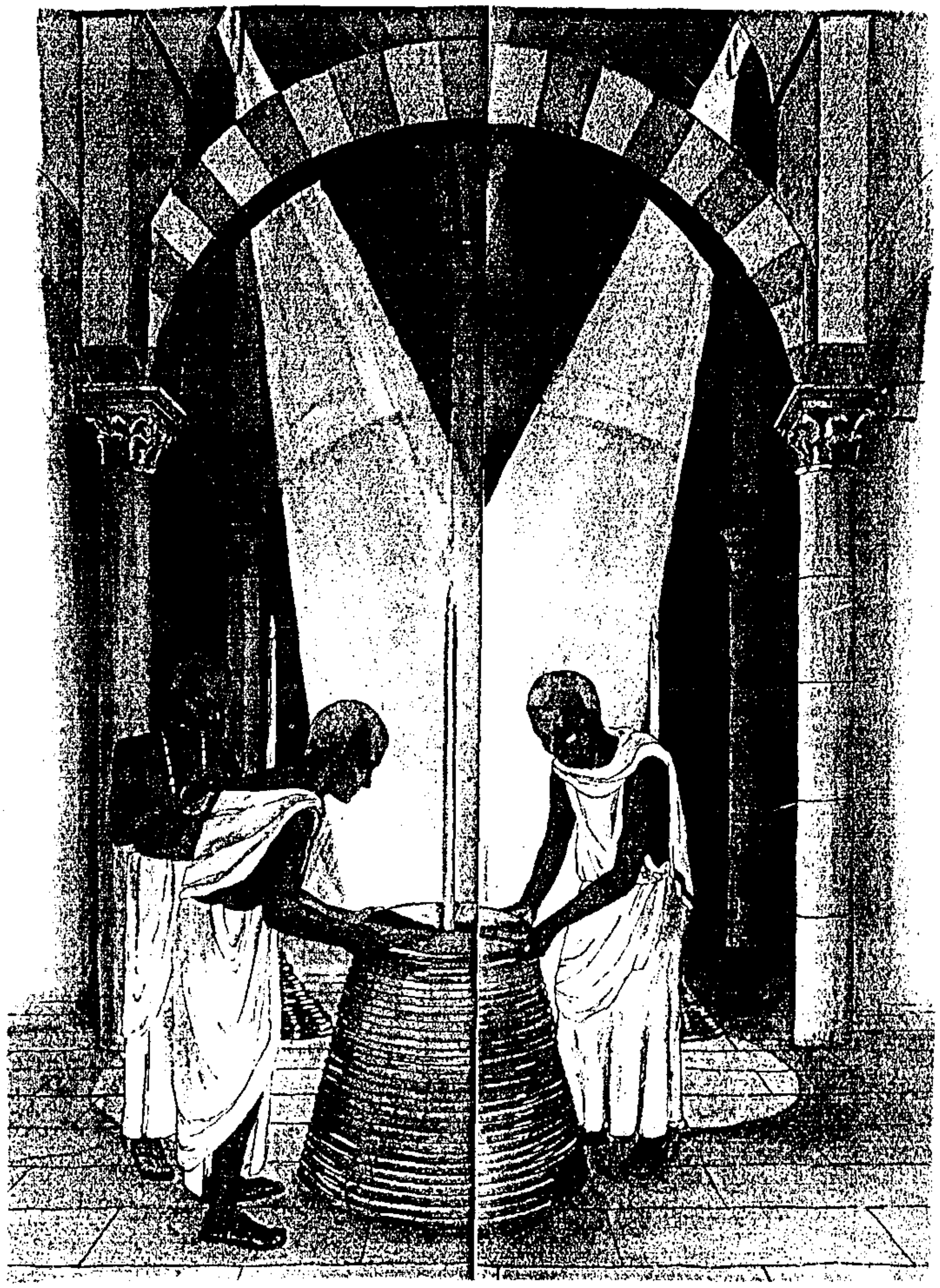
One way in which to win is to try and Box your last dots in, So your opponent can't get to it with out crossing the lines.

If you try and reduce the dots by drawing circuls and boxing in you have less working out to win.

If its an even number the Second player wins and if its an odd number the First player wins.

You get more chance if you go Second because the game usually ends in 6, 7, or 8.

THE TOWER OF HANOI

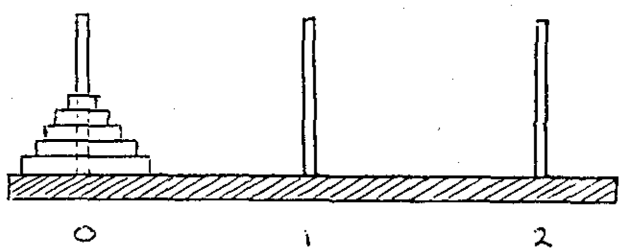


The story goes that once in Hanoi there existed the temple of Brahma. The God Brahma had set the monks of the temple a problem which would take them from the beginning until the end of the world.
There were 64 gold discs on a diamond needle the height of a man. There were also two other such needles and the problem was to move all the discs onto one of the other needles following two rules:

1. Only one disc may be moved at a time.
2. No disc may be placed above a smaller one.

When the problem was completed then Brahma would end the world with a clap of thunder !!

We shall consider a much smaller tower, one with just 5 rings on it.



How many moves do you think you would need to move all the discs from pole 0 to pole 2 ?

Play the game with the squares of card which are inside your envelope. Play a few times to see if you can improve each time.

What is your best score ?

Do you think that you could improve on this ?
Compare your score with some of your friends and talk about how you did it.

Can you write down any rules to help anyone to solve this problem quickly ?

Try a tower with a different number of rings.

Start with small numbers and work up, perhaps you will find it helpful to record your results in a table like this:

No. of rings	Least no. of moves
1	1
2	
3	
4	
5	
6	
7	

ETC...

Someone has said that it would have taken the Priests over

18 000 000 000 000 000 000 moves.

Do you think that this is true ?

How long do you think that this number of moves would take ?

THE TOWER OF HANOI

This is another game that I have included mainly to demonstrate the necessity of strategy in problem solving. It is again possible for pupils to quickly discover an informal strategy for moving the discs which is equivalent to the strategy derived through the rather complicated mathematical analysis. The puzzle itself was introduced to Europe in 1883 by the Frenchman, Professor Lucas. There are various versions of the myth which accompanies the problem, but they are all based on the same idea.

There is a method which enables us to know exactly where each disc is after any given number of moves. The number of the move is first expressed as a sum of the powers of 2 and we then use ternary representations in place of each power. These ternary numbers are not the base three equivalent values, but the values given in the table on the following page. The derivation of these numbers need not be gone into here, since I do not feel that it is of value to the current discussion.

Digit	Even	Odd
1	1	2
2	21	12
4	122	211
8	2111	1222
16	12222	21111
32	211111	122222
64	1222222	2111111

ETC!

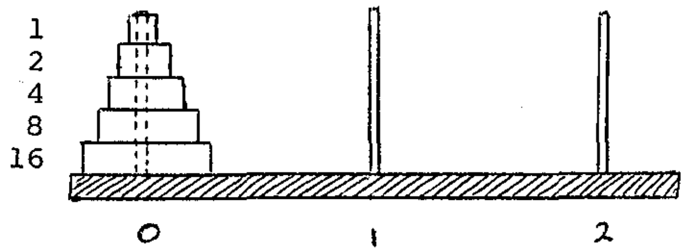
When the ternary numbers are added, modulo three, then the digit in each column indicates the position of a particular disc, with the right hand digit representing the smallest disc and the left hand digit the largest. If the sum does not give sufficient digits then leading zeros have to be added.

Using this method as a basis for the five disc tower, as discussed on the sheet, after say 11 moves, we have:

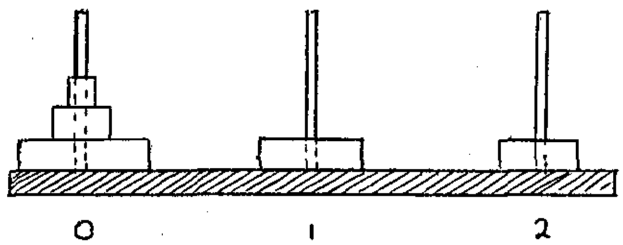
1	2
2	12
+8	+1222
=11	=1200

This result totally describes the position of the five disc tower after 11 moves.

16	8	4	2	1
0	1	2	0	0



The number 01200 means that disc 1 is on pole 0, disc 2 is also on pole 0, disc 4 is on pole 2, disc 8 on pole 1 and disc 16, the largest, is on pole 0.



This particular method also always ensures that any even power of two disc is only placed next to, or between, odd power of two discs, and vice versa. The complete analysis of all the moves for the five disc tower, using this method, is shown in the table given on the following page.

THE 32 MOVES TO SOLVE THE FIVE RING TOWER PROBLEM.

Move	Disc				
	16	8	4	2	1
0	0	0	0	0	0
1	0	0	0	0	2
2	0	0	0	1	2
3	0	0	0	1	1
4	0	0	2	1	1
5	0	0	2	1	0
6	0	0	2	2	0
7	0	0	2	2	2
8	0	1	2	2	2
9	0	1	2	2	1
10	0	1	2	0	1
11	0	1	2	0	0
12	0	1	1	0	0
13	0	1	1	0	2
14	0	1	1	1	2
15	0	1	1	1	1
16	2	1	1	1	1
17	2	1	1	1	0
18	2	1	1	2	0
19	2	1	1	2	2
20	2	1	0	2	2
21	2	1	0	2	1
22	2	1	0	0	1
23	2	1	0	0	0
24	2	2	0	0	0
25	2	2	0	0	2
26	2	2	0	1	2
27	2	2	0	1	1
28	2	2	2	1	1
29	2	2	2	1	0
30	2	2	2	2	0
31	2	2	2	2	2

It can also be shown that the minimum number of moves needed for an N disc tower is 2^N-1. The simplest proof of this is by induction. The proof takes account of the fact that to move N discs, we move N-1 discs from pole 0 to pole 1, then make a single move of the largest disc from pole 0 to pole 2, and then proceed to again move the N-1 discs, but this time from pole 1 to pole 2. This is probably a relationship that pupils can be encouraged to discover through the study of simple cases, leading to those of greater difficulty. This would enable them to decide if the claimed value of 18 000 000 000 000 000 000, as given on the sheet, is a reasonable estimate for the number of moves for the 64 disc tower that the priests were set.

$$2^{64}-1 = 18\ 446\ 744\ 073\ 709\ 551\ 615$$

Someone once worked out that if the priests made one move per second, for 24 hours a day and 365 days a year, then it would only take them

58 454 204 609 060 YEARS !!! and a bit.

Again this is a problem which the pupils could be asked to think about. This would allow them to estimate and discuss various quantities and ideas.

This is yet another game which may be used at a variety of levels. During the trials, it was used with first year secondary pupils, to get them to think about the moves whilst following the rules carefully, hoping that they would develop a pattern in their moves, through to more able fifteen year olds who were set the problem of discovering the relationship 2^N-1 from using the idea of simple cases and working from there.

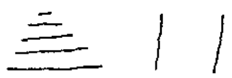
The game achieved considerable success at both levels and example of pupils' work are given below. The majority of the older pupils could see the 'vertical' relationship that the number of moves for N discs was related to the number of moves for N-1 discs, by doubling and adding one, but then became rather confused, as one would expect, when trying to predict the number of moves required for the 64 disc problem. This was due partly to the calculator notation and partly to the length of the problem. However, some did in fact find the 'horizontal' relationship between the number of discs and the number of required moves, $N \mapsto 2^N-1$, rather than the vertical one discussed above. It was of course these pupils who could successfully predict, both quickly and accurately, the number of moves for any size problem.

It takes longer with more discs.

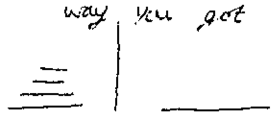
Using 64 rings
the least number
of moves =
$8{\cdot}797739 \times 10^{18}$

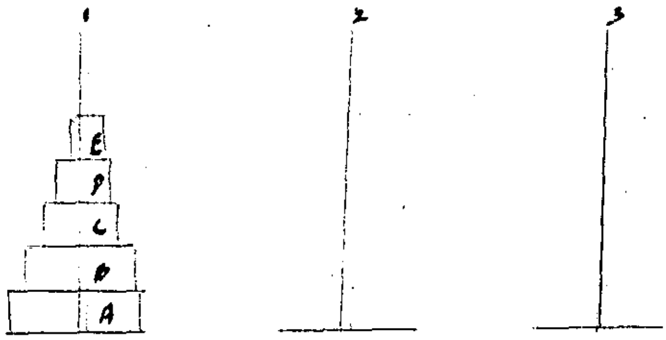
We arrived at this
number by x the
previous number by 2
and adding 1.

The Solution is as follows, you start like this



then you move them so they look like this - then you double back the way you got there





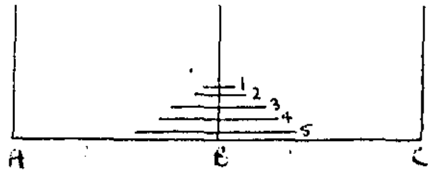
1) E-3
2) D-2
3) E-2
4) C-1
5) E-3
6) D-1
7) E-1
8) B-2
9) E.2
10) D.3
11) E3
12) C2
13) E1
14) B2
15) E2
16) A.1
17) E.3
18) D.1
19) E.1
20) C.3
21) E.2
22) D-3
23) E-3
24) B-1
25) E.1
26) D.2
27) E-2
28) C-1
29) E-3
30) D.1
31) E3

Number of ring	Number of move
1	1
2	3
3	7
4	15
5	31

for 64 ring 1.8446744x10[illegible]

No. of rings	Least number of moves.
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1023

If you use a pattern it takes
31 goes to do the 5 tower and
7 goes for the 3 tower.



Starting Position

By a process of trial and error, we established the shortest way of transferring all five pieces of wood (numbered 1-5) onto a different peg and without moving more than one piece at a time or allowing a larger block to fall on top of a smaller block.

The formula is:

1 – A	1 C
2 – C	2 B
1 – C	1 B
3 – A	4 A
1 – B	1 – A
2 – A	2 – C
1 – A	1 – C
4 – C	3 – A
1 – C	1 – B
2 – B	2 – A
1 – B	1 A
3 – C	31 moves
1 – A	
2 – C	
1 – C	
5 – A	
1 – B	
4 – A	
1 – A	
3 B	

We also established a table of results

No. of blocks moved	No. of moves taken
3	7
4	15
5	31
6	63
7	127
64	$1.844674406 \times 10^{19} - 1$

To find out how many moves it would take with 64 counters is

$2 \times 2 = 4$

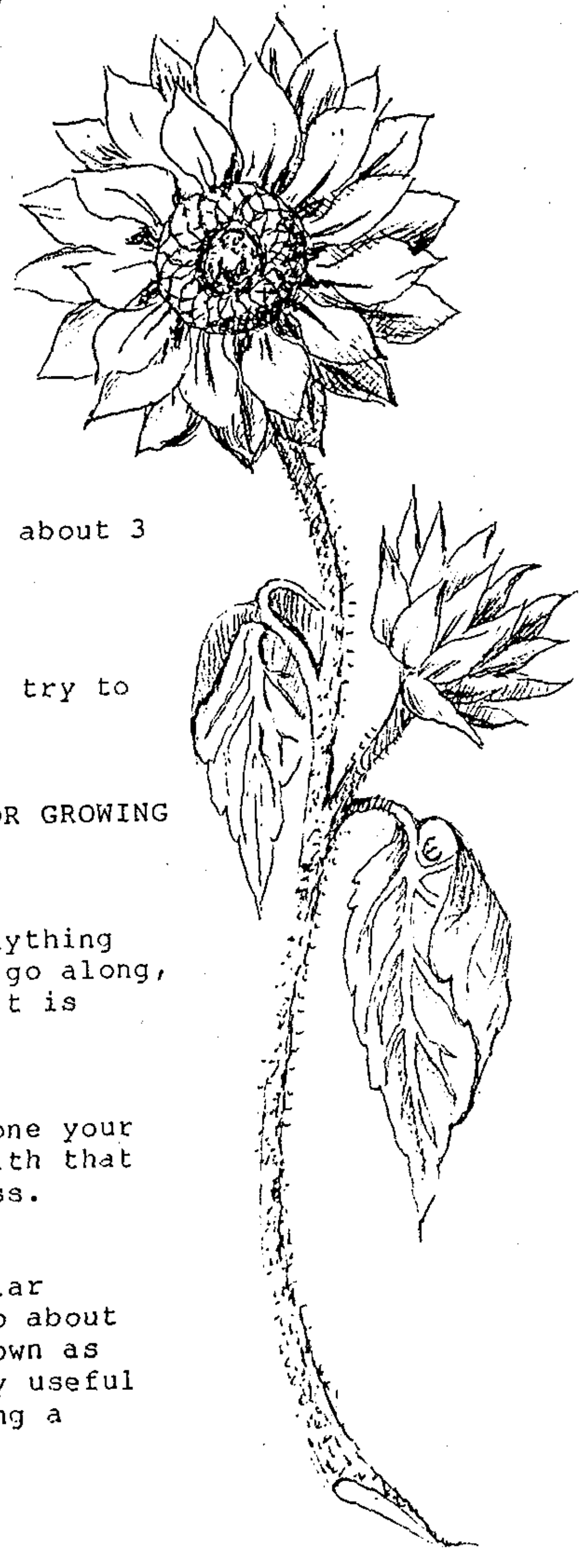
$4 \times 2 = 8$

$8 \times 2 = 16$

you times each answer by 2 63 times less 1.

$= 1.84467 \times 10^{19} - 1$

SUNFLOWER



This is a COMPUTER GAME.

It is best to work in groups of about 3 or 4 for this activity.

You have to follow the computer instructions very carefully and try to grow a sunflower.

TRY TO BEAT THE WORLD RECORD FOR GROWING A SUNFLOWER !

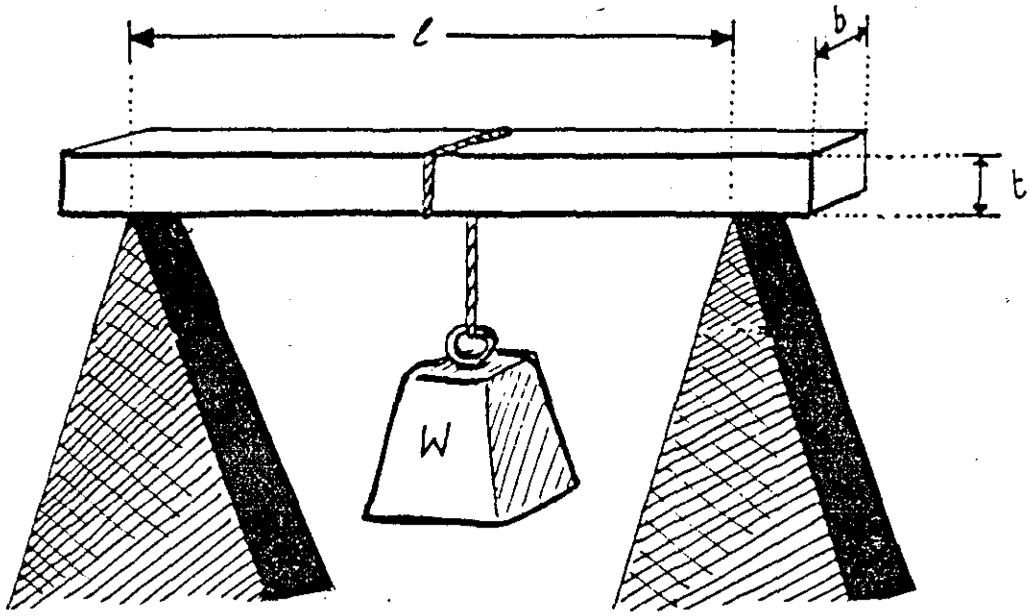
Write down in your own words anything that you do or think of as you go along, even if you do not think that it is important at the time.

When you think that you have done your best, compare your sunflower with that of the other groups in the class.

If you had to carry out a similar investigation, how would you go about it ? This approach would be known as your STRATEGY. It is often very useful to have a strategy when tackling a problem in mathematics.

NOW TRY TO SOLVE THE BRIDGE PROBLEM !!!

THE BRIDGE PROBLEM



The table drawn below shows the maximum weight that can be theoretically supported by wooden planks with various dimensions. The results are written in order, from the strongest plank to the weakest.

Length l(cm)	Breadth b(cm)	Thickness t(cm)	Max. Weight W(kg.wt)
2	40	4	160
1	20	4	160
2	30	4	120
1	20	3	90
2	20	4	80
1	30	2	60
4	40	3	45
1	20	2	40
2	10	4	40
2	30	2	30
3	30	2	20
3	10	3	15
4	30	2	15
1	20	1	10
4	40	1	5

Try to discover a rule by which the strength of any size plank can be found from its dimensions. Did you use the strategy that you worked out in the sunflower problem ?

Can you now predict the maximum weight that a wooden beam with breadth 4cm, thickness 6cm and length 3cm can support ?

SUNFLOWER

Although this is not really a game, I have included it because I feel that it has all the attributes of a good game, and it is an excellent exercise to demonstrate that strategy exists, what a strategy is and the importance of using stategies in problem solving. The strategy in any such investigation as this, must be to adopt a scientific methodology and initially investigate the effect of each factor individually. Only then can the higher order interactions be investigated.

Pupils will soon find that they need to record their tests and then try to spot a pattern, make a hypothesis and test it out. They must then use the results of their further tests to adopt, adapt or scrap their hypothesis. They continue in this way until they achieve a new world record height for growing a sunflower. Again the social skills of working together in a group are developed and the discussion may well be heated and certainly enthusiastic. The investigation suddenly becomes a class competition. The ideal set-up for this exercise is one computer for each small group, of say 3 or 4, in the class.

The three chemicals may in fact be broadly classified as a macro nutrient, a micro nutrient and a poison, however, under no circumstances should pupils be told of this. In the latest version of this software, which has been produced by the ITMA collaboration, six different growth models are available, and one is chosen at random each time the program is used. However, a teacher may select a particular model if continuation work is required or if all groups are to work on the same model. With all of these models, there are in fact no interactions between the chemicals, and each acts separately on the growth of the sunflower. On some of the previous games, see Subgame and Multigame, I have discussed the fact that the computer may or may not be used. However, with this game, the computer is an absolute necessity to give the immediate feedback on the tests in this complex problem.

When this software is used, we also benefit from the conceptual reinforcement of the compactness of decimals, a major worry in the teaching of the topic. Pupils discuss and use values between say, 0.1 and 0.2 . This, in my view, has to be a hugh positive bonus with this 'game'. Often the discussion with regard to such values is meaningless and rather false in the pupils' experience, whereas, this problem relates the values to their immediate needs.

I have included The Bridge Problem with this 'game' since I feel the strategy developed with the sunflower 'game' is most applicable to this 'real' problem. Perhaps this problem, presented in the traditional manner, would cause difficulty, but presented to a group of able pupils of age around 14, with no further help, then they are able to solve it reasonably quickly and with relative ease and confidence. Although I have stated that this problem is more suited to able children, the sunflower 'game' has no such limitation.

This game and related problem were not used during the trials of this package, however, I have used both, in conjunction with each other, during the original ITMA trials of this software and later during the trials of the second Shell Centre module 'Language and Functions of Graphs', from where The Bridge Problem originates. The task was set to a very able group of fourteen and fifteen year old pupils, and after spending about thirty minutes on the Sunflower problem, they then completely solved The Bridge Problem in about twenty to thirty minutes, with no more guidance given other than that on the sheet. In my view, this would not have been achieved without the use of the Sunflower 'game' as an introduction to the particular type of strategic approach required. This problem had a long lasting effect on the approaches used by this particular group in further problem solving work in mathematics.

When The Bridge Problem is tackled, pupils will probably start by discussing their intuitive ideas on the possible variation relationships which may arise. Having just completed the Sunflower game, the obvious approach will be to test just one variable at a time, whilst keeping the other two fixed. By selecting suitable sets of data, the following three relationships will be found:

$$W \propto \frac{1}{L}$$

$$W \propto B$$

$$W \propto T^2$$

These three relationships may then be combined to give:

$$W \propto \frac{BT^2}{L}$$

$$W = \frac{kBT^2}{L}$$

where k is the constant of proportionality.

Then by substitution, of any single set of data, we arrive at the relationship:

$$W = \frac{BT^2}{2L}$$

This equation may then be used to predict the strength of any bridge, given its dimensions; the strength of the bridge given on the sheet itself being 24 kg.wt.

PICO CENTRO

Pico Centro is a number game.

Let's imagine that I have 10 cards numbered from 0 to 9. I then choose just 3 of them and make them into a three digit number. You have to guess my number, but I will give you some help after each guess by telling you your Pico Centro score. Your Pico score tells you how many digits you have which are used in my number and your Centro score then tells you how many of these you have put in the correct position.

Eg

MY NUMBER IS 285

If your guess is 364 then your score is PICO 0, CENTRO 0

If your guess is 253 then your score is PICO 2, CENTRO 1

If your guess is 258 then your score is PICO 3, CENTRO 1

Play this game with a friend using a pencil and a piece of paper. One of you make up the number and one of you try to work it out.

How many goes do you usually need to get it right ?

You can make the game harder if you wish by:

a) Using 4 digits.

b) Using digits more than once in a number.

Perhaps you know another game which is similar to this.

Do you use the same strategies in both games ?

PICO CENTRO

This game is essentially a number version of the popular and commercially available game of Mastermind. Most pupils have played Mastermind previously, and soon catch on to the ideas behind Pico Centro because of this. The game is really included to develop mathematical and logical thinking. The strategy is, of course, a guess and check process using all the previous guesses and clues to make each further guess. This, in my view, links up with many of the problem solving heuristic strategies which were discussed in chapter two. These links make it a very useful game to demonstrate the value of these strategies. Pupils catch on to this strategy and realise both the importance of it and the need for some form of representation to record their past guesses. They usually use some form of table for this recording, and then use the entries in the table to develop their ideas.

There are many variations of this game which may be used in the classroom, including Mastermind itself. The variations may be generated by increasing the number of digits, allowing digits to be repeated, using shapes rather than numbers or perhaps changing the scoring system to the one used in Mastermind or indeed any other.

During the trials, Pico Centro was played mainly with younger children, first year secondary age. The main emphasis was placed on them playing the game and developing the idea that they must always use all the information which they have available to them at the time. Clearly from the pupils' comments, given on the following page, this was achieved, although one pupil clearly felt that there was an easier way, by looking at the markers face!

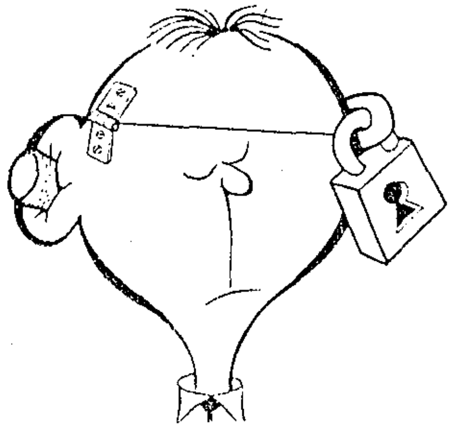
I Look up to see which numbers I have right and were the hiest numbers are in the Pico and centro. It took me 10 goes.

	Pico	Centro
111	1	1
914	2	2
912	2	2
913 ✓		

Go down the numbers in threes until you get three picos.
You usually ~~have~~ need 4 to 7 you to guess.
Mastermind with the coulors is like this game

I think it is a good idea when you are geussing to look at the other Persons ~~face~~ face Some times you can tell if you are near the number

The quickes way to win the game is to look at the Pico centro colume and see what Numbers you get. the most numbers you get in the colume and try to Figure out which numbers the are and stick to them.



COVER UP

Cover Up is a game for 2 players.

This type of game can be played with many different sets of shapes. We are going to play it with either a set of TETROMINOES or PENTOMINOES. If you play with the tetrominoes then you have one set each, but different colours of course. If you play with the pentominoes then you have one set between you and the game starts by either dealing these out or by taking turns to pick a shape until all the shapes have been shared out equally.

The rules are the same for either game.

Once you have your set of shapes, you and your partner take turns to place a shape on to a grid. The first person who can not go is the LOSER!!

The grid is 6X6 if you play with the tetrominoes and 8X8 if you play with the pentominoes.

When you play these games, play several times to get the hang of the rules. Try to play against a few different people.

Try to write down any ideas that you have on winning the game. The way that you play the game is called your STRATEGY, this is like your plan of attack.
Have you heard the word strategy before ?
It is often useful to have a strategy in maths when you tackle a problem and try to solve it.

You will find everything that you need to play the game of COVER UP in your envelopes.

COVER UP

Cover Up is the first of three shape based games which I have included in the package. All three of the games link up well with work on polyominoes, particularly this game and the game Colours.

Although there are two games suggested on this game sheet, they are I feel, essentially the same. If we first look at the tetromino game then we see that the board of play has only 36 squares to cover and the number of squares to place is 40, 2 sets of 5 tetrominoes. This means that not all the pieces can be placed, so there must eventually be a winner. In fact from the figures above it would appear that 9 pieces may be placed, however, I would claim that in practice, it is only possible to place at most 8 pieces. This would imply that the second player should win unless the board is further reduced by the blocking off of squares which are not playable for the second player. It is therefore my view, that the game thus reduces to one of creating unplayable spaces for the opposite player and generating an N move game where N is odd or even according to whether the player is the first or second to go.

The pentomino game can be thought of in a similar manner, with 64 squares being available and only 60 to play. However, whilst accepting that this implies that a winner may not be obtained, in practice only 11 pentominoes may be placed at most. This implies that the game is initially a first player game with 9 spare squares and thus both players will start from this position and try to reduce the game to a lower number of moves, again an odd or even number according to whether the player is the first or second to go.

The idea of creating unplayable spaces may be carried out either by creating spaces with less squares than the number on a playing piece, or by creating a space in which you may place one of your own pieces but in which the other player cannot. This second tactic is probably more important in the pentomino game. If you have more of this second type of trapped space than your opposition, then you should be in a commanding position.

When children play this game, they soon become aware that certain pieces are harder to play as the game goes on and tend to play the 'awkward shaped' pieces early on in the game. Initially, the games are often played in a very hap-hazard manner, but the ideas of trapping and blocking soon become apparent. Judging from the pupils' comments on the game, they feel safer if they play the so called 'awkward pieces' first and start in the middle of the board, although with further experience of the game, I am sure that the blocking idea, as mentioned by some pupils, would become more important. This game may also be used to discuss and develop various ideas on reflections, rotations and translations.

When it is your go you have
to get rid of your bits but
try to stop the other person
putting down their bits

You KEEP LOSEing if you USE the wrong PEices at the wrong time. you should of used the awkward peices First.

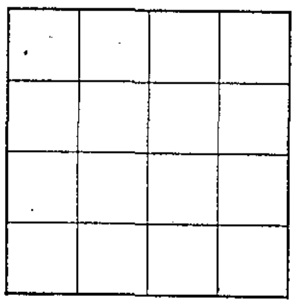
SAVE A BIT IF YOU CAN PUT IT IN A SPACE BUT THE OTHER PLAYER CAN NOT GO THERE.

to win the game you have to stop the other person from going.

When you are playing this game you should start in the middle thats the best place to start

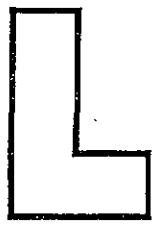
THE L SHAPED GAME

This is a game for 2 players and you need the following pieces:



The board

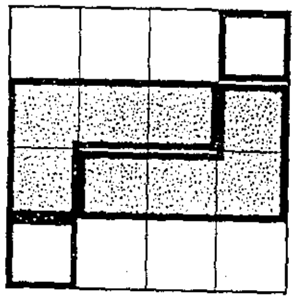
Two neutral pieces



Two 'L's' of different colours

You will find all the pieces in your envelope.

Take the pieces and arrange them in this way:



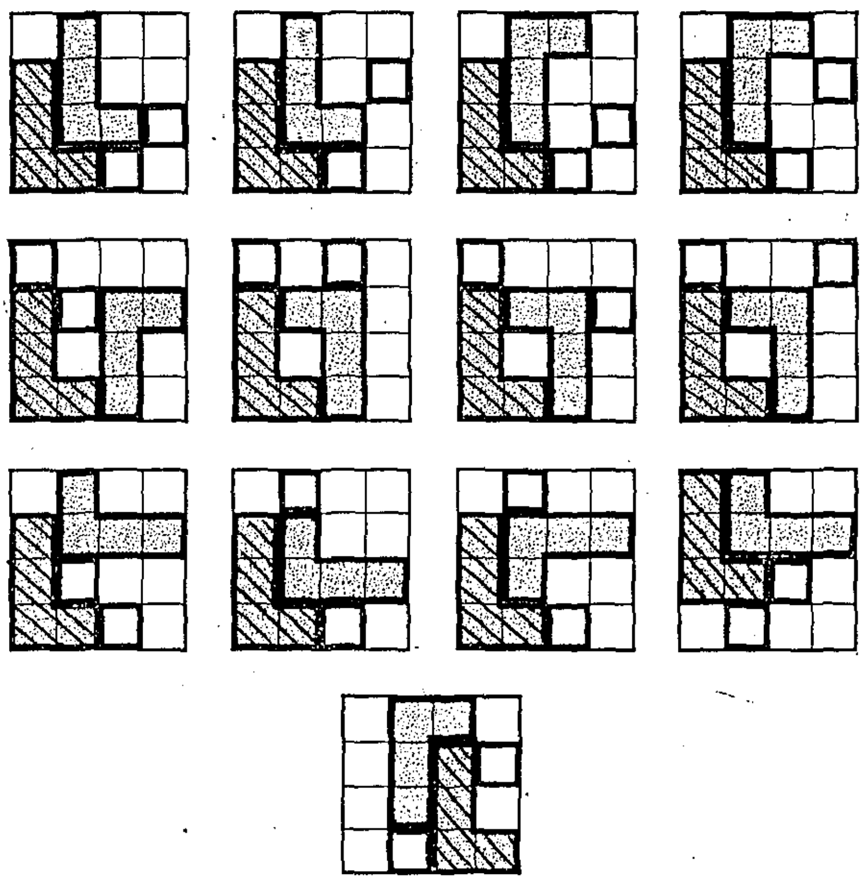
This is the starting position of the game.
Now decide who is going to go first.
Each player, in turn, must pick up his own 'L' and put it in a different position. The player can then, if they wish, move any one of the neutral pieces. The 'L' piece can be turned around or over or both when it is being moved.

The winner is the first person to stop their opponent from being able to go.

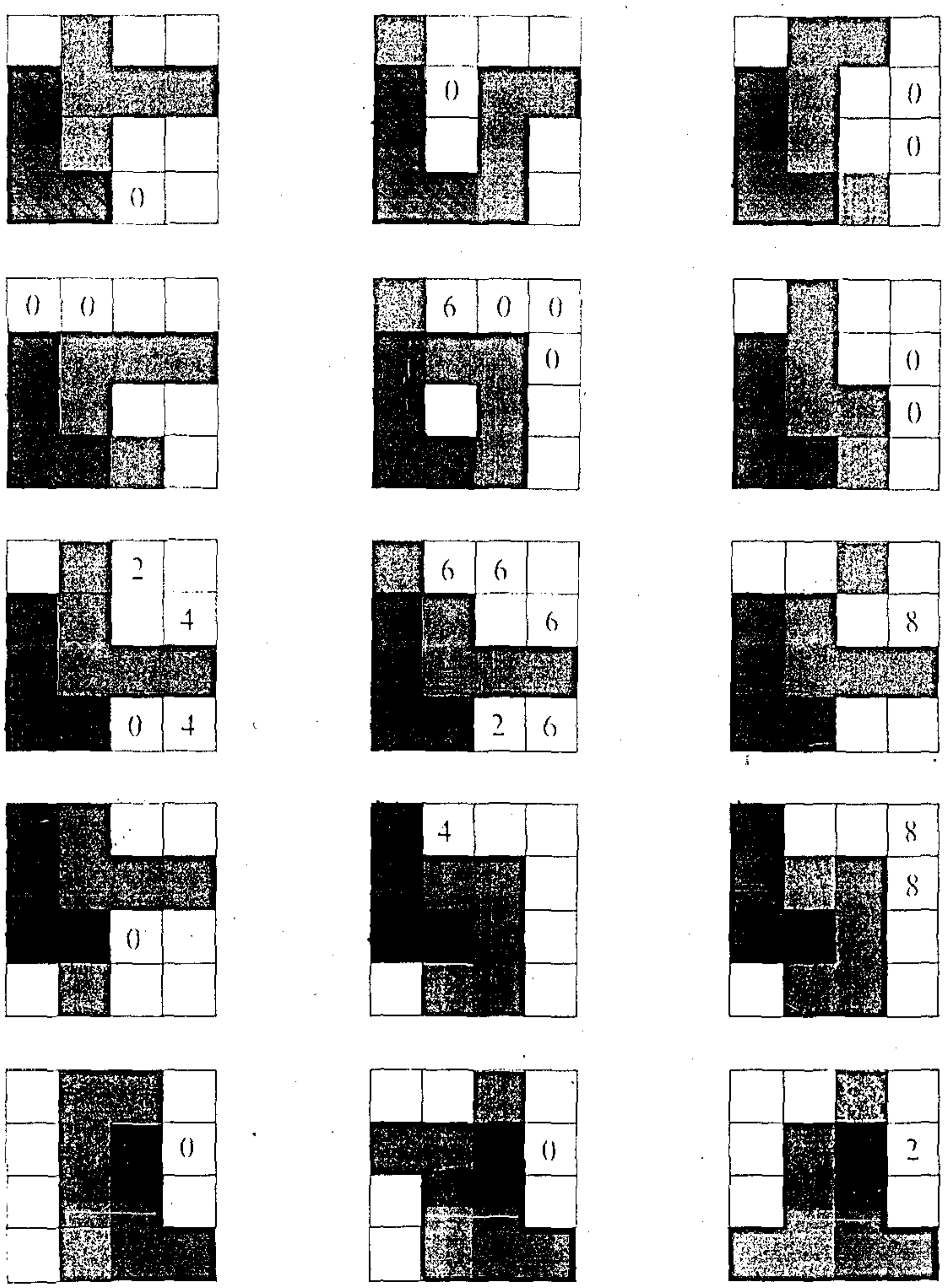
See if you can write down any ideas that you have about this game. Can you suggest any winning STRATEGIES ?

THE L SHAPED GAME

This game, the second involving ideas on shape, was invented by Edward de Bono and may be seen as either a game of strategy or as one to generally develop mathematical thinking. It generates, in my view, quite a nice investigation in searching for winning positions. However, such an investigation would only be introduced after the game had been played many times, generally understood and informal strategy development had taken place. This game is offered by SMP Book D and in the related teachers' guide, they give 13 winning positions, ignoring reflections and rotations of course:



A good player, I suppose, could be said to be one who has memorised each position and plays towards them. Berkelamp, Conway and Guy, in 'Winning Ways', Volume 1 (1982), give 29 strategic positions from which a win will be obtained, provided 'good' play follows, otherwise the game should end in a draw by infinite repetition, unless you allow your opponent to leave you with one of the 29 strategic positions.



In each of the diagrams it is the turn of the hatched, or darker, L and the numbers show both the position of the second neutral square and the 'remoteness' of that position, this being the number of moves to a win, all values having to be even, since the player who has just moved, has left this strategic position and hence should win. In their analysis, Berkelamp et al, state that these are the only such strategic positions and that there are 1006 positions which should lead to losing the game if left by any player, and the remaining 1261 positions are neutral and leaves the game open to either player. Their analysis, does in fact show 15 winning positions, those of remoteness 0.

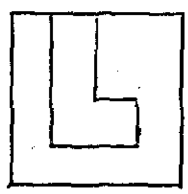
Trying to remember all 29 positions and playing towards them does not really define a strategy since the important aspect is to reach one of these and then to proceed with 'good' play. N.E.Goller has found a simple and elegant strategy by which a player can guarantee at least a draw from the initial position, and from many others:

Place your L piece either:

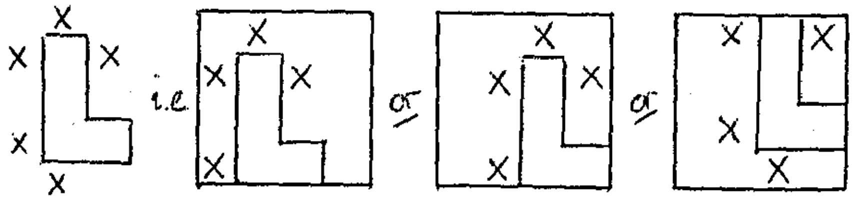
(a) so that it occupies three of the central four squares or

(b) so that it occupies two of those squares, and no neutral piece occupies any other square marked X.

(a)



(b)



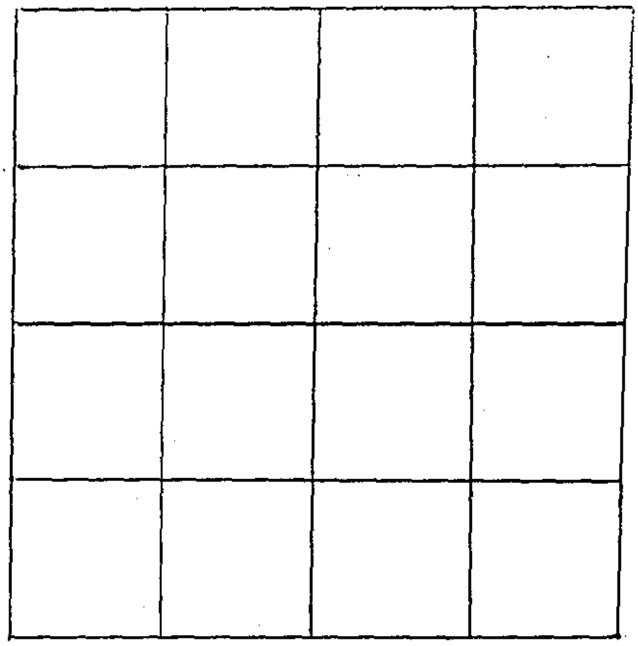
It is possible for the first player to move into such a position on the first move of the game, but then the second player may do likewise, hence, it is likely that if both players are playing to Goller's strategy, then the game will be a draw.

When pupils first play this game they tend to get into loops and only stumble on winning positions by chance, but after a while they again develop the idea of creating unplayable space for their opponent, in the same way as they did for the game of Cover Up, although with this game, it is certainly not as easy.

The use of this game in the trials was more to develop general mathematical and strategic thinking rather than looking at the number of winning positions or stating any particular strategies. There is therefore no pupils' work included with my comments on this game.

COLOURS

This game is played on a 4X4 grid and it involves players trying to work out a pattern set by one of the players who acts as the pattern maker.



It is probably best to play in groups of about four.

The pattern maker uses 4 colours and shades in 4 squares of each colour. However, each colour must form a tetromino. This means that all 4 squares of any colour must be joined together by full edges, there are only 5 ways of joining 4 squares together in this way.

Perhaps you should try to find all 5 ways before you start to play this game.

When the pattern maker has set his pattern the other players have to work it out. This is done by requesting information about certain rows or columns of the pattern. The players take turns to secretly ask the pattern maker for information. The players may only ask about 1 row or column at a time. The pattern maker gives information simply by saying how many of each colour there are in the row or column, this information is again passed back to the player without the other players finding out what it is.

All players start with 8 points in each round and they lose one point each time they request information, therefore the quicker you guess the colour square pattern the more points you get. The players can make a guess only when it is their turn and if they are not correct they have to count this as a go and lose the point for a go, as normal.

The game ends when all players have either got the pattern correct, have no points left or given up, in which case they score 0 anyway. The pattern maker changes for each round and a game ends when all players have set a pattern. The WINNER is then the player with the most points.

Play a few games and see if you can find any helpful ideas on how to play the game well. Discuss these with the others in your group and one of you write them down.

COLOURS

This is the last of the shape based games, using in this case, tetrominoes. However, unlike the previous two it is not a game of entrapment but one of investigation and deduction, closer to the ideas involved in the game Pico Centro rather than Cover Up or The 'L' Shaped Game. Like Pico Centro there is a need to record previous requests and the outcomes of these requests. This is perhaps best done by using a blank four by four grid and noting the details given after a request either above the column or to the side of the row. This leaves the grid itself for the anticipated pattern.

Example:

		3 Blue 1 Orange ↓		
Red	Red	Blue	Green	← 2 Red, 1 Blue, 1 Green
		Blue	Green	
		Blue	Green	
Orange	Orange	Orange	Green	← 3 Orange, 1 Green

At first, pupils simply make requests with little regard to their previous acquisitions of information, but later they learn the 'develop a hypothesis and test it' strategy, again such an important idea in the heuristic approach. With this game, pupils may develop both attack and defence strategies.

At first, pupils find the ideas of this game difficult and seem to be reluctant to start the game before they have clearly formulated their ideas on the rules of it. However, once they have set into the game, then they quickly pick up the ideas on how to play it. Pupils often benefitted more from this game if they were trying to discover the Colours pattern in pairs. This led to a greater degree of discussion and pupils would, strangely enough, often start to discuss their strategies with their 'opponent' players, who were also trying to guess the pattern. They became very interested in the processes of deciding which row or column should be asked and what could be concluded from the resulting information. To this extent, the game must be seen as a hugh success, developing ideas on guess and check, finding suitable recording procedures, logical thinking, looking for patterns, etc., in fact almost all of the general heuristic strategies which I have previously outlined.

The majority of pupils preferred to start their pattern build up from a corner point, choosing a row and a column running along the edge of the pattern. All pupils quickly realised the importance of using both the rows and the columns very early on in the game, usually choosing one of each with their first two requests, and most were able to put the pattern together after about four goes, although this depended upon the general abilty of the children. This game was used mainly with above average ability pupils. The main defensive strategy that came out of the classroom trials was that the player setting the pattern should not use the tetromino which consisted of the four squares joined together in a line since if a player requested this row or column then it made the rest of the game very easy.

Clearly this particular game could be linked up with the teaching of shape or simply as a reinforcement of the importance of the heuristic strategies.

1) We started first picking a corner. It takes on average 3 goes and 1 guess.

2) The designer should try not to draw a line of 4 if possible.

In five goes you can guess the grid by starting at a the corner and working your way up diagonal where the line cross.

Take a column and then a row so they will cross, do the same again and you'll be able to work it out.

The strategy of the game is to go for the corner's and then work inwards, it takes about 4 goes to get your colour's, and to work on the ~~sd~~ side parts first like R1 and C4.

Use rows and columns in turn. Always work out what you can before asking for the next piece of information this will tell you which to ask for. You should be able to get it after about 4 goes but if you are lucky then 3.

DESIGNER

This is a game for 3 or more players about pattern and symmetry.

Think about the types of symmetry that a square has.

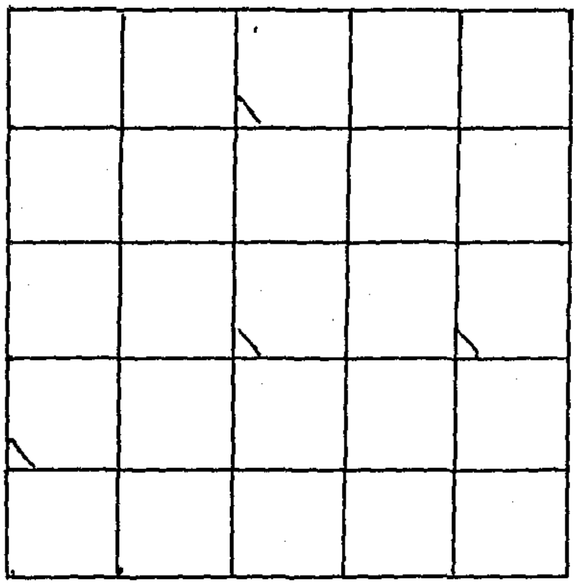
Each player has a 5X5 grid. One player is the designer and has to draw a pattern on their grid by using the symbols *,+ and # . The designer must put a symbol in each square and the pattern must have some form of symmetry.

Eg

*	*	+	*	*
*	#	#	#	*
+	+	*	+	+
*	#	#	#	*
*	*	+	*	*

The design can have as much or as little symmetry that the designer wishes, but it must have some.

The other players now have to work out what the design is. Players make enquiries by putting a mark in any box where they wish to be told the symbol. Their grid is then passed to the designer without the other players seeing it and the designer puts in the requested symbols.



There are no turns in this game and players can request information whenever they like. There is also no limit on the number of turns for any one player or the number of boxes being requested. When a player thinks that they have worked out the design then they fill in all the other boxes and give it to the designer. If a player feels that they can not work out the design, then they can give up by handing in an unfinished grid to the designer.

The game is over when the designer has all the grids. He then scores all the grids. All players who gave up score 0. The other players get +1 for each correct guess and -1 for each wrong guess. The boxes that you requested are not counted in the scoring.

The designer gets a score equal to twice the difference between the best and worst scores, less a penalty of 5 for the first 'give up' and 10 for each other 'give up'.

Play the game in groups of about four, taking turns to be the designer. At the end of a round, when every player has had a turn at being the designer, see who has the best score. Play a few rounds and see if you improve.

Discuss within your group any methods, or strategies, that you used in trying to find the design. Write down your ideas as a group.

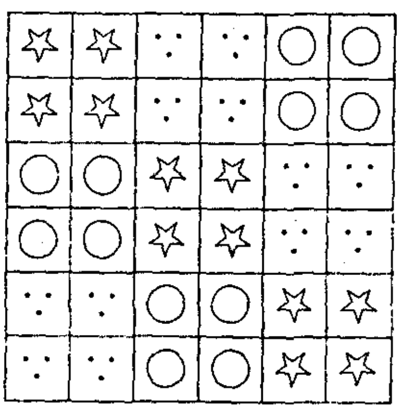
NOW HAVE ANOTHER ROUND

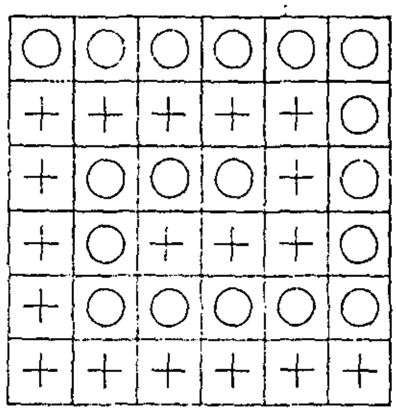
DESIGNER

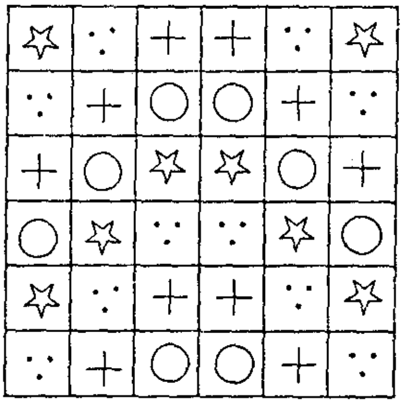
Designer is my adapted version of a game invented by Sid Sackson and called by him, Patterns II. It is described in his 'Gamut of Games' (1969) as a game of induction rather than deductive logic and indeed this type of game, as Sackson claims, is a rarity in the world of games, but not unique. However, I would claim that there is a certain amount of deductive logic involved in the game as well as inductive thought when Designer is played in the classroom.

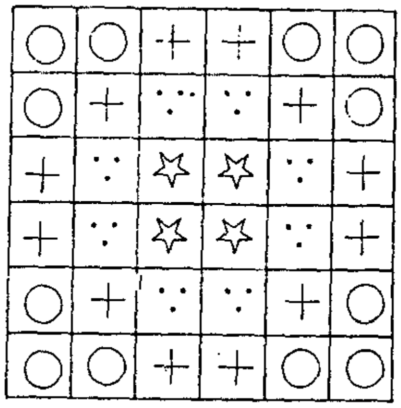
Many games and pastimes have flimsy analogies with induction, this being the rather strange but quite scientific procedure which leads scientists to a multitude of general conclusions. The games of Poker and Bridge use various observational clues to frame probable hypotheses about an opponent's hand and the game of Battleships, perhaps, has a slightly stronger analogy with induction, as do Fleet and Colours, both given in this collection. However, the first true game of induction, Eleusis, was invented by Robert Abbott and was first given in 'Scientific America' (June 1959) and later explained in greater detail in his own book 'Abbott's New Card Games' (1963). This was soon followed by a similar game, Delphi, given by Martin Kruskal in 1962. In both of these games a secret rule exists which dictates the order in which playing cards are placed. Players have then to guess the rule inductively and, not unlike scientists, test their conjectures.

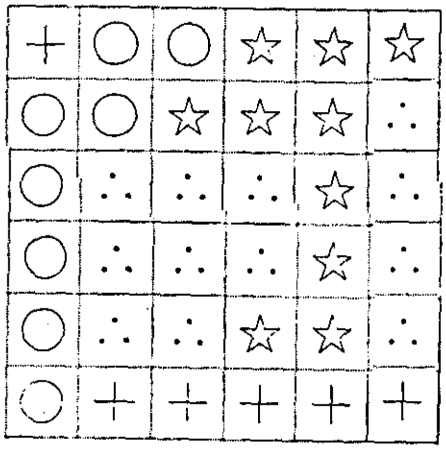
Patterns II, and thus Designer, differ from these other two games in many ways but the similarity lies in the scientific approach and the related problems involving the inductive process. The only difference between Sackson's game and my version is that I have restricted the design to have some form of symmetry and also reduced the game in size. The game, Patterns II, uses a 6X6 grid with four different symbols. Clearly, I have made these adjustments to the game to make it more suitable for classroom use. Sackson had no restriction on the symmetrical aspect of the game, although his brilliant scoring system, taken from Abbott's original game of Eleusis, encourages the designer to give some order to the pattern since their own score is maximised if at least one player discovers the pattern but at the same time another player does not. Clearly, a clever designer would try to judge the ability of the players when setting the pattern, and it is for this reason that I have restricted the game in the way that I have. Martin Gardner, in 'Mathematical Circus' (1982), offers the patterns given below as being suitable for Sackson's game, but two of them would not be suitable for mine.



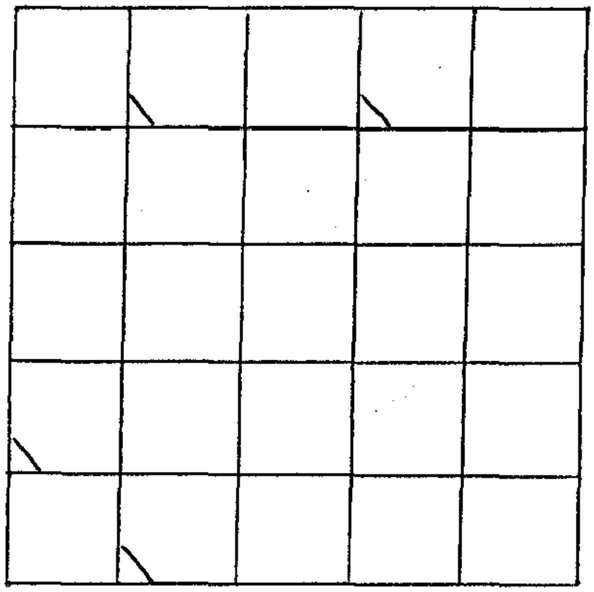








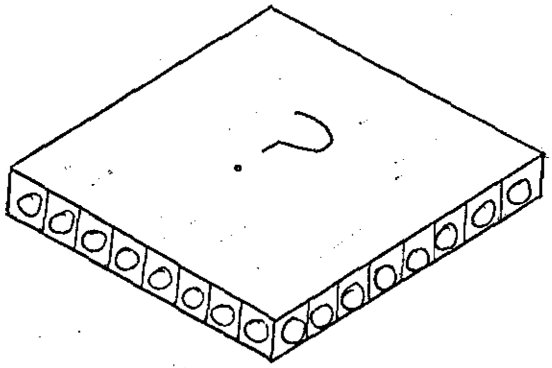
When this game is used in the classroom, the teacher may of course restrict the game to certain types of symmetry if they so wish, or indeed move to the broader rules offered by Sackson in Patterns II. Clearly, with this game, there is a certain strategy in which points should be chosen initially. I would suggest, for the game of Designer, that the enquiries shown on the following page may be a suitable first move.



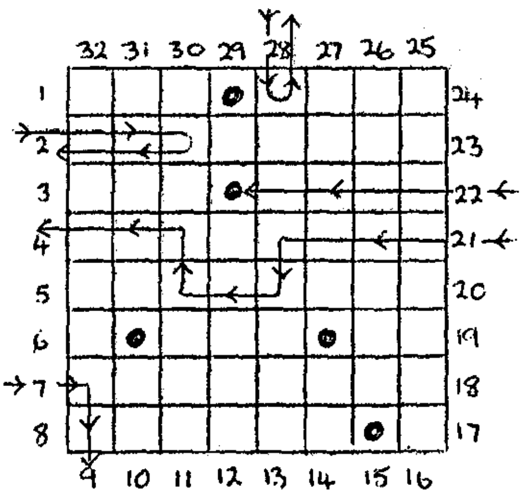
These four squares check various line, point and rotational symmetries of the design. Further enquiries would then depend upon the results of the initial findings.

This was another game which was not used during the trials but its value both as an inductive and conceptual type of game should be clear from the discussion given above. Once again the game probably benefits from being played in pairs since this increases both self confidence and discussion. This will probably then lead to a greater understanding of the important ideas involved in the game.

THE BLACK BOX GAME



THIS IS A GAME FOR 2 PLAYERS.



Each player has a board which is 8 squares by 8 squares and there are numbers around the outside. Each player hides 5 atoms on their board. The object of the game is to discover where your opponent has hidden his atoms. This is done by firing rays into the box and then being told what happens to them.

Rays enter the box through one of the 32 entrance positions and they travel either across or up and down. The path of a ray is governed by the following rules:

1. Rays that hit an atom are absorbed (they do not come out).

2. Rays that come within one square of an atom in a diagonal direction (so that they would pass next to the atom if they continued) are deflected at right angles away from the atom.

3. Rays aimed between two atoms (one square gap between them) are reflected back the way they came from.

4. Rays that enter next to an atom are reflected back out again.

5. Rays otherwise travel in straight lines.

Therefore each time that a player fires a ray into the box, the other player will tell them where the ray has come out or tell them that the ray has been absorbed.
Check these rules out on the diagram of the board on the first sheet.

Play the BLACK BOX GAME with your partner to get to know the rules.
When you are sure of the rules, play against each other trying to guess your partner's box before they guess yours.

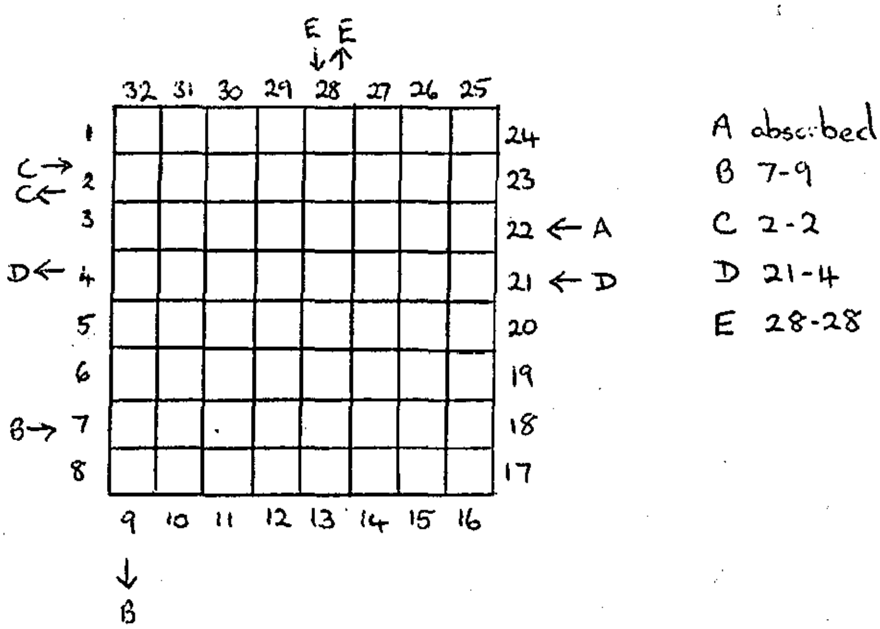
Are there any worthwhile tactics in this game ?

Write down any ideas that you have for good attacks or good defence.

THE BLACK BOX GAME

This game is similar to the previous two, Colours and Designer, in many respects. It is again a game of mathematical deduction, with players trying to work out an unknown pattern by obtaining clues. Unlike the previous two games, however, direct information is not given on the pattern, but only related information with regard to where a given ray will come out of the box.

This game again demonstrates the need for various heuristic strategies and the ideas here should be clear from the discussion of the other games. The method of recording, a common problem in many of the deductive type games, is probably best carried out on a blank grid simply labelling the in and out point of each ray and giving each a label, either a number or preferably a letter, so as not to get confused with the numbered entrances. The blank grid may then be used to develop the hypothesised pattern. The grid below shows the recording of the guesses, and clues given, for the board and atoms shown on the sheet:



When this game is played in the classroom, it is probably best to play two against two since this increases the discussion element in the game. This idea of working in pairs for games tends also to increase pupil confidence, particularly when it comes to taking a chance with the guess and check type situation. It is also a nice idea with this game, as with Hunt The Hurkle, to ask pupils to estimate the expected number of guesses required to pinpoint all five atoms. The guesses here will probably not be as wildly out as they were for Hunt The Hurkle, but the pupils will probably find that they need less guesses than they thought to complete the game, although this is for a totally different reason than for the search type Hurkle game.

Various defence and attack strategies will be developed after the game has been played several times, although no real mathematical analysis is possible here. From personal experience of playing the game, I have always found the best attacking strategy to be one of firing rays around the outside of the box, since this seems either to give quick information about the position of the atoms or reduces the size of the box under investigation. I suppose the corresponding defence strategy would be not to place atoms on the outermost two bands. I have also in the past found the protection of one atom by some of the others to be a particularly good defensive ploy. This may result in the opposing player finding four of the atoms very quickly but then being frustrated in their search for the final atom. However, such a tactic may only really be played once against any particular opponent and therefore is not really a long term strategy.

This particular game was not used during the trials and my comments are based on previous classroom use and the playing of the game myself. Variations on the game could be produced by either altering the size of the box or changing the number of atoms. The game could be developed further to allow rays to be sent along the diagonals of the box or it could be played in three dimensions, although this would require the introduction of further rules. In my view, the game could do with a good scoring system, with an unusual basis, rather than the 'see who can guess it first' idea, which I have suggested. This could be a nice problem to set to the pupils after the game has been studied in the classroom.

This game is now in fact another which has become available commercially and is known by exactly the same name as I have used.

NIM

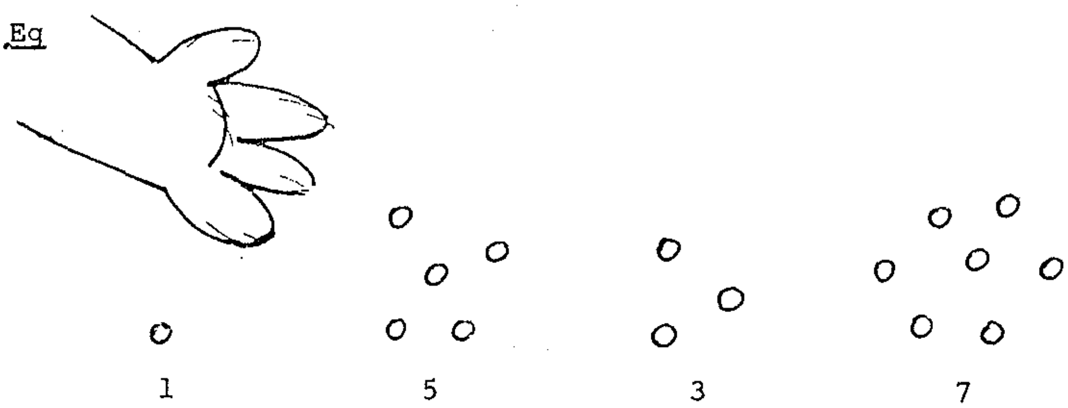
The game NIM takes no time at all to learn, the rules are very straight forward.

IT IS A GAME FOR 2 PLAYERS

You take a bundle of counters, beads, matchsticks or anything else you can get hold of. You then divide the bundle into any number of piles and it does not matter how many piles there are or how many there are in each pile.

EASY SO FAR EH!

Eg



Each player now takes turn to remove as many counters as they like from any ONE of the piles. They can if they wish take the whole pile, the only rule is that they must take at least 1 counter.

The WINNER is the person who picks up the last counter.

Play the game with a friend and see if you can discover any WINNING STRATEGIES.

Write down in your own words any ideas or strategies that you have for playing NIM.

NIM

I find the game of Nim both amazing and fascinating, and would recommend it to anyone who enjoys a game and has not yet played it. The word Nim is an anglicised form of the German word meaning to take and the game is one of a very large family of this type of game. It is another game of strategy which may be, and has been, totally analysed.

The game itself has, of course, an infinite number of starting points and this makes the existence of a single strategy even harder to comprehend for the novice, particularly the school child. Every state of the game may be classified as either safe or unsafe for the person who produces that state. A state is only safe if the sum, modulo 2, of the binary representations of the numbers in each pile is zero. This does, of course, have various particular implications and the game has been analysed in great detail by many different authors. These analyses would produce many immediately recognisable safe and unsafe positions, however, all are guided by this single idea.

Example:

In the example given on the sheet, there are four piles containing 1,5,3 and 7 stones respectively.

1	1
5	101
3	11
7	111
Total modulo 2 =	000

Hence this is a safe state.

There are many more unsafe states than there are safe. This can be proven with a little logical thought being applied to the above rule. If the number of stones in each pile is selected at random then the binary representation of that number will be a random string of binary digits. Therefore any particular binary digit will be odd or even with equal chance. If we now consider further, that the number of piles is also taken at random, then this implies that the total for any column of the modulo 2 sum is also equally likely to be odd or even. Now, the modulo 2 total will only be zero if all columns are even, hence, for a game of Nim where the highest pile of stones is 7, i.e. all numbers may be expressed using three binary digits, then the probability that the total is zero is 0.5 raised to the power 3 =0.125 i.e. 1/8 . Hence, in a game with no more than 7 stones in any pile, any random state is seven times more likely to be unsafe than it is safe. It is interesting to note that the number of piles has no bearing on this matter.

The winning strategy for any player must therefore be to always leave a safe position since it can be proved that the only states available from such a state are in fact unsafe. The player must then restore the safe state, this always being possible from any unsafe state.

The analysis of this game therefore shows that if the starting state of the game is safe then the second player wins and if it is unsafe then the first player wins, assuming both are expert players. If a player is faced with a safe state, set by the opposing player, then as with many games of strategy, the best thing to do is as little as possible, trying to extend the game hoping that this gives the opposing player more time, and therefore more chance, to make a mistake, which may then be capitalised upon.

This game was not used a great deal during the trials, although when it was, it provoked considerable thought and discussion. Pupil strategies usually developed around the idea of producing two equal piles, although some took this idea further and realised that if the game of Nim could be split into two further, but equal, Nim games, then this was also a winning position. Other pupils gave certain states of the game which would lead to the equal piles situation.

1

1) Reduce to 2 piles
2) Make numbers even when it is your partners go next.

2

if there are 2 piles, the person who goes first wins, with the exception of when there are 2 equal amounts in each group, if this happens, then the first person loses.

3

if there are 3 groups, make them 3,2,1 on your opponents go, then you have won.

You should cut the groups down to two, Then you should make the two groups have the same number of counters in each, with your opponent having the next move. After his move you should take the same amount as his last move which will eventually result in a 1 to 1 situation with you opponent to move.

It is difficult to be able to see what to do to make your chances of winning higher until there are only very few dots left.

If you are given with these combinations you can win -

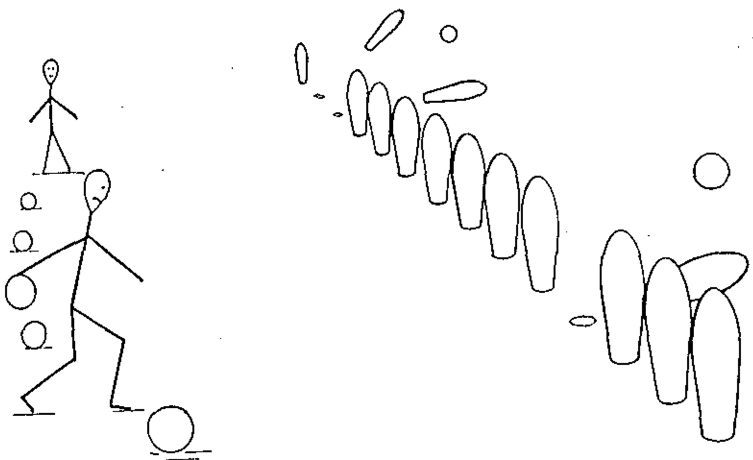
2, 2, 1

2, 1, 1,

A general combination is any ~~m~~ number and then an even number of ones.
e.g. (28, 1, 1, 1, 1) (5, 1, 1,)

KAYLES

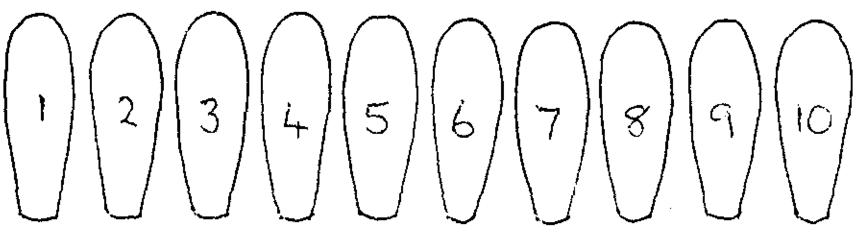
This is an old English skittles game.



The players become so good at this game that they can bowl to knock down any single skittle or any pair of skittles next to each other that they choose. It is impossible to knock down two skittles unless they are standing next to each other, even the world champion can not do this. The winner is the person to knock down the last skittle.

If you were an expert Kayles player, how would you play the game ?

Here are 10 skittles:



Play against a friend, remembering that you can knock down just 1 skittle or 2 next to each other.

Play the game lots of times and find out who is the Kayles Champion. See if you and your partner can come up with some winning ideas. If you do, then write them down in your own words.

HAPPY BOWLING !

KAYLES

Kayles is a game that was invented by the English puzzler Henry Ernest Dudeney and given in his first book, 'The Canterbury Puzzles' in 1907. He posed this as a possible situation which may have arisen in a 14th century skittles game of the same name.

This game may be analysed in the same way as Nim, and in fact once again all states may be classified as either safe or unsafe for the player who has produced that state. However, the mathematical analysis and determination of safety of this game is much more complicated than it is for Nim. The complication is clearly caused by the additional rule of restricting the number taken away at any one time.

The determination of safety for any state still depends upon the modulo 2 sum of binary representations being zero. However, the binary representations are not the direct equivalents, as in Nim, they are in fact the k or Grundy values, named after P.M.Grundy who was one of the first to show how such values provide a strategy for a large family of Nim type games. The derivation of these numbers is complicated and would add nothing to the present discussion if included, therefore this will not be considered. However, the interested reader may find this in 'Puzzles and Paradoxes' (T.H.O'Beirne 1965)

The table of k-values shows no regularity until after 70, when there appears, rather curiously, a periodicity of order 12. The k-values for numbers over 70 can be found by using the equivalent modulo 12 number in the second table, whereas the k-value for numbers up to 70 may be found in the first table, both tables being given on the following pages.

k-VALUES FOR ROWS OF UP TO 70 IN THE GAME OF KAYLES.

Number	k-value	Number	k-value	Number	k-value
1	1	26	10	51	1000
2	10	27	1000	52	1
3	11	28	101	53	100
4	1	29	100	54	111
5	100	30	111	55	10
6	11	31	10	56	1
7	10	32	1	57	100
8	1	33	1000	58	10
9	100	34	110	59	111
10	10	35	111	60	100
11	110	36	100	61	1
12	100	37	1	62	10
13	1	38	10	63	1000
14	10	39	11	64	1
15	111	40	1	65	100
16	1	41	100	66	111
17	100	42	111	67	10
18	11	43	10	68	1
19	10	44	1	69	1000
20	1	45	1000	70	110
21	100	46	10		
22	110	47	111		
23	111	48	100		
24	100	49	1		
25	1	50	10		

k-VALUES FOR ROWS OF MORE THAN 70 IN THE GAME OF KAYLES.

Modulo 12 value.	k-value
0	100
1	1
2	10
3	1000
4	1
5	100
6	111
7	10
8	1
9	1000
10	10
11	111

If we look at the table of k-values we see that no number has a k-value of zero, and hence, any single set in Kayles will represent an unsafe position, therefore the first player should always win, again assuming both are strategic players. This is because the first player can produce a safe state simply by removing one or two skittles from the centre of the row to create two new equal rows. This will then always produce an answer of zero in the modulo 2 sum, hence, whatever the second player does to one pile, the first player simply does the same to the other, this again creates an even number of equal piles.

Although the game, as I have presented it here, only has one pile or set, it can be seen that if a skittle is removed from the middle then we have two sets or piles. This leads to the more general idea of applying the Kayles rules to a starting position of more than one row, as with the game of Nim. With reference to the game of skittles this is equivalent to taking any number of sets with any number of skittles within each set and simply lining them up in one long row with a skittle space between each set. Martin Gardner in 'Mathematical Carnival' (1975) gives various safe doubles and triples for sets of less than nine, and suggests the learning of these could be extremely useful in the playing of the game. The safe doubles, apart from two equal sets, are 1-4, 1-8, 2-7, 3-6, 4-8 and 5-9. The safe triples can quickly be calculated by choosing one digit from each of the following three sets: 1-4-8, 2-7 and 3-6, this in fact gives twelve resulting safe triples. Sam Loyd, the famous American puzzler, gave a further alternative form of the game. This involved setting up the skittles to form a circle, this in effect has the added idea that a player may knock down the two initial end skittles in a single row, if they so wish.

Although the mathematical analysis of this particular game is more complicated that that of Nim, it is probably an easier game to introduce and use in the mathematics classroom. This is probable due to the idea that a single row of ten skittles is used. Perhaps, if I had presented the game of Nim in one of its simpler cases, then this would have been seen as the easier. During the trials this particular game was played mainly by the younger pupils within the secondary age range, whereas Nim was played by the older pupils. Most pupils eventually discovered particular winning positions, but few actually stated the complete strategy for the actual game of Kayles given on the sheet. I am sure that if this problem had been given to the older pupils, who played the game of Nim, then they would have been able to state such a strategy and this would have clarified their ideas on the game of Nim itself.

The Best way to start with two numbers and go to ~~one~~ two spaces

I would start in the middle and work up like this

Two spaces in middle.

1 2 3 4 X X 7 8 9 10

I would go first.

the person who goes first if they put 1x then the other person willwin. but if you start with 2 x then the first person willwin

the one who goes first wins

IF YOU GET 1 AND 1
OR 2 AND 2 OR
SOMETHING LIKE THIS
YOU WIN AND IF THE
OTHER PERSON GETS
IT YOU LOSE.

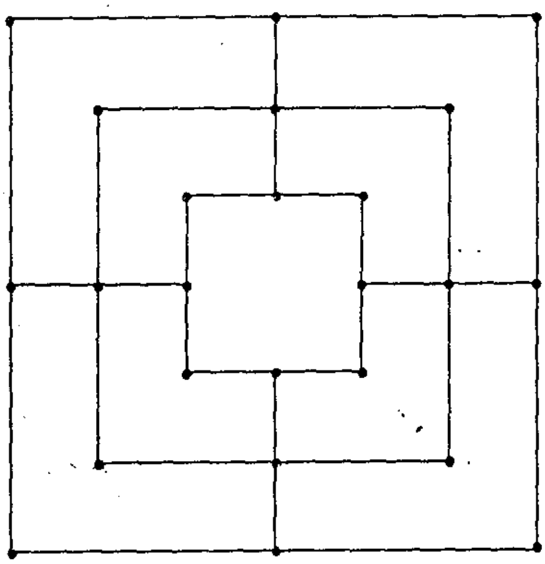
NINE MEN'S MORRIS

This is a game for 2 players.

This game was played in England in Shakespeare's time when the 'board' was cut in turf and wooden pegs were used as counters. After very wet weather, the 'board' often became unsuitable. This game was in fact mentioned in Shakespeare's 'A Midsummer Night's Dream'.

A version of the game is still played today. The shepherd boys in Lesotho mark a board in the sand and play with stones and beans, but this is being frowned upon because it distracts them from their work ! The Lesotho version is known as Marabaraba and is played with 12 pieces each.

The version that we are going to play uses 9 pieces each, hence the name. The board used is like this:

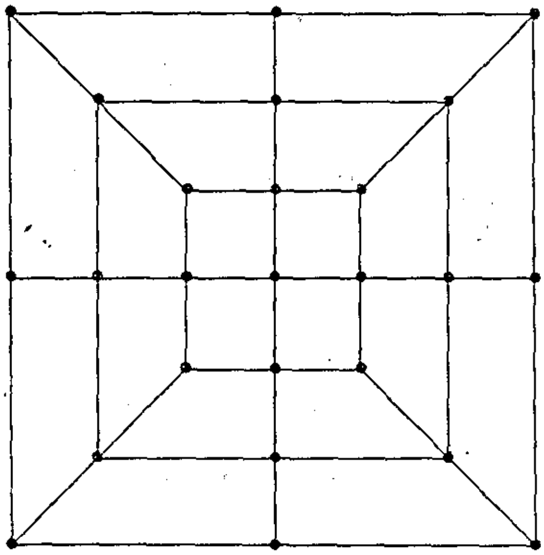


Enter the men alternately, one at a time, on any vacant point. Each time a player forms a row (or a mill) of three men along any line, he can remove one of his opponent's pieces, except those already in a mill.

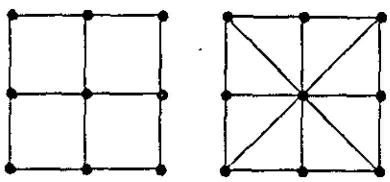
When all the men have been entered, continue to take turns by moving a man to an adjacent vacant point along a line, trying to create more mills, thus reducing the number of your opponent's pieces. A Player wins by either blocking all his opponent's men so that they can not move, or by reducing them to 2 pieces.

Games like this go back a long way and are very widespread. A 'board' has been found from an Egyptian temple dated about 1400 BC.

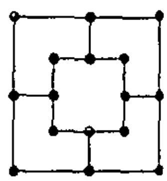
Marabaraba (12 men's morris) has already been mentioned but versions of three and six men's morris can also be played.



A Marabaraba board



Forms of Three Men's Morris



Six Men's Morris

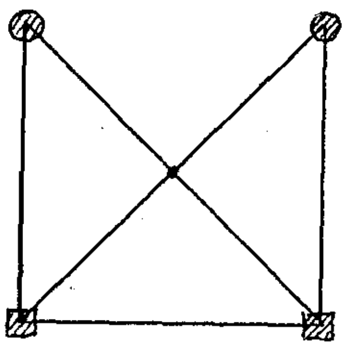
Play 9 men's morris several times against a friend.

Can you find any winning ways or strategies ?

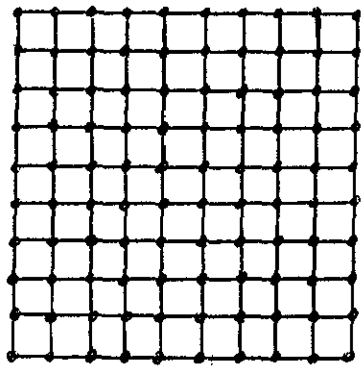
Write down any ideas that you have about this game.

NINE MEN'S MORRIS

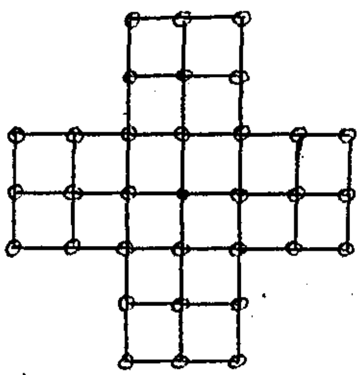
The game of Nine Men's Morris, like many given in this package, is one of a large family of similar games. The game itself is a very old one and it is surprising how little known it is, considering this and the fact that there are so many versions and variations of the game. The family of games which this particular game comes from are positional type games and include the already mentioned Noughts and Crosses together with the well known games of Solitaire and Go and less well known versions of Nine Men's Morris called Wei-ch'i, Pong Nan K'i, Mu Torere and Achi. All of these games have similar rules but are played on different boards, some of which are shown below.



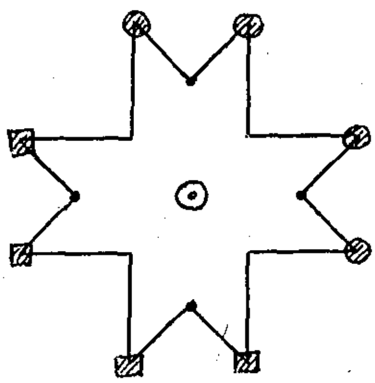
PONG NAN K'I



GO

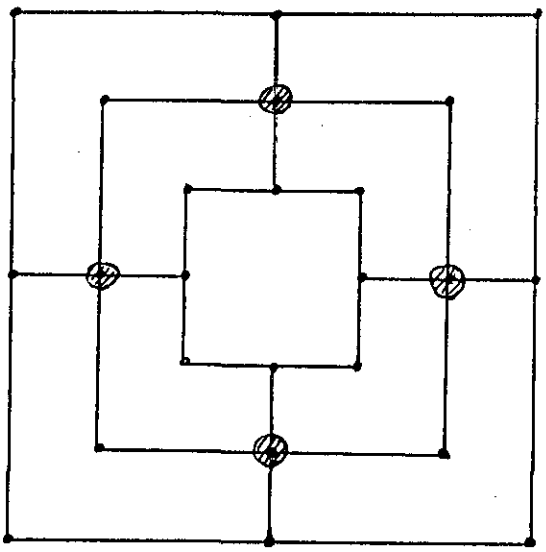


SOLITAIRE



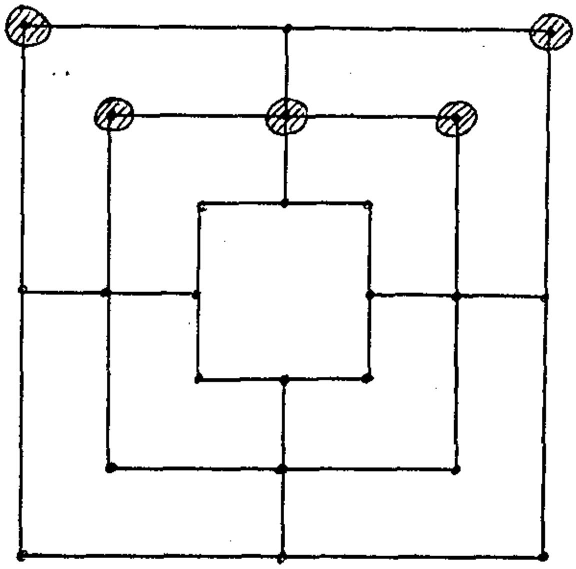
MU TORERE

Looking now at the actual game itself, it does not take a great deal of time playing the game to realise that there are certain strategic points, or sets of points, which are of greater value than others. This game is given in SMP Book F and in the related discussion, which appears in the corresponding teacher's guide, the strategic importance of certain points is in fact discussed. Since play in the second stage of the game involves moving to adjacent vacant points, then it is suggested that the four points shown below are of greatest value. These points should therefore be chosen with priority during the first stage, since there is a greater chance of finding a vacant point to move to from these positions.



If both players adopt this strategy then they will each have two of these points, but the advantage will still lie with the first player since they are always one move ahead and this has a major effect on the play, due to the features of the game.

The ideal situation to be in during the second stage of the game allows a mill to be formed with every single move. Such a position is shown on the following page.



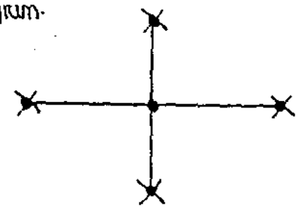
The counter in the middle is simply moved backwards and forwards, creating a new mill each time. Any successful strategy must therefore take account of these ideas in both stages of the game.

When this game is used in the classroom, it may either be built up to, using some of the smaller versions of the game, or it may be introduced immediately in its own right. During the trials, this again proved to be a popular game and pupils quickly developed their ideas on it, seeing a similarity with Noughts and Crosses. Some pupils had seen this game previously, since it is now available commercially, and one boy produced a magnetic pocket version which generated several playground extension games, again showing the enthusiasm that the work in the mathematics classroom can generate.

The majority of pupils were not really aware of the first stage strategic points, choosing instead to show a preference for corner points or points around the centre square. I should imagine that this is something to do with their linking of this game to the ideas in Noughts and Crosses. However, most of the pupils quickly became aware of the idea of blocking their opponents whilst at the same time trying to form mills themselves, so that they could reduce the number of opposition counters. In general all pupils handled this 'balance' with a great deal of confidence and expressed their ideas well on how they thought someone should play the game. Some pupils managed to successfully use and express the main strategy for the second stage of the game.

Remember that you have to stop you parter from getting a mill at but try to get one yourself. When you have a mill keep your counters close together so you can mak up a mill again quickly and easily.

To win 2 mills you can put your counters on 4 places making sure theirs a dot in the middle. See diagram.
If you move your counter on the spare space you can get 2 mills and take away 2 men.



Once you've made a mill keep moveing one of them prefable the middle one backwards and forwards and keep taking your opanents counters off.

The best place to start is the corners because then you can have more direction in order to make a Mill.

Put your counters in the corners so you can get a mill two diffrent ways. Put your counters so most of them are in the middle but make sure some of them are on the outside. Then you can easliy block them so they can't move. If you keep all your counters round one square then the other person can block you.

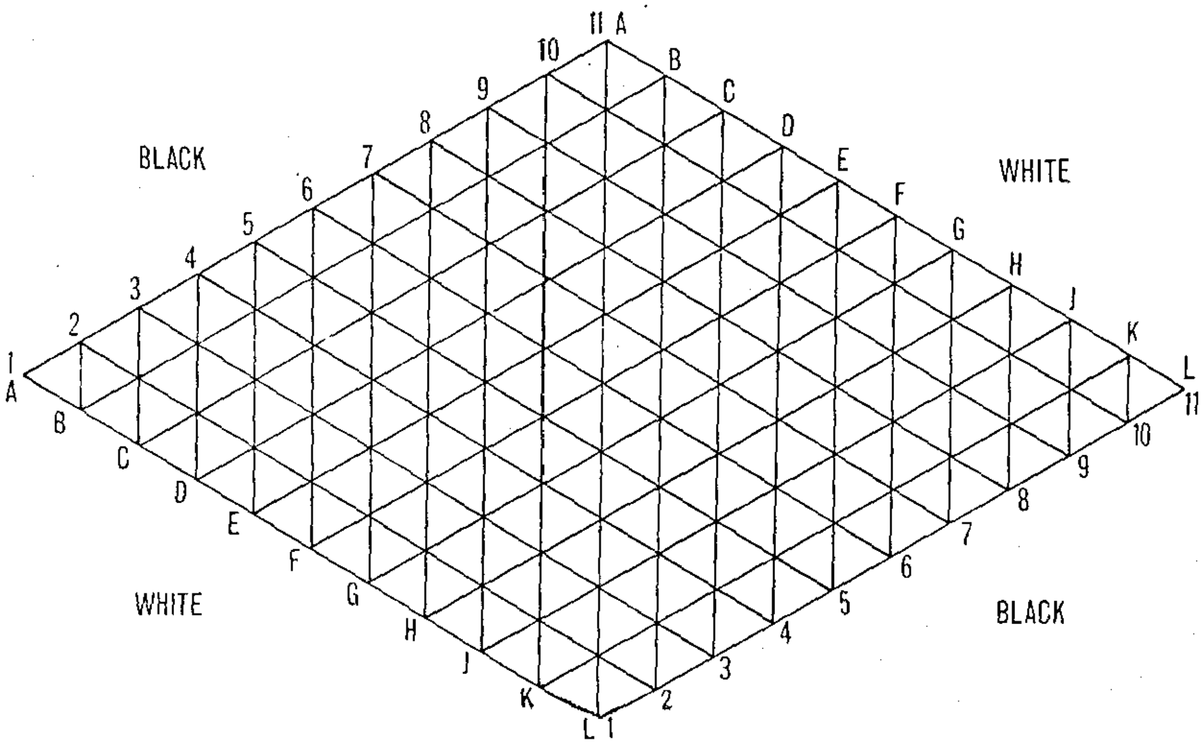
In the gam n.m.m it is best to get a mill, surround your mill, and do not let your appoinent get into the guard and then un do and redo your mill getting your appoinants pieces off every time

I think that once some one has got a line of 3 then they take one of your piece's off it is ~~easy~~ more easy for them to win because they have 9 pieces to play with and you only hafe 8 so then they have more chances to win. This game is a similar version of O's and X's who ever has the four corner on the outside square ~~[illegible]~~ Loses

HEX

Hex is a board game for 2 players which was invented in Denmark in the 1940's.

It is played on a diamond shaped board which is divided into triangles.

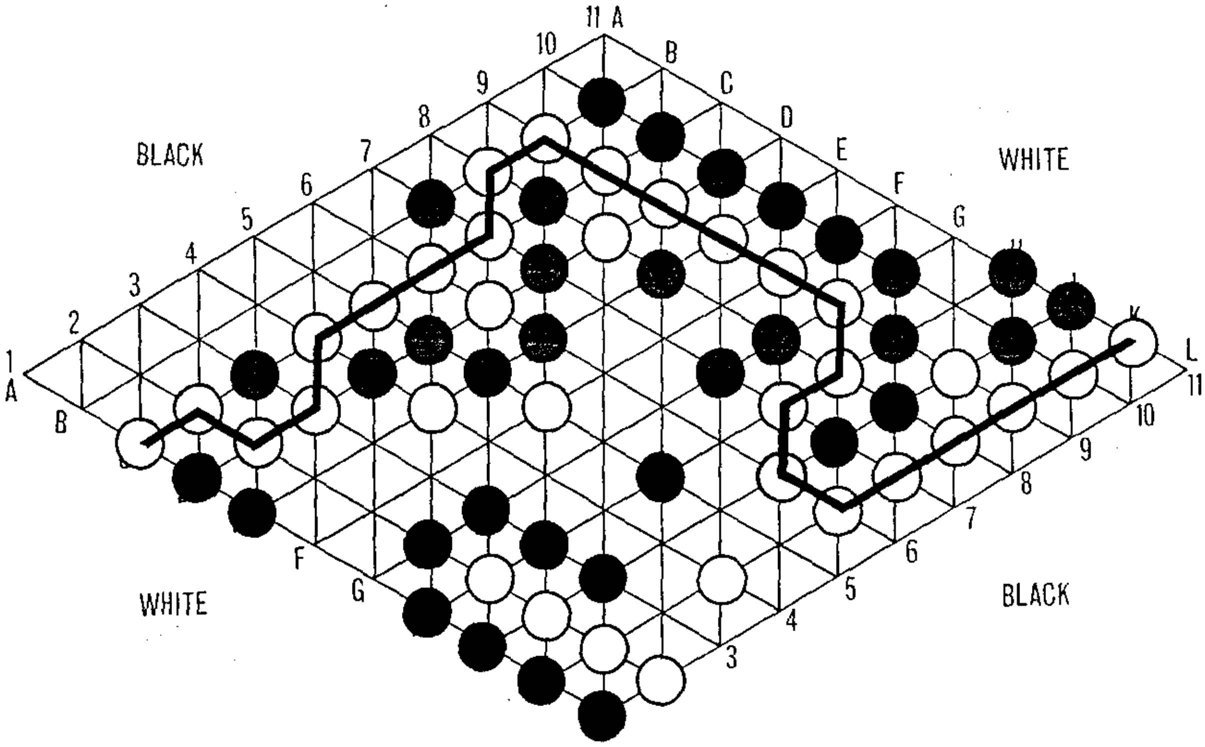


Usually the board has 10 triangles along each side but the game could be played on boards which are either smaller or larger.

Each player has a set of counters, one player has black and the other white. Players take turns to place their counters on any of the grid points.

Once a counter has been played it may not be moved and there is *no* capturing of counters. Each player has to build a continuous chain of counters joining together the two sides marked with their colour. The first player to do this is the WINNER !!!

Here is a game that white has already won.



Play the game with a friend and see if you can work out any winning ways. If you have any ideas then write them down trying to explain them carefully.

HEX

Hex is another relatively new game, being invented by Piet Hein, the Danish mathematician, in 1942. There is in fact very little literature on this particular game and there is no complete analysis available on it as yet. However, there has been some work carried out on it by various people. Martin Gardner accredits the American mathematician, John Nash, with the proof that Hex is a first player game and Beck et al (1969) give several theorems and lemmas on the game, but these are far from a full analysis.

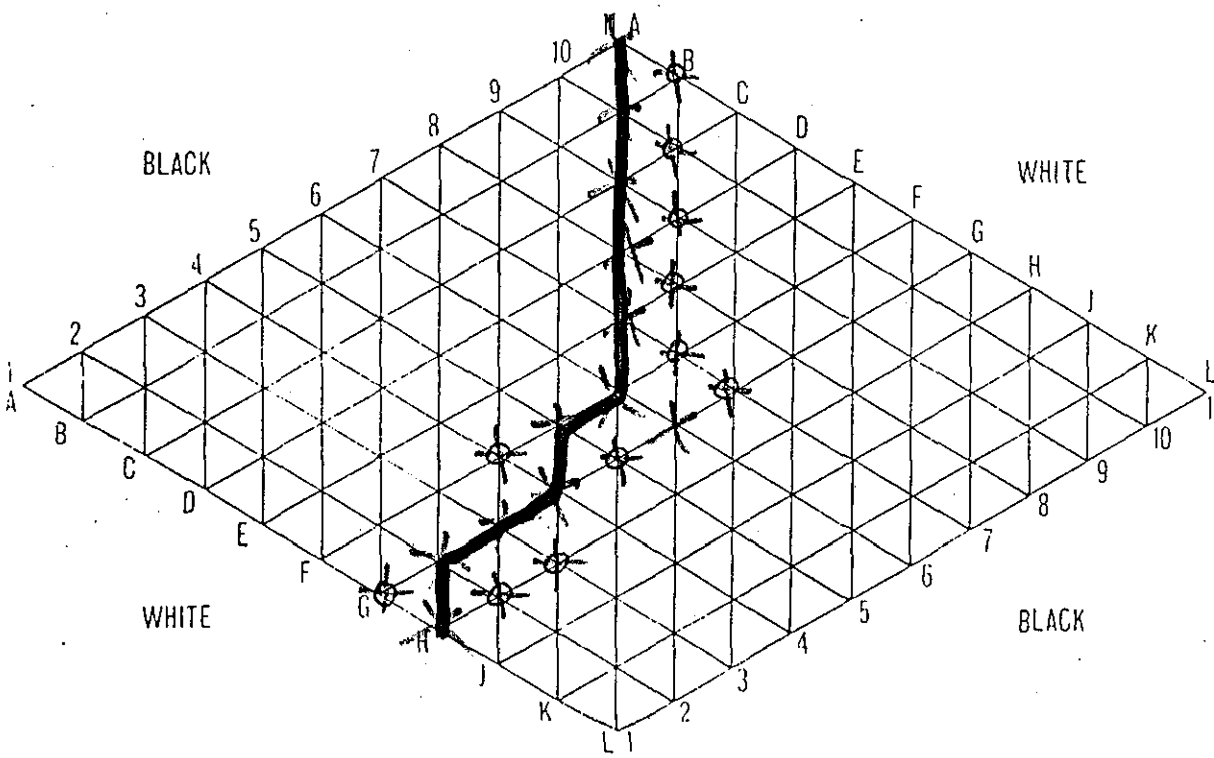
Hex is another game which is, perhaps, rather deceiving when first encountered in the sense that it gives an impression of being relatively straight forward to play and likely to end in a draw each time that it is played between two good players. However, both of these facts are false. A winner must be obtained for each and every game; even if the pieces were placed at random, at some stage they would still offer a win for one of the players. The second fact is soon rejected after playing the game a few times, although the approach of the inexperienced pupil player is probably dictated by the idea that there is not much to the game. However, they soon start to develop both attacking and defensive strategies.

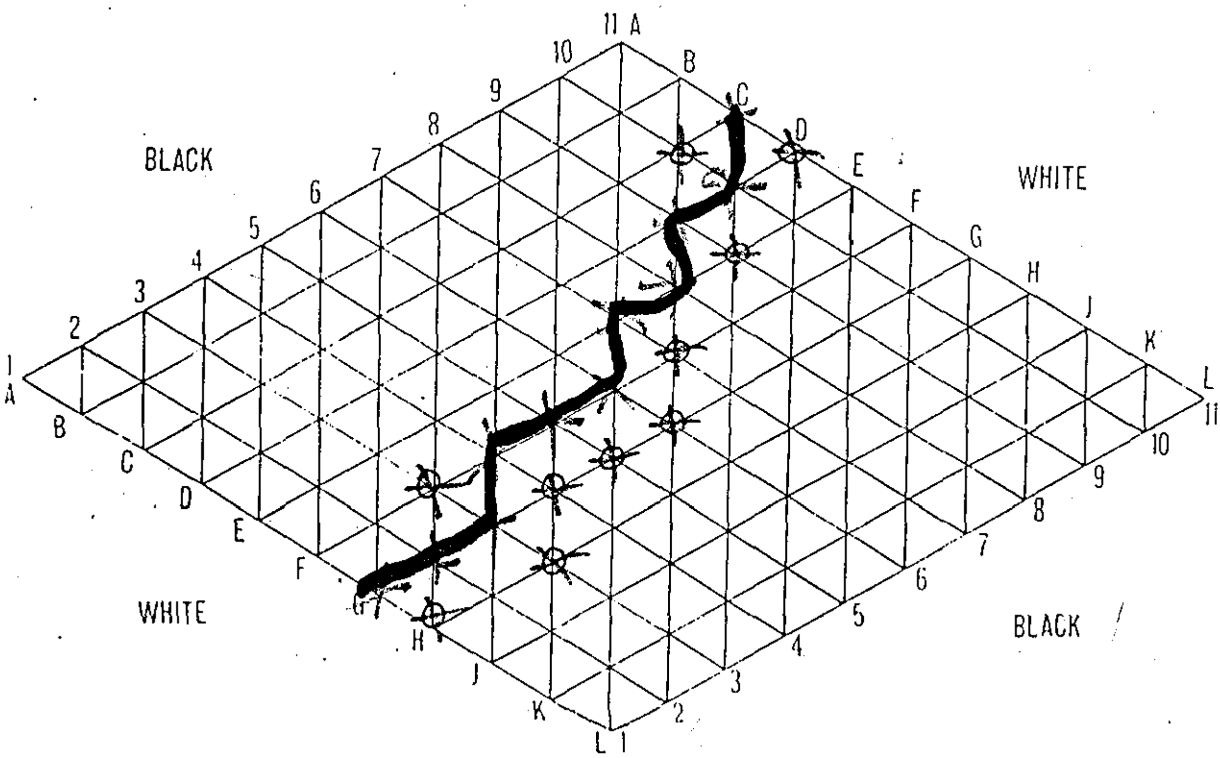
When Hex was used in the classroom, the inital games were very short, with the pupils trying to set off across the board to form their continuous chain as quickly as possible, almost ignoring the other player. This clearly led to the game being won by the first player. The second players then generally tried blocking the points immediately in front of the last first player counter but this strategy was unsuccessful since the first player could always move around the blocking counter. The usual strategy to follow from this stage was to defend the opposition 'finishing' line by trying to form a barrier which was difficult to get around. If this barrier was completely across the path of the opposition then this was in fact a win in itself. This again is not an absolute strategy since either player may play in any position at any time.

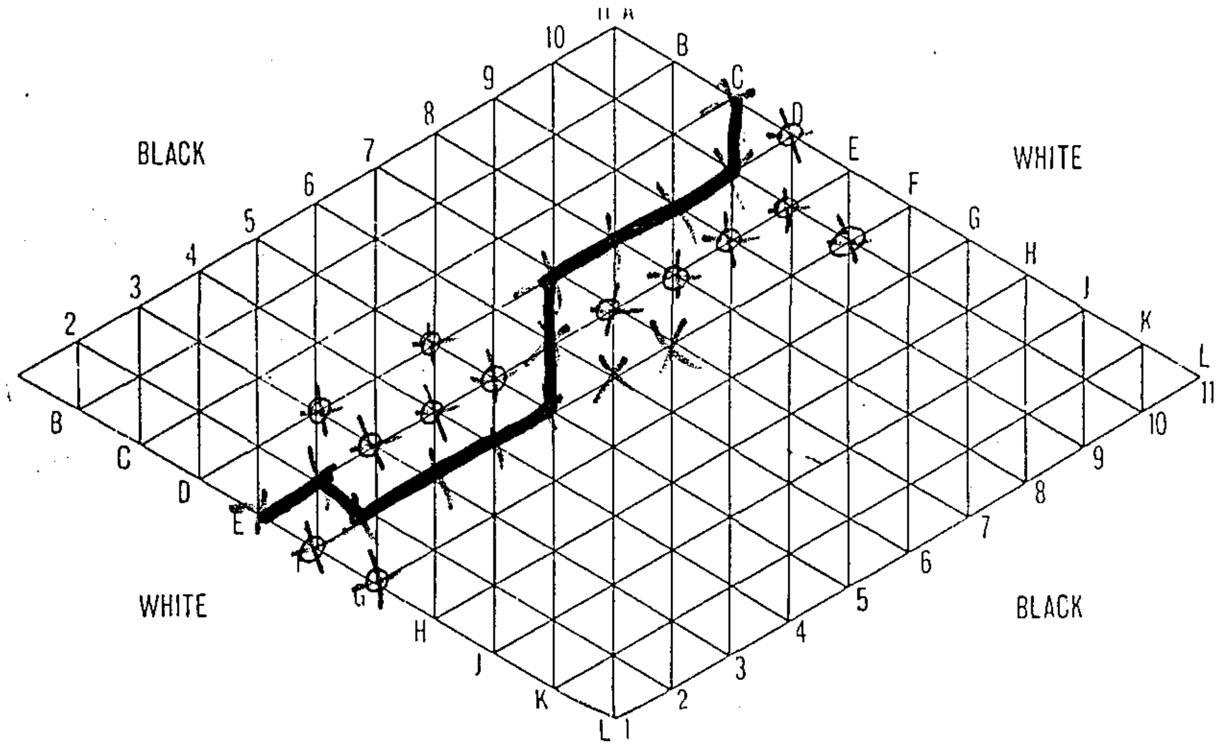
Once the pupils realised that there was more to the game than simply racing from one side to another, the playing of the games became a much slower, thoughtful and strategic affair. The player who goes first still has the upper hand and is looking for the break-through all the time and the second player tries to defend at all costs, hoping that the first player will make a suitable mistake, thus allowing the advantage to move to them.

This proved to be another popular and interesting game in the classroom, and as with many of the others the pupils showed more interest, perhaps, in discussing and working out the game within their group than actually winning. This, in my view, is a major step forward in the teaching of mathematics. If we can get pupils interested in discussing their ideas and methods, and genuinely working together on a problem, then this is a much superior learning situation than each trying to simply get the correct answer on their own. This situation may be discussed with the pupils to emphasis the point and it is one which they tend to both understand and appreciate.

The majority of pupil work on this game involved them in discussing various approaches and strategies and putting these into action, rather than giving their strategies in a written form. I have, therefore, included a series of Hex games, taken in chronological order, from one particular group, who although initially set out playing the game one against one, ended up playing with teams of three against each other. The series of games show the development of pupil ideas which I have discussed above.



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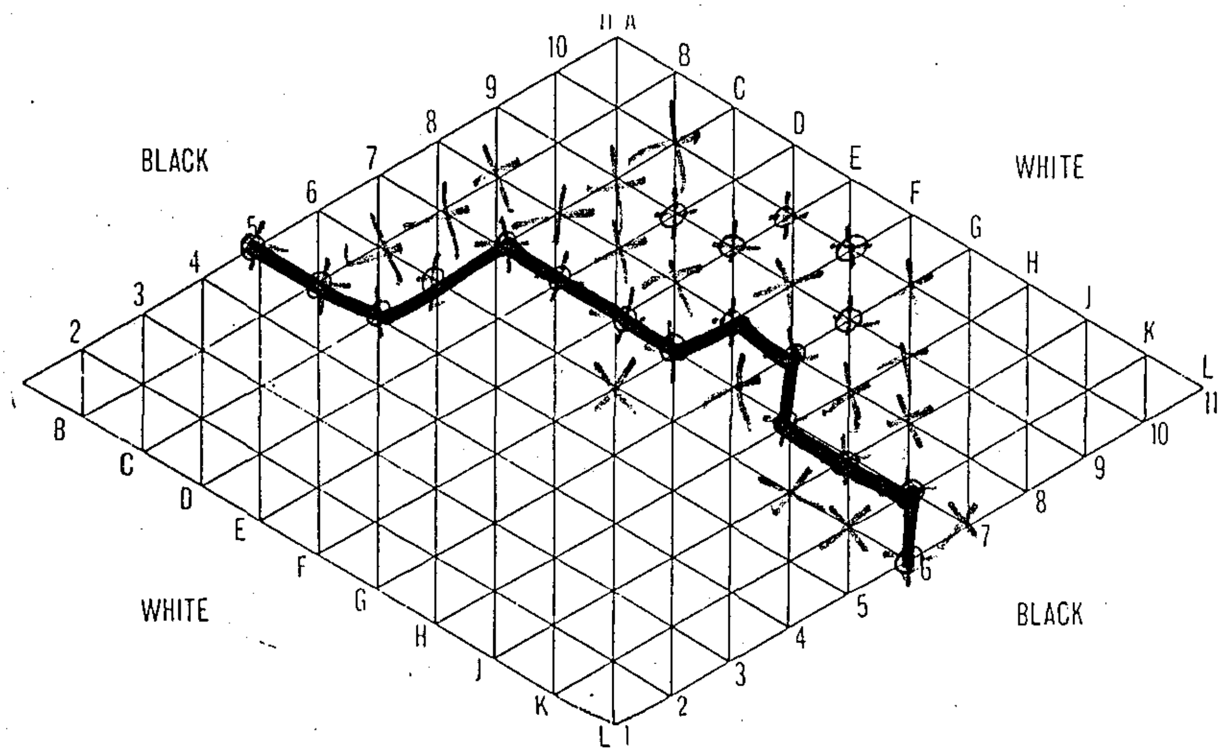
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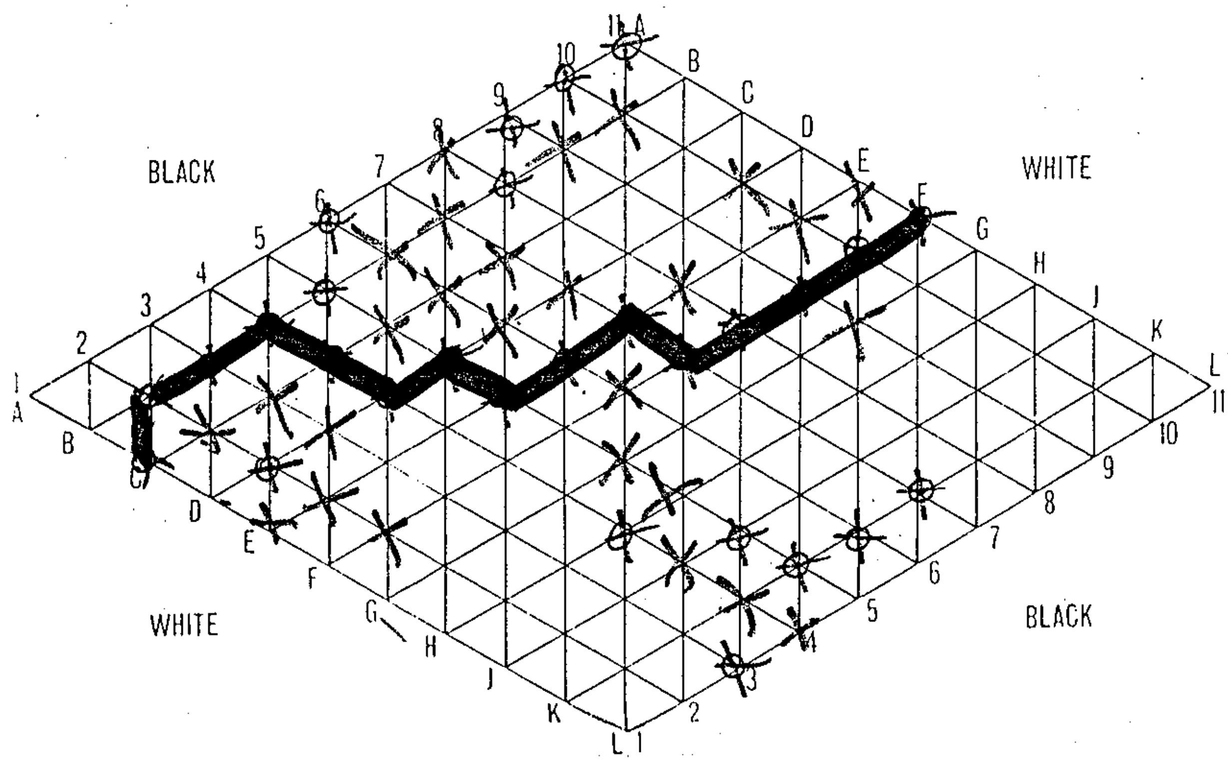
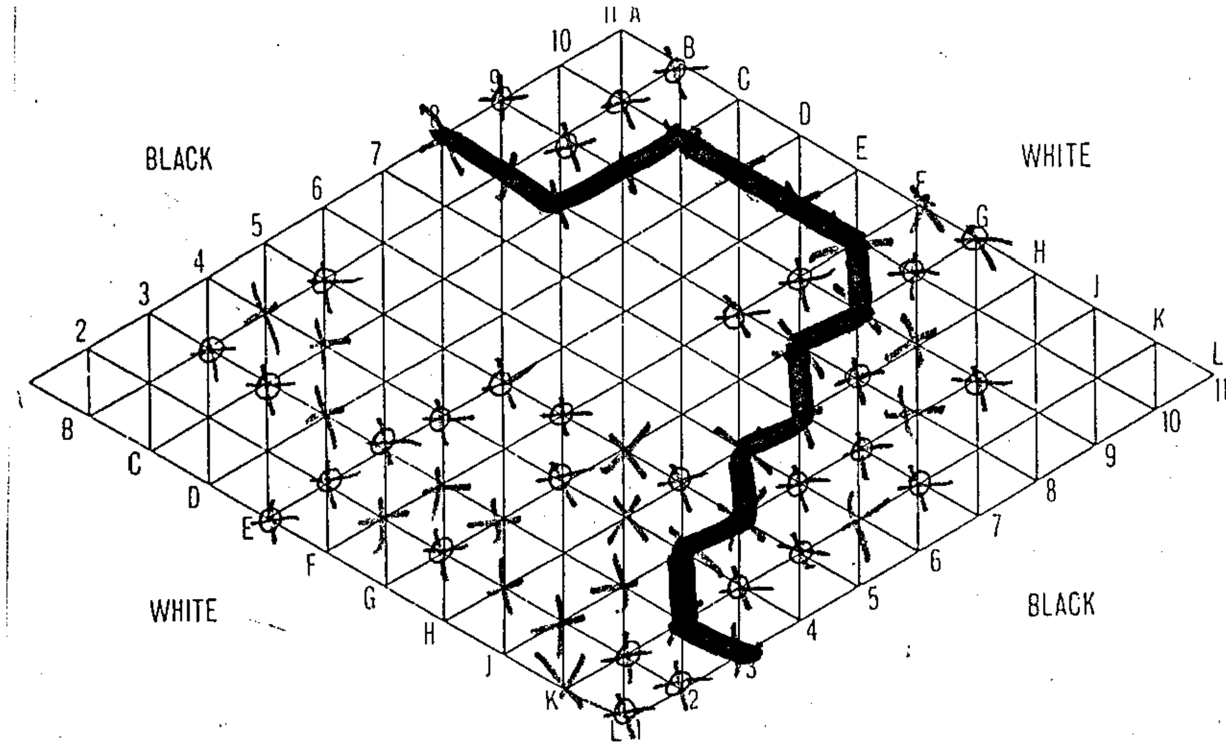




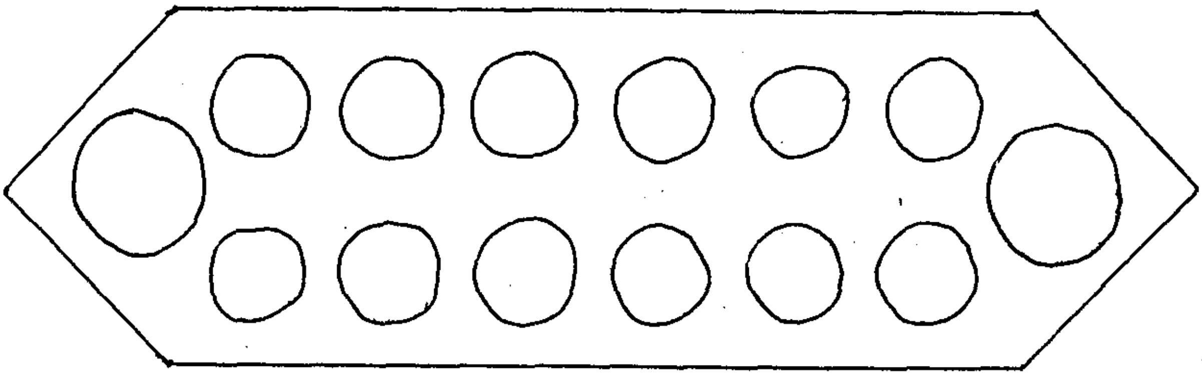
If you make a long blocking line, your opposition can't get past. Also if you start around no 6-7 then spread your dots I think you have a better chance of winning because you only have to join them up.

WARI

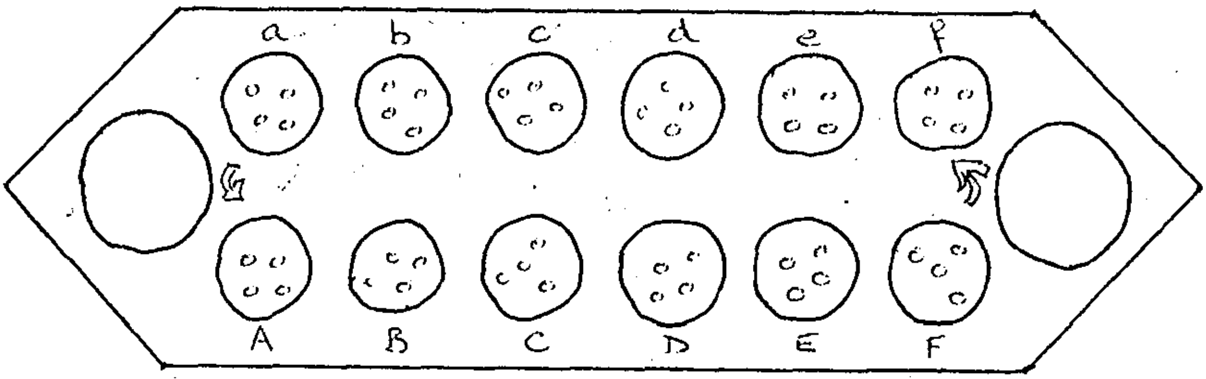
A GAME FOR TWO PLAYERS

Wari is a version of an ancient game known as MANCALA. It has been described as the national game of Africa and many claim that it is the oldest board game that still exists. It is believed to have been popular during the Empire Age of Ancient Egypt well over 3000 years ago. Although there are over 200 versions of Mancala, Wari is one of the most widely played versions and originated in West Africa.

A Wari board looks like this:



The game is played by placing and moving beads around the cups. The board is set up by putting 4 beads in each of the 12 small cups.



The 6 cups labelled with capital letters all belong to one player and the other 6 belong to the opposing player.

The first player picks up all the beads from any of his cups and puts 1 bead in each cup moving anticlockwise around the 12, starting from the cup which is just to the right of the one which they have just emptied. The second player then does the same thing.

Play continues until a player puts the last bead from one of his cups into an opponents cup which had contained just 1 or 2 beads (ie it now contains 2 or 3). The player now wins the beads in this cup and puts them in his large cup at the left hand end of his row of small cups. He also wins the beads in the previous cup if this is an opposition cup and it now contains EXACTLY 2 or 3 beads, and so for the cup previous to that and so on.

If a cup has more than 11 beads and this is the cup that you want to empty, then when you have put one bead in each cup, going around the board, you jump the cup that you are emptying.

A player may not leave all his opponent's cups empty, they must play to leave at least 1 bead if at all possible, if this is not possible then the player takes all the remaining beads and the game is over.

THE WINNER IS THE PLAYER WHO HAS WON THE MOST BEADS.

Play the game against a friend several times so that you get used to it.

Can you write down any suggestions as to how to play the game and be good at it ? This is known as a STRATEGY. You can often do a lot better in a game if you have a strategy. This is also true when you are trying to solve problems in mathematics, science or any other area.

WARI

This is another game which I have included more for general reasons, rather than anything specific. Wari is a very old African game and exists in a multitude of forms. There is no chance element at all in Wari and so, I suppose, theoretically a 'best play' can be calculated for any position of the game, and hence a complete analysis and approach could be given from the starting position. However, one need not stretch the imagination to extreme limits to realise both the complexity and vastness of such a solution.

In the absence of a full strategy, we need not despair, since informal strategies and tactics become reasonably clear after playing the game several times. For example, a cup containing just two beads is obviously vulnerable from an opposition cup which contains the correct number of beads to reach that cup. The weakness must be defended and this may be done by either increasing the number of beads in the cup by emptying an allied cup to the left of the vulnerable one, or by emptying a cup so as to add a bead to the threatening cup so that it can no longer place its last bead in the vulnerable cup. An alternative, but short term, and hence unsatisfactory, solution is simply to empty the vulnerable cup, however, this cup can quickly be made vulnerable once again by dropping in a further bead.

An aggressive game calls for a build up of beads in the cups at the right hand end. If the number of beads in any cup is so low that the last bead will not land in an opposition cup, then it is inoffensive. A good attacking strategy is clearly to place several opposition cups under threat simultaneously.

This is quite an advanced type of game and is, maybe, more suitable for older or more able pupils, however it can be played by all and I do feel that a slightly simplified version could even be played by very young children to emphasise something as basic as counting.

This final game in the package was not used at all during the trials, like most of the general type games, and hence, no discussion of pupil work or activity on the game is included. Since I have stated that its inclusion is for general purposes then it may be used at any time to develop logical and mathematical thinking and would probably not be related to any specific topic in a mathematics course.

THE FINAL QUESTION

The final question to this work must be directed at the value of the material which I have produced, and whether or not I have achieved what I set out to. These questions cannot really be answered separately since the success of my work depends totally upon the value of the material. I have previously stated, in chapter 3, that good teaching material is that which works well for both teacher and pupil alike. I would claim that the material has achieved a great deal of success in the development of the following areas of mathematics teaching:

* group work and discussion;
* the awareness of the existence and importance of strategy in tackling mathematical problems;
* the general heuristic strategies;
* general enjoyment and enthusiasm in mathematical work;
* pupils' individual confidence;
* a variety of concepts;
* the integration of the various aspects of mathematics teaching.

Clearly if my package has helped to develop all of these aspects, as I have claimed, then it has been a huge success and a valuable contribution to mathematical education in general. However, the final judgement must lie both with the reader and user.

When this work was introduced to children they usually struggled to verbalise their own ideas and hence I would claim that an increase in this type of activity is therefore necessary. Charles and Lester (1984) talk of various types of problems: drill; simple translation; complex translation; process; applied and puzzle. It is really the last three that I have directed my work at since the others have always been given sufficient treatment. Obviously I would not claim that playing games alone will develop all the aspects which I have mentioned above and I think that I have made this clear throughout my discussion, nor would I claim that games of strategy, and indeed my teaching package, is the only type of work which would achieve this. What I am claiming however, is that I feel the package is another useful tool in the battle towards a greater understanding and enjoyment of mathematics and perhaps a tool which was not previously available in such a form.

In the previous chapter I have mentioned how the particular games were or may be used. I would now like to talk briefly, and in more general terms, about how the package was used during the trials. The majority of the sheets in the package were used to initiate the classroom activities through the teacher and the sheets were not normally given out to the pupils but instead acted as teachers' guides. When constructing the sheets, I deliberately tried to strike a balance between provoking ideas and leaving the game totally up to the pupils. I certainly did not want to prescribe exactly what would be discovered during the activities or how the teacher should use the games; however I did want to try to set the atmosphere and general type of approach to allow the development of group discussion, self questioning and strategic thought. It was, in my view, very important that the use of the material and general approach were both correct and in line with the views that I have stated in the earlier chapters of this work.

I did not give any guidelines as to the appropriate age group for the games since it is my belief that material to develop the ideas that I have suggested should be sufficiently open-ended to be appropriate to a wide range of both ages and abilities. However, the majority of games were developed with secondary age pupils in mind, although in chapter 4, I have often made reference to the suitability of some of the games to children from a younger age group.

Throughout this work, I have concentrated on the idea of strategy. I have tried to emphasise, both in my discussion and on the game worksheets, how important I think it is to have a strategy when solving a problem in any area of life, and that a greater degree of success is achieved when such a strategy is adopted. It is therefore my view that it is extremely important for a mathematics teacher to have such a strategy in their own approach. This strategy must be the long term development of the necessary problem solving techniques, the heuristic strategies. It is therefore important to continually emphasise these to the pupils. Such an emphasis takes time, it cannot be rushed since the ultimate aim is the development of a breadth of knowledge and a depth of understanding of the subject. In my view and experience, it is the mathematics teacher who has a long term strategy, a self justification and who continually questions their own work, who achieves the greatest success in the mathematics classroom. They are the ones who achieve their aims of teaching a true understanding and enjoyment of the subject.

As for the question of examination results, tight time schedules, syllabi and schemes of work all preventing such an approach, then I would state that a true understanding of the general strategies can be applied to many new ideas with great success and therefore the schemes of work and examination results will look after themselves. It is also important to realise that problem solving techniques provide valuable tools that can be used to discover and develop new concepts. There is, in my view, no need therefore to feel such a restricting pressure, but a strong self belief and justification is necessary.

During the trials, considerable discussion took place on all the games, by the staff involved in the trials, and the majority of them were tested to some extent. However, the general type games tended not to be used. I think that the main reason for this is because this type of game is best used after the other games, involving more specific strategies, have been studied over a long period of time; since this was not the case, then the other games were used in preference. I would not really claim that any game has been fully tested but I have indicated the extent to which they were used during the trials.

The trials took place during a time of severe industrial action within schools and this prevented me from using any mathematics department outside of my own school. However, all the teachers who regularly taught mathematics within the school, about ten in all, offered their help and assistance in testing the package, the trials were therefore broad in the sense of involving a good range of individual teachers and pupils even if they were rather narrow in the sense of all being within a single school.

When the games are played in the classroom, the teacher can never predict with absolute confidence the outcome of the lesson since much of the lead, quite correctly in this type of work, has to come from the pupils' ideas and responses. Something different may well crop up each time the material is used with a different teacher or group of pupils. For these reasons I have tried to include sufficient pupil work in chapter 4 to indicate the variety of ideas which arose during the trials. I have deliberately included work from all ability pupils and usually in its original form unless it was not suitable to be photocopied, as happened in a few cases due to coloured crayons being used in which case I rewrote the work exactly as the pupils had. It was clear, in my view, that pupils did not have to come up with the best strategy or an in depth mathematical analysis to benefit from this material; the true benefit came from their discussion, development and expression of ideas and strategy.

I think that perhaps it may have been better to restrict the trials to a limited number of the games and carry out more detailed trials on these rather than to scan trial the majority of the games in the package, although I do not really see this as a major problem in my work. Obviously I would not claim that this area of study and research has been fully exhausted since there are many further areas of related research which may be picked up from this study alone. I would suggest that the following related areas may be suitable for further work in the future:

* In depth and broader classroom trials of the games in the package;
* Further study of the mathematics relating to the various games;
* Games, their use and occurrence in the more popular mathematics schemes which are used in schools;
* Pupil assessment in such work;
* A review and study of the commercially available games which are suitable for use in the mathematics classroom.

Research in this last area would be greatly assisted by Sid Sackson's list and brief description of over 200 commercially available games given at the end of his 'A Gamut Of Games' (1982). Although I am suggesting that there are further related areas of study, I would also state that my own original intentions have more than been covered and hence completed. I set out to state my case for the use of games of strategy in mathematics teaching and to develop a teaching package which would integrate the ideas into a practical, relevant, suitable and interesting mathematics course which would allow me to achieve my stated aims for mathematics teaching, this I feel I have done.

I would like to conclude by stating that children need to understand and be able to use mathematics; they have therefore to be taught problem solving strategies through experience, these are not basic skills to be treated in a traditional algorithmic fashion. Teachers therefore have to listen, interact and evaluate and never ignore, interfere or judge. With these ideas in mind it may be noted that 'games increase student interaction and hypothesis testing within the group' (Meyer 1983), a statement which is fully supported by my reported trials. 'The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies, ... therefore appropriate curriculum material to teach problem solving should be developed for all levels. Most current material strongly emphasises an algorithmic approach to the learning of mathematics and as such they are inadequate to support or implement fully a problem solving approach'. (NCTM 1980) I believe that the work contained in this dissertation makes a move to correct this situation.

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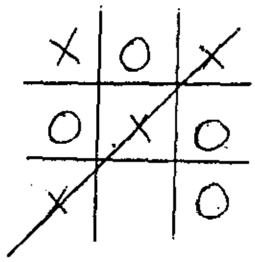
APPENDIX A

THE GAMES



NOUGHTS AND CROSSES

You have probably all played this game before but play it again several times against your partner, taking turns to go first.

Try to write down some rules for trying to win, this is called a *STRATEGY*.

If you had a choice would you go first or second ?

Why ?

Can you write anything else about this game ?

9+3+6=?

15 TO WIN

1+4+8=?

[1] [2] [3] [4] [5] [6] [7] [8] [9]

Try this game using the cards in your envelope.

Spread the cards out so that all the numbers are showing.

You and your partner now take turns to choose a card from the set. You must leave your cards in front of you with the numbers showing so that your opponent can see them. The idea is that you try to get a set of any 3 which add up to 15, the first to do this WINS!!

Play the game several times.

You may notice something familiar about this game.

Write down any ideas that you have about this game.

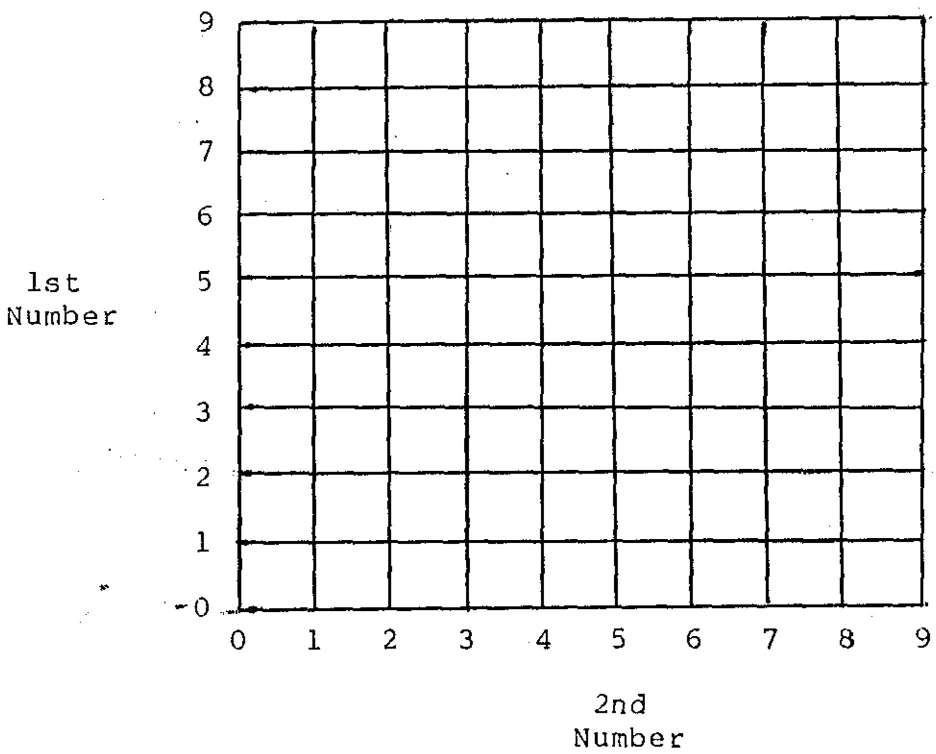
3+1+4=?

5+3+7=?

FLEET

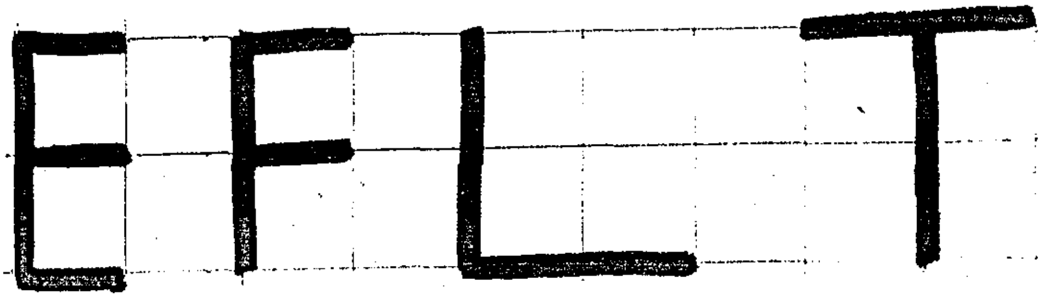
This is a game which is very similar to BATTLESHIPS, but there is one major difference which I will leave you to discover for yourself.

Fleet is played on a grid like this:



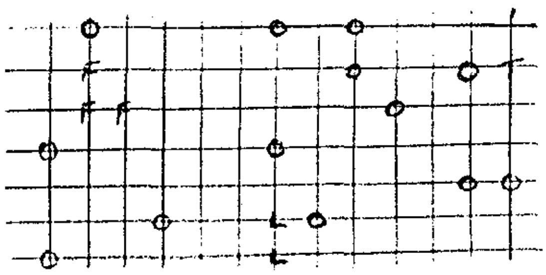
You have to hide your FLEET and try to find the fleet of your opponent.

Write the word FLEET on your grid using letters like these:



You then take turns to guess a position using coordinates. You will need 2 grids, one to hide your own fleet and one to help you find the opposition fleet.
When you have made a guess put an o on the point if you have not hit and put the letter that you have hit on the point if you have been successful.

Eg



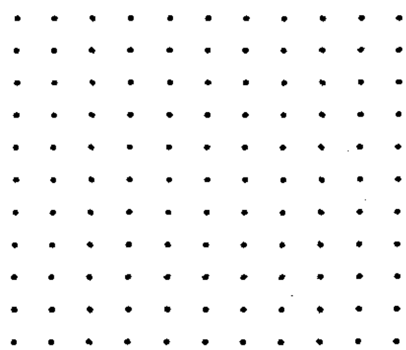
The WINNER is the first to destroy the opposition FLEET.

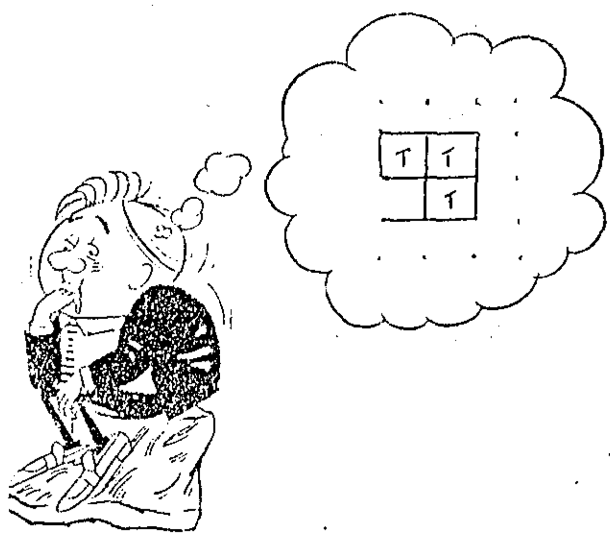
DOTS AND BOXES

This is a very popular game which you may have played before.
It is a game for 2 players and you take turns to join up any 2 dots next to each other, either with a vertical or horizontal line. The idea is to complete more boxes than your partner. If you complete a box then you put your initial in it. The winner is the player with most boxes when the playing sheet is full.

Play the game on a sheet which will give 10 boxes by 10 boxes. Notice how many dots there are in each row and column, can you explain this ?

Can you write down any ideas on how to play the game well ?





CENTURY

Have a go at this game!

You can choose any number from 1 to 10 to add to the running total of the game. Starting at zero, you and your partner take turns adding to the total and the one who gets the CENTURY (Total=100) WINS!!!

Play the game several times to see who can score the most centuries.

Discuss with your partner any ideas you have on this game and try to write about them in your own words.

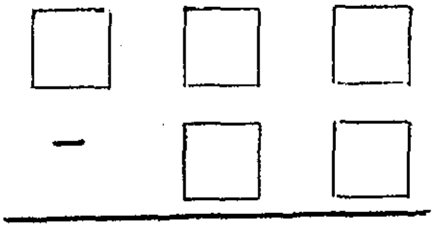
Now, you and your partner use your ideas to play against another pair.
Did you win ?

100

SUBGAME

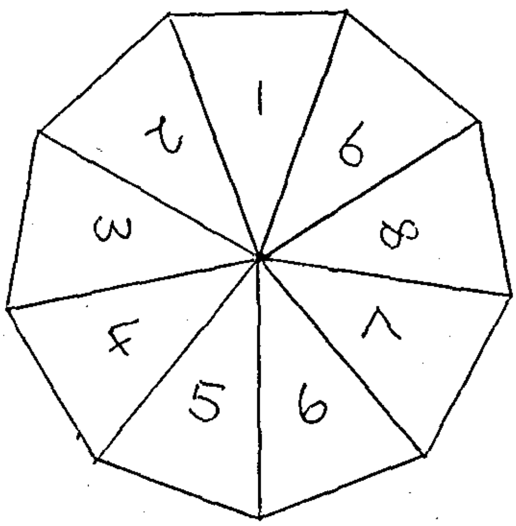
You may have played SUBGAME against the computer, if you haven't, then don't worry, here are the RULES.

You are given a subtraction sum like this:



You are then given five random numbers between 1 and 9, one at a time. (These are called digits.) Each time that you are given a digit, you have to decide which box to put it in, it cannot be moved after you have placed it in a particular box. The idea is that you have to get the highest answer that you can. The winner is the person who gets an answer which is bigger than anyone else's. You can either play this game against the computer or your friends.

Now that you all know the rules, play the game in pairs with the help of a 9 sided spinner like this one:



Using the Subgame grid and the spinner (to produce the numbers) see who can get the highest answer for the subtraction sum.
When you have played the game quite a lot write down in your own words the method that you used to try to win. A method like this is called your STRATEGY.

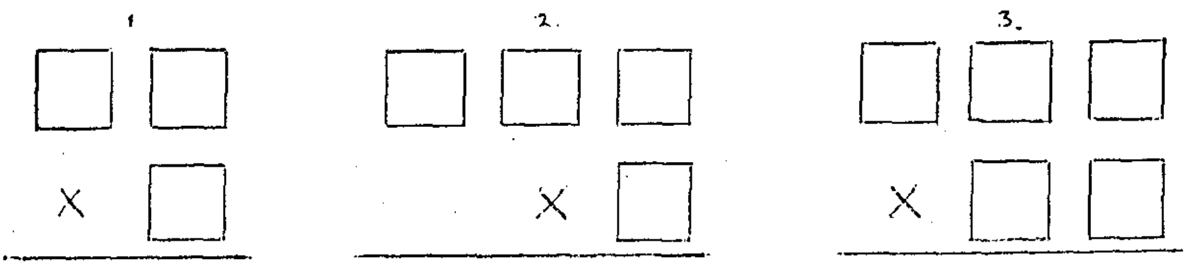
How would you change your strategy if you were trying to lose the game, in other words, you were trying to get a lower answer for your subtraction sum than your friends ?

MULTIGAME

MULTIGAME is an exciting game about multiplying that you can play against the computer or your friends.

You are given some random numbers between 1 and 9 and you have to make them multiply together to give an answer bigger than that of your opponent.

If you play this game against the computer, then there are three levels of difficulty:



However, there are some more rules that you need to know before you can play this game. You only get one number at a time and you have to decide which box to put this into before you get the next one. Once you have decided on a box then it has to stay there. The same numbers may come up more than once !!!

Play the game against either the computer or your friends, you will of course have to work out a way to get your random numbers for the game. You may like to start with level 1.

Play the game lots of times.

Now that you have played the game on level 1 several times, can you write down your ways of trying to beat your friends ?

If all the boxes were empty and you got a 1 where would you put it ?

How about a 2 ?

What about the other digits up to 9 ?

What if you got one of these numbers and that box was full, which box would you use instead ?

Put your ideas into a table like this one:

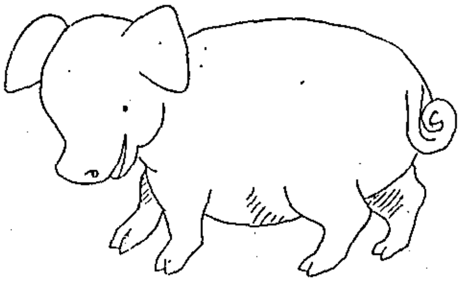
Number	1st choice	2nd choice	3rd choice
1			
2			
3			
4			
5			
6			
7			
8			
9			

This table shows your STRATEGY or plan of attack.

Can you outline a strategy for playing the game on level 2 ?

What about level 3 ?

Can you support your strategies mathematically ?



PIG

A dice game for two or more players.

The aim of the game is to be the first player to score over 100 .
You will need 2 dice and a piece of paper to score on.

When it is your turn, roll both dice. If you score a six with either dice (or both) your score is zero for that turn and you must pass the dice on to the next player. If neither of your two numbers is a six then add the two numbers together. You must now make a decision, either you record this total as your score for this turn and pass the dice on to the next player or you roll the two dice again. On this second throw the same rules apply, if neither number is a six then the two numbers are added to your score for the turn so far. However, if either number (or both) is a six then your score for the turn is immediately set to zero and your turn is over. So long as you do not roll a six you may continue rolling the two dice and adding to your score for the turn. The winner is the first player to reach a total score of 100.

Play the game a few times against your friends either in pairs or small groups.

Keep your score carefully.

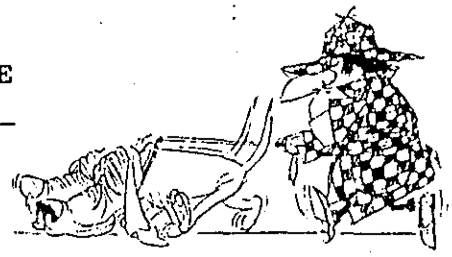
Eg.

SCORES ON DICE	SCORE FOR TURN	TOTAL FOR GAME
(4+2),(1+5),(4+3),(6+4)	0	0
(3+5)	8	8
(4+4),(1+3)	12	20
(6+5)	0	20
(4+1),(2+2),(3+4),(5+2)	23	43

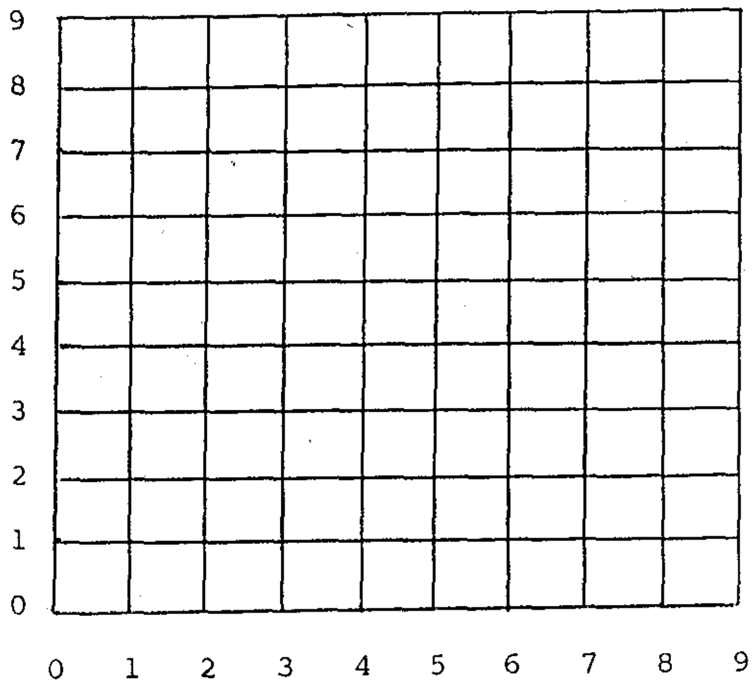
ETC...

Can you suggest any ways to improve your chances of winning ? This is called a STRATEGY. Does your strategy make sure that you never lose ? Write down your ideas on the game.

HUNT THE HURKLE

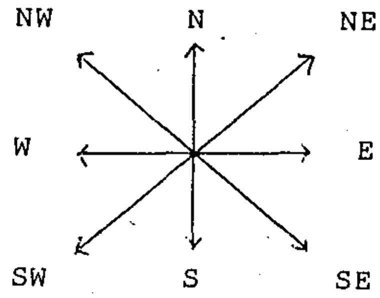


THIS IS A GAME FOR 2 PLAYERS.



The *HURKLE* is a strange creature and it hides in a grid and it's position is one of the grid points. These positions are given as a pair of coordinates.
Eg (5,3) meaning across to 5 then up to 3 .

One player is to be the Hurkle and the other is the hunter. The Hurkle decides where it is going to hide and the hunter has to make guesses to find the position of the Hurkle. After each guess the Hurkle gives the hunter a clue by saying the direction that the hunter has to travel to find him. If the direction is N,S,W OR E then this is the exact direction whereas the other 4 directions are only rough directions.



Guess how many goes you would need to catch the Hurkle if you were the hunter ?

What about if the numbers went up to 99 each way ?
What if they went up to 999 ?
When you play the game, write down your guesses and also the direction that you have to follow, this will help you to remember what guesses you have already made. Here are the details of a game that Mark and Sue have already started:

Sue is the hunter.
Mark is the Hurkle.

Sue	Mark
(8,2)	SW
(2,1)	NE

ETC...

Play the game several times and keep a note of how many goes each game took.

Game	1	2	3	4	5	6	7	8	9	10
No. of goes										

What is the average number of goes ?

How close was your guess ?

Write down anything that you have found out or worked out about this game.

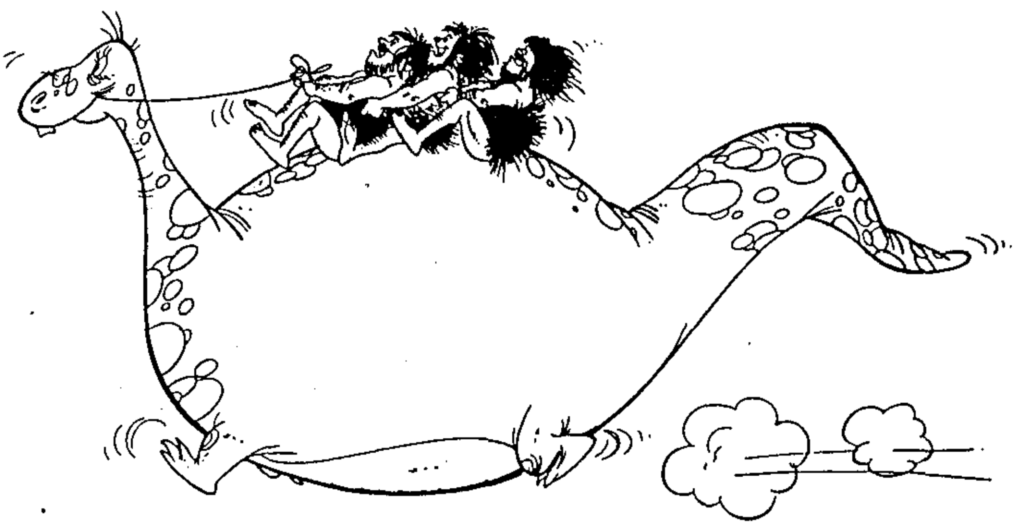
Would you like to change your guess about how many goes you would need if the numbers went up to

a) 99 ?
b) 999 ?

What if the grid was not square, perhaps 0-24 across and 0-1999 up, now how many goes would you need ?

Is there any way to work this out ?

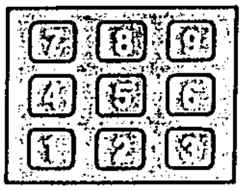
HAPPY HUNTING!!!



THE KEYBOARD GAME

THIS IS A CALCULATOR GAME !

In this game you only use the numbers from 1 to 9 .



Start with 31 on the display.

The first player picks any number from 1 to 9 .
They then subtract this from the display.

The second player now picks a number but their number must be next to the key that the first player used.
The second player then subtracts their number.

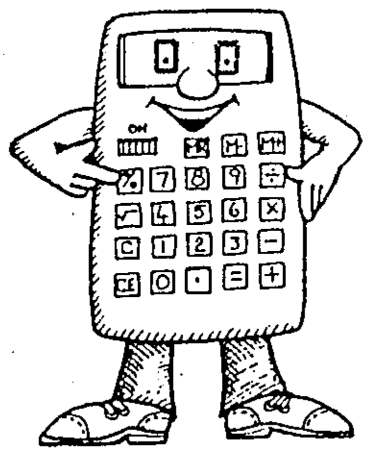
Then the first player has to go again but using the same rule about only being able to use a key which is next to the one that the other player used.

The game keeps going like this and the first player to end up with a minus number is the LOSER !!!

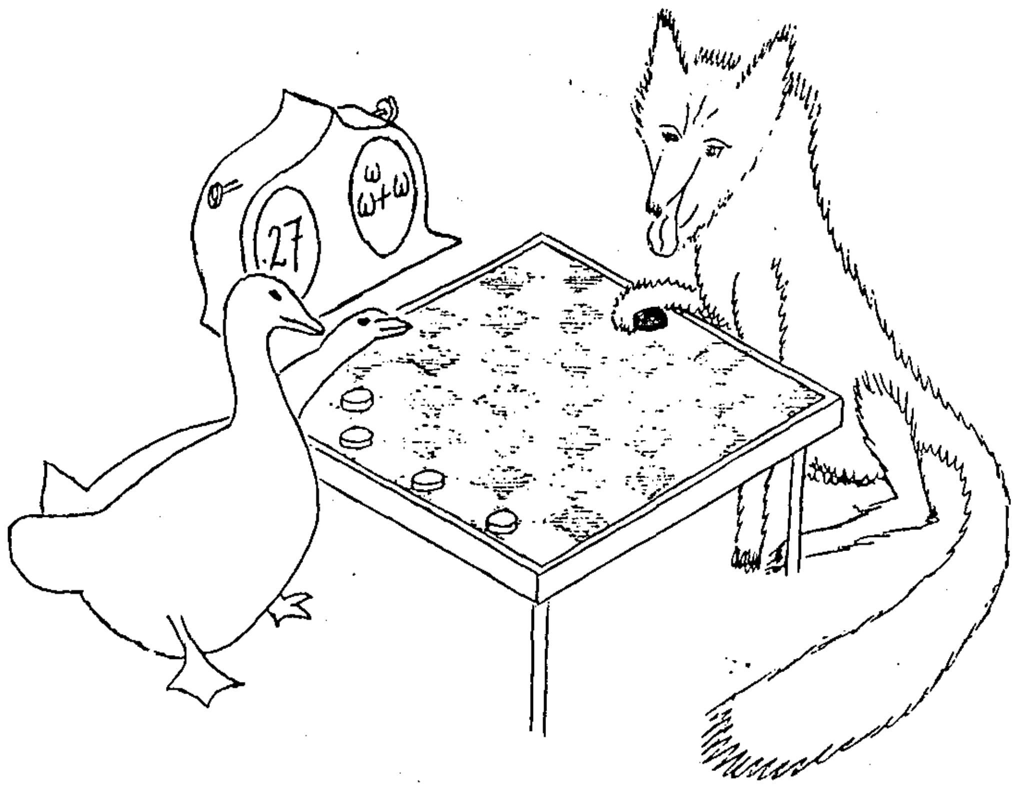
Can you find any ways of making sure that you win ?
If so write them down and try to explain why they help you to win.

Is this CHEATING ?

What if you start the game with a different number ?



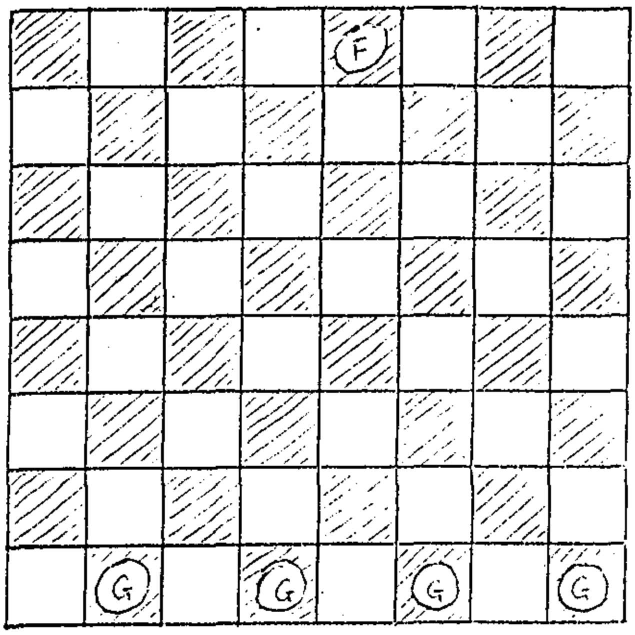
FOX AND GEESE



This is a game for 2 players.

It is played on an ordinary chess board using just the black squares. The Fox has just 1 counter Ⓕ and there are 4 Geese counters Ⓖ .

Play starts like this:



Sometimes we let the fox start on any black square on the back row. The Geese can only move diagonally forwards but the Fox can move both forwards and backwards. There is no jumping or taking. The Geese must try to trap the Fox and the Fox tries to break through the Geese and therefore get free.

Let the fox go first and take turns to move.
Play the game several times changing over between being Fox and Geese.

Can you find any winning strategies ?
Write down your ideas on this game.

SPROUTS

Sprouts is a mathematical game which was invented by Professor J.H.Conway and M.S.Paterson in 1967 and is now quite popular.

It is a game for two players.

Mark 3 dots anywhere on a sheet of paper.

Each player now in turn draws a line joining a dot either to itself or to another dot and places a new dot on this line. Here are some possible starts:

No line may cross either itself or any other line and no dot may have more than 3 lines leaving it. The last player who is able to go is the winner.

Play the game several times with the person next to you. Sprouts may not be as easy as you first think!

Think about some of the following questions:

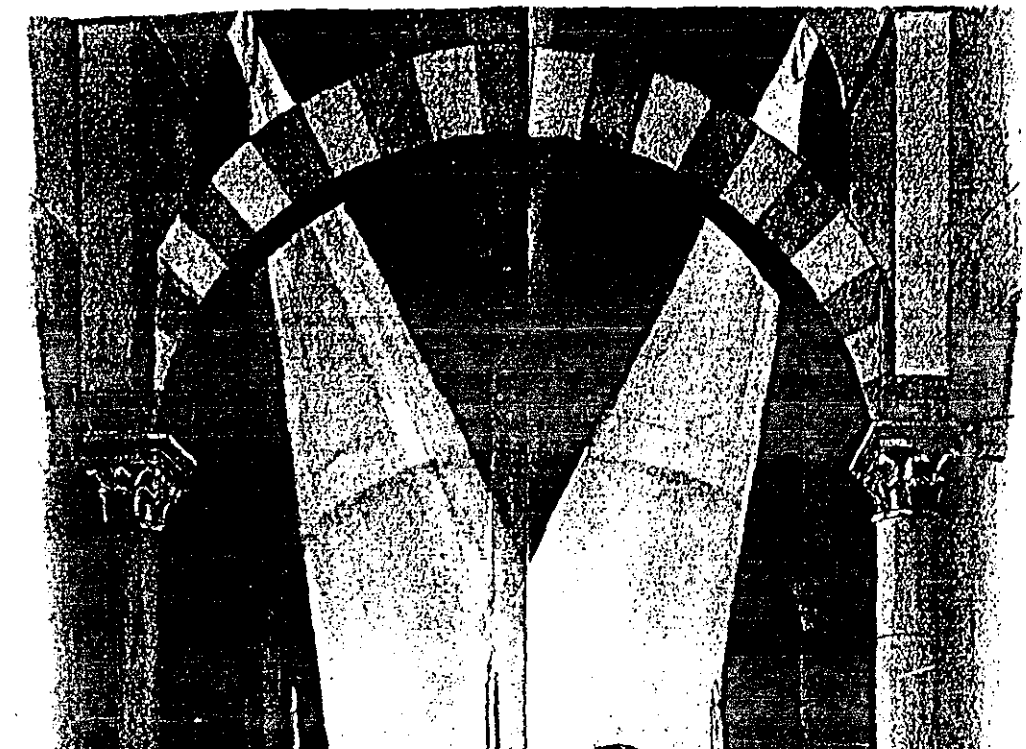
a) How many goes did it take to get a winner ?

b) How quick could you win if you were an expert playing against someone who had never played the game before ?

c) What would happen if you changed the number of starting dots ?

Try to work a few things out about this game and write about the ideas that you have. Can you suggest any winning strategies ?

THE TOWER OF HANOI

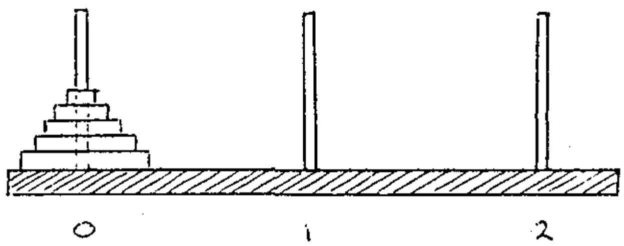


The story goes that once in Hanoi there existed the temple of Brahma. The God Brahma had set the monks of the temple a problem which would take them from the beginning until the end of the world.
There were 64 gold discs on a diamond needle the height of a man. There were also two other such needles and the problem was to move all the discs onto one of the other needles following two rules:

1. Only one disc may be moved at a time.
2. No disc may be placed above a smaller one.

When the problem was completed then Brahma would end the world with a clap of thunder !!

We shall consider a much smaller tower, one with just 5 rings on it.



How many moves do you think you would need to move all the discs from pole 0 to pole 2 ?

Play the game with the squares of card which are inside your envelope. Play a few times to see if you can improve each time.

What is your best score ?

Do you think that you could improve on this ?
Compare your score with some of your friends and talk about how you did it.

Can you write down any rules to help anyone to solve this problem quickly ?

Try a tower with a different number of rings.

Start with small numbers and work up, perhaps you will find it helpful to record your results in a table like this:

No. of rings	Least no. of moves
1	1
2	
3	
4	
5	
6	
7	

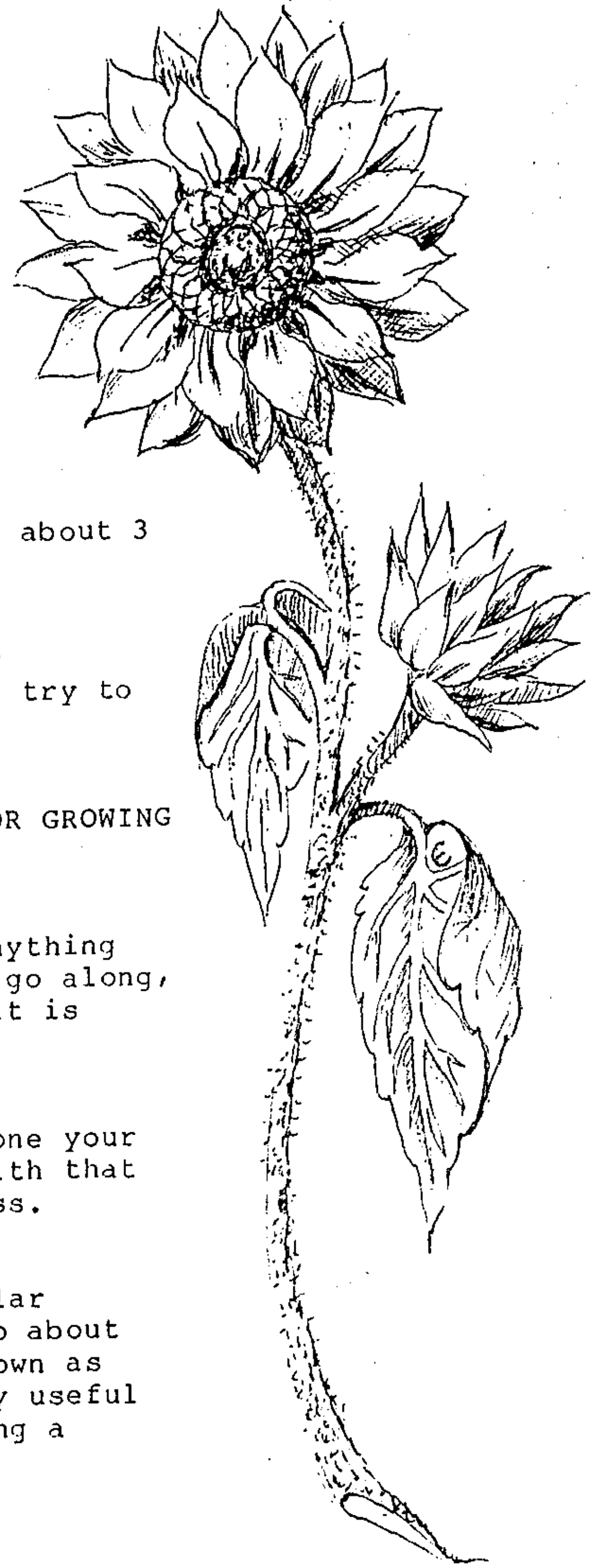
ETC...

Someone has said that it would have taken the Priests over

18 000 000 000 000 000 000 moves.

Do you think that this is true ?

How long do you think that this number of moves would take ?



SUNFLOWER

This is a COMPUTER GAME.

It is best to work in groups of about 3 or 4 for this activity.

You have to follow the computer instructions very carefully and try to grow a sunflower.

TRY TO BEAT THE WORLD RECORD FOR GROWING A SUNFLOWER !

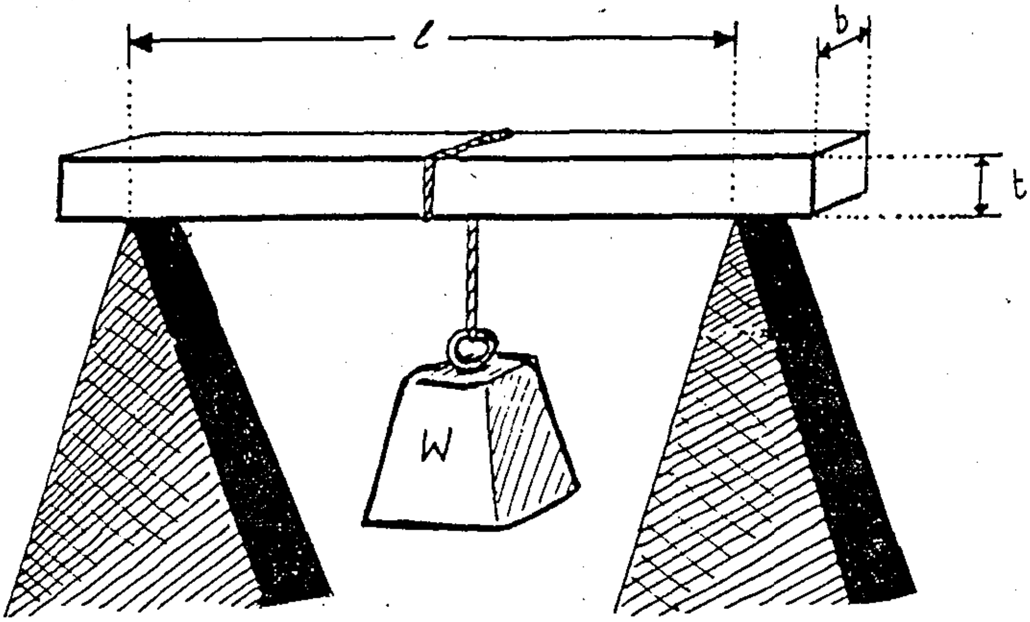
Write down in your own words anything that you do or think of as you go along, even if you do not think that it is important at the time.

When you think that you have done your best, compare your sunflower with that of the other groups in the class.

If you had to carry out a similar investigation, how would you go about it ? This approach would be known as your STRATEGY. It is often very useful to have a strategy when tackling a problem in mathematics.

NOW TRY TO SOLVE THE BRIDGE PROBLEM !!!

THE BRIDGE PROBLEM



The table drawn below shows the maximum weight that can be theoretically supported by wooden planks with various dimensions. The results are written in order, from the strongest plank to the weakest.

Length l(cm)	Breadth b(cm)	Thickness t(cm)	Max. Weight W(kg.wt)
2	40	4	160
1	20	4	160
2	30	4	120
1	20	3	90
2	20	4	80
1	30	2	60
4	40	3	45
1	20	2	40
2	10	4	40
2	30	2	30
3	30	2	20
3	10	3	15
4	30	2	15
1	20	1	10
4	40	1	5

Try to discover a rule by which the strength of any size plank can be found from its dimensions. Did you use the strategy that you worked out in the sunflower problem ?

Can you now predict the maximum weight that a wooden beam with breadth 4cm, thickness 6cm and length 3cm can support ?

PICO CENTRO

Pico Centro is a number game.

Let's imagine that I have 10 cards numbered from 0 to 9. I then choose just 3 of them and make them into a three digit number. You have to guess my number, but I will give you some help after each guess by telling you your Pico Centro score. Your Pico score tells you how many digits you have which are used in my number and your Centro score then tells you how many of these you have put in the correct position.

Eg

MY NUMBER IS 285

If your guess is 364 then your score is PICO 0, CENTRO 0

If your guess is 253 then your score is PICO 2, CENTRO 1

If your guess is 258 then your score is PICO 3, CENTRO 1

Play this game with a friend using a pencil and a piece of paper. One of you make up the number and one of you try to work it out.

How many goes do you usually need to get it right ?

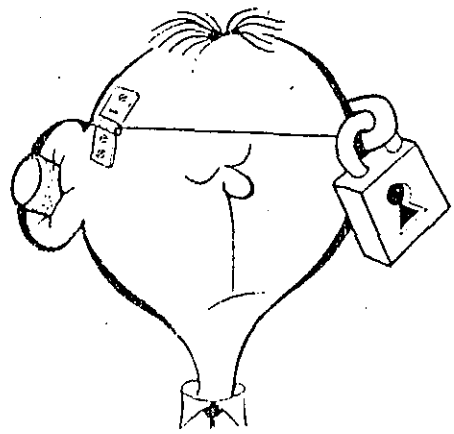
You can make the game harder if you wish by:

a) Using 4 digits.

b) Using digits more than once in a number.

Perhaps you know another game which is similar to this.

Do you use the same strategies in both games ?



COVER UP

Cover Up is a game for 2 players.

This type of game can be played with many different sets of shapes. We are going to play it with either a set of TETROMINOES or PENTOMINOES. If you play with the tetrominoes then you have one set each, but different colours of course. If you play with the pentominoes then you have one set between you and the game starts by either dealing these out or by taking turns to pick a shape until all the shapes have been shared out equally.

The rules are the same for either game.

Once you have your set of shapes, you and your partner take turns to place a shape on to a grid. The first person who can not go is the LOSER!!

The grid is 6X6 if you play with the tetrominoes and 8X8 if you play with the pentominoes.

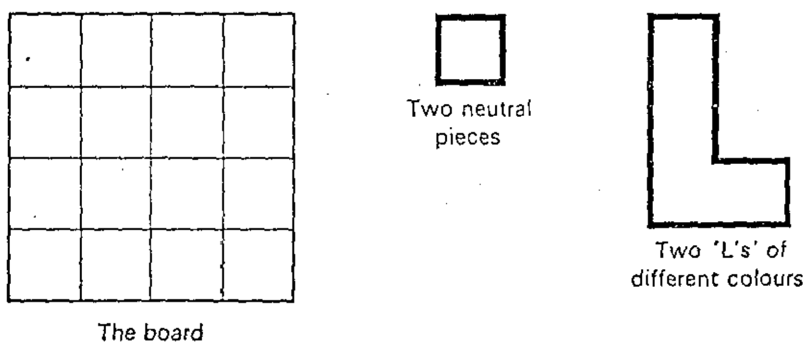
When you play these games, play several times to get the hang of the rules. Try to play against a few different people.

Try to write down any ideas that you have on winning the game. The way that you play the game is called your STRATEGY, this is like your plan of attack.
Have you heard the word strategy before ?
It is often useful to have a strategy in maths when you tackle a problem and try to solve it.

You will find everything that you need to play the game of COVER UP in your envelopes.

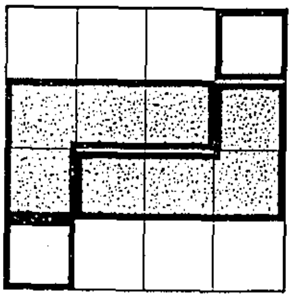
THE L SHAPED GAME

This is a game for 2 players and you need the following pieces:



You will find all the pieces in your envelope.

Take the pieces and arrange them in this way:



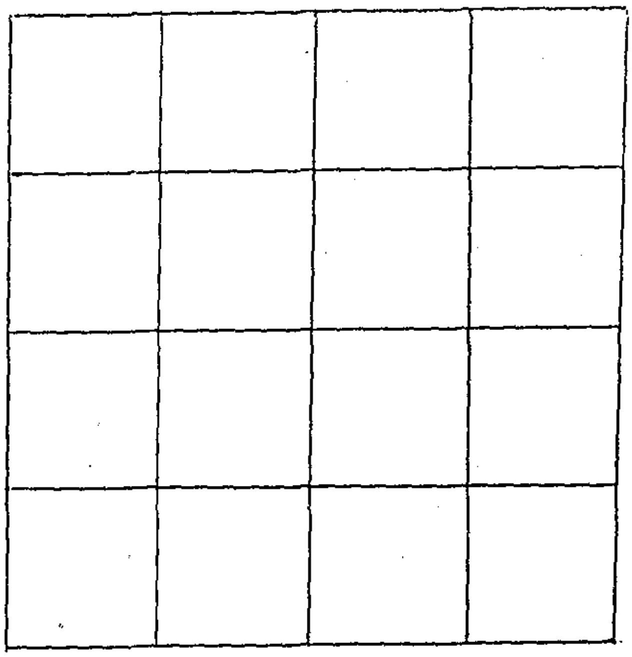
This is the starting position of the game.
Now decide who is going to go first.
Each player, in turn, must pick up his own 'L' and put it in a different position. The player can then, if they wish, move any one of the neutral pieces. The 'L' piece can be turned around or over or both when it is being moved.

The winner is the first person to stop their opponent from being able to go.

See if you can write down any ideas that you have about this game. Can you suggest any winning STRATEGIES ?

COLOURS

This game is played on a 4X4 grid and it involves players trying to work out a pattern set by one of the players who acts as the pattern maker.



It is probably best to play in groups of about four.

The pattern maker uses 4 colours and shades in 4 squares of each colour. However, each colour must form a tetromino. This means that all 4 squares of any colour must be joined together by full edges, there are only 5 ways of joining 4 squares together in this way.

Perhaps you should try to find all 5 ways before you start to play this game.

When the pattern maker has set his pattern the other players have to work it out. This is done by requesting information about certain rows or columns of the pattern. The players take turns to secretly ask the pattern maker for information. The players may only ask about 1 row or column at a time. The pattern maker gives information simply by saying how many of each colour there are in the row or column, this information is again passed back to the player without the other players finding out what it is.

All players start with 8 points in each round and they lose one point each time they request information, therefore the quicker you guess the colour square pattern the more points you get. The players can make a guess only when it is their turn and if they are not correct they have to count this as a go and lose the point for a go, as normal.

The game ends when all players have either got the pattern correct, have no points left or given up, in which case they score 0 anyway. The pattern maker changes for each round and a game ends when all players have set a pattern. The WINNER is then the player with the most points.

Play a few games and see if you can find any helpful ideas on how to play the game well. Discuss these with the others in your group and one of you write them down.

DESIGNER

This is a game for 3 or more players about pattern and symmetry.

Think about the types of symmetry that a square has.

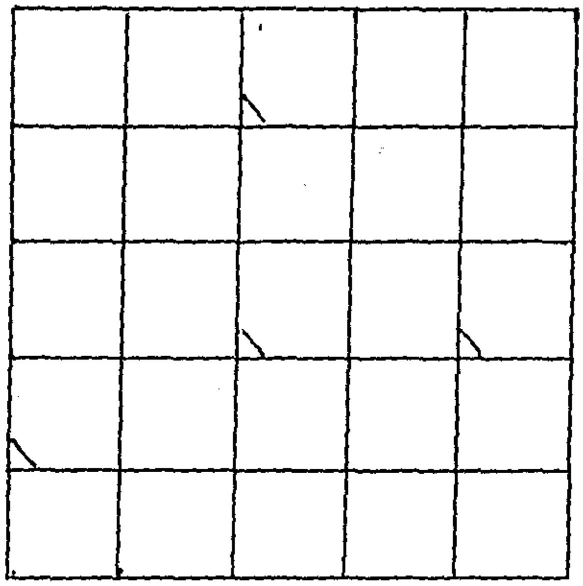
Each player has a 5X5 grid. One player is the designer and has to draw a pattern on their grid by using the symbols *,+ and # . The designer must put a symbol in each square and the pattern must have some form of symmetry.

Eg

*	*	+	*	*
*	#	#	#	*
+	+	*	+	+
*	#	#	#	*
*	*	+	*	*

The design can have as much or as little symmetry that the designer wishes, but it must have some.

The other players now have to work out what the design is. Players make enquiries by putting a mark in any box where they wish to be told the symbol. Their grid is then passed to the designer without the other players seeing it and the designer puts in the requested symbols.



There are no turns in this game and players can request information whenever they like. There is also no limit on the number of turns for any one player or the number of boxes being requested. When a player thinks that they have worked out the design then they fill in all the other boxes and give it to the designer. If a player feels that they can not work out the design, then they can give up by handing in an unfinished grid to the designer.

The game is over when the designer has all the grids. He then scores all the grids. All players who gave up score 0. The other players get +1 for each correct guess and -1 for each wrong guess. The boxes that you requested are not counted in the scoring.

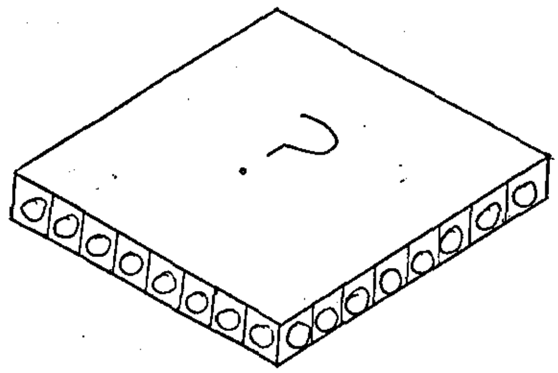
The designer gets a score equal to twice the difference between the best and worst scores, less a penalty of 5 for the first 'give up' and 10 for each other 'give up'.

Play the game in groups of about four, taking turns to be the designer. At the end of a round, when every player has had a turn at being the designer, see who has the best score. Play a few rounds and see if you improve.

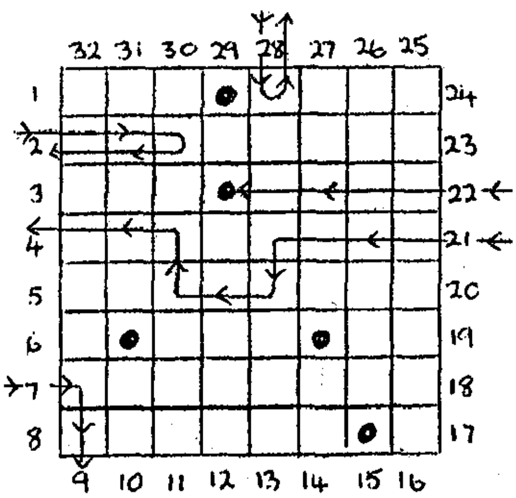
Discuss within your group any methods, or strategies, that you used in trying to find the design. Write down your ideas as a group.

NOW HAVE ANOTHER ROUND

THE BLACK BOX GAME



THIS IS A GAME FOR 2 PLAYERS.



Each player has a board which is 8 squares by 8 squares and there are numbers around the outside. Each player hides 5 atoms on their board. The object of the game is to discover where your opponent has hidden his atoms. This is done by firing rays into the box and then being told what happens to them.

Rays enter the box through one of the 32 entrance positions and they travel either across or up and down. The path of a ray is governed by the following rules:

1. Rays that hit an atom are absorbed (they do not come out).

2. Rays that come within one square of an atom in a diagonal direction (so that they would pass next to the atom if they continued) are deflected at right angles away from the atom.

3. Rays aimed between two atoms (one square gap between them) are reflected back the way they came from.

4. Rays that enter next to an atom are reflected back out again.

5. Rays otherwise travel in straight lines.

Therefore each time that a player fires a ray into the box, the other player will tell them where the ray has come out or tell them that the ray has been absorbed.
Check these rules out on the diagram of the board on the first sheet.

Play the BLACK BOX GAME with your partner to get to know the rules.
When you are sure of the rules, play against each other trying to guess your partner's box before they guess yours.

Are there any worthwhile tactics in this game ?

Write down any ideas that you have for good attacks or good defence.

NIM

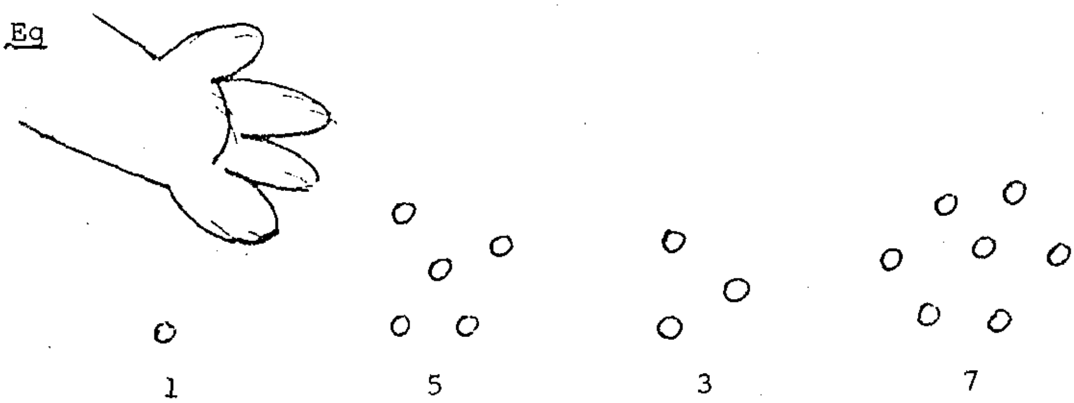
The game NIM takes no time at all to learn, the rules are very straight forward.

IT IS A GAME FOR 2 PLAYERS

You take a bundle of counters, beads, matchsticks or anything else you can get hold of. You then divide the bundle into any number of piles and it does not matter how many piles there are or how many there are in each pile.

EASY SO FAR EH!

Eg



Each player now takes turn to remove as many counters as they like from any ONE of the piles. They can if they wish take the whole pile, the only rule is that they must take at least 1 counter.

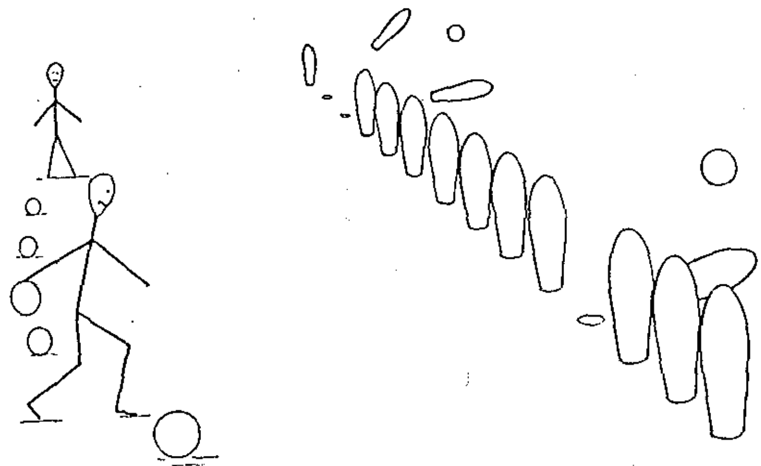
The WINNER is the person who picks up the last counter.

Play the game with a friend and see if you can discover any WINNING STRATEGIES.

Write down in your own words any ideas or strategies that you have for playing NIM.

KAYLES

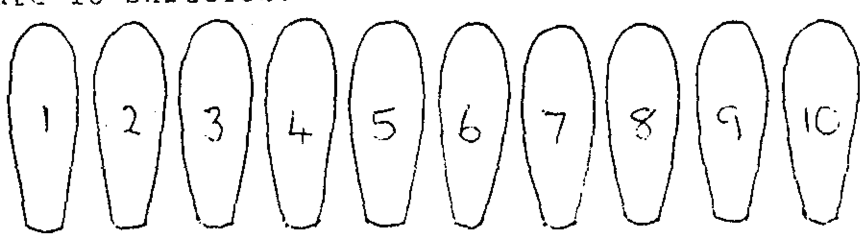
This is an old English skittles game.



The players become so good at this game that they can bowl to knock down any single skittle or any pair of skittles next to each other that they choose.It is impossible to knock down two skittles unless they are standing next to each other, even the world champion can not do this. The winner is the person to knock down the last skittle.

If you were an expert Kayles player, how would you play the game ?

Here are 10 skittles:



Play against a friend, remembering that you can knock down just 1 skittle or 2 next to each other.

Play the game lots of times and find out who is the Kayles Champion. See if you and your partner can come up with some winning ideas. If you do, then write them down in your own words.

HAPPY BOWLING !

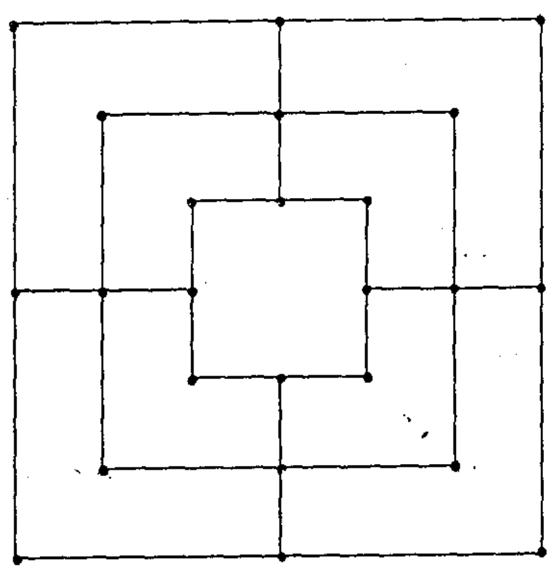
NINE MEN'S MORRIS

This is a game for 2 players.

This game was played in England in Shakespeare's time when the 'board' was cut in turf and wooden pegs were used as counters. After very wet weather, the 'board' often became unsuitable. This game was in fact mentioned in Shakespeare's 'A Midsummer Night's Dream'.

A version of the game is still played today. The shepherd boys in Lesotho mark a board in the sand and play with stones and beans, but this is being frowned upon because it distracts them from their work ! The Lesotho version is known as Marabaraba and is played with 12 pieces each.

The version that we are going to play uses 9 pieces each, hence the name. The board used is like this:

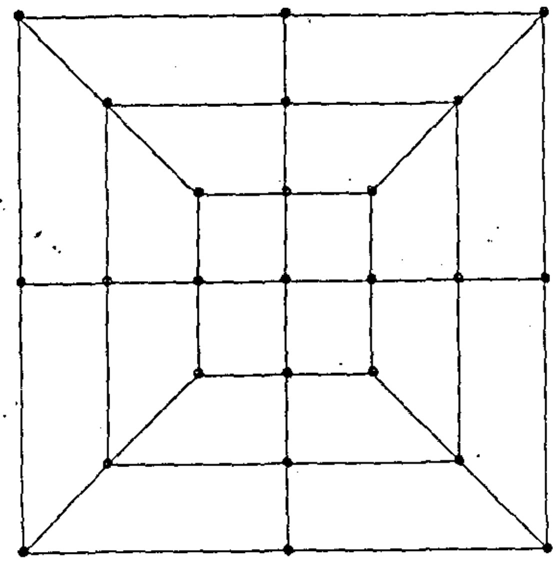


Enter the men alternately, one at a time, on any vacant point. Each time a player forms a row (or a mill) of three men along any line, he can remove one of his opponent's pieces, except those already in a mill.

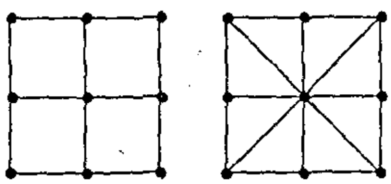
When all the men have been entered, continue to take turns by moving a man to an adjacent vacant point along a line, trying to create more mills, thus reducing the number of your opponent's pieces. A Player wins by either blocking all his opponent's men so that they can not move, or by reducing them to 2 pieces.

Games like this go back a long way and are very widespread. A 'board' has been found from an Egyptian temple dated about 1400 BC.

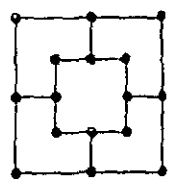
Marabaraba (12 men's morris) has already been mentioned but versions of three and six men's morris can also be played.



A Marabaraba board



Forms of Three Men's Morris



Six Men's Morris

Play 9 men's morris several times against a friend.

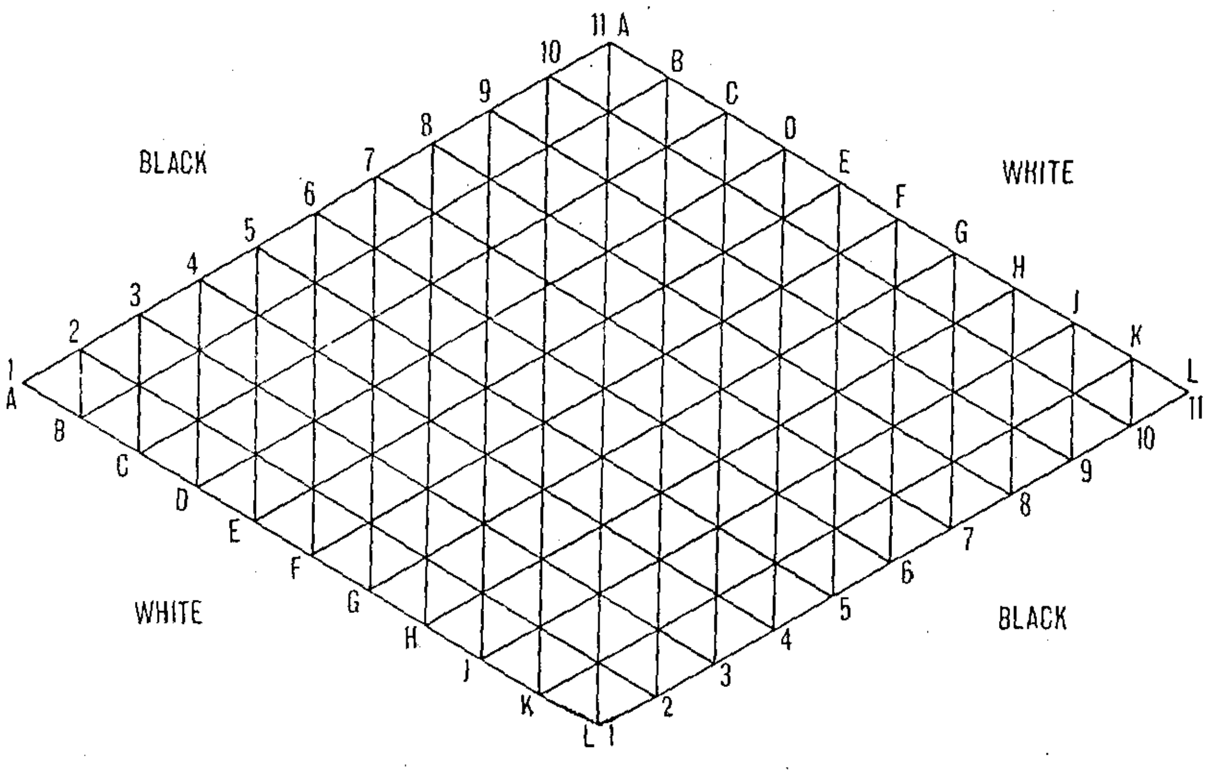
Can you find any winning ways or strategies ?

Write down any ideas that you have about this game.

HEX

Hex is a board game for 2 players which was invented in Denmark in the 1940's.

It is played on a diamond shaped board which is divided into triangles.

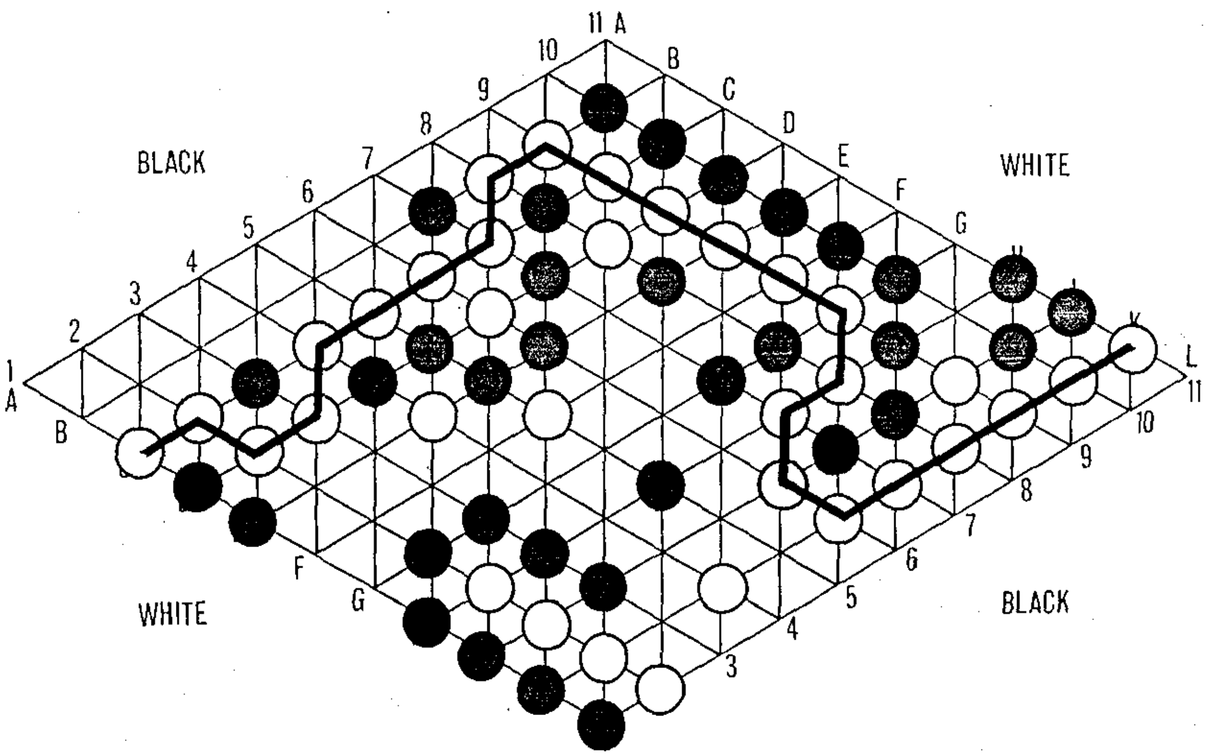


Usually the board has 10 triangles along each side but the game could be play'ed on boards which are either smaller or larger.

Each player has a set of counters, one player has black and the other white. Players take turns to place their counters on any of the grid points.

Once a counter has been played it may not be moved and there is no capturing of counters. Each player has to build a continuous chain of counters joining together the two sides marked with their colour. The first player to do this is the WINNER !!!

Here is a game that white has already won.



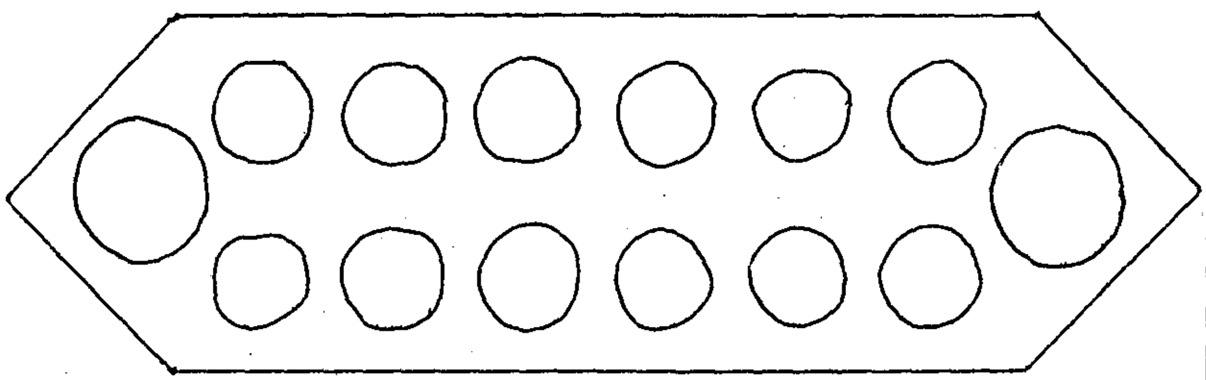
Play the game with a friend and see if you can work out any winning ways. If you have any ideas then write them down trying to explain them carefully.

WARI

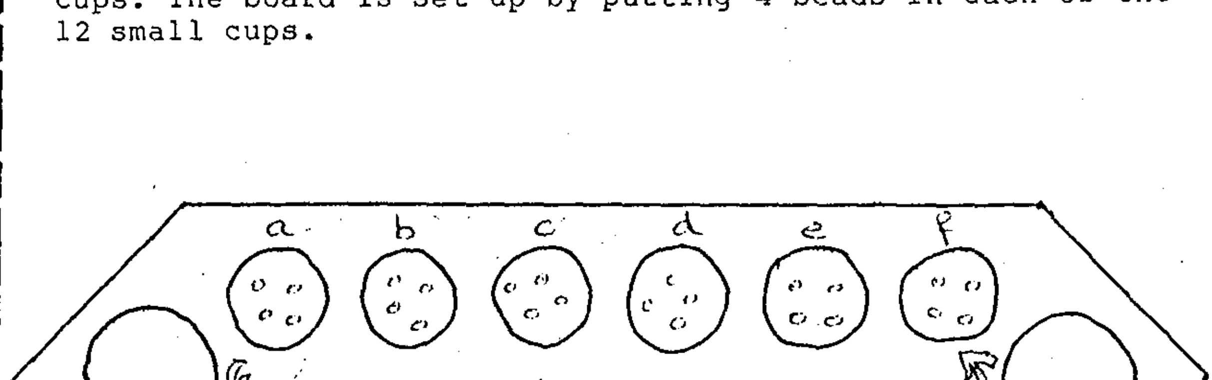
A GAME FOR TWO PLAYERS

Wari is a version of an ancient game known as MANCALA. It has been described as the national game of Africa and many claim that it is the oldest board game that still exists. It is believed to have been popular during the Empire Age of Ancient Egypt well over 3000 years ago. Although there are over 200 versions of Mancala, Wari is one of the most widely played versions and originated in West Africa.

A Wari board looks like this:



The game is played by placing and moving beads around the cups. The board is set up by putting 4 beads in each of the 12 small cups.



The 6 cups labelled with capital letters all belong to one player and the other 6 belong to the opposing player.

The first player picks up all the beads from any of his cups and puts 1 bead in each cup moving anticlockwise around the 12, starting from the cup which is just to the right of the one which they have just emptied. The second player then does the same thing.

Play continues until a player puts the last bead from one of his cups into an opponents cup which had contained just 1 or 2 beads (ie it now contains 2 or 3). The player now wins the beads in this cup and puts them in his large cup at the left hand end of his row of small cups. He also wins the beads in the previous cup if this is an opposition cup and it now contains EXACTLY 2 or 3 beads, and so for the cup previous to that and so on.

If a cup has more than 11 beads and this is the cup that you want to empty, then when you have put one bead in each cup, going around the board, you jump the cup that you are emptying.

A player may not leave all his opponent's cups empty, they must play to leave at least 1 bead if at all possible, if this is not possible then the player takes all the remaining beads and the game is over.

THE WINNER IS THE PLAYER WHO HAS WON THE MOST BEADS.

Play the game against a friend several times so that you get used to it.

Can you write down any suggestions as to how to play the game and be good at it ? This is known as a STRATEGY. You can often do a lot better in a game if you have a strategy. This is also true when you are trying to solve problems in mathematics, science or any other area.

APPENDIX B

Material for the games.

Many of the games in this package could be described as pencil and paper games which require no further material. Other games are best presented with prepared playing sheets; the type of game to which I refer would include Dots and Boxes, The Black Box Game, Subgame, Multigame, Fleet and Hunt the Hurkle. The prepared sheets save an unnecessary amount of pupil time being spent on the preparation of grids, this time may then be spent much more profitably on playing and discussing the game. Finally, there are some games in the package which require boards, playing pieces, computer software etc. In this appendix, I have simply listed the necessary material for each game and given suggestions of any sheets, boards etc. which may be required. These materials may well be printed onto cardboard and cut out to make the games, as they were during the trials.

Nought and Crosses

This is a simple pencil and paper type game.

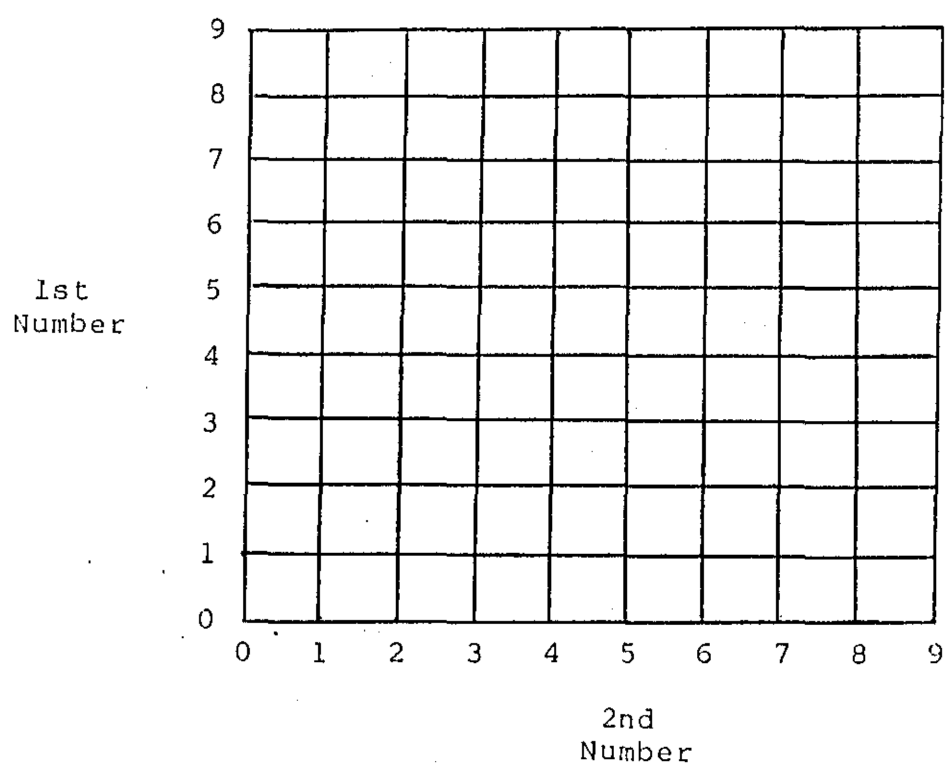
15 To Win

This is a simple game to make and the following grid may be printed onto suitable material and cut up to form the pieces.

1	2	3
4	5	6
7	8	9

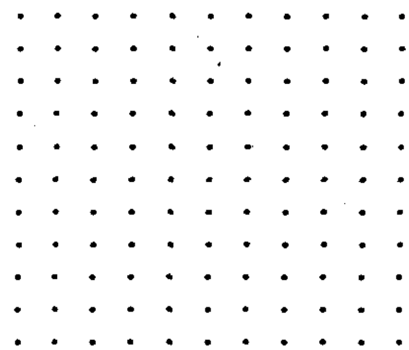
Fleet

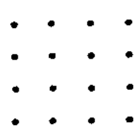
This game is best presented with prepared sheets containing a number of blank grids.

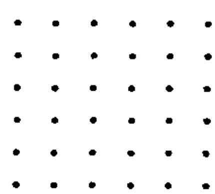


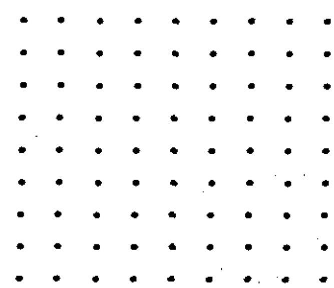
Dots and Boxes

This is another game which is best presented using prepared sheets. It is a good idea to have some smaller grids available as an introduction to the game.







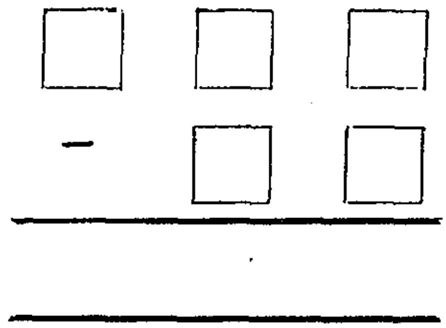


Century

This is a simple pencil and paper type game.

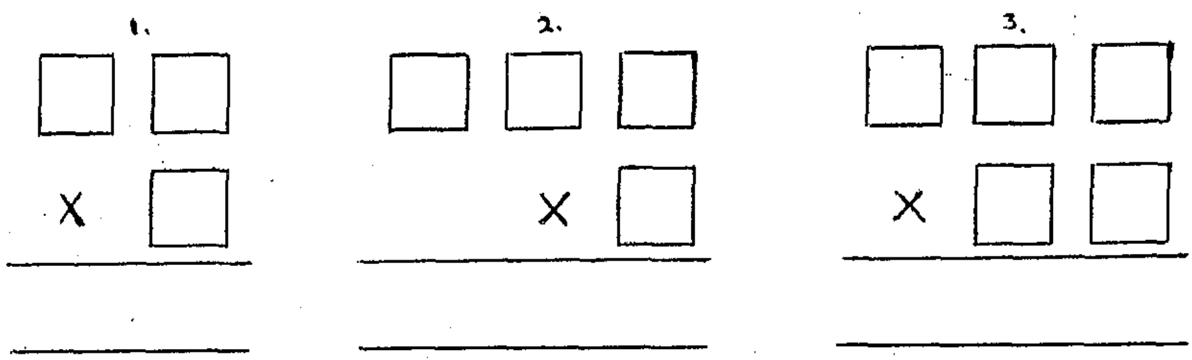
Subgame

This game may be played either with or without the computer. The software has been developed by the ITMA collaboration and is available from Longmans. If the game is played without a computer then some form of random number generator will be required, this may involve using a spinner, as shown on the sheet, calculator or table of random numbers. Whichever way the game is played, I have found in the past that pupils work best on a prepared sheet of blank Subgame grids.



Multigame

Clearly my comments on the material for this game are the same as for Subgame, the software this time being available from Nottinghamshire County Council Computer Education Centre. The sheets would give a number of blank grids for the appropriate level.

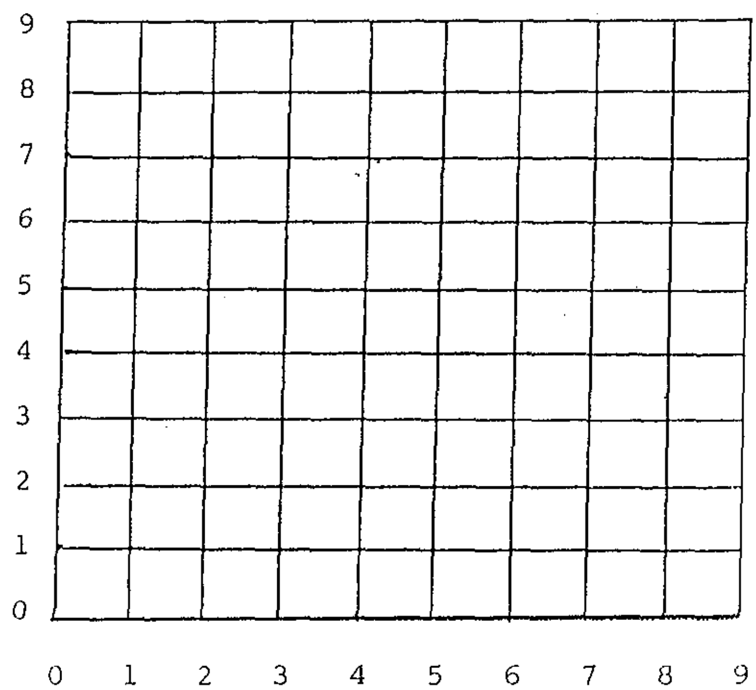


Pig

This is a simple game to prepare and only requires two dice together with pencil and paper for scoring.

Hunt the Hurkle

Although this is a game involving the logical development of an efficient search method and extends to extremely large search areas, the pupils do need the small grids to play the game initially.

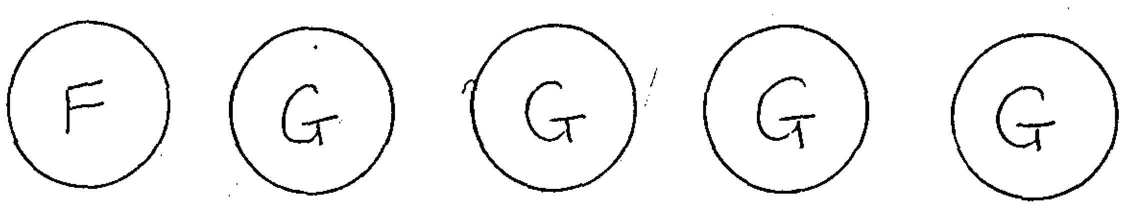


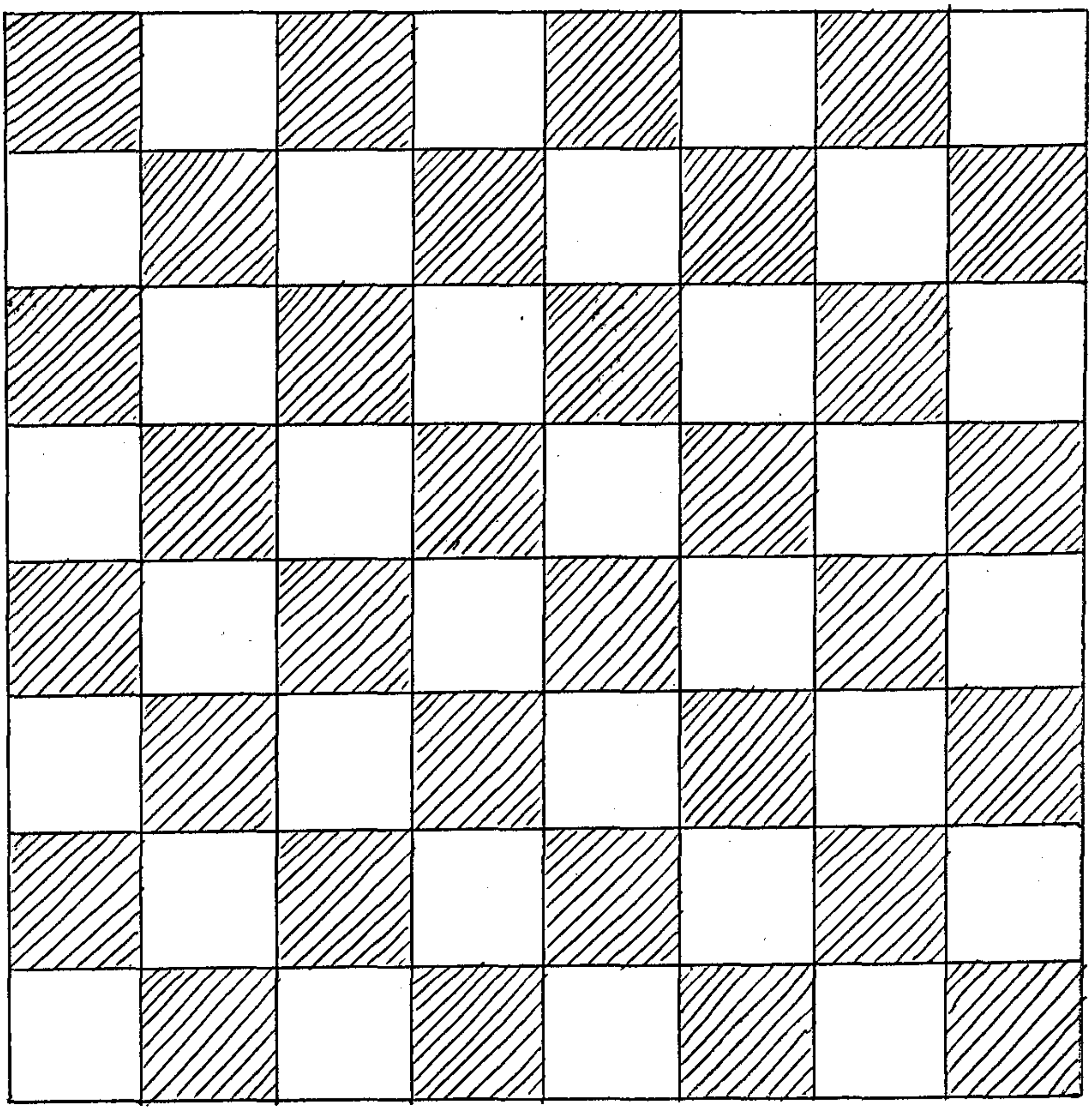
The Keyboard Game

This game requires one calculator for each pair playing the game.

Fox and Geese

This game requires a chess board, four pawns and one other playing piece. However, this can be made with relative ease using the pieces shown below and the board on the following page.



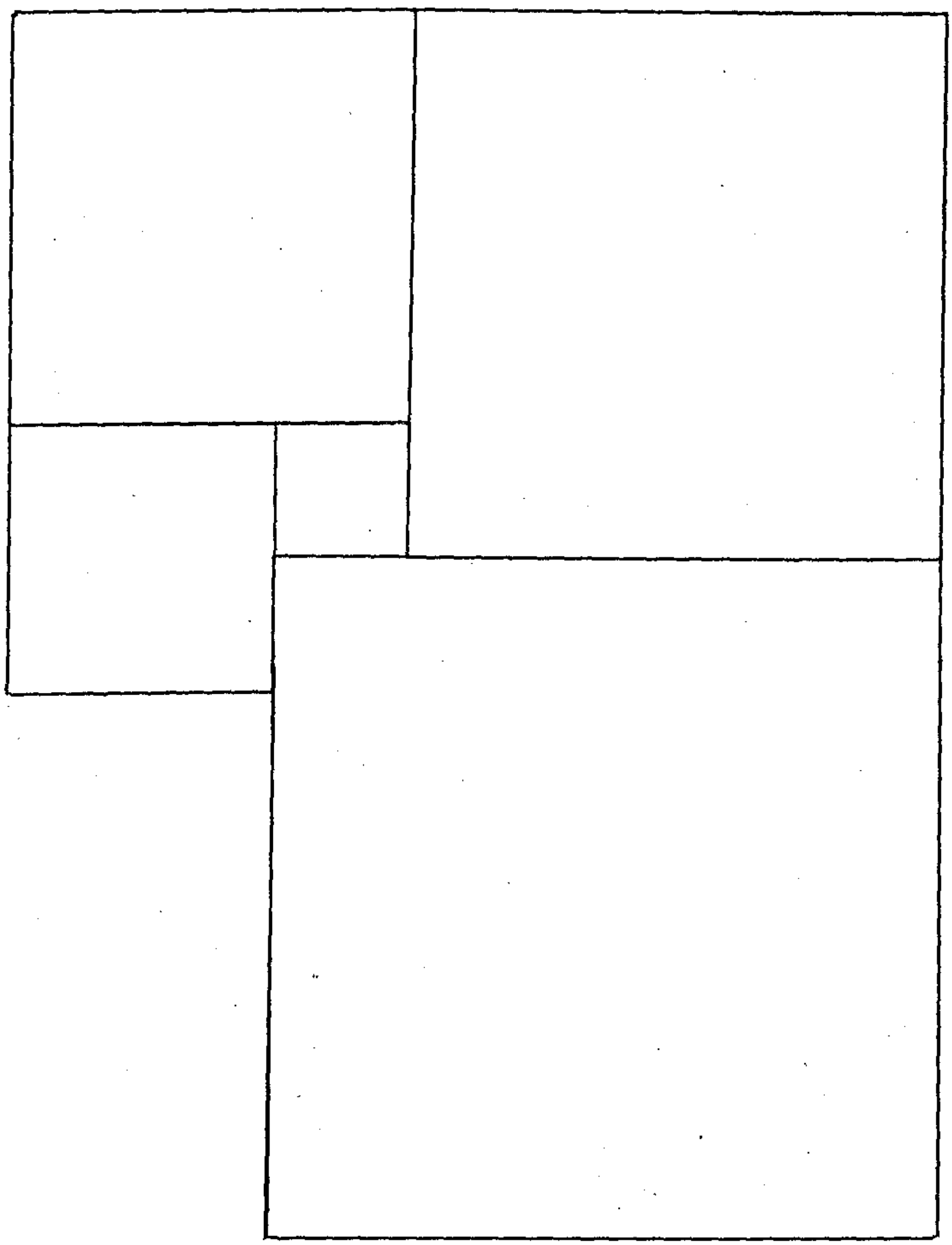


Sprouts

This is another simple pencil and paper type game.

The Tower of Hanoi

It is an absolute must with this game to have some form of practical material. This material could comprise a set of different size weights or discs which are often found hanging around a mathematics department. Alternatively, the pieces could be made out of card using the design given below.



Sunflower

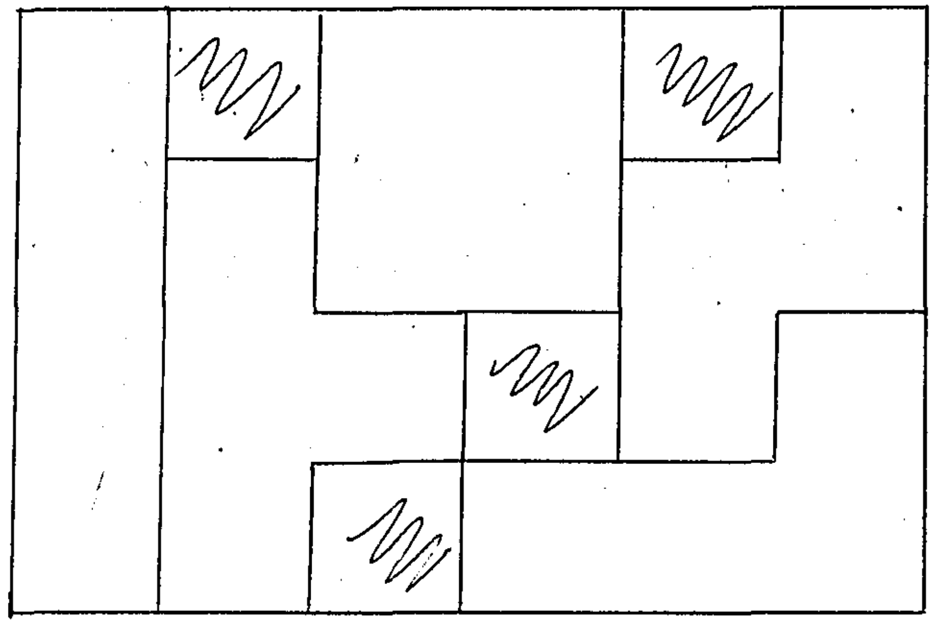
This game cannot be played without the relevant software which is again ITMA material available from The Shell Centre. It is also included in their second module. There is now support software available for the bridge problem which is also included in the Shell Centre Module.

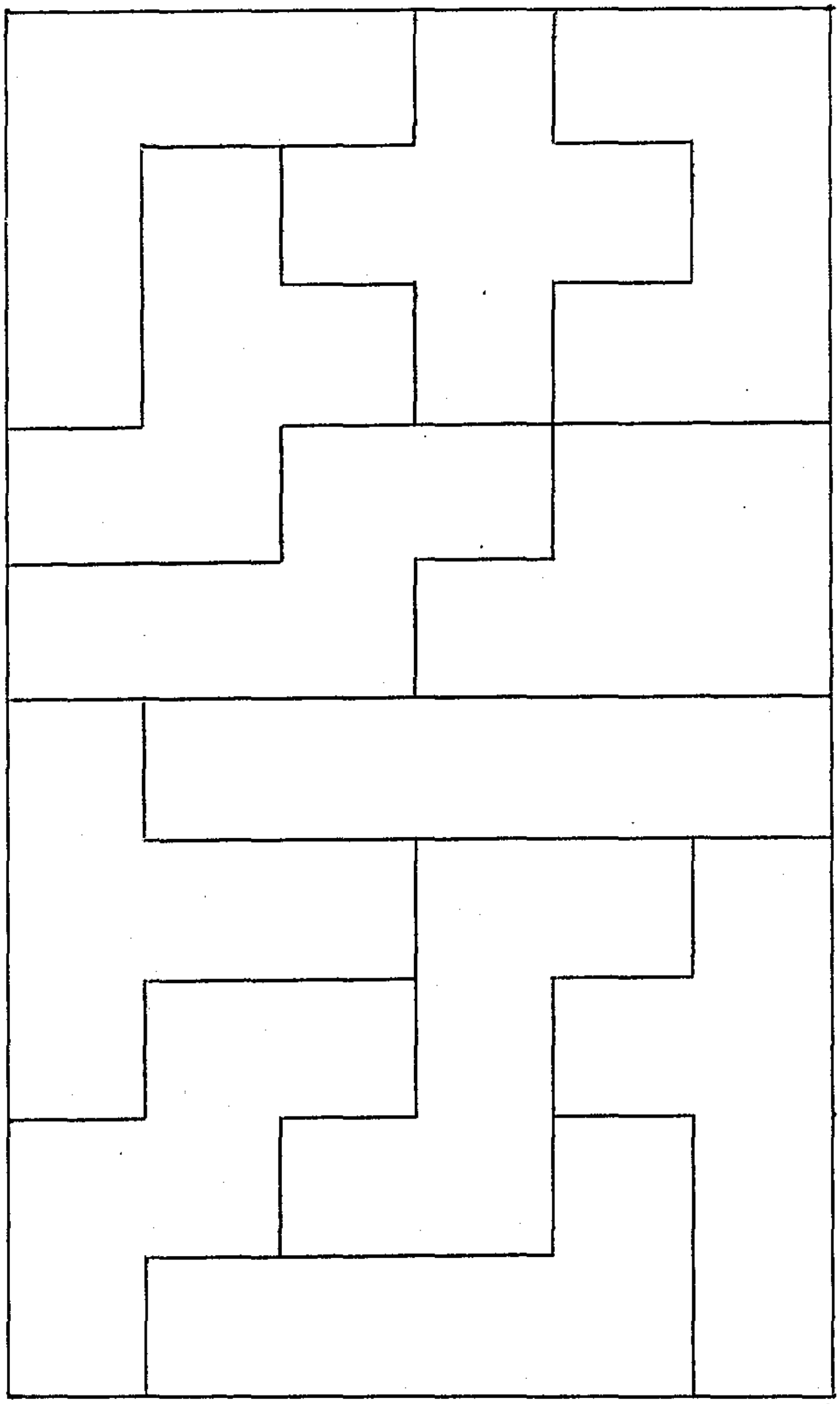
Pico Centro

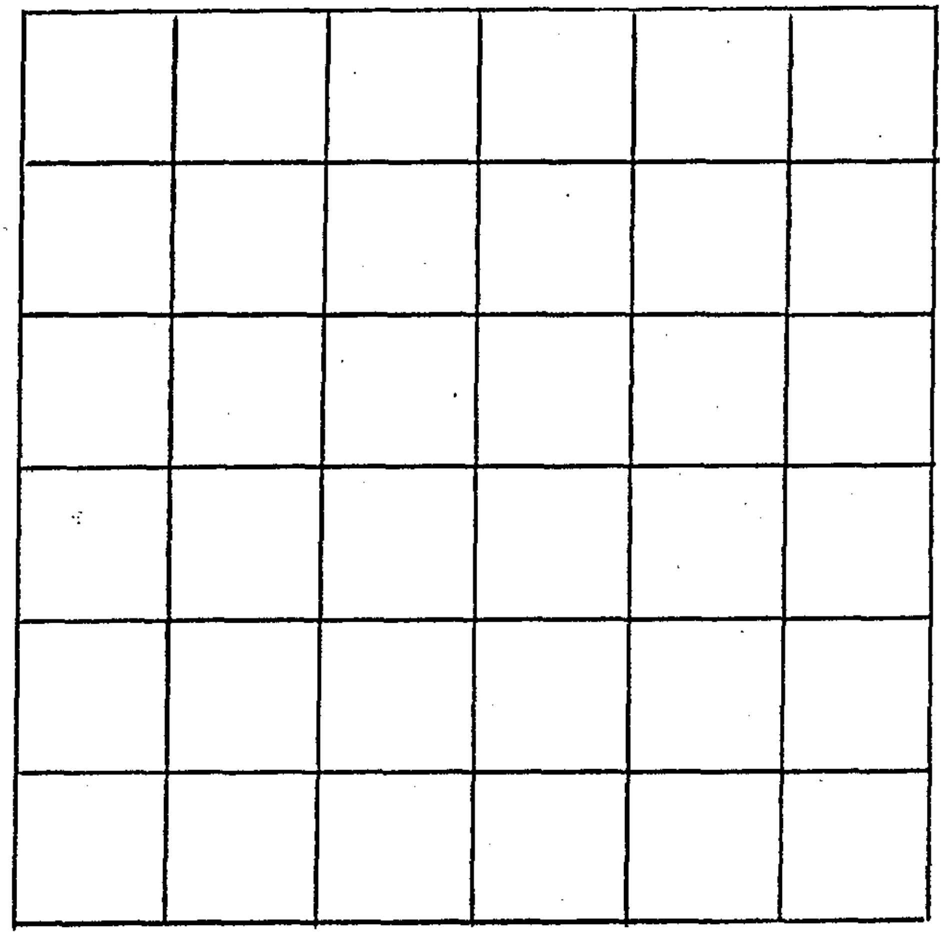
This game like many included in the package only requires pencil and paper. However, its obvious links with Mastermind may make it profitable to have a few sets of this latter game available in the classroom.

Cover Up

This game will require the board and pieces as given below together with the Fox and Geese board for the pentomino version of the game. Again it is probably best to have these printed onto some form of cardboard.

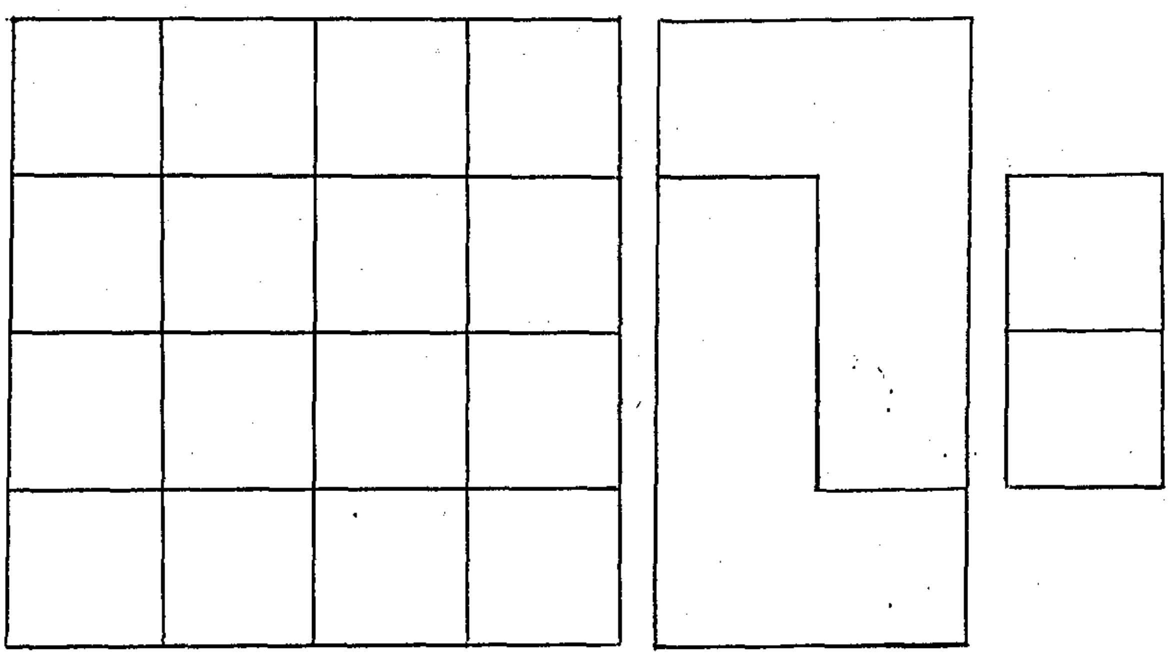






The 'L' Shaped Game

This game again requires a board and pieces and my comments on the material for the previous game are also relevant here.



Colours

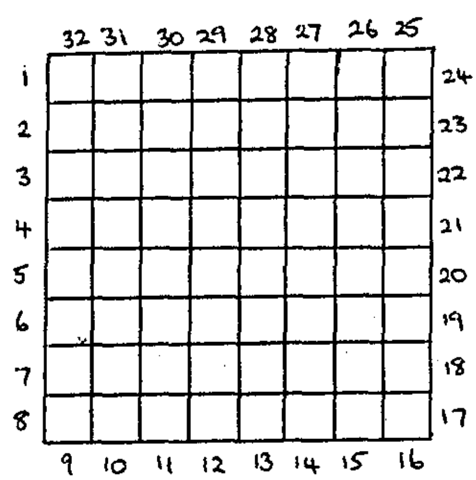
Although this game is played on a grid, I see very little point in preparing grid sheets due to the simplicity of the grid used in this particular game. Therefore all that is required is coloured pencils and paper.

Designer

This is another game which is played on a basic and relatively small grid and my comments are the same as for the game Colours.

The Black Box Game

With this game, I feel that it is worthwhile preparing grid sheets beforehand. Each pupil could be given a sheet with about 8 blank grids on it.



Nim

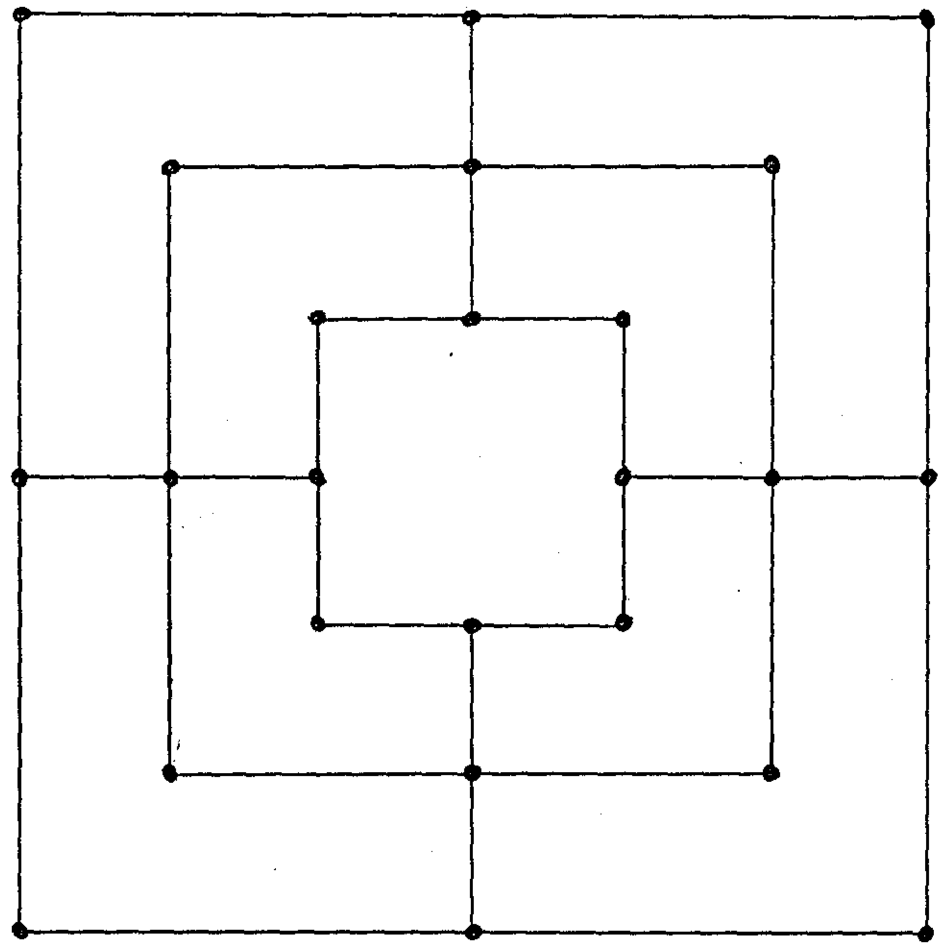
In the past, I have found the easiest way to prepare this game is by simply giving the pupils a piece of scrap paper each and getting them to tear it up and then screw it into little balls. This also helps to emphasise the arbitrary nature of the starting position.

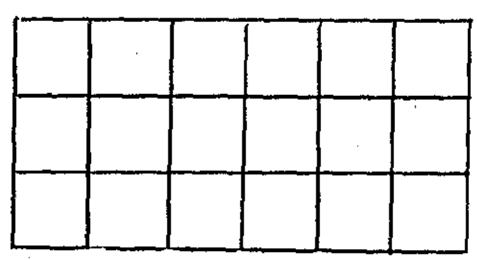
Kayles

This again is best played as a pencil and paper game. The pupils can simply list the numbers and then strike them out.

Nine Men's Morris

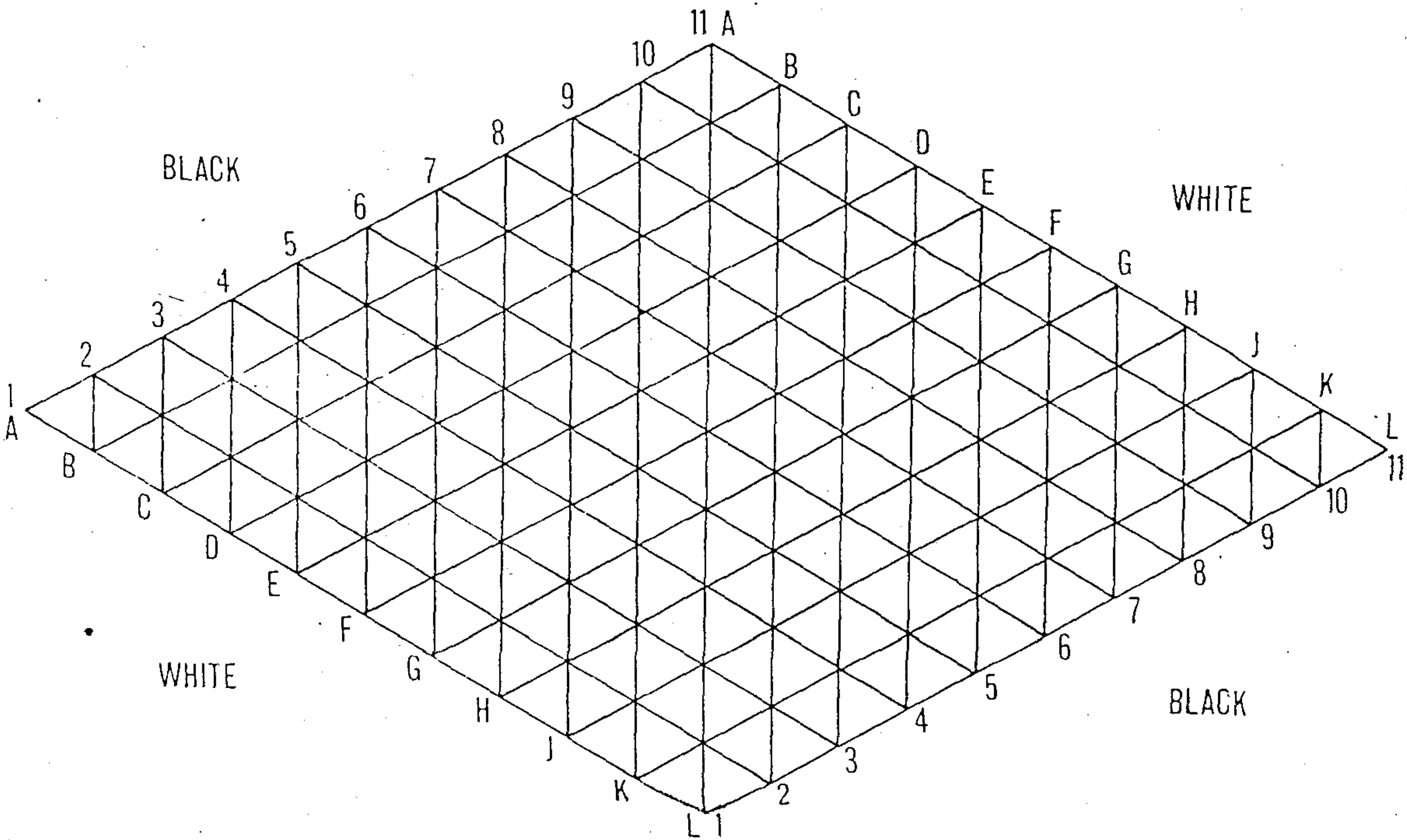
Sets of this game are available commercially but they are again easy to produce on card. There is a suitable size set, including both board and pieces, given on the following page.

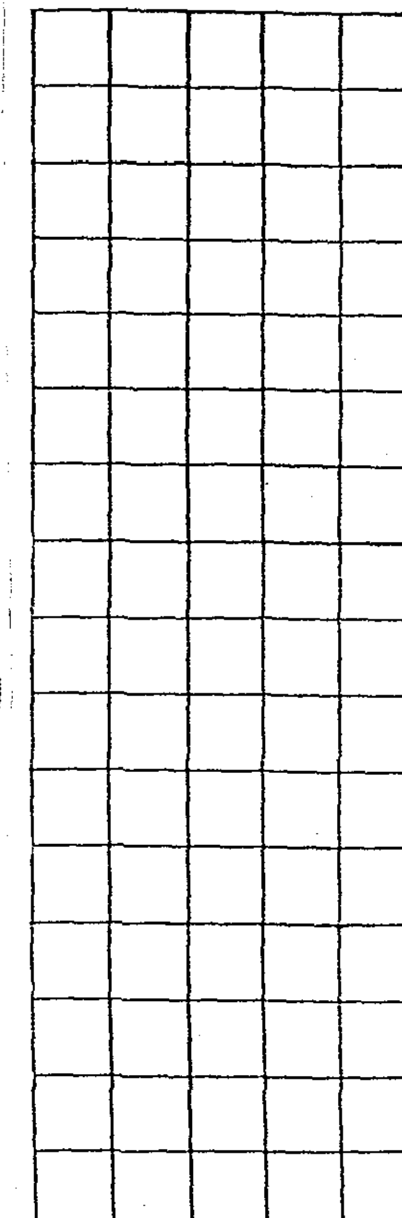




Hex

This game may either be played with the board and pieces cut out of cardboard or may be played on a prepared sheet containing many grids which the pupils may write on. I would suggest that the first is the more economical since an average size class will probably play a large number of games in a relatively short time. A suitable set is given on the following page.


BLACK
WHITE
WHITE
BLACK
1 2 3 4 5 6 7 8 9 10 11
A B C D E F G H J K L
A B C D E F G H J K L
1 2 3 4 5 6 7 8 9 10 11



Wari

The pieces for this game may be made in a similar way to those in Nim and the board printed on card unless you can get the craft department to oblige and make you some nice wooden Wari boards which you could then use with small cubes or something similar.