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Programmed learning of factors and equations for schools

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PROGRAMMED LEARNING

OF

FACTORS AND EQUATIONS

FOR

SCHOOLS

by

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A Master's Dissertation submitted in partial fulfilment of the requirements for the award of the degree of M.Sc. in Mathematical Education of the Loughborough University of Technology, January 1984.

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ABSTRACT

Algebra is one of those topics in mathematics often feared at school and not soundly understood. At the foundations of this subject lie the the concept of a factor and the ability to solve equations. This project attempts to introduce these two very important items into the classroom as a programmed text.

The theory behind programmed learning is looked at in a historical way. Stimulus-response psychology and Gestalt psychology are examined in relation to programmed learning and the type of programme used.

Current trends in recent years have been towards pupil-centred learning. With this and the advent of the micro-computer the potential for programmed learning in schools is discussed.

In producing a programmed text certain points need considering. The type of model used, the initial and terminal behaviour required, the progress of a student through the text, the validation and other points are considered for this particular text.

Originally this programmed text was not seen as a preparation to a particular examination level. However, because of the content it is considered suitable for the top 20% - 30% of the ability range of 4th/5th year. The text contains four sections:

Section 1 - Factors
Section 2 - Simple Equations
Section 3 - Quadratic Equations
Section 4 - Simultaneous Equations

Each section has a terminal test designed to contain all the

objectives of that section and to test the text's effectiveness. Results of these terminal tests are included and some suggested revisions are added.

Programmed texts need continual revision to improve their effectiveness and update their examples. This project attempts to introduce ideas in a practical way and initially at a concrete level.

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CHAPTER 1

INTRODUCTION

Algebra is perhaps one of the areas in mathematics most central to the philosophy and thinking contained in mathematical education. It is often the first time an individual learning mathematics is asked to make that conceptual leap forward in abstraction. Algebra is often used as a derogatory remark by people describing unhappy experiences of school mathematics. It can be the time when an individual feels that mathematics has ceased to be something real and has become a mere series of exercises using meaningless techniques. I can still remember my own childhood bewilderment at the astonishing discovery that letters could be added together and my equal confusion that $a + b$ sometimes equalled c and sometimes did not!!

Mathematics teachers, perhaps, overlook the difficulty that students have with algebra and the fact that many students never fully understand the items of abstraction. The fact that pupils perform certain standard routines, although important in themselves, do not guarantee that the concept of abstraction has been fully grasped and more complex generalisations prematurely made by the teacher. The balance between those parts of the curriculum that is meaningful now and those parts of the curriculum that are a preparation for further study has been a continual dilemma.

Nearly eighty years ago, Baker and Bourne (1904) wrote in their preface to 'Elementary Algebra' :

> "The object of the writers has been to provide a text book of practical interest and utility, fulfilling the latest requirements of various examining bodies, and following to a great extent the recommendations of the Mathematical Association."

The aim of making the mathematical curriculum more meaningful, then, is not new, but the decision to make some step forward in algebra is often governed by considerations other than pure understanding. Cockcroft (1982) states his opinion very clearly:

> "We believe it should be a fundamental principle that no topic should be included unless it can be developed sufficiently for it to be applied in ways which pupils can understand."

He does then go on to admit that, for example, algebraic manipulations required at A-level must be started at an O-level standard even if not directly relevant. This project in one small way has attempted to fulfil some of these aims. The topics are from the staple diets of most examination boards and the content fairly traditional. However, the overall aims of the topics are to solve problems (albeit contrived) and the techniques involved must be seen as secondary to this. Hopefully, of course the effect will be that on completion of the topics students will be prepared in those algebraic techniques in other situations. This project does not claim to explain the items of abstraction and generalisation and, indeed, assumes some knowledge of these concepts.

One of the problems continually met in the teaching of algebra is that letters in mathematics are used for many different purposes. Hart (1981) describes the use of letters in school mathematics in a hierarchial way. When preparing work in algebra the use of letters is often seen as using a variable long before the concept of generalisation has been reached. She describes the misuse thus:

> "The blanket term 'variable' in generalised arithmetic is a common practice which has served to observe both the meaning of the term itself and the very real differences in meaning that can be given to letters."

Generally speaking the project rarely uses letters for reasons beyond what Hart calls 'Letter used as a specific unknown'.

There has been movement recently towards packages that offer opportunity for individual learning. The formation of a programmed learning package that operates purely on an individual basis to the exclusion of other forms of activity has probably yet to be produced even if one thought that this was totally desirable. This text is seen as a vehicle to be used in teaching certain algebraic topics and not as a text that should replace the classroom teacher in any way. Cockcroft (1982) points to some difficulties associated with individual schemes:

> "One is that of providing sufficient opportunity for oral work and discussion. Another is the difficult of devising materials from which all pupils can learn satisfactorily and of ensuring that the necessary

> interconnections are established between the topic which is being studied and other pieces of mathematics."

This project has not attempted to link any of the topics investigated with others outside the project. It is also not exclusive in its explanation as it is assumed that the material will be presented to a class by an experienced mathematics teacher conversant with the outcomes. This is not to detract from the philosophy of programmed learning. On the contrary, programmed material is a very powerful tool when used in conjunction with normal classroom practice.

One of the misconceptions often made by a student approaching his first programmed text is that 'practice makes perfect'. There has been the analogy made between music and mathematics describing both as subjects that that 'need practice'. However, recent reports on mathematics have not indicated this. Hart (1983) asks the question:

> "Is there really any point in teaching something we know most children will not understand? One reason for doing this is that the child will become familiar with the idea and understand it later. We have no proof of this, in fact our results show that the understanding does not 'come'."

Simple repetition within a programmed text is not then enough. Often the need for practice is confused with the need for a longer time to assimilate ideas. A pupil failing to understand an idea may not need more repetitive practice but more time to imternalise the problem. As Cockcroft (1982) remarks:

> "A concept which some may comprehend in a single lesson may require days or even weeks if worked by others, and be inaccessible, at least for the time being, to those who lack understanding of the concepts on which it depends."

The advantages of a programmed text lie in not only the format giving opportunity for practice, but also the accessibilty of a structured explanation to the student available for him for as long as he wishes. In this way a student can take the time he needs to assimilate an idea and not be fully governed by the pace of the class teacher.

On many occasions a pupil's difficulty lies in the language of the problem and not the mathematics it contains. As Leedham and Unwin (1965) state:

> "It is infuriating when a pupil can solve the 'pure' examples with 100 per cent success, but is completely incapable of transferring this skill to cases where the example is dressed up as a problem."

The algebraic skills are a means to the solution of a problem. Some time has been given to the transference from a problem to a mathematical model. Although this is a skill it is undoubtably more difficult to programme. Holt and Marjoram (1973) discuss the differences in people's perceptions of a problem. They point out:

> "Success in mathematics learning, it seems, depends to an astonishing extent on learned skills and innate abilities, the marshalling and control of which are left almost totally to chance."

Programmed learning lends itself readily towards 'skills' and with more difficulty towards the problem solving aspect. The ability to perceive a problem and to 'make a start' on a solution is something where the programmed text may need supplementing from a normal classroom situation involving, perhaps, group discussion.

CHAPTER 2

HISTORICAL PERSPECTIVE

2.1 Programmed learning is often thought of as a recent development. The origins of programmed instruction, however, must be as old as learning itself. As soon as some thought is given to the transmission of knowledge from one person to another, and the process by which the transmission takes place, a crude form of 'planned instruction' exists. Lysaught and Williams (1963) claim:

> "One of the earliest programmers was Socrates, who developed a programme for geometry which was recorded by Plato in the dialogue 'Meno'. "

Indeed, forerunners of programmed learning as we know it today are too numerous to list. The very process of selecting material for students on the basis of what they already know is a technique employed by most practising teachers. It is these and other fundamental techniques that have been classified and researched during this century and have given rise to a finely tuned process known as Programmed Learning.

Expansion in educational opportunity has resulted in more public interest in the process itself. A brief historical background into programmed learning can be achieved by considering the main schools of thought in this era.

2.2 Stimulus-Response Psychology

Early experiments that considered how the learning process took place were carried out by I. P. Pavlov. Although he was concerned with conditioning rather than learning, his work does illustrate how an intermediate process can be used in transmitting an idea. Critics of programmed learning would point to its mechanistic nature and the conditioning that takes place and claim that it is not true education. Pavlov showed that, by training, dogs could be made to salivate to a bell as opposed to salivate at the sight of food. He felt that the animal had learned that the sight of food soon led to it being given something to eat. Pavlov attempted to show that it would be possible to condition the dog to salivate to any chosen stimulus.

This may seem a crude feature in the field of human learning, and indeed Pavlov's work receives little credit from modern scholars, but the association between ideas that Pavlov investigated can be found in many programmed learning texts.

The theory of modern programmed learning has its foundations in the work of E. L. Thorndike. Like Pavlov he investigated behaviour of animals but not solely in their conditioning. In his books 'Animal Intelligence' (1911) and 'Educational Psychology' (1913) he put forward his 'law of effect'. This, according to Deterline (1962), is where:

> "behaviour can be thought of as a trial and error process in which the 'connections' are strengthened between a stimulus (situation) and a response (behaviour) only if success or satisfaction follow the response."

Interestingly enough he claimed that pure repetition of a process was not enough and some positive record is necessary. Today it may be said that this is a statement of the obvious and most practising teachers are aware of this. However, the accepted view of education at that time had been that simple rote learning and practice were enough.

Probably the best remembered name in behavioural psychology is that of B. F. Skinner. Like Thorndike his work was concerned with learning by reinforcement. His experiments led to the important conclusion that in promoting a desired learning pattern, positive reinforcement is more effective than any aversive reinforcement. He suggested that learning should take place through small steps regularly and quickly reinforced by asking a student for a correct response. The early 'automatic teacher' devised by Pressey but abandoned in the 1930's was perhaps the first movement towards a teaching machine. Skinner attempted to put some of his theory into practice by creating a teaching machine along the lines of his learning theory. Pressey's machine was objective in the required response, whereas Skinner produced a machine in 1954 that required a correct response which thus made use of immediate reinforcement.

Skinner's view of programmed instruction is evident in many of the programmes of today, particularly in respect to the step-size (frames) and the frequency and positioning of rewards for the student. Programmes modelled on the ideas of Skinner are often called 'linear programmes', that is they follow a required line of thought, building one small step on top of another.

A brief account of cognitive psychology would not be complete without some mention of the work of Jean Piaget. The work carried out by Piaget on the stages of cognitive development has influenced greatly educational thinking and classroom practice in all subjects His work seeked to model the way children build up a picture of the world. In this he would investigate when and how concepts were formed by children. He put forward the idea of intellectual development being made in stages which he called:

1.	Sensory-motor	(0 - 2 years)
2.	Pre-conceptual	(2 - 4 years)
3.	Intuitive thinking	(4 - 7 years)
4.	Concrete operations	(8 - 11 years)
5.	Formal operations	(11 - 15 years)

Although it is not proposed to investigate these stages here, it is important to realise when preparing material for classroom use that, according to Piaget, adolescents and even adults do revert to earlier stages when faced with unfamiliar circumstances.

2.3 Gestalt Psychology

The previous scholars of the process of learning have agreed on their views of the way we learn. Generally speaking the learning process is seen as something that can be broken down into small steps which, by the correct positioning of positive rewards and stimuli, the required response can be achieved. This is by no means to be believed the only view of the learning process. An historical perspective would be incomplete without reference to the Gestalt theory of learning. Critics of programmed learning argue against the idea of learning being made up of small steps. The Gestalt view is that we learn by an appreciation of the whole. Kohler's (1957) experiments on apes showed in his view that they solved many problems not by trial and error but by seeing the whole situation. He introduced the term 'insightful learning' and criticised the linear approach to learning as being incomplete.

The word 'Gestalt' means 'pattern' and Gestalt psychologists believe that there exists an inate ability to organise stimuli into patterns and then learn by them. In that way the breaking down to small steps may actually hinder the learning process. Stones (1968) illustrates this with an analogy to music:

> "The argument is that the significance of a situation or pattern of stimuli is in the total pattern, not in its separate elements. The significance of a piece of music is not in the individual notes but in the whole composition."

Katz (1951) allied the conditioning theory to 'bricks and mortar' with no perception of the 'intended architecture'. While Gestalt theory may be believable there have been criticisms to suggest that insightful learning is achieved by relying on previous experience. Lovell (1969) suggests:

> "We should do well not to belittle the effect of previous learning and experience when children are organising their sense fields. The more familiar a child is with given material the more easily will he be able to group it and incorporate new and relevant material."

The Gestalt theory recognises the needs of creative thought and insight and perhaps illustrates the weaknesses of programmed instruction. However, to transmit existing knowledge in an efficient manner to a wide ability range, programmed learning undoubtedly has a role.

2.4 Linear Programming

Linear programming has derived from the work of Skinner. The important aspect of a linear programme is that it is made up of small steps arranged in order so that the student must work through each step in order to progress. Regular reinforcement is present, requiring one response from the student. Leedham and Unwin (1965) describe it this way:

> "A linear programme is a self-instructional medium aimed at a given class of students. All the students will work through the same substantive material. A programme is carefully arranged so that there is a very gradual progression from easy to difficult."

2.5 Branching Programmes

Branching programmes developed from the work of Crowder at the Educational Science Division of U.S. Industries, Santa Barbara. This process of programming differs from Skinner's in a number of ways. The path through a linear programme is unique whereas in a branching programme there are a number of ways to proceed. In a linear programme a stimulus requires one answer and errors are unhelpful, whereas in a branching programme Crowder would give alternative answers and then errors could be used in a constructive way by re-routing a student to another part of a programme depending upon his error. In this way an error could be looked upon as an opportunity to clarify a misconception. Crowder (1960) says:

> "We suspect that human learning takes place in a variety of ways and that these ways vary with the abilities and present knowledge of different students."

The consequence of branching does mean, of course, that a brighter student can work through material more quickly whereas those who need 'remedial' help find themselves guided through other parts of the programme. Such programmes will be more complicated to write as the responses to stimuli have to be foreseen.

The multi-choice feature of a branching programme means that the student may not be involved in creating a lengthy response to the given stimuli. A linear programme requires this involvement and may therefore guarantee a more active learning environment.

Programmes are available in many forms, for many types of students, on many subject matters. This has been a brief overview of the history and roots of programmed instruction. Although criticisms exist about programmed instruction, it remains an effective way of transmitting knowledge in an interactive way.

CHAPTER 3

CURRENT PRACTICES IN SCHOOLS

3.1 During the last twenty years the growth of information in Mathematical Education has been vast. The way mathematics is presented to children in school has changed beyond recognition. The development of systems like the 'Schools Mathematics Project' (SMP) and the 'Scottish Mathematics Group' (SMG) has revolutionised the format of school mathematics. There has been a forceful movement towards making mathematics more meaningful to the learner. There are numerous schemes available to schools within the mathematics curriculum that put the learner as the central feature rather than the content. The content of the material is seen as somethin that needs to be put into context rather than a bulk of knowledge needed to be learnt for learning's sake. Presentation of material has been changing in its physical appearance as well.

Workcards and worksheets, designed to explain one small point, have become more popular. The purpose of this format of material is seen as a way of introducing individualised learning. The teacher is freed from his more formal role and can use his time on individuals, while the pupils are allowed to progress at their own pace. SMP Cards and SMG Worksheets attempt to do this, breaking material down into small steps. Many of these schemes are following the philosophy of programmed learning and embody pupil-centred learning to promote in-depth learning. Work done at Chelsea College investigating the ways children learn in mathematics has found evidence that children learn algebra by many methods. In their publication 'Children's Understanding of Mathematics' (1981) they found evidence to suggest that children tackle objective problems with methods that have little to do with what has been taught. The project says:

> "This may be because mathematics teaching is often seen as an initiation into rules and procedures which though very powerful (and therefore attractive to teachers) are often seen by children as meaningless."

The project tested algebra at different levels, much of which were items where letters were specific unknowns as opposed to generalised number. Surprisingly, the majority of 13, 14 and 15 year olds were not able to cope consistently with the general use of letters and seemed to be at a 'concrete level' of understanding. The suggestion here is that most algebra teaching should be based on concrete operations.

While recognising these facts this programmed learning text does attempt to cover syllabus items of GCE/CSE examinations, and consequently some generalisation will be necessary. However, each section begins with a problem that hopefully the pupil can relate to and see some purpose in completing the work contained in the section. While understanding has been a central feature of this programmed text certain items have been curtailed where the theoretical content was thought too formal. For example, in the solution of quadratics it was decided to give the formula and miss out the stage of 'completion of square'. It may be possible at a later stage to redirect more able pupils to other frames.

From Plowden to Cockcroft there has been much emphasis placed upon learning by doing, or learning through experience.

Importance is placed upon understanding of concepts as opposed to performing meaningless tasks.

3.2 Technology and Programmed Learning

In the recent 'technological revolution' the availability of computers to schools has flourished. Almost every school has some use available to it of a computer. The micro-computer has many uses in schools some of which have hardly been explored. One consequence of their presence is a new interest in programmed learning. The speed and the ability to cope with interaction will have far reaching effects on individualised learning in schools. Literature on micro use in schools has expanded enormously.

Although slow at the beginning, educational software is now becoming available from most publishers. It must not be thought that the micro-computer is just merely another teaching machine. Carefully prepared software, using attractive visual stimuli, could revolutionise the way programmed material is presented. The television screen can bring alive learning situations and in the medium the pupils are often very used to. The use of the computer and other hardware will enable unparalleled opportunities to change our teaching methodology. Programmed learning will have its place in the future.

CHAPTER 4

PRACTICAL CONSIDERATIONS

4.1 Background of Project

Originally this project was designed to cover most of the algebraic syllabuses covered in the late part of secondary schools. This initial aim was made somewhat more modest when the size of such a project reached a testing stage. With the changing scene in the educational arena it was thought unwise to direct this project towards any specific examination syllabus. However, the content of those areas taught in Chapter 5 covers what would be expected by most O-level syllabuses on those limited subjects.

Initially the project was conceived in eight headings, namely:

1. Substitution (Use of letters)
2. Factors
3. Linear Equations
4. Simultaneous Equations (Two equations with two unknowns)
5. Quadratic Equations
6. Graphical solution of equations
7. Re-arranging formulae
8. Problems leading to equations.

The field testing of this project was to take place in the 4th/5th year of the secondary school. It was considered likely that the pupils would have covered (1) and to make the programmed test more manageable it was decided to prepare a detailed test for (2), (3), (4) and (5), incorporating (8) within the relevant sections

1. FACTORS
2. SIMPLE EQUATIONS
3. QUADRATIC EQUATIONS
4. SIMULTANEOUS EQUATIONS

4.2 Target Population

This project covers a part of the mathematical syllabuses used at all levels in a secondary school. However, when the depth of areas is taken into account it would be unrealistic to presume that the majority of the 4th/5th year of a secondary school could hope to complete the programmed text. In most schools the target population would be in the 4th/5th year and comprise approximately the top 20% - 30% of the ability range. Although the sections could be used in either the 4th or 5th year, it seems sensible to use 1 and 2 in the 4th year and 3 and 4 in the 5th year. In practice, then, the target population in most schools at present would be their O-level sets and perhaps their top CSE sets. A student who had completed the text would then be in a position to answer questions on those areas in most CSE/O examinations.

4.3 Programming Model

In considering the type of model this project should be based upon, a number of aspects were taken into account. The style and type of programme depends on many variables including the material, the maturity of the student and the competence of the writer. As a first attempt a linear programme would seem more straight forward to write and yet would loose the 'skipping' advantages of branching. Leith (1964) remarks:

> "It seems likely, at the present time, that building up concepts and teaching unfamiliar topics, especially

> for the mature learner, may be best accomplished by constructed response programmes."

This project is meant to be used within a classroom as the medium of teaching and it is essential that there is an active participation by the learner with the material met. As the topics involved would be, for the most part, unfamiliar it was thought wise to progress upon a linear path, with as many required responses from the student as possible. The project would be split into sections so making a very crude form of 'skipping' possible for the student if the teacher felt it necessary.

The construction of a full branching programme was thought at this stage both unwise and possibly inappropriate. The step-size involved should be small, therefore involving the understanding of one task at a time. It should be pointed out, however, that small steps can mean slow progression and often lead to boredom and the disenchantment of the learner. This can often be avoided by including enough stimuli that demand response, thereby supplying the reinforcement to learning. Pocztar (1972) describes the need for dividing the material into small steps:

> "to ensure maximum effectiveness by increasing the number of reinforcers."

He continues:

> "the subject must be divided into units of information in order that progress may be made by small steps, and the largest possible number of reinforcers supplied."

The interaction of the student with the material is commented on in 4.7

4.4 Criterion Behaviour

Essentially this programme is to be used as an aid with pupils of either O-level calibre or CSE candidates aspiring to Grade One. In the testing of this project classes of this sort were chosen so a fairly high success rate should be expected. The level of success, however, may be somewhat lower as the content of the projec is something most candidates find difficult and hard to grasp. Most teachers in a trial examination for O-level predict likely success when a pupil obtains 40% or more. It seems reasonable to use this as a minimum level of success for most of the candidates. For this target population the criterion behaviour is expressed as 90/40, meaning that 90% of pupils should achieve a success rate of 40% or more.

Criterion behaviour is often applied to individual objectives. In this project, however, it will be applied to the four terminal tests of each section of the text.

4.5 Initial Behaviour

Algebra is probably one of the areas where children find most difficulty. It is likely that all the target population will have met some algebra by their 4th year, and possibly some of the areas tested. However, schools are not consistent on whom are regarded as O/Good CSE candidates at the end of the 3rd year. To include an initial test purely to decide on the effectiveness of the programmed text can be demoralising to the students and can have destructive consequences. The majority of techniques met in this text will be new to the candidates. For these reasons it was thought inappropriate to give a pre-test as many candidates would frequently score 0 and it would likely have a damaging effect.

A knowledge of the following ideas and meaning of words is thought essential:

The use of letters to represent numbers

Substitution of numbers into statements such as
$3x$, $x + y$, xy, etc.....

Indices

Correcting numbers to decimal places

The commutative law

The distributive law

The meaning of the words:

expressions

product

term

denominator

multiple

consecutive

perimeter

area

coefficient.

4.6 Terminal Behaviour

A clear understanding of what the objectives of a programme should be is essential before beginning the construction of the programme. It is important to be able to say exactly what a student should be able to do after completing a programmed text. This claim is often tested by some series of tests written to cover all the concepts covered. Stones (1966) says a programmer should start by:

> "deciding what we consider should be in the repertoire of the pupil at the end of the programme."

With this particular programme the objectives were taken from present O-level/CSE syllabuses. The terminal tests were written with these objectives in mind. For convenience the objectives of this project are given under the four section headings of Chapter 5.

Section 1 FACTORS

On completion of this section the pupils should be able to:

(a) Find the set of factors of a given number.

(b) Factorise a number into prime factors.

(c) Factorise algebraic products.

(d) Factorise expressions containing common factors.

(e) Factorise expressions of four terms using grouping.

(f) Recognise and factorise general quadratic expressions.

(g) Recognise and factorise the difference of two squares.

Section 2 SIMPLE EQUATIONS

On completion of this section the pupils should be able to:

(a) Manipulate simple equations by the addition of numbers to both sides.

(b) Manipulate simple equations by the multiplication of numbers to both sides.

(c) Manipulate simple equations by adding unknown terms to both sides.

(d) Arrive at a solution using (a), (b) and (c).

(e) Solve simple equations containing brackets.

(f) Check to see whether a given solution is correct.

(g) Arrive at a simple equation from a given problem.

(h) Solve a simple problem, giving an answer in a meaningful way.

(i) Manipulate and solve simple equations involving unknown terms in the denominator.

Section 3 QUADRATIC EQUATIONS

On completion of this section the pupil should be able to:

(a) Recognise a quadratic equation.

(b) Manipulate a quadratic equation into its 'normal' form.

(c) Solve a general quadratic equation by factors.

(d) Write down the formula for solving quadratic equations.

(e) Solve a general quadratic equation by using the formula.

(f) Arrive at a quadratic equation from a given problem.

(g) Solve a quadratic problem, giving an answer in a meaningful way.

(h) Interpret the value of $b^2 - 4ac$ from the formula, giving the number of real solutions to a given quadratic equation.

Section 4 SIMULTANEOUS EQUATIONS

On completion of this section the pupil should be able to:

(a) Use the method of elimination on two linear simultaneous equations in two unknowns by substitution.

(b) Use the method of elimination on two linear simultaneous equations in two unknowns by the multiplication of a constant before subtracting the equations.

(c) Use the method of elimination on two linear simultaneous equations in two unknowns by multiplication of a constant before adding the equations.

(d) Recognise the unknowns to eliminate what would solve the last manipulation.

(e) Re-arrange simultaneous equations into the form $ax + by = c$ where a, b and c are real numbers.

(f) Arrive at simultaneous equations from a given problem.

(g) Solve a problem involving simultaneous equations giving the answer in a meaningful way.

4.7 Student Progress

The inclusion of regular stimuli requiring responses is essential to the effectiveness of a programmed text. The introduction of a programmed text does not necessarily mean that the teacher is redundant. It is seen as essential with this particular project that there is an active participation by the teacher in the presentation of the text. Although the format of any programme should be what an individual can progress at his own speed, it should not be thought that a programme will ever successfully replace a teacher. The use of a programmed text may minimise many of the disadvantages of the traditional approach to new material. Traditional classroom instruction involves a great deal of passive listening on the part of the student even if regular teacher-pupil contact is made. Hughes (1962) argues:

> "Even in classroom discussion, while one student is responding the others may not be paying attention. In programmed instruction, on the other hand, there is more assurance that learning is taking place, because the student is forced to participate actively at each step in the programme by continually making responses."

4.8 Validation

In the construction of a programmed learning text the process of validation is vital. The role of a programme should be different from that of merely a textbook. A student will rely heavily on a programme for clarification, so that it is important that it is written with the leaner as the central feature of the programme and not the content. Feedback and revision of the programme will be an ongoing exercise. A specialist writing on his own subject will often overestimate what a student can cope with and not recognise a point that will cause difficulty.

The process of validation will be needed at all stages. 'Field testing' is used as the programme is written or immediately afterwards. It normally involves a few students who are presented with the text, one frame at a time and their responses resolved verbally. This process is very expensive in time but will often enable immediate alterations to be made before more complete testing. Because of the lack of time this project has not used this kind of testing in any formal way, although students were consulted at the stage of writing certain frames. A second type of validation is often called 'master validation'. Here the programme is given a more thorough test by involving more pupils and having them complete the programme with a terminal test or take tests at particular points in the programme. An analysis of errors can then provide information enabling the writer to change certain frames or introduce extra ones.

The criterion behaviour will need to be achieved but is unlikely on the first testing. However, a high error rate may be explained by many different reasons. Step-size may be too big and frames may have to be changed accordingly. The programme itself may have had inappropriate stimuli to the type of questions in the terminal test. Alternatively, the terminal test itself may not truly reflect the objectives of the programme.

Error analysis of each frame stimuli has not been attempted in this project. The degree of success of the pupil, and hence the effectiveness of the text, is measured in the results of the terminal test. Comments on the results are contained in Chapter 6

CHAPTER 5

PROGRAMMED TOPICS

INSTRUCTIONS

1. These booklets introduce some of the basic ideas of algebra. Some of the ideas you will have met before; others will be new.

2. The booklets are divided into FRAMES. Often each FRAME will hold some new idea. Try to understand each FRAME before progressing to the next.

3. Usually each FRAME will give you questions to answer. Try to answer these yourself BEFORE looking at the answers which follow a dotted line.

4. If you are unable to answer a question, re-read the FRAME containing the questions and then ask your teacher.

5. At the end of each booklet you will be given a test involving questions similar to those in the booklet.

6. When you have finished using the booklets you may be asked to give YOUR OPINION on them. Think of this as you are working through them and try to remember any improvements you would make in their presentation.

SECTION 1

Factors

FRAME 1 INTRODUCTION

Answer the following questions as quickly as you can without any help from calculators or tables.

1. A very generous teacher takes his class into a restaurant and orders 25 meals costing £2-77 each and 25 drinks costing 23p each. How much does he spend?

2. Find $\sqrt{1764}$

3. Calculate $28^2 - 18^2$

ANSWER 1

1. This problem could have been solved by finding

25 X £2-77	=	£69-25
25 X £0-23	=	£ 5-75
		£75-00

It would be quicker to notice 25 X £2-77 + 25 X £0-23 is the same as 25 X (£2-77 + £0-23)

= 25 X £3

= £75

2. $\sqrt{1764}$

$= \sqrt{2 \times 882}$

$= \sqrt{2 \times 2 \times 441}$

$= \sqrt{2 \times 2 \times 3 \times 147}$

$= \sqrt{2 \times 2 \times 3 \times 3 \times 49}$

$= \sqrt{2 \times 2 \times 3 \times 3 \times 7 \times 7}$

$= 2 \times 3 \times 7$

$= \underline{42}$

3. $28^2 - 18^2$ could be worked out by finding

$$\begin{aligned} 28^2 &= 784 \\ 18^2 &= \underline{324} - \\ &\quad \underline{460} \end{aligned}$$

It would be quicker to notice

$$\begin{aligned} 28^2 - 18^2 &= (28+18)(28-18) \\ &= 46 \times 10 \\ &= \underline{460} \end{aligned}$$

All the ideas used in these questions come from understanding FACTORS and FACTORISATION.

FRAME 2 FACTORS

A FACTOR of a number (say z) is any whole number that divides exactly into z.

For example, 3 is a factor of 12 since 3 divides exactly into 12.

In the same way 1 is a factor of 12

2 is a factor of 12

3 is a factor of 12

4 is a factor of 12

6 is a factor of 12

12 is a factor of 12

So {1, 2, 3, 4, 6, 12} is the set of factors of 12 as it contains <u>all</u> whole numbers that divide exactly into 12.

Notice that 1 and the number itself are always factors.

Find the set of factors for each of the following numbers

1. 6: 2. 18: 3. 24: 4. 68:

ANSWER 2

1. {1, 2, 3, 6}
2. {1, 2, 3, 6, 9, 18}
3. {1, 2, 3, 4, 6, 8, 12, 24}
4. {1, 2, 4, 17, 34, 68}

FRAME 3 TO FACTORISE A NUMBER

We FACTORISE a number by writing it as a product of two or more other whole numbers (i.e. its factors). If the number 12 is written as 2 X 6 we say 12 has been FACTORISED or has been written in FACTOR FORM.

There are often many different ways of factorising a number and no unique answer.

3 X 4, 1 X 2 X 6, 2 X 2 X 3 are all ways of writing 12 factorised.

The use of 1 when factorising is unnecessary at this stage and is rarely included. Thus 1 X 2 X 6 would be written simply as 2 X 6.

FRAME 4 PRIME NUMBERS

A number that has no factors (except 1 and itself) is called a PRIME number. For example, 7 is a prime number as no other whole number divides into it. There are six prime numbers between 10 and 30. See if you can find them.

ANSWER 4

11, 13, 17, 19, 23, 29

FRAME 5 PRIME FACTORS

In FRAME 3 we looked at ways of factorising 12. Without using 1 there are the following possibilities:-

3 X 4 .... (1)

2 X 6 .... (2)

2 X 2 X 3 .... (3)

If we look at (1) you will see that 4 will factorise further, so that 3 X 4 becomes 3 X 2 X 2. In the same way, (2) factorises further so that 2 X6 becomes 2 X 2 X 3.

You will notice that (1) and (2) have become the same as (3). Although (1), (2) and (3) are all 12 factorised, only (3) contains prime numbers only. 2 and 3 are known as PRIME FACTORS of 12 and 2 X 2 X 3 is said to be 12 FULLY FACTORISED or in PRIME FACTOR FORM. When you are asked to factorise a number you should always try to express the number fully factorised. For example, 45 can be written as 5 X 9 or 3 X 15 or 3 X 3 X 5. Only 3 X 3 X 5 is in the fully factorised form (any order is satisfactory, e.g. 3 X 5 X 3 is correct).

Write the following numbers in prime factor form

1. 36: 2. 18: 3. 100: 4. 57: 5. 132

ANSWER 5

1. 2 X 2 X 3 X 3
2. 2 X 3 X 3
3. 2 X 2 X 5 X 5
4. 3 X 19
5. 2 X 2 X 3 X 11

FRAME 6

In FRAME 1, one of the questions asked was to find $\sqrt{1764}$. Using the ideas of FRAME 5, 1764 is written in prime factor form and then the square root is found. The ideas used in the other two questions will be looked at in the following frames. To do this in a more general way, we will first look at the idea of factors in algebra.

FRAME 7

As you should know, in algebra we represent numbers by letters and we use expressions such as

	ab	for	a X b
or	uyz	for	u X y X z
or	a^2	for	a X a
or	a^2b^2	for	a X a X b X b

a X b	is	ab	factorised
u X y X z	is	uyz	factorised
a X a	is	a^2	factorised
a X a X b X b	is	a^2b^2	factorised

In the same way factorise

1. uy 2. abc 3. a^2b 4. f^2g^2

ANSWERS 7

1. u X y
2. a X b X c
3. a X a X b
4. f X f X g X g

FRAME 8 COMMON FACTORS

In Frame 1 look at Example 1. You will see that 25 X £2-77 + 25 X £0-23 was worked out more easily by calculating 25(£2-77 + £0-23) rather than working out 25 X £2-77 and 25 X £0-23 separately and then adding. This idea is known as finding a COMMON FACTOR. In this case it was 25. In algebra you will need to factorise expressions like $ab + ac$. This expression has two terms, ab which factorises to $a \times b$, and ac which factorises to $a \times c$. In each case the factor 'a' appears and we call it a common factor.

$$ab \quad + \quad ac \qquad \ldots\ldots\ldots\ldots\ldots \ (1)$$

$$= \boxed{a} \times b \quad + \quad \boxed{a} \times c \qquad \ldots\ldots\ldots\ldots\ldots \ (2)$$

COMMON FACTOR

$$= \boxed{a} \times (b + c) \qquad \ldots\ldots\ldots\ldots\ldots \ (3)$$

$$= a \times (b+c) \qquad \ldots\ldots\ldots\ldots\ldots \ (4)$$

Line (4) means that 'a' multiplies both 'b' and 'c'. So as $a \times (b+c) = ab + ac$, 'a' and '(b+c)' are factors of $ab + ac$ in the same way as 2 and 3 are factors of 6. You would write $a \times (b+c)$ as $a(b+c)$ and this would be $ab + ac$ factorised.

Examine the following expressions for a common factor and write them factorised:

1. $mn + mp$	2. $rs - rt$
3. $sv - uv$	4. $ms - sp$

ANSWERS 8

1. $m(n + p)$	2. $r(s - t)$
3. $v(s - u)$	4. $s(m - p)$

FRAME 9

Now try to factorise $a^3b + a^2c$ in the same way.

	a^3b	$+\ a^2c$	.... 9.1
$=$	$a \times a \times a \times b$	$+\ a \times a \times c$	Factorise each term
$=$	$\boxed{a} \times a \times a \times b$	$+\ \boxed{a} \times a \times c$	Find a common factor
$=$	$\boxed{a} \times (a \times a \times b$	$+\ a \times c)$	Introduce the brackets
$=$	$a(a^2b$	$+\ ac)$	.... 9.2

Although we have attempted to factorise 9.1, the expression 9.2 is not fully factorised. The bracket in 9.2 is not a prime factor, i.e. $a^2b + ac$ can be factorised again.

When looking for a common factor, you should look for as many common factors as possible. So the correct procedure for factorising 9.1 would look like this:-

	a^3b	$+\ a^2c$	
$=$	$a \times a \times a \times b$	$+\ a \times a \times c$	Factorise each term
$=$	$\boxed{a} \times \boxed{a} \times a \times b$	$+\ \boxed{a} \times \boxed{a} \times c$	Find the common factors
$=$	$\boxed{a} \times \boxed{a} \times (a \times b$	$+\ c)$	Introduce the brackets
$=$	$a^2(ab + c)$		

Notice that $a^2(ab + c)$ is quite acceptable and it is not necessary to write the expression as $a \times a \times (ab + c)$

The important feature here is the common factor 'a^2' and not the fact that $a^2(ab + c) = a \times a \times (ab + c)$.

FRAME 10

In Frame 9 we looked at the importance of finding the largest common factor.

Here is another example. Follow the same procedure as before.

Example 10.1

$$fg^2h \quad - \quad f^2gh^2$$

$$= f \times g \times g \times h \quad - \quad f \times f \times g \times h \times h \qquad \text{........(i)}$$

$$= \boxed{f} \times \boxed{g} \times g \times \boxed{h} \quad - \boxed{f} \times f \times \boxed{g} \times \boxed{h} \times h \qquad \text{........(ii)}$$

$$= \boxed{f} \times \boxed{g} \times \boxed{h} \times (g \quad - \quad f \times h) \qquad \text{........(iii)}$$

$$= fgh\ (g - fh)$$

$fg^2h - f^2gh^2$ has been factorised, the factors being

f, g, h, $(g - fh)$.

When you are familiar with this technique you will find it unnecessary to include all the stages in the working, for example lines (ii) and (iii) above.

Factorise the following expressions, writing out their working fully as in Example 10.1

1. $ab^3 + b^2c$
2. $a^2b^2 - a^2c$
3. $r^2 + rh$
4. $x^2yz^2 - xyz^3$

ANSWERS 10

1. $$ab^3 \quad + \quad b^2c$$

$$= a \times b \times b \times b \quad + \quad b \times b \times c$$

$$= a \times \boxed{b} \times \boxed{b} \times b \quad + \quad \boxed{b} \times \boxed{b} \times c$$

$$= \boxed{b} \times \boxed{b} \times (a \times b \quad + \quad c)$$

$$= \underline{b^2(ab + c)}$$

2. $a^2b^2 \quad - \quad a^2c$

$= a \times a \times b \times b \quad - \quad a \times a \times c$

$= \boxed{a} \times \boxed{a} \times b \times b \quad - \quad \boxed{a} \times \boxed{a} \times c$

$= \boxed{a} \times \boxed{a} \times (b \times b \quad - \quad c)$

$= \underline{a^2(b^2 - c)}$

3. $\pi r^2 \quad + \quad \pi rh$

$= \pi \times r \times r \quad + \quad \pi \times r \times h$

$= \boxed{\pi} \times r \times r \quad + \quad \boxed{\pi} \times r \times h$

$= \boxed{\pi} \times \boxed{r} \times (r \quad + \quad h)$

$= \underline{\pi r(r + h)}$

4. $x^2yz^2 \quad - \quad xyz^3$

$= x \times x \times y \times z \times z \quad - \quad x \times y \times z \times z \times z$

$= \boxed{x} \times x \times \boxed{y} \times \boxed{z} \times \boxed{z} \quad - \quad \boxed{x} \times \boxed{y} \times \boxed{z} \times \boxed{z} \times z$

$= \boxed{x} \times \boxed{y} \times \boxed{z} \times \boxed{z} \times (x \quad - \quad z)$

$= \underline{xyz^2(x - z)}$

FRAME 11

The common factors that we have been considering could be ordinary numbers as in FRAME 2. For example following the steps described in FRAMES 9 and 10

Example 11.1

$7ab \quad + \quad 21d$

$= 7 \times a \times b \quad + \quad 3 \times 7 \times d$

$= \boxed{7} \times a \times b \quad + \quad 3 \times \boxed{7} \times d$

$= \boxed{7} \times (a \times b \quad + \quad 3 \times d)$

$= 7(ab + 3d)$

Example 11.2

$$\begin{array}{llcl} & 45y & - & 36z \\ = & 3 \times 3 \times 5 \times y & - & 2 \times 2 \times 3 \times 3 \times z \\ = & \boxed{3} \times \boxed{3} \times 5 \times y & - & 2 \times 2 \times \boxed{3} \times \boxed{3} \times z \\ = & \boxed{3} \times \boxed{3} \times (5 \times y & - & 2 \times 2 \times z) \\ = & 9(5y - 4z) & & \end{array}$$

Notice that in Example 11.2 9 is not written as 3 X 3 as this is thought not necessary for the same reasons as explained in FRAME 9. The importance is to show that 9 is a common factor and not that 9 = 3 X 3.

Factorise 1. $3a + 6b$ 2. $4ab - 12d$ 3. $12a^2 + 16b^2$

ANSWERS 11

1. $3(a + 2b)$ 2. $4(ab - 3d)$ 3. $4(3a^2 + 4b^2)$

FRAME 12

Now when you look for a common factor you must make sure you have all the common factors including numbers and letters.

Example 12.1

$$\begin{array}{llcl} & 3a^2b & + & 21ac \\ = & 3 \times a \times a \times b & + & 3 \times 7 \times a \times c \\ = & \boxed{3} \times \boxed{a} \times a \times b & + & \boxed{3} \times 7 \times \boxed{a} \times c \\ = & \boxed{3} \times \boxed{a} \times (a \times b & + & 7 \times c) \\ = & 3a\,(ab + 7c) & & \end{array}$$

Try to factorise these questions using all the steps shown in the Example 12.1

1. $5ab^2 + 10bc$ 2. $15ab^2c - 40bcd$

ANSWERS 12

1. $5ab^2 \quad + \quad 10bc$

$= 5 \times a \times b \times b \quad + \quad 2 \times 5 \times b \times c$

$= \boxed{5} \times a \times \boxed{b} \times b \quad + \quad 2 \times \boxed{5} \times \boxed{b} \times c$

$= \boxed{5} \times \boxed{b} \times (a \times b \quad + \quad 2 \times c\,)$

$= 5b\ (ab + 2c)$

2. $15ab^2c \quad - \quad 40bcd$

$= 3 \times 5 \times a \times b \times b \times c \quad - \quad 2 \times 2 \times 2 \times 5 \times b \times c \times d$

$= 3 \times \boxed{5} \times a \times \boxed{b} \times b \times \boxed{c} \quad - \quad 2 \times 2 \times 2 \times \boxed{5} \times \boxed{b} \times \boxed{c} \times d$

$= \boxed{5} \times \boxed{b} \times \boxed{c} \times (3 \times a \times b \quad - \quad 2 \times 2 \times 2 \times d\,)$

$= 5bc\ (3ab - 8d)$

FRAME 13

Sometimes when you find a common factor it will appear as though there is nothing left in a term. For example:

$9a^2b \quad + \quad 3b$

$= 3 \times 3 \times a \times a \times b \quad + \quad 3 \times b$

$= \boxed{3} \times 3 \times a \times a \times \boxed{b} \quad + \quad \boxed{3} \times \boxed{b}$ 13.1

$= \boxed{3} \times \boxed{b} \times (3 \times a \times a \quad + \quad ?\ )$ 13.2

As 13.2 shows there appears as though there is nothing left to go into the second term in the brackets.

As you saw in FRAME 4 the number 1 can be included at every stage of factorising but normally it is unnecessary to do so. In this case 13.1 could be written as:

$$\boxed{3} \times 3 \times a \times a \times \boxed{b} \quad + \quad \boxed{3} \times \boxed{b} \times 1$$

which makes 13.2 :

$$\boxed{3} \times \boxed{b} \times (3 \times a \times a \quad + \quad 1)$$

$$= 3b\ (3a^2 + 1)$$

Here are a few examples for you to try:

1. $2ab + 2$ 2. $a^2b + a$ 3. $3b + 12b^2$ 4. $16a^2b - 4ab$

ANSWERS 13

1. $2(ab + 1)$ 2. $a(ab + 1)$ 3. $3b(1 + 4b)$ 4. $4ab(4a - 1)$

FRAME 14

So far we have dealt with common factors from only two terms. A common factor can be often found from more than two terms.

Here are some examples:

(a) $21mp \quad - \quad 7m \quad + \quad 14m^2$

$= 3 \times 7 \times m \times p \quad - \quad 7 \times 1 \times m + 2 \times 7 \times 1 \times m \times m$

$= 3 \times \boxed{7} \times \boxed{m} \times p \quad - \quad \boxed{7} \times 1 \times \boxed{m} + 2 \times \boxed{7} \times 1 \times \boxed{m} \times m$

$= \boxed{7} \times \boxed{m} \times (3 \times p \quad - \quad 1 \quad + 2 \times 1 \times m\ )$

$= \underline{7m\ (3p - 1 + 2m)}$

(b) $4ab^2 - 2a^2bc + 6ab$

$= \underline{2ab\ (2b - ac + 3)}$

(c) $p^3q^2r + p^2q^2r^2 + pq^2r^3$

$= \underline{pq^2r\ (p^2 + pr + r^2)}$

(d) $8x^6 + 16x^4 + 48x^3$

$= \underline{8x^3\ (x^3 + 2x + 6)}$

(e) $16abcx - 28bcdx - 20cdex$

$= \underline{4cx\ (4ab - 7bd - 5de)}$

Now try factorising these expressions:

1. $pt + qt + rt$
2. $an + bn - cn$
3. $abc^2 + ab^2 - a^2b$
4. $18mn^2 - 6m^2n + 12mn$

ANSWERS 14

1. $t\ (p + q + r)$
2. $n\ (a + b - c)$
3. $ab\ (c^2 + b - a)$
4. $6mn\ (3n - m + 2)$

FRAME 15 GROUPING

A common factor should always be sought first. However, there are expressions which have no common factors but which can be factorised by GROUPING. Terms are collected together into groups which do possess a common factor.

For example:

$$ax + bx + ay + by \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 15.1$$

15.1 has no common factor but we can factorise the expression by writing 15.1 as a product of two expressions.

Firstly, the terms in 15.1 are collected together into groups, each group containing a common factor. (N.B. - there is more than one way of doing this)

collected

$$\overbrace{ax + bx} \qquad + \qquad \overbrace{ay + by}$$

Then the groups can be factorised to give:

$$x(a + b) \qquad + \qquad y(a + b) \quad \ldots\ldots\ldots \quad 15.2$$

The purpose of this last step is that the contents of the sets of brackets appearing in 15.2 are the same.

15.2 is then an expression of two terms, namely $x(a + b)$ and $y(a + b)$.

$(a + b)$ is a common factor in 15.2 so that it is possible to factorise further:

$$\begin{aligned}
& x(a + b) && + && y(a + b) \\
= \; & x \times (a + b) && + && y \times (a + b) \\
= \; & x \times \boxed{(a + b)} && + && y \times \boxed{(a + b)} \\
= \; & \boxed{(a + b)} && \times && (x + y) \\
= \; & (a + b)(x + y) && && \ldots\ldots\ldots\ldots\ldots\ldots \quad 15.3
\end{aligned}$$

So $(a + b)$ and $(x + y)$ are the two factors of the original expression 15.1.

In practice the solution would be reduced to:

$$
\begin{aligned}
& ax + bx + ay + by \\
= \; & x(a + b) + y(a + b) \\
= \; & \underline{(a + b)(x + y)}
\end{aligned}
$$

Notice the essential point that the contents of the brackets appearing in 15.2 <u>must</u> be the same as otherwise the step to 15.3 would be impossible.

<u>FRAME 16</u>

Here are a couple of examples:

(a)
$$
\begin{aligned}
& sx + 5tx + sy + 5ty \\
= \; & x(s + 5t) + y(s + 5t) \quad \text{Factorise the groups} \\
= \; & \underline{(s + 5t)(x + y)}
\end{aligned}
$$

(b)
$$
\begin{aligned}
& 2mn - 4np + rm - 2pr \\
= \; & 2n(m - 2p) + r(m - 2p) \\
= \; & \underline{(m - 2p)(2n + r)}
\end{aligned}
$$

Factorise these expressions:

1. $ax - ay + bx - by$ 2. $am + 2al + bm + 2bl$

ANSWERS 16

1. $ax - ay + bx - by$

$a(x - y) + b(x - y)$

$\underline{(x - y)(a + b)}$

2. $am + 2al + bm + 2bl$

$a(m + 2l) + b(m + 2l)$

$\underline{(m + 2l)(a + b)}$

FRAME 17

$$a^2l + b^2m + b^2l + a^2m \quad \ldots\ldots\ldots\ldots\ldots \quad 17.1$$

If you look at 17.1 you will see that the terms are arranged so that there are no common factors in the first pair. When this happens it will be necessary to re-arrange the terms so that the procedure described in FRAME 15 can be carried out. For example, the middle two terms of 17.1 can be exchanged so that the expression becomes:

$$a^2l + b^2l + b^2m + a^2m$$

(Notice that there may be more than one way of re-arranging)

The expression can then be factorised by grouping:

$$l(a^2 + b^2) + m(b^2 + a^2)$$

$$= l(a^2 + b^2) + m(a^2 + b^2) \quad \ldots\ldots \quad b^2+a^2 = a^2 + b^2$$

$$= \underline{(a^2 + b^2)(l + m)}$$

FRAME 18

The examples so far have not involved taking a negative number as a factor. Sometimes this is unavoidable.

Look at $10pq - 5pr - 2sq + sr$ 18.1

When you factorise the first pair of terms of 18.1, the expression becomes:

$$5p(2q - r) - 2sq + sr$$

Look carefully at the last two terms of this expression. To factorise fully we will need the second bracket to be $(2q - r)$. To make this possible, a factor of $-s$ must be taken, giving:

$5p(2q - r) - s(2q - r)$. Notice how $-s \times -r$ gives sr.

The factorising is then completed by

$$(2q - r)(5p - s)$$

To summarise this example:

$$10pq - 5pr - 2sq + sr$$

$$5p(2q - r) - s(2q - r)$$

$$\underline{(2q - r)(5p - s)}$$

FRAME 19

Factorise these expressions:

1. $ax + ay + bx + by$
2. $am - 2al + bm - 2bl$
3. $3zw^2 - y^2t + 3zt - y^2w^2$
4. $4ax - 5bx + 5by - 4ay$

ANSWERS 19

1. $ax + ay + bx + by$

$= a(x + y) + b(x + y)$

$= \underline{(x + y)(a + b)}$

2. $am - 2al + bm - 2bl$

$= a(m - 2l) + b(m - 2l)$

$= \underline{(m - 2l)(a + b)}$

3. $3zw^2 - y^2t + 3zt - y^2w^2$

$= 3zw^2 + 3zt - y^2t - y^2w^2$

$= 3z(w^2 + t) - y^2(t + w^2)$

$= 3z(w^2 + t) - y^2(w^2 + t)$ .... $t + w^2 = w^2 + t$

$= \underline{(w^2 + t)(3z - y^2)}$

4. $4ax - 5bx + 5by - 4ay$

$= x(4a - 5b) - y(-5b + 4a)$

$= x(4a - 5b) - y(4a - 5b)$..... $-5b + 4a = 4a - 5b$

$= \underline{(4a - 5b)(x - y)}$

FRAME 20 QUADRATIC EXPRESSIONS

Expressions containing a term of x^2 are called QUADRATICS in x. These are very important expressions in mathematics as they crop up in many problems involving equations. To solve equations containing quadratic expressions (dealt with in FRAMES 61-75) you will need to know how to factorise these expressions. Generally you will find a quadratic expression has three terms although sometimes it only has one or two terms. A term containing x^2 must be present, however, for the expression to be a quadratic in x. Here are some examples of quadratics. In this case the letter x is being used every time, although any letter would be satisfactory.

$3x^2 + 6$ 20.1

$2x^2 + 3x$ 20.2

$x^2 - 2x + 1$ 20.3

$3x - x^2$ 20.4

$4 + 3x - x^2$ 20.5

$x^2 + 2$ 20.6

$3x + 2 + 5x^2$ 20.7

All these expressions are quadratic because they contain a term with x^2

Notice that 20.1, 20.2, and 20.4 can all be factorised by finding a common factor. 20.3, 20.5, 20.6, and 20.7 are all expressions without a common factor and the procedure described as grouping could not be applied. Indeed, 20.6 and 20.7 cannot be factorised at all into real factors. The purpose of the next frame is to describe ways of factorising quadratic expressions that are possible to factorise. (i.e. 20.3 and 20.5)

FRAME 21

The general quadratic expression x is

$ax^2 + bx + c$ 21.1

where a, b and c are numbers. For the present we will concentrate on the cases where $a = 1$

i.e. $x^2 + bx + c$

b and c can then take any value including 0.

Look at the expression $x^2 + 3x + 2$ 21.2

21.2 has no common factor and there is no obvious way of grouping the expressions as there are only three terms.

If we write the $3x$ in 21.2 as $x + 2x$, 21.2 becomes

$$x^2 + x + 2x + 2$$

This expression can be factorised as shown in FRAMES 15 - 19.

The factorising of 21.2 then looks like this:

$$\begin{aligned} & x^2 + 3x + 2 \\ = \; & x^2 + x + 2x + 2 \\ = \; & x(x + 1) + 2(x + 1) \\ = \; & \underline{(x + 1)(x + 2)} \end{aligned}$$

therefore the factors of 21.2 are $x + 1$ and $x + 2$.

FRAME 22

Notice that in 21.2 $3x$ could have been replaced by $2x + x$ instead

$$\begin{aligned} \text{i.e.} \quad & x^2 + 3x + 2 \\ = \; & x^2 + 2x + x + 2 \\ = \; & x(x + 2) + 1(x + 2) \\ = \; & \underline{(x + 2)(x + 1)} \end{aligned}$$

So the order of replacement does not matter but choosing $2x$ and x is important. Had we tried anything else as a replacement for $3x$ we would not have been able to factorise 21.2. For example, suppose we chose $4x - x$ as a replacement

$$\begin{aligned} & x^2 + 3x + 2 \\ = \; & x^2 + 4x - x + 2 \\ = \; & x(x + 4) + \; ? \end{aligned}$$

It is now impossible to proceed as we are unable to make the second bracket the same as the first (See FRAME 15).

Factorise the following taking the suggested replacements for the middle term:

1. $x^2 + 5x + 6$ .... replace with $3x + 2x$
2. $x^2 + 7x + 12$ .... replace with $4x + 3x$
3. $x^2 + 3x - 18$ .... replace with $6x - 3x$

ANSWERS 22

1.
$$
\begin{aligned}
& x^2 + 5x + 6 \\
= {} & x^2 + 3x + 2x + 6 \\
= {} & x(x + 3) + 2(x + 3) \\
= {} & \underline{(x + 3)(x + 2)}
\end{aligned}
$$

2.
$$
\begin{aligned}
& x^2 + 7x + 12 \\
= {} & x^2 + 4x + 3x + 12 \\
= {} & x(x + 4) + 3(x + 4) \\
= {} & \underline{(x + 4)(x + 3)}
\end{aligned}
$$

3.
$$
\begin{aligned}
& x^2 + 3x - 18 \\
= {} & x^2 + 6x - 3x - 18 \\
= {} & x(x + 6) - 3(x + 6) \\
= {} & \underline{(x + 6)(x - 3)}
\end{aligned}
$$

FRAME 23

The question you should now ask is "How do we know what replacement will work ?" Look at the replacement used in FRAME 22.

With:

$x^2 + 3x + 2$ we used $2x + x$ .... (i)

$x^2 + 5x + 6$ we used $3x + 2x$ .... (ii)

$x^2 + 7x + 12$ we used $4x + 3x$ .... (iii)

$x^2 + 3x - 18$ we used $6x - 3x$ .... (iv)

The replacements have to add up to the middle term of the quadratic. There are numerous pairs of numbers that can do this; only one pair will work in factorising.

The numbers used in the replacement must also be factors of the third term.

In (i) $2x + x$ is used because $2 \times 1 = 2$

In (ii) $3x + 2x$ is used because $3 \times 2 = 6$

In (iii) $4x + 3x$ is used because $4 \times 3 = 12$

In (iv) $6x - 3x$ is used because $6 \times (-3) = -18$

Each time the numbers used multiply to give the third term of the quadratic expression.

FRAME 24

Suppose we wish to factorise:

$$x^2 + 3x - 10.$$

We must replace 3x by a pair that will add up to 3x. There are many possible pairs, for example:

$2x + x$

$4x - x$

$5x - 2x$

$-6x + 9x$ etc.......

Look at the pairs of numbers suggested. We need a pair that will multiply to give -10. Only 5x - 2x will do this as $5 \times (-2) = 10$.

$$\begin{aligned} \text{So} \quad & x^2 + 3x - 10 \\ &= x^2 + 5x - 2x - 10 \\ &= x(x + 5) - 2(x + 5) \\ &= \underline{(x + 5)(x - 2)} \end{aligned}$$

Factorise the following expressions, putting the correct replacement first:

1. $x^2 + 7x + 6$
2. $x^2 + 8x + 12$
3. $x^2 + 2x - 24$
4. $x^2 - 5x - 14$

ANSWERS 24

1. $x^2 + 7x + 6$

$= x^2 + 6x + x + 6$ (As $6 \times 1 = 6$)

$= x(x + 6) + 1(x + 6)$

$= \underline{(x + 6)(x + 1)}$

2. $x^2 + 8x + 12$

$= x^2 + 2x + 6x + 12$ (As $2 \times 6 = 12$)

$= x(x + 2) + 6(x + 2)$

$= \underline{(x + 2)(x + 6)}$

3. $x^2 + 2x - 24$

$= x^2 + 6x - 4x - 24$ (As $6 \times (-4) = -24$)

$= x(x + 6) - 4(x + 6)$

$= \underline{(x + 6)(x - 4)}$

4. $x^2 - 5x - 14$

$= x^2 - 7x + 2x - 14$ (As $(-7) \times 2 = -14$)

$= x(x - 7) + 2(x - 7)$

$= \underline{(x - 7)(x + 2)}$

FRAME 25

You may have noticed by now that the working in the previous questions is not really necessary. Once you have found the numbers that will serve as a replacement to the middle term, these numbers are the ones that appear in the final factor, e.g.:

In $x^2 + 7x + 6$ we used $1x + 6x$ which gave the factors $(x + 1)(x + 6)$

In $x^2 + 3x + 12$ we used $2x + 6x$ which gave the factors $(x + 2)(x + 6)$

In $x^2 + 2x - 24$ we used $6x - 4x$ which gave the factors $(x + 6)(x - 4)$

In $x^2 - 5x - 14$ we used $-7x + 2x$ which gave the factors $(x - 7)(x + 2)$

It is possible to state your answers without any working.

Try to do this with the following questions:

1. $x^2 + 7x + 10$
2. $x^2 + 18x + 81$
3. $x^2 - 9x + 18$
4. $x^2 - 4x - 21$

ANSWERS 25

1. $(x + 2)(x + 5)$
2. $(x + 9)(x + 9)$
3. $(x - 3)(x - 6)$
4. $(x + 3)(x - 7)$

FRAME 26

In FRAME 21 we began looking at quadratics of the form $x^2 + bx + c$ (i.e. where $a = 1$). We will now look at how to factorise expressions where $a \neq 1$.

Consider $2x^2 - 5x - 3$ 26.1

This time the replacement of $-5x$ in 26.1 has to be made by two numbers that multiply to give not just the third term -3 but the number -6 obtained from $2 \times (-3)$. So first multiply the numbers in positions a and c, i.e. $2 \times (-3) = -6$.

You then need to look for a replacement of $-5x$ using numbers that are factors of -6.

$-6x + x$ replaces $-5x$ as $-6 \times 1 = -6$.

To factorise 26.1 we would take these steps.

$$\begin{aligned} & 2x^2 - 5x - 3 \\ = \; & 2x^2 - 6x + x - 3 \\ = \; & 2x(x - 3) + 1(x - 3) \\ = \; & \underline{(x - 3)(2x + 1)} \end{aligned}$$

FRAME 27

Let us look again at factorising a general quadratic, e.g.:

$$6x^2 + 17x + 5$$

The replacement of $17x$ has to use numbers that are factors of 30 as $6 \times 5 = 30$.

So $17x$ is replaced by $15x + 2x$ as $15 \times 2 = 30$.

Therefore

$$\begin{aligned} & 6x^2 + 17x + 5 \\ = \; & 6x^2 + 15x + 2x + 5 \\ = \; & 3x(2x + 5) + 1(2x + 5) \\ = \; & \underline{(2x + 5)(3x + 1)} \end{aligned}$$

Take as another example:

$$\begin{aligned} & 6x^2 - x - 1 \\ = \; & 6x^2 + 2x - 3x - 1 \quad (\text{as } 6 \times (-1) = -6 \text{ and } 2 \times (-3) = -6) \\ = \; & 2x(3x + 1) - 1(3x + 1) \\ = \; & \underline{(3x + 1)(2x - 1)} \end{aligned}$$

Notice that in this case there is no simplification of the working as in the case where the coefficent of x^2 was one (See FRAME 25).

Factorise the following:

1. $3x^2 + 5x + 2$
2. $5x^2 - 11x + 2$
3. $7x^2 - 3x - 4$

ANSWERS 27

1. $$\begin{aligned} & 3x^2 + 5x + 2 && (3 \times 2 = 6) \\ &= 3x^2 + 2x + 3x + 2 && (2 \times 3 = 6) \\ &= x(3x + 2) + 1(3x + 2) \\ &= \underline{(3x + 2)(x + 1)} \end{aligned}$$

2. $$\begin{aligned} & 5x^2 - 11x + 2 && (5 \times 2 = 10) \\ &= 5x^2 - 10x - x + 2 && ((-10) \times (-1) = 10) \\ &= 5x(x - 2) - 1(x - 2) \\ &= \underline{(x - 2)(5x - 1)} \end{aligned}$$

3. $$\begin{aligned} & 7x^2 - 3x - 4 && (7 \times (-4) = -28) \\ &= 7x^2 - 7x + 4x - 4 && ((-7) \times 4 = -28) \\ &= 7x(x - 1) + 4(x - 1) \\ &= \underline{(x - 1)(7x + 4)} \end{aligned}$$

FRAME 28 DIFFERENCE OF TWO SQUARES

A special case of quadratics is where the middle term is zero and the remaining two numbers are squares subtracted.

e.g. (i) $x^2 - 9$

(ii) $x^2 - 25$

(iii) $4x^2 - 49$

(iv) $9a^2 - 4b^2$

All these are known as the DIFFERENCE OF TWO SQUARES.

Parts (i) and (ii) can be factorised as described in FRAMES 23-25.

(i)

$$= x^2 + 0x - 9$$

$$= x^2 + 3x - 3x - 9 \quad (\text{As } 3 \times (-3) = -9)$$

$$= x(x + 3) - 3(x + 3)$$

$$= \underline{(x + 3)(x - 3)}$$

(ii) $x^2 - 25$

$$= x^2 + 5x - 5x - 25 \quad (\text{As } 5 \times (-5) = -25)$$

$$= x(x + 5) - 5(x + 5)$$

$$= \underline{(x + 5)(x - 5)}$$

Part (iii) can be factorised as described in FRAMES 26-27.

(iii) $4x^2 - 49$

$$= 4x^2 + 0x - 49 \quad (4 \times (-49) = -196)$$

$$= 4x^2 + 14x - 14x - 49 \quad (\text{As } 14 \times (-14) = -196)$$

$$= 2x(2x + 7) - 7(2x + 7)$$

$$= \underline{(2x + 7)(2x - 7)}$$

All these examples (i) - (iii) could be factorised immediately by noticing that each expression is made from squared terms, i.e.:

$$x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$$

$$x^2 - 25 = x^2 - 5^2 = (x + 5)(x - 5)$$

$$4x^2 - 49 = (2x)^2 - 7^2 = (2x + 7)(2x - 7)$$

Notice that each time the factors are made from the sum and difference of the terms that are squared. So:

(iv) $9a^2 - 4b^2$

$= (3a)^2 - (2b)^2$

$= \underline{(3a + 2b)(3a - 2b)}$

FRAME 29

Any expression that can be written as the difference of two numbers squared can be factorised as the sum and the difference of the two numbers. In the following expressions write them as the difference of two squared numbers and then factorise:

1. $x^2 - 16$
2. $p^2 - 25q^2$
3. $16a^2 - 25b^2$
4. $4p^2 - q^2r^2$
5. $100a^2 - 81b^2$

ANSWERS 29

1. $x^2 - 16$

   $= x^2 - 4^2$

   $= \underline{(x + 4)(x - 4)}$

2. $p^2 - 25q^2$

   $= p^2 - (5q)^2$

   $= \underline{(p + 5q)(p - 5q)}$

3. $16a^2 - 25b^2$

   $= (4a)^2 - (5b)^2$

   $= \underline{(4a + 5b)(4a - 5b)}$

4. $4p^2 - q^2r^2$

   $= (2p)^2 - (qr)^2$

   $= \underline{(2p + qr)(2p - qr)}$

5. $100a^2 - 81b^2$

$= (10a)^2 - (9b)^2$

$= \underline{(10a + 9b)(10a - 9b)}$

FRAME 30 SUMMARY

When factorising an expression remember to carry out these steps:

1. Check for common factor first.

2. Factorise the remaining expression if possible. It could involve: GROUPING

QUADRATICS

DIFFERENCE OF TWO SQUARES.

FRAME 1	Examples of numerical shortcuts in arithmetic.
FRAME 2	Definition of a FACTOR.
FRAME 3	To factorise a number.
FRAME 4	PRIME numbers.
FRAME 5	PRIME Factors.
FRAME 6	Consolidation.
FRAME 7	Factorising algebraic terms of the type a^2b.
FRAMES 8 - 14	Common Factors.
FRAMES 15- 19	Factorising by Grouping.
FRAMES 20- 27	Factorising Quadratics.
FRAMES 28- 29	Difference of Two Squares.

SECTION 2

Simple Equations

FRAME 31 SIMPLE EQUATIONS

Can you find three consecutive odd numbers which add up to 39? You might try to do this by guesswork until you find three numbers that work. However, this could take a long time.

This is an example of a problem which can be solved using SIMPLE EQUATIONS (sometimes called LINEAR EQUATIONS).

Here is another example. Imagine you are offered a job where you are paid £5 for each day you work but fined £2.50 for each day you fail to work. After 20 days you are paid £70. Find how many days you had worked.

Again you may try to do this by guesswork.

Before we return to this and other SIMPLE PROBLEMS (they are not all so simple!!) we will look at SIMPLE EQUATIONS and the methods of solving them.

FRAME 32 THE SOLUTION OF A SIMPLE EQUATION

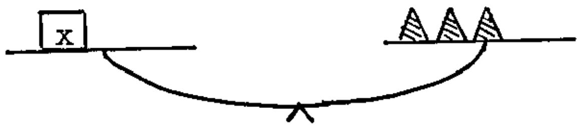
Think of an equation as a balance with the equals sign as the point of the balance.

Here $x = 3$ .... Eq. 32.1

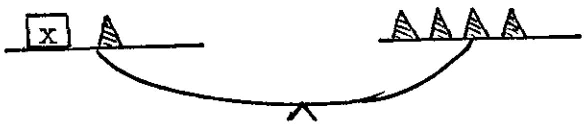
An equation in this form is said to be a SOLUTION as it gives the value of the unknown.

FRAME 33

Look at Eq. 32.1 again. This equation can be represented by



If we add a  to each side the balance should be maintained.



This is represented by

$x + 1 = 4$ Eq.33.1

Equation 33.1 is said to be EQUIVALENT to 32.1 as $x + 1 = 4$ can be obtained from $x = 3$ by adding 1 to each side.

In the same way

$x + 3 = 6$ (adding 3)

$x + 4 = 7$ (adding 4)

$x - 1 = 2$ (adding -1)

are all equivalent to $x = 3$.

FRAME 34

This diagram shows $x = 2$.

If we make each side of the balance three times as big the result will look like this

This represents the equation $3x = 6$.

So $3x = 6$ and $x = 2$ are equivalent to one another.

In the same way

$$2x = 4 \qquad \text{(multiplying by } 2\text{)}$$

$$6x = 12 \qquad \text{(multiplying by } 6\text{)}$$

$$\frac{1}{3}x = \frac{2}{3} \qquad \text{(multiplying by } \frac{1}{3}\text{)}$$

are all equivalent to $x = 2$.

FRAME 35

To solve an equation you must end up with an equation in solution form (See FRAME 32).

For example, look at

$x - 3 = 12$ .... Example 35.1

This can be solved by adding 3 to both sides. This is shown by the following working:

$x - 3 = 12$ .... (1)

$x - 3 + 3 = 12 + 3$ .... (2) (+3)

$\underline{x = 15}$ .... (3)

(+3) is placed at the end of (2) to show the reader what has happened. You may prefer to miss out (2) altogether giving

$x - 3 = 12$

$\underline{x = 15}$ (+3)

Always underline a solution equation.

Look at the next example:

$x + 5 = 17$ .... Example 35.2

$x + 5 - 5 = 17 - 5$ (-5)

$\underline{x = 12}$

Notice here we added -5 to both sides.

FRAME 36

Each time you finish a solution you can check to see your answer is true by using the original equation.

In Example 35.1 $x - 3 = 12$

$x = 15$ was the solution.

The check is to replace x by 15

i.e. $15 - 3 = 12$.

In Example 35.2 $x + 5 = 17$

$x = 12$ was the solution.

To check this replace x by 12

$12 + 5 = 17$.

Solve the following equations showing your working as in FRAME 35 and then check your answers.

1. $x + 2 = 5$
2. $x - 13 = 12$
3. $x + 14 = 20$
4. $x - 2 = 12$
5. $x + 3 = 2$

ANSWERS 36

1.
$$x + 2 = 5$$
$$x + 2 - 2 = 5 - 2 \qquad (-2)$$
$$\underline{x = 3}$$

Check: $3 + 2 = 5$

2.
$$x - 13 = 12$$
$$x - 13 + 13 = 12 + 13 \qquad (+13)$$
$$\underline{x = 25}$$

Check: $25 - 13 = 12$

3.
$$x + 14 = 20$$
$$x + 14 - 14 = 20 - 14 \qquad (-14)$$
$$\underline{x = 6}$$

Check: $6 + 14 = 20$

4.
$$x - 2 = 12$$
$$x - 2 + 2 = 12 + 2 \qquad (+2)$$
$$\underline{x = 14}$$

Check: $14 - 2 = 12$

5.
$$x + 3 = 2$$
$$x + 3 - 3 = 2 - 3 \qquad (-3)$$
$$\underline{x = -1}$$

Check: $-1 + 3 = 2$

FRAME 37

Another form of equation you need to be able to solve is given below:

$$3x = 12 \qquad \text{........... Eq. 37.1}$$

In this case adding a number to each side would not achieve anything. For example, suppose we add -3 to each side of 37.1. This would give $3x - 3 = 9$.

Notice that this does not solve the equation.

Suppose we take Eq. 37.1 and multiply each side by $\frac{1}{3}$

$$3x = 12$$
$$\frac{1}{3} \times 3x = \frac{1}{3} \times 12$$
$$\underline{x = 4}$$

Check: $3 \times 4 = 12$

Notice that to remove the 3 from Eq. 37.1 we multiplied by $\frac{1}{3}$

Look at another example:

$$5x = 15 \quad \ldots\ldots\ldots\ldots\ldots \text{Eq. } 37.2$$

In this case we wish to remove the 5.

To do this we must multiply by $\frac{1}{5}$

i.e. $5x = 15$

$$\frac{1}{5} \times 5x = \frac{1}{5} \times 15$$

$$\underline{x = 3}$$

Check: $5 \times 3 = 15$

Solve these equations in the same way:

1. $2x = 8$
2. $4x = 16$
3. $7x = 56$
4. $6x = 42$

<u>ANSWERS 37</u>

1. $2x = 8$

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 8$$

$$\underline{x = 4}$$

Check: $2 \times 4 = 8$

2. $4x = 16$

$$\frac{1}{4} \times 4x = \frac{1}{4} \times 16$$

$$\underline{x = 4}$$

Check: $4 \times 4 = 16$

3. $7x = 56$

$$\frac{1}{7} \times 7x = \frac{1}{7} \times 56$$

$$\underline{x = 8}$$

Check: $7 \times 8 = 56$

4. $6x = 42$

$$\frac{1}{6} \times 6x = \frac{1}{6} \times 42$$

$$\underline{x = 7}$$

Check: $6 \times 7 = 42$

FRAME 38

$$\frac{x}{4} = 20 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{Eq. } 38.1$$

In the case of Eq. 38.1 to remove the 4 you multiply both sides by 4.

i.e.

$$\frac{x}{4} = 20$$

$$4 \times \frac{x}{4} = 4 \times 20$$

$$\underline{x = 80}$$

Check: $\frac{80}{4} = 20$

Try these:

1. $\frac{x}{3} = 4$ 2. $\frac{x}{6} = 12$ 3. $\frac{x}{7} = 3$

ANSWERS 38

1.

$$\frac{x}{3} = 4$$

$$3 \times \frac{x}{3} = 3 \times 4$$

$$\underline{x = 12}$$

Check: $\frac{12}{3} = 4$

2.

$$\frac{x}{6} = 12$$

$$6 \times \frac{x}{6} = 6 \times 12$$

$$\underline{x = 72}$$

Check: $\frac{72}{6} = 12$

3. $$\frac{x}{7} = 3$$

$$7 \times \frac{x}{7} = 7 \times 3$$

$$\underline{x = 21}$$

Check: $\frac{21}{7} = 3$

FRAME 39

Suppose we have an equation of the form:

$\frac{3x}{4} = 9$ .... Eq. 39.1

Eq. 39.1 can be solved by using both the methods of FRAMES 38 and 37.

$$\frac{3x}{4} = 9$$

$$4 \times \frac{3x}{4} = 4 \times 9 \qquad \text{(using FRAME 38)}$$

$$3x = 36$$

$$\frac{1}{3} \times 3x = \frac{1}{3} \times 36 \qquad \text{(using FRAME 37)}$$

$$\underline{x = 12}$$

Check: $\frac{3}{4} \times 12 = 9$

Solve the following equations:

1. $\frac{2x}{3} = 8$ 2. $\frac{4x}{7} = 28$ 3. $\frac{2x}{5} = 7$

ANSWERS 39

1. $$\frac{2x}{3} = 8$$

$$3 \times \frac{2x}{3} = 3 \times 8$$

$$2x = 24$$

$$\frac{1}{2} \times 2x = \frac{1x}{2} \times 24$$

$$\underline{x = 12}$$

Check: $\frac{2}{3} \times 12 = 8$

2. $$\frac{4x}{7} = 28$$

$$7 \times \frac{4x}{7} = 7 \times 28$$

$$4x = 196$$

$$\frac{1}{4} \times 4x = \frac{1}{4} \times 196$$

$$\underline{x = 49}$$

Check: $\frac{4}{7} \times 49 = 28$

3. $$\frac{2x}{5} = 7$$

$$5 \times \frac{2x}{5} = 5 \times 7$$

$$2x = 35$$

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 35$$

$$\underline{x = 17\tfrac{1}{2}}$$

Check: $\frac{2}{5} \times 17\frac{1}{2} = 7$

FRAME 40

If we combine the ideas introduced in FRAMES 35 and 37 you will see how to solve equations of the form:

$$3x - 2 = 7 \quad \ldots\ldots\ldots\ldots\ldots \text{ Eq. 40.1}$$

In Eq. 40.1 we first remove -2 in the same way as introduced in FRAME 35 (adding 2 to both sides)

$$3x - 2 + 2 = 7 + 2$$

$$3x = 9$$

Now we can remove the 3 in the same way as shown in FRAME 37 (multiplying both sides by $\frac{1}{3}$)

$$3x = 9$$

$$\frac{1}{3} \times 3x = \frac{1}{3} \times 9$$

$$\underline{x = 3}$$

Check: $3 \times 3 - 2 = 9 - 2$

$= 7$

The position of the unknown x (or any other letter) is not important. What is important is the way the numbers are removed so that the equation is left in solved form. Let us look at another example:

$$5 = 2 - 3x$$

$$5 - 2 = 2 - 3x - 2 \qquad \text{(add } -2 \text{ to each side)}$$

$$3 = -3x$$

$$-\frac{1}{3} \times 3 = -\frac{1}{3} \times -3x \qquad \text{(multiply } -\frac{1}{3} \text{ to each side)}$$

$$\underline{-1 = x}$$

Check: $2 - 3(-1)$

$= 2 + 3$

$= 5$

FRAME 41

Here are some examples:

Example 41.1

$$4x + 2 = 10$$

$$4x + 2 - 2 = 10 - 2 \qquad \text{(add } -2\text{)}$$

$$4x = 8$$

$$\frac{1}{4} \times 4x = \frac{1}{4} \times 8 \qquad (\times \frac{1}{4})$$

$$\underline{x = 2}$$

Check: $4 \times 2 + 2$

$= 8 + 2$

$= 10$

Example 41.2

$$7y - 2 = 26$$

$$7y - 2 + 2 = 26 + 2 \qquad \text{(add } 2\text{)}$$

$$7y = 28$$

$$\frac{1}{7} \times 7y = \frac{1}{7} \times 28 \qquad (\times \frac{1}{7})$$

$$\underline{y = 4}$$

Example 41.3

$$5 = 3 + 2x$$

$$5 - 3 = 3 + 2x - 3 \qquad \text{(add } -3\text{)}$$

$$2 = 2x$$

$$\frac{1}{2} \times 2 = \frac{1}{2} \times 2x \qquad (\times \tfrac{1}{2})$$

$$\underline{1 = x}$$

Check: $3 + 2x \times 1$

$= 3 + 2$

$= 5$

Example 41.4

$$15 - 3x = 9$$

$$15 - 3x - 15 = 9 - 15 \qquad \text{(add } -15\text{)}$$

$$-3x = -6$$

$$-\frac{1}{3} \times -3x = -\frac{1}{3} \times -6 \qquad (\times -\tfrac{1}{3})$$

$$\underline{x = 2}$$

Check: $15 - 3(2)$

$= 15 - 6$

$= 9$

Example 41.5

$$4 + \frac{2x}{3} = 18$$

$$4 + \frac{2x}{3} - 4 = 18 - 4 \qquad \text{(add } -4\text{)}$$

$$\frac{2x}{3} = 14$$

$$3 \times \frac{2x}{3} = 3 \times 14 \qquad (\times 3)$$

$$2x = 42$$

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 42 \qquad (\times \tfrac{1}{2})$$

$$\underline{x = 21}$$

Check: $4 + \frac{2}{3} \times 21$

$= 4 + 2 \times 7$

$= 4 + 14$

$= 18$

FRAME 42

Here are some problems for you to try. Remember the letter that is used is not important and can vary from question to question.

1. $3x + 5 = 11$	2. $9x - 5 = 22$
3. $5a - 26 = 54$	4. $3m + 15 = 27$
5. $1 + 7x = 8$	6. $3 - 2a = 1$
7. $\frac{2b}{3} = 4$	8. $\frac{3x}{2} - 1 = 2$
9. $14 = 2 - 3x$	10. $\frac{3x}{4} - 7 = 1$

ANSWERS 42

1. $x = 2$	2. $x = 3$
3. $a = 16$	4. $m = 4$
5. $x = 1$	6. $a = 1$
7. $b = 6$	8. $x = 2$
9. $x = -4$	10. $x = 10\frac{2}{3}$

FRAME 43

Look carefully at the following equation:

$$5x + 4 = 16 - 3x \quad \ldots\ldots\ldots\ldots\ldots \text{Eq. } 43.1$$

The difference between this equation and all the others you have seen is that the unknown (in this case x) occurs twice. When an equation looks like this the first step is to try to get the terms involving the unknown together. In the case of Eq. 43.1, by adding 3x to both sides the x's will disappear from the right hand side. By adding -5x to both sides the x's will disappear from the left hand side. Either of these would collect the terms containing x together. Trying the first way, we get the following:

$$5x + 4 = 16 - 3x$$

$$5x + 4 + 3x = 16 - 3x + 3x \qquad (+3x)$$

$$8x + 4 = 16$$

Now the equation looks the same as those described in FRAMES 40 and 41.

Continuing:

$$8x + 4 = 16$$

$$8x + 4 - 4 = 16 - 4 \qquad (\text{add } -4)$$

$$8x = 12$$

$$\frac{1}{8} \times 8x = \frac{1}{8} \times 12 \qquad (\times \frac{1}{8})$$

$$\underline{x = 1\tfrac{1}{2}}$$

Notice from now on the checking will become more complicated but probably more necessary.

Check: Left-hand side (L.H.S.) $5x + 4$

$$= 5 \times 1\tfrac{1}{2} + 4$$

$$= 7\tfrac{1}{2} + 4$$

$$= 11\tfrac{1}{2}$$

Right-hand side (R.H.S.) $16 - 3 \times 1\tfrac{1}{2}$

$$= 16 - 4\tfrac{1}{2}$$

$$= 11\tfrac{1}{2}$$

<u>FRAME 44</u>

Here are some examples like the problem of FRAME 43:

<u>Example 44.1</u>

$$10x + 3 = 4x + 21$$

$$10x + 3 - 4x = 4x + 21 - 4x \qquad (-4x)$$

$$6x + 3 = 21$$

$$6x + 3 - 3 = 21 - 3 \qquad (-3)$$

$$6x = 18$$

$$\frac{1}{6} \times 6x = \frac{1}{6} \times 18 \qquad (\times \tfrac{1}{6})$$

$$\underline{x = 3}$$

Check: L.H.S. $10 \times 3 + 3$

$$= 30 + 3$$

$$= \underline{33}$$

R.H.S. $4 \times 3 + 21$

$$= 12 + 21$$

$$= \underline{33}$$

<u>Example 44.2</u>

$$9x - 1 = 3x + 13$$

$$9x - 11 - 3x = 3x + 13 - 3x \qquad (-3x)$$

$$6x - 11 = 13$$

$$6x - 11 + 11 = 13 + 11 \qquad (+11)$$

$$6x = 24$$

$$\frac{1}{6} \times 6x = \frac{1}{6} \times 24 \qquad (\times \tfrac{1}{6})$$

$$\underline{x = 4}$$

Check: L.H.S. $9 \times 4 - 11$

$= 36 - 11$

$= \underline{25}$

R.H.S. $3 \times 4 + 13$

$= 12 + 13$

$= \underline{25}$

Example 44.3

$$3 - x = 14 + 10x$$

$$3 - x + x = 14 + 10x + x \qquad (+x)$$

$$3 = 14 + 11x$$

$$3 - 14 = 14 + 11x - 14 \qquad (-14)$$

$$-11 = 11x$$

$$\frac{1}{11} \times (-11) = \frac{1}{11} \times 11x \qquad (\times \frac{1}{11})$$

$$\underline{-1 = x}$$

Check: L.H.S. $3 - (-1)$

$= 3 + 1$

$= \underline{4}$

R.H.S. $14 + 10(-1)$

$= 14 - 10$

$= \underline{4}$

Here are some problems for you to try:

1. $9x + 1 = 5x + 9$ 2. $8x - 12 = 5x - 3$
3. $17x - 40 = 8x - 13$ 4. $4 - 2x = 19 + 3x$

ANSWERS 44

1. $x = 2$ 2. $x = 3$ 3. $x = 3$ 4. $x = -3$

FRAME 45

Look back to Examples 41.1 and 41.4 and see how they were solved. Both of these examples could be solved differently by noticing that each side of the equation had a common factor,

e.g. $4x + 2 = 10$

The left hand side will factorise to give

$$2(2x + 1) = 10$$

It is possible to multiply both sides by $\frac{1}{2}$

$$\frac{1}{2} \times 2(2x + 1) = \frac{1}{2} \times 10$$

$$2x + 1 = 5$$

This then solves to give

$$2x = 4 \qquad (-1)$$

$$\underline{x = 2} \qquad (\times \tfrac{1}{2})$$

Although this method has saved little work in this example, using factors can make the numerical work easier.

Take Example 44.4:

$$15 - 3x = 9$$

$$3(5 - x) = 9$$

$$\frac{1}{3} \times 3(5 - x) = \frac{1}{3} \times 9 \qquad (\times \tfrac{1}{3})$$

$$5 - x = 3$$

$$5 - x - 5 = 3 - 5 \qquad (-5)$$

$$-x = -2$$

$$(-1) \times (-x) = (-1) \times (-2) \qquad (\times (-1))$$

$$\underline{x = 2}$$

FRAME 46

Sometimes an equation will be given with parts already factorised. If the brackets 'are in the way' they must be multiplied out first before solving.

i.e. $10(x + 1) = 7(x + 4)$ 46.1

In this case 46.1 has had both sides factorised but no common factor exists on each side. In this case 46.1 should be multiplied out and then the equation will look like those described in FRAME 44.

$$
\begin{aligned}
10x + 10 &= 7x + 28 & & \\
3x + 10 &= 28 & &(-7x) \\
3x &= 18 & &(-10) \\
\underline{x} &\underline{= 6} & &(\times \tfrac{1}{3})
\end{aligned}
$$

Check: L.H.S. $10 \times 6 + 10$
$= 60 + 10$
$= \underline{70}$

R.H.S. $7 \times 6 + 28$
$= 42 + 28$
$= \underline{70}$

Notice in this case we have missed out some lines of working, the statements on the right being sufficient. This will speed up your solution of equations.

Remember to look for a common factor BEFORE you multiply the brackets out.

For example:

$$8(x + 1) = 2(x + 16)$$

Here there is a common factor of 2.

So multiply by $\frac{1}{2}$ before you remove the brackets.

$$\frac{1}{2} \times 8(x + 1) = \frac{1}{2} \times 2(x + 16)$$

$$4(x + 1) = x + 16$$

$$4x + 4 = x + 16$$

$$3x + 4 = 16 \qquad (-x)$$

$$3x = 12 \qquad (-4)$$

$$\underline{x = 4} \qquad (\times \tfrac{1}{3})$$

Check: L.H.S. $8(4 + 1)$

$= 8 \times 5$

$= \underline{40}$

R.H.S. $2(4 + 16)$

$= 2 \times 20$

$= \underline{40}$

Try these:

1. $5(x + 4) = 3(x + 12)$ 2. $9(1 - x) = 8(x - 1)$
3. $20(x - 3) = 16(x + 2)$ 4. $2(x - 1) = 33-3(2x + 1)$

<u>ANSWERS 46</u>

1. $x = 8$ 2. $x = 1$ 3. $x = 23$
4. $x = 4$

<u>FRAME 47</u>

Look at the equation below:

$$\frac{x + 2}{5} = \frac{x + 8}{2} \quad \ldots\ldots\ldots\ldots\ldots \quad 47.1$$

This equation involves fractions. In these cases it is best to remove the fractions first. In 47.1 this can be done by multiplying both sides by 10:

$$\overset{2}{\cancel{10}} \times \frac{(x + 2)}{\underset{1}{\cancel{5}}} = \overset{5}{\cancel{10}} \times \frac{(x + 8)}{\underset{1}{\cancel{2}}}$$

$$2(x + 2) = 5(x + 8)$$

This equation now looks like those described in FRAME 46.

10 was chosen as it is the smallest number of which 5 and 2 are both factors.

Notice also that brackets were introduced when each side was multiplied by 10.

Let us look at another example:

$$\frac{2x + 3}{8} = \frac{x - 2}{12}$$

The smallest number with 8 and 12 as factors is 24. So we multiply both sides by 24:

$$\overset{3}{\cancel{24}} \times \frac{(2x + 3)}{\underset{1}{\cancel{8}}} = \overset{2}{\cancel{24}}\frac{(x - 2)}{\underset{1}{\cancel{12}}}$$

$$3(2x + 3) = 2(x - 2)$$

This is then solved as described in FRAME 46.

FRAME 48

Here are some worked examples:

Example 48.1

$$\frac{x - 5}{5} = \frac{x - 2}{8}$$

$$\overset{8}{\cancel{40}} \times \frac{(x - 5)}{\underset{1}{\cancel{5}}} = \overset{5}{\cancel{40}} \times \frac{(x - 2)}{\underset{1}{\cancel{8}}} \qquad (\times 40)$$

$$8(x - 5) = 5(x - 2)$$

$$8x - 40 = 5x - 10$$

$$3x - 40 = -10 \qquad (-5x)$$

$$3x = 30 \qquad (+40)$$

$$\underline{x = 10} \qquad (\times \tfrac{1}{3})$$

Check: L.H.S. $\dfrac{10 - 5}{5}$

$= \dfrac{5}{5}$

$= \underline{1}$

R.H.S. $\dfrac{10 - 2}{8}$

$= \dfrac{8}{8}$

$= \underline{1}$

Example 48.2

$$\frac{3x - 8}{4} = \frac{2x - 3}{5}$$

$$\overset{5}{\cancel{20}} \times \frac{(3x - 8)}{\underset{1}{\cancel{4}}} = \overset{4}{\cancel{20}} \times \frac{(2x - 3)}{\underset{1}{\cancel{5}}} \qquad (\times 20)$$

$$5(3x - 8) = 4(2x - 3)$$

$$15x - 40 = 8x - 12$$

$$7x - 40 = -12 \qquad (-8x)$$

$$7x = 28 \qquad (+40)$$

$$\underline{x = 4} \qquad (\times \tfrac{1}{7})$$

Check: L.H.S. $\dfrac{3 \times 4 - 8}{4}$

$= \dfrac{12 - 8}{4}$

$= \dfrac{4}{4}$

$= \underline{1}$

R.H.S. $\dfrac{2 \times 4 - 3}{5}$

$= \dfrac{8 - 3}{5}$

$= \dfrac{5}{5}$

$= \underline{1}$

Example 48.3

$$\frac{x - 2}{6} = \frac{4x + 1}{8}$$

$$\overset{4}{\cancel{24}} \times \left(\frac{x - 2}{\underset{1}{\cancel{6}}}\right) = \overset{3}{\cancel{24}} \times \left(\frac{4x + 1}{\underset{1}{\cancel{8}}}\right) \qquad (\times 24)$$

$$4(x - 2) = 3(4x + 1)$$

$$4x - 8 = 12x + 3$$

$$-8 = 8x + 3 \qquad (-4x)$$

$$-11 = 8x \qquad (-3)$$

$$\underline{-1\tfrac{3}{8} = x} \qquad (\times \tfrac{1}{8})$$

Check: L.H.S. $\dfrac{-1\frac{3}{8} - 2}{6}$

$= \dfrac{-3\frac{3}{8}}{6}$

$= \underline{\dfrac{-9}{16}}$

R.H.S. $\dfrac{4(-1\frac{3}{8}) + 1}{8}$

$= \dfrac{\frac{-11}{2} + 1}{8}$

$= \underline{\dfrac{-9}{16}}$

Try these questions:

1. $\dfrac{x - 5}{7} = \dfrac{x - 3}{9}$ 2. $\dfrac{3x - 5}{5} = \dfrac{5x - 1}{12}$

3. $\dfrac{x - 2}{9} = \dfrac{2x + 5}{12}$ 4. $\dfrac{4x - 5}{7} = \dfrac{5x - 4}{11}$

ANSWERS 48

1. $x = 12$ 2. $x = 5$ 3. $x = -11\frac{1}{2}$

4. $x = 3$

FRAME 49

Here are some further examples, with more than one fraction on each side:

$$\frac{3x - 2}{8} + \frac{2}{3} = \frac{2 - 5x}{4} \quad \text{...........} \quad \text{Eq. 49.1}$$

In this case we must find a number that has 8, 3 and 4 as factors. The smallest number is 24. By multiplying both sides by 24 all the denominators should disappear.

$$\overset{3}{\cancel{24}} \times \frac{(3x - 2)}{{}_1\cancel{8}} + \overset{8}{\cancel{24}} \times \frac{2}{{}_1\cancel{3}} = \overset{6}{\cancel{24}} \times \frac{(2 - 5x)}{{}_1\cancel{4}}$$

$$\begin{aligned}
3(3x - 2) + 16 &= 6(2 - 5x) \\
9x - 6 + 16 &= 12 - 30x \\
9x + 10 &= 12 - 30x \\
39x + 10 &= 12 && (+ 30x) \\
39x &= 2 && (- 10) \\
x &= \frac{2}{39} && (\times \tfrac{1}{39})
\end{aligned}$$

In this example each step is numbered:

$$\frac{3}{4} - \frac{2(x + 1)}{5} = \frac{(2x - 9)}{2}$$

$$\text{1.} \quad \overset{5}{\cancel{20}} \times \frac{3}{{}_1\cancel{4}} - \overset{4}{\cancel{20}} \times \frac{2(x + 1)}{{}_1\cancel{5}} = \overset{10}{\cancel{20}} \times \frac{(2x - 9)}{{}_1\cancel{2}} \quad (\times 20)$$

$$\begin{aligned}
& & 15 - 8(x + 1) &= 10(2x - 9) \\
&\text{2.} & 15 - 8x - 8 &= 20x - 90 \\
& & 7 - 8x &= 20x - 90 \\
&\text{3.} & 7 &= 28x - 90 && (+ 8x) \\
&\text{4.} & 97 &= 28x && (+ 90) \\
&\text{5.} & \frac{97}{28} &= x && (\times \tfrac{1}{28}) \\
& & 3\tfrac{13}{28} &= x
\end{aligned}$$

FRAME 50

If you look carefully at the last example of FRAME 49 you will see that the important steps of solving the equation have been numbered. Here is the list with a description of each one:

1. Multiply by the least number to remove all denominators.
2. Multiply the brackets out.
3. Collect the terms containing the unknown together.
4. Collect the numbers to the other side.
5. Remove the number in front of the unknown (known as a coefficient).

These five steps applied in order will solve any simple equation.

FRAME 51 MIXED EXAMPLES

1. $x + 3 = 8$
2. $3x = 27$
3. $4a + 2 = 6$
4. $9b - 5 = 22$
5. $6g - 5 = 4g + 7$
6. $a + 2(a + 1) = 8$
7. $2(x - 1) = 33 - 3(2x + 1)$
8. $\frac{x - 4}{3} = \frac{x - 11}{10}$
9. $\frac{2x}{3} = 4$
10. $\frac{4a}{3} - \frac{1}{2} = - \frac{9}{2}$
11. $\frac{2(x + 3)}{3} + \frac{3}{4} = 4(1 - x)$
12. $\frac{2 - x}{3} = 24 - \frac{6(3x + 1)}{7}$

ANSWERS 51

1.	$x = 5$	2.	$x = 9$
3.	$a = 1$	4.	$b = 3$
5.	$g = 6$	6.	$a = 2$
7.	$x = 4$	8.	$x = 1$
9.	$x = 6$	10.	$a = -3$
11.	$x = \frac{15}{56}$	12.	$x = 10\frac{2}{47}$

FRAME 52

Look at the original problems described in FRAME 31. In the first one we need three consecutive odd numbers that add up to 39.

Let the first number be x.

The next odd number would be found by adding 2, so the next odd number is $x + 2$. In the same way the next odd number is $x + 4$. So the three odd numbers are:

$$x, \quad x + 2, \quad x + 4$$

They add up to 39

Therefore
$$\begin{aligned} x + (x + 2) + (x + 4) &= 39 \\ x + x + 2 + x + 4 &= 39 \\ 3x + 6 &= 39 \\ 3x &= 33 \quad (-6) \\ x &= 11 \quad (\times \tfrac{1}{3}) \end{aligned}$$

So the numbers are

11, 13, 15

FRAME 53

In the other problem from FRAME 31 let the number of days worked be represented by x. Therefore the number of days you did not work would be 20 - x.

Your pay for days worked would be x X £5.

Your fine for days not worked would be (20 - x) X £2.5

As you earn £70 this means:

$$x \times £5 - (20 - x) \times £2.5 = £70$$
$$5x - 2.5(20 - x) = 70$$
$$5x - 50 + 2.5x = 70$$
$$7.5x - 50 = 70$$
$$7.5x = 120 \quad (+ 50)$$
$$x = 16 \quad (\times \tfrac{1}{7.5})$$

Therefore you worked for 16 days.

FRAME 54

To solve a simple problem you must convert the problem to a simple equation. Then once you have solved the equation you must state the answer to the problem. Here is a suggested list of steps to take:

1. Let a letter stand for the item which has to be found (or something connected to it), stating clearly the units (if any) to be used.
2. From the question try to write each statement in the problem in terms of the unknown.
3. Using these statements make an equation and solve it.
4. Answer the question as a statement.
5. Check the answer from the facts given in the question.

FRAME 55

Here are some examples:

Example 55.1

A certain number is multiplied by 4, and 9 is then added to the obtained product giving a result of 33. Find the number.

SOLUTION (Check the steps given in FRAME 54)

1. Let the number be x (no units)
2. Multiply by 4 $\longrightarrow$ $4x$

   Add 9 $\longrightarrow$ $4x + 9$
3. The result is 33 $\longrightarrow$ $4x + 9 = 33$

   $4x = 24 \quad (-9)$

   $x = 6 \quad (\times \frac{1}{4})$
4. The required number is 6
5. $4 \times 6 + 9$

   $= 24 + 9$

   $= 33$ Checked.

Example 55.2

I spend 36p on crisps and lemonade. Cans of lemonade cost 15p and packets of crisps cost 7p each. If I buy one can of lemonade, how many packets of crisps do I buy?

SOLUTION

1. Let x be the number of packets of crisps that I buy (no unit
2. Cost of the crisps $\longrightarrow$ $7x$ pence

   Cost of the lemonade $\longrightarrow$ 15 pence

   Amount spent $\longrightarrow$ 36 pence
3. $7x + 15 = 36$

   $7x = 21 \quad (-15)$

   $x = 3 \quad (\times \frac{1}{7})$
4. I buy 3 bags of crisps.

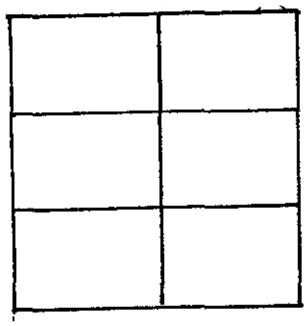
5. $15 + 3 \times 7$

$= 15 + 21$

$= 36$ Checked.

Example 55.3

Six paving stones are arranged in a square array as illustrated:



If each stone has a length 20cm. greater than its width, find:

(a) the dimension of each stone

(b) the area of ground space covered by the six stones

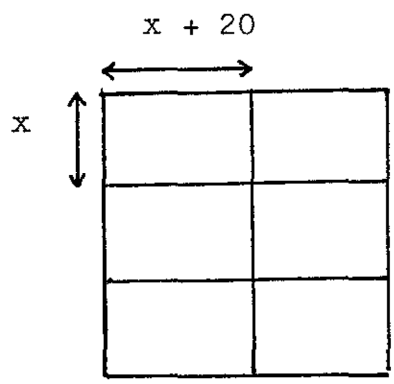
(c) the perimeter of this ground space.

SOLUTION

1. Let the width of one stone be x cm.

2. Width $\longrightarrow$ x cm.

Length $\longrightarrow$ $x + 20$ cm.



3. As the diagram is a square

$$2(x + 20) = 3x$$

$$2x + 40 = 3x$$

$$40 = x \quad (- 2x)$$

4. (a) Dimensions of each stone are: length 60 cm.

width 40 cm.

(b) Length of the side of a square 120 cm.

Therefore area of groundspace is 120×120

$= \underline{14400 \text{ cm.}^2}$

(c) Perimeter of this groundspace $= 4 \times 120$

$= 480$ cm.

5. Check with the diagram.

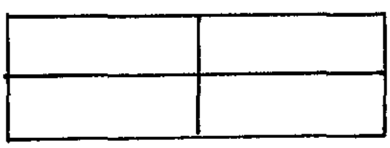
FRAME 56

Here are some problems for you to try:

I. A number is multiplied by 8 and 11 is then subtracted from the obtained product giving a result of 29. Find the number.

II. Tom has a certain number of marbles. He plays a game and wins; he finds that he has three times as many. He then plays a second game and loses eight of them. He finds that he has 16 left. How many did he have originally?

III. Four wooden planks, each having a length of 80 cm. longer than its width, are arranged as illustrated in the diagram.



If the perimeter of the floor space they occupy is 480 cm., find (a) the dimensions of each plank

(b) the area of the floor space that the planks are occupying together.

ANSWERS 56

I. 1. Let the number be x (no units)

2. Multiply by 8 $\longrightarrow$ $8x$

Subtract 11 $\longrightarrow$ $8x - 11$

3. $8x - 11 = 29$

$8x = 40$ $(+ 11)$

$\underline{x = 5}$ $(\times \frac{1}{8})$

4. The required number is 5

5. $8 \times 5 - 11$

$= 40 - 11$

$= 29$ Checked.

II. Letting x represent the number of marbles he originally had, the equation formed should be $3x - 8 = 16$

The solution is 8 marbles.

III. If x cm. represents the width, then the equation formed is $4x + 4(x + 80) = 480$.

Solutions are (a) 20 cm. width

100 cm. length

(b) area is 8000 cm.^{2}

FRAME 57

In all the cases so far the denominators of the fractions have not involved the unknown. In some cases this will not be so. The same five steps as described in FRAME 50 can be applied but more care must be taken with the first step.

For example: $\frac{2}{t} = \frac{3}{t + 1}$

At first this does not look like a simple equation. Indeed, at this point you would not be sure what kind of equation it is. To remove the denominators in this case we multiply by $t(t + 1)$

$$\frac{2}{\cancel{t}_1} \times \overset{1}{\cancel{t}}(t + 1) = \frac{3}{\cancel{t + 1}_1} \times t\overset{1}{\cancel{(t + 1)}} \quad (\times\ t(t + 1))$$

$$2(t + 1) = 3t$$

Now you see we are left with a simple equation:

$$2t + 2 = 3t$$

$$\underline{2 = t} \quad (-\ 2t)$$

FRAME 58

Here are some more examples of the sort described in FRAME 57

Example 58.1

$$\frac{x - 1}{x + 1} = \frac{2x - 3}{2x + 3}$$

$$\frac{(x - 1)}{\cancel{(x + 1)}_1} \times \overset{1}{\cancel{(x + 1)}}(2x + 3) = \frac{(2x - 3)}{\cancel{(2x + 3)}_1} \times (x + 1)\overset{1}{\cancel{(2x + 3)}}$$

$$(\times\ (x + 1)(2x + 3))$$

$$(x - 1)(2x + 3) = (2x - 3)(x + 1)$$

$$2x^2 + 3x - 2x - 3 = 2x^2 + 2x - 3x - 3$$

$$2x^2 + x - 3 = 2x^2 - x - 3$$

$$x - 3 = -x - 3 \qquad (-2x^2)$$

$$2x - 3 = -3 \qquad (+x)$$

$$2x = 0 \qquad (+3)$$

$$\underline{x = 0} \qquad (\times \tfrac{1}{2})$$

Example 58.2

$$\frac{1}{z} + \frac{1}{z+1} = \frac{2}{z-1}$$

$$\frac{1}{\cancel{z}_{1}} \times \cancel{z}^{1}(z+1)(z-1) + \frac{1}{\cancel{z+1}_{1}} \times z\cancel{(z+1)}^{1}(z-1) = \frac{2}{\cancel{z-1}_{1}} \times z(z+1)\cancel{(z-1)}^{1}$$

$$(\times z(z+1)(z-1))$$

$$(z+1)(z-1) + z(z-1) = 2z(z+1)$$

$$z^2 - 1 + z^2 - z = 2z^2 + 2z$$

$$2z^2 - 1 - z = 2z^2 + 2z$$

$$-1 - z = 2z \qquad (-2z^2)$$

$$-1 = 3z \qquad (+z)$$

$$\underline{-\tfrac{1}{3} = z} \qquad (\times \tfrac{1}{3})$$

Solve these equations:

1. $\dfrac{2y - 1}{2y + 4} = \dfrac{3y - 1}{3y + 2}$

2. $\dfrac{3}{x - 1} + \dfrac{1}{x + 1} = \dfrac{4}{x}$

ANSWERS 58

1. $$\frac{(2y-1)}{\cancel{(2y+4)}_{1}} \times \cancel{(2y+4)}^{1}(3y+2) = \frac{(3y-1)}{\cancel{(3y+2)}_{1}} \times (2y+4)\cancel{(3y+2)}^{1}$$

$$(\times (2y+4)(3y+2))$$

$$\begin{aligned}
(2y-1)(3y+2) &= (3y-1)(2y+4) \\
6y^2 + 4y - 3y - 2 &= 6y^2 + 12y - 2y - 4 \\
y - 2 &= 10y - 4 && (-6y^2) \\
-2 &= 9y - 4 && (-y) \\
2 &= 9y && (+4) \\
\underline{\tfrac{2}{9} = y} & && (\times \tfrac{1}{9})
\end{aligned}$$

2. $$\frac{3}{\cancel{(x-1)}_{1}} \times x\cancel{(x-1)}^{1}(x+1) + \frac{1}{\cancel{(x+1)}_{1}} \times (x-1)\cancel{(x+1)}^{1} = \frac{4}{\cancel{x}_{1}} \times \cancel{x}^{1}(x-1)(x+1$$

$$(\times x(x-1)(x+1))$$

$$\begin{aligned}
3x(x+1) + x(x-1) &= 4(x-1)(x+1) \\
3x^2 + 3x + x^2 - x &= 4x^2 - 4 \\
4x^2 + 2x &= 4x^2 - 4 \\
2x &= -4 && (-4x^2) \\
\underline{x = -2} & && (\times \tfrac{1}{2})
\end{aligned}$$

FRAME 59

If you look at the Examples in FRAME 58 you will see that they do not look like simple equations. In fact many equations of this type turn out to be NOT simple equations but 'Quadratic Equations' as described in the following chapter.

e.g. $\frac{3}{x-2} + \frac{8}{x+3} = 2$

$$\frac{3}{\cancel{x-2}_{1}} \times \overset{1}{\cancel{(x-2)}}(x+3) + \frac{8}{\cancel{x+3}_{1}} \times (x-2)\overset{1}{\cancel{(x+3)}} = 2(x-2)(x+3)$$

$$(\times (x-2)(x+3))$$

$$3(x+3) + 8(x-2) = 2(x-2)(x+3)$$

$$3x + 9 + 8x - 16 = 2x^2 + 2x - 12$$

$$11x - 7 = 2x^2 + 2x - 12 \text{...Eq59.}$$

You will see that although the equation started by looking something like those equations in FRAME 58, Eq. 59.1 has a term $2x^2$ involved which means it is not a simple equation. It is, in fact, a quadratic equation the solutions of which are dealt with in the next chapter.

FRAME 60 SUMMARY

To solve a simple equation:

1. Multiply by the least number to remove all denominators.
2. Multiply the brackets out.
3. Collect the terms containing the unknown together.
4. Collect the numbers to the other side.
5. Remove the coefficient.

FRAME 31	Two simple problems.
FRAMES 32 - 42	Solution of simple equations (With unknown terms together).
FRAMES 43 - 44	Solution of simple equations (With unknown terms on both sides).
FRAME 45	Solution using factors.
FRAME 46	Simple equations including brackets.
FRAMES 47 - 49	Simple equations including constant denominators.
FRAME 50	Five steps to a solution.
FRAME 51	Mixed examples.
FRAMES 52 - 53	Solution of problems from FRAME 31.
FRAME 54	Solution of simple problems.
FRAMES 55 - 56	Simple problems.
FRAMES 57 - 58	Simple equations with general denominators.
FRAME 59	Non-simple equations.

SECTION 3

Quadratic Equations

FRAME 61 QUADRATIC EQUATIONS

So far we have looked at equations which can be solved by re-arrangement.

In FRAME 20 we looked at quadratic expressions and now we look at equations which contain a quadratic expression. Such equations are known as

QUADRATIC EQUATIONS

and they cannot be solved in the same way as simple equations.

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

where a, b, and c are numbers.

FRAME 62

Look at the equation:

$$x^2 - 3x - 10 = 0$$

If we try to solve this in the same way as before it would follow:

$$x^2 - 3x - 10 = 0$$

$$x^2 - 3x = 10 \qquad (+10)$$

This has failed to solve the equation as a value of x has not been found (See FRAME 32).

When you meet a quadratic equation you should arrange it in the standard form described in FRAME 61. The solution of this is described in the following FRAMES.

FRAME 63

One of the most important points about a quadratic equation is the fact that solutions often come in pairs. In fact a quadratic equation can have (a) two solutions,

(b) one solution,

(c) no solution.

We are going to look at two ways of solving quadratic equations. Firstly by FACTORISATION and secondly by FORMULA.

FRAME 64 SOLUTION BY FACTORISATION

Look again at the equation:

$x^2 - 3x - 10 = 0$ .... (i)

First factorise the left hand side as described in FRAMES 21 - 27, giving:

$(x + 2)(x - 5) = 0$ .... (ii)

In line (ii) we have two numbers (namely $x + 2$ and $x - 5$) multiplied together to give zero. If two numbers multiply together to give zero it must mean that either one or both of the numbers are themselves zero,

e.g. 5×0, 0×8, 0×0 etc. --------

This means from line (ii):

$x + 2 = 0$ or $x - 5 = 0$

$\Longrightarrow x = -2$ or $\Longrightarrow x = 5$

Therefore the original equation in line (i) has the solution:

$x = \underline{-2 \text{ or } 5}$

Check

$$x^2 - 3x - 10 \qquad \text{or} \qquad x^2 - 3x - 10$$

$$= (-2)^2 - 3(-2) - 10 \qquad = 5^2 - 3(5) - 10$$

$$= 4 + 6 - 10 \qquad = 25 - 15 - 10$$

$$= 0 \qquad = 0$$

FRAME 65

Here are some more examples:

Example 65.1

$x^2 + 5x - 24 = 0$

$(x - 3)(x + 8) = 0$ Factorise the left-hand side.

Therefore $x - 3 = 0$ or $x + 8 = 0$

$\Longrightarrow \underline{x = 3}$ or $\underline{x = -8}$

(Notice 2 solutions)

Example 65.2

$$x^2 - 8x + 16 = 0$$

$$(x - 4)(x - 4) = 0$$

Therefore $x - 4 = 0$

$$\Longrightarrow \quad \underline{x = 4}$$

(Notice 1 solution)

Example 65.3

$$x^2 - 4x = 21$$

$$x^2 - 4x - 21 = 0 \qquad (-21)$$

$$(x - 7)(x + 3) = 0$$

Therefore $x - 7 = 0$ or $x + 3 = 0$

$$\Longrightarrow \quad \underline{x = 7} \quad \text{or} \quad \underline{x = -3}$$

Now try to solve these (You may find it useful to check your answers):

1. $x^2 - 6x + 5 = 0$
2. $x^2 + 4x + 3 = 0$
3. $x^2 + 10x + 16 = 0$
4. $x^2 - 10x - 39 = 0$
5. $x^2 - 6x = 16$
6. $x^2 = 5x + 6$

ANSWERS 65

1. $x = 1$ or 5
2. $x = -1$ or -3
3. $x = -2$ or -8
4. $x = -3$ or 13
5. $x = -2$ or 8
6. $x = -1$ or 6

FRAME 66

As described in FRAME 26 a quadratic function may have any number of x^2, i.e. $2x^2 - 5x - 3 = 0$ is a quadratic equation. The left-hand side factorises so the equation becomes (see 26.1):

$$(x - 3)(2x + 1) = 0$$

$$\Longrightarrow \quad x - 3 = 0 \quad \text{or} \quad 2x + 1 = 0$$

$$\Longrightarrow \quad \underline{x = 3} \quad \text{or} \quad \underline{x = -\tfrac{1}{2}}$$

Solve the following quadratic equations by writing in the standard way and then using factors:

1. $2x^2 - 3x + 1 = 0$
2. $7x^2 + 8x + 1 = 0$
3. $3x^2 - 16x + 5 = 0$
4. $4x^2 + 5x - 6 = 0$
5. $8 + 10x = 8x^2 + 5$
6. $12 - x - x^2 = 0$

ANSWERS 66

1. $2x^2 - 3x + 1 = 0$

$$(2x - 1)(x - 1) = 0$$

$$\Longrightarrow \quad 2x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\Longrightarrow \quad \underline{x = \tfrac{1}{2}} \quad \text{or} \quad \underline{x = 1}$$

2. $7x^2 + 8x + 1 = 0$

$$(7x + 1)(x + 1) = 0$$

$$\Longrightarrow \quad 7x + 1 = 0 \quad \text{or} \quad x + 1 = 0$$

$$\Longrightarrow \quad \underline{x = -\frac{1}{7}} \quad \text{or} \quad \underline{x = -1}$$

3. $$3x^2 - 16x + 5 = 0$$
$$(3x - 1)(x - 5) = 0$$
$$\Longrightarrow \quad 3x - 1 = 0 \quad \text{or} \quad x - 5 = 0$$
$$\Longrightarrow \quad \underline{x = \tfrac{1}{3}} \quad \text{or} \quad \underline{x = 5}$$

4. $$4x^2 + 5x - 6 = 0$$
$$(4x - 3)(x + 2) = 0$$
$$\Longrightarrow \quad 4x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$\Longrightarrow \quad \underline{x = \tfrac{3}{4}} \quad \text{or} \quad \underline{x = -2}$$

5. $$8 + 10x = 8x^2 + 5$$
$$10x = 8x^2 - 3 \qquad (-8)$$
$$0 = 8x^2 - 10x - 3 \qquad (-10x)$$
$$0 = (4x + 1)(2x - 3)$$
$$\Longrightarrow \quad 4x + 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$\Longrightarrow \quad \underline{x = -\tfrac{1}{4}} \quad \text{or} \quad \underline{x = \tfrac{3}{2}}$$

6. $$12 - x - x^2 = 0$$
$$12 - x = x^2 \qquad (+x^2)$$
$$12 = x^2 + x \qquad (+x)$$
$$0 = x^2 + x - 12 \qquad (-12)$$

This is to avoid a negative sign with the x^2.

$$0 = (x - 3)(x + 4)$$
$$\Longrightarrow \quad x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$
$$\Longrightarrow \quad \underline{x = -3} \quad \text{or} \quad \underline{x = -4}$$

FRAME 67

The solution of the quadratics described in FRAMES 64-66 have been dependent on the factorisation of the quadratic function. There are occasions when the function will not easily factorise:

e.g. $2x^2 + 5x + 1 = 0$ .... Eq. 67.1

It is possible to solve a quadratic equation without using factors. To do this you can use a formula which will solve all quadratic equations that have real solutions.

FRAME 68 SOLUTION BY FORMULA

To solve a quadratic equation of the form :

$$ax^2 + bx + c = 0$$

where a, b, and c are numbers, you use the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{....... Eq. 68.1}$$

THIS IS AN EXTREMELY IMPORTANT FORMULA. LEARN IT !!

FRAME 69

Let us solve Eq. 67.1 using the formula from FRAME 68.

The standard form of a quadratic is $ax^2 + bx + c = 0$

Eq. 67.1 is $2x^2 + 5x + 1 = 0$

This means: $a = 2$, $b = 5$, $c = 1$.

Substituting these values into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25 - 4(2)(1)}}{4}$$

$$= \frac{-5 \pm \sqrt{25 - 8}}{4}$$

$$= \frac{-5 \pm \sqrt{17}}{4}$$

$$= \frac{-5 \pm 4.123}{4} \qquad \text{(Working to 3 decimal places)}$$

$$= \frac{-5 + 4.123}{4} \quad \text{or} \quad \frac{-5 - 4.123}{4}$$

$$= \underline{-0.219} \quad \text{or} \quad \underline{-2.281}$$

So these are the solution (approximate because of the square root) of the original equation 67.1.

FRAME 70

Using the formula solve these quadratic equations. Give your answers correct to 3 decimal places.

1. $2x^2 + 7x + 1 = 0$
2. $3x^2 + 6x + 2 = 0$
3. $2x^2 + 4x + 1 = 0$

ANSWERS 70

1. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In this case $a = 2, \quad b = 7, \quad c = 1$

$$x = \frac{-7 \pm \sqrt{49 - 4(2)(1)}}{4}$$

$$= \frac{-7 \pm \sqrt{49 - 8}}{4}$$

$$= \frac{-7 \pm \sqrt{41}}{4}$$

$$= \frac{-7 + \sqrt{41}}{4} \quad \text{or} \quad \frac{-7 - \sqrt{41}}{4}$$

$$= \underline{-0.149} \quad \text{or} \quad \underline{-3.351}$$

2. In this case $a = 3$, $b = 6$, $c = 2$.

$$x = \frac{-6 \pm \sqrt{36 - 4(3)(2)}}{6}$$

$$= \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$= \frac{-6 \pm \sqrt{12}}{6}$$

$$= \frac{-6 + \sqrt{12}}{6} \quad \text{or} \quad \frac{-6 - \sqrt{12}}{6}$$

$$= \underline{-0.423} \quad \text{or} \quad \underline{-1.577}$$

3. In this case $a = 2$, $b = 4$, $c = 1$.

$$x = \frac{-4 \pm \sqrt{16 - 4(2)(1)}}{4}$$

$$= \frac{-4 \pm \sqrt{16 - 8}}{4}$$

$$= \frac{-4 \pm \sqrt{8}}{4}$$

$$= \frac{-4 + \sqrt{8}}{4} \quad \text{or} \quad \frac{-4 - \sqrt{8}}{4}$$

$$= \underline{-0.293} \quad \text{or} \quad \underline{-1.707}$$

FRAME 71

The equations we looked at in FRAME 70 contained positive values for a, b, and c. The same formula applies with negative values of a, b, and c but more care has to be taken. Here are some examples:

Example 71.1

$$x^2 - 5x + 1 = 0$$

In this case $a = 1$, $b = -5$, $c = 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(1)}}{2}$$

$$= \frac{5 \pm \sqrt{25 - 4}}{2}$$

$$= \frac{5 \pm \sqrt{21}}{2}$$

$$= \frac{5 + \sqrt{21}}{2} \quad \text{or} \quad \frac{5 - \sqrt{21}}{2}$$

$$= \underline{4.791} \quad \text{or} \quad \underline{0.209}$$

Example 71.2

$$3x^2 - 2x - 6 = 0$$

In this case $a = 3$, $b = -2$, $c = -6$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{6}$$

$$= \frac{2 \pm \sqrt{4 + 72}}{6}$$

$$= \frac{2 \pm \sqrt{76}}{6}$$

$$= \frac{2 + \sqrt{76}}{6} \quad \text{or} \quad \frac{2 - \sqrt{76}}{6}$$

$$= \underline{1.786} \quad \text{or} \quad \underline{-1.120}$$

Example 71.3

$$x^2 + 4x - 9 = 0$$

In this case $a = 1$, $b = 4$, $c = -9$.

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-9)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 + 36}}{2}$$

$$= \frac{-4 \pm \sqrt{52}}{2}$$

$$= \frac{-4 + \sqrt{52}}{2} \quad \text{or} \quad \frac{-4 - \sqrt{52}}{2}$$

$$= \underline{1.606} \quad \text{or} \quad \underline{-5.606}$$

FRAME 72 PROBLEMS LEADING TO QUADRATIC EQUATIONS

Sometimes problems will create quadratic equations. The same steps should be taken when solving problems, as described in FRAME 54.

Example 72.1

The length of a room is 4 metres longer than its width. The height of the room is 3 metres and its volume is $1000m^3$. Find the length and width of the room.

Let x represent the width.

Units: metres.

Dimensions of the room: length is $(x + 4)$ m.

width is x m.

height is 3 m.

Volume $= (x + 4) \times x \times 3$

Also Volume $= 1000m^3$.

Therefore $(x + 4) \times x \times 3 = 1000$

$$3x(x + 4) = 1000$$

$$3x^2 + 12x = 1000$$

$$3x^2 + 12x - 1000 = 0 \qquad (-1000)$$

In this case $a = 3$, $b = 12$, $c = -1000$.

$$x = \frac{-12 \pm \sqrt{144 - 4(3)(-1000)}}{6}$$

$$= \frac{-12 \pm \sqrt{144 + 12000}}{6}$$

$$= \frac{-12 \pm \sqrt{12144}}{6}$$

$$= \frac{-12 + \sqrt{12144}}{6} \quad \text{or} \quad \frac{-12 - \sqrt{12144}}{6}$$

$$= \underline{16.367} \quad \text{or} \quad \underline{-20.367}$$

In this case -20.367 is not a realistic answer as we are dealing with lengths.

Therefore $x = 16.367$m.

So the width of the room is 16.367m.

length of the room is 20.367m.

Example 72.2

The distance from London to Bournemouth is 160km. If a train were 16km/h slower, it would take 20 minutes longer on the journey. Find the average speed of the train.

(Speed $= \dfrac{\text{Distance}}{\text{Time}}$)

Let xkm/h represent the speed of the train.

Time to travel 160km is $\dfrac{160}{x}$ hours

Time to travel 160km at slower speed $= \dfrac{160}{x-16}$ hours.

As 20 minutes is $\frac{1}{3}$ of an hour:

$$\frac{160}{x-16} - \frac{1}{3} = \frac{160}{x}$$

$$480x - x(x - 16) = 480(x - 16)$$

$$480x - x^2 + 16x = 480x - 7680$$

$$0 = x^2 - 16x - 7680$$

$$0 = (x - 96)(x + 80)$$

$$\Longrightarrow \quad x - 96 = 0 \quad \text{or} \quad x + 80 = 0$$

$$\Longrightarrow \quad x = 96 \quad \text{or} \quad x = -80$$

Again in this case -80 can be ignored as unrealistic.

Therefore the speed of the train is <u>96km/hr</u>.

FRAME 73

Form the quadratic equations from these problems and then solve them by either factors (if possible) or using the formula.

1. The length of a rectangle is 3m greater than its width and the area of the rectangle is $108m^2$. By letting x represent the width find:

(a) the length in terms of x

(b) an equation in x

(c) the value of x

(d) the length of the rectangle.

2. The sides of a right angled triangle are $(2x + 1)$cm, $2x$ cm and $(x - 1)$cm. Find x. (Hint: Pythagoras Theorem)

3. The product of two numbers, differing by 7, is 60. Find them.

4. A motorist has to travel 160km. His average speed is 8km/hr. slower than he estimated and he takes 1 hour longer on the journey. Find his actual average speed.

5. A man spends £20 on cigars. Had they been 10p cheaper, he could have bought 10 more. How many did he buy?

ANSWERS 73

1. (a) $(x + 3)$m.

(b)
$$\begin{aligned} x(x+3) &= 108 \\ x^2 + 3x &= 108 \\ x^2 + 3x - 108 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} (x-9)(x+12) &= 0 \\ x &= 9 \quad \text{or} \quad -12 \end{aligned}$$

(d)
$$\begin{aligned} \text{Length} &= 9 + 3 \\ &= \underline{12\text{m}} \end{aligned}$$

2. Quadratic equation
$$\begin{aligned} x^2 - 6x &= 0 \\ x(x-6) &= 0 \\ x &= 0 \quad \text{or} \quad 6 \end{aligned}$$

Solution $\underline{6}$

3. If you let the larger of the two be x
$$\begin{aligned} x(x-7) &= 60 \\ x^2 - 7x &= 60 \\ x^2 - 7x - 60 &= 0 \\ x &= -5 \quad \text{or} \quad 12 \end{aligned}$$

Solutions are -5 and -12

or $\underline{12 \text{ and } 5}$

4. If you let the actual average speed be xkm/hr.

   Quadratic: $x^2 + 8x - 1280 = 0$

   Solution: 32km/hr.

5. If you let the number of cigars bought be x

   Quadratic: $x^2 + 10x - 2000 = 0$

   Solution: 40 cigars

FRAME 74

We have seen examples of quadratic equations that have one and two solutions. Let us now look at an example where there is no real solution,

e.g. $x^2 + 2x + 5 = 0$ .... Eq. 74.1

Eq. 74.1 has no real solution. Certainly it is not easily factorised so we will try the formula.

In this case $a = 1$, $b = 2$, $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$\sqrt{-16}$ is not obtainable. There is no such real number. In this case there is no value of x which solves Eq. 74.1

FRAME 75

In Eq. 68.1 the value of $b^2 - 4ac$ shows how many real solutions there will be

If $b^2 - 4ac > 0 \Longrightarrow$ Two real solutions

$b^2 - 4ac = 0 \Longrightarrow$ One real solution

$b^2 - 4ac < 0 \Longrightarrow$ No real solution

It is often worth checking the value of $b^2 - 4ac$ before proceding with a solution.

By looking at the value of $b^2 - 4ac$ in the following equations say how many real solutions the equations have (there is no need to find them).

1. $2x^2 - 5x + 2 = 0$
2. $2x^2 - 2x + 5 = 0$
3. $x^2 - 7x - 18 = 0$
4. $4x^2 - 4x + 1 = 0$
5. $x^2 - 2x + 8 = 0$

ANSWERS 75

1. $b^2 - 4ac = 25 - 16$
   $= 9 \Longrightarrow$ 2 solutions
2. $b^2 - 4ac = 4 - 40$
   $= -36 \Longrightarrow$ 0 solution
3. $b^2 - 4ac = 49 + 72$
   $= 121 \Longrightarrow$ 2 solutions
4. $b^2 - 4ac = 16 - 16$
   $= 0 \Longrightarrow$ 1 solution
5. $b^2 - 4ac = 4 - 32$
   $= -28 \Longrightarrow$ 0 solution

FRAME 76 SUMMARY

FRAMES 61 - 62	Quadratic equations cannot be solved by re-arrangement.
FRAME 63	Number of solutions of quadratic equations.
FRAMES 64 - 66	Solution of quadratic equations by factors.
FRAMES 67 - 71	Solution of quadratic equations by formula.
FRAMES 72 - 73	Problems leading to quadratic equations.
FRAMES 74 - 75	The use of the value of $b^2 - 4ac$ in quadratic equations.

SECTION 4

Simultaneous Equations

FRAME 77 SIMULTANEOUS EQUATIONS

A man waiting for ice-cream in a queue at the cinema overheard the two customers ahead of him. The first one had bought 2 choc-ices and 3 lollipops which had cost him £1.34. The second customer had bought 3 choc-ices and 1 lollipop and had paid £1.03. How much should the man expect to pay for 1 choc-ice and 1 lollipop?

This is an example of a problem that will lead to a pair of SIMULTANEOUS EQUATIONS. You may like to try to solve this problem now. We will return to it in FRAME 90.

FRAME 78

Look carefully at this equation:

$$x + y = 6$$

The difference between this equation and others you have met is that it contains more than one unknown. One equation with two unknowns has no unique answer.

For example, if $x + y = 6$ you could have

$$x = 1 \text{ and } y = 5$$
$$\text{or } x = 2 \text{ and } y = 4$$
$$\text{or } x = -8 \text{ and } y = 14$$

The list of solutions is endless.

However, if as well as $x + y = 6$,

we are given $x - y = 4$,

then $x = 5$ and $y = 1$ is the only solution. This is because, although you may be able to find countless pairs of numbers that add to give 6, only one pair will add to give 6 AND subtract to give 4.

Equations that happen at the same time are called SIMULTANEOUS EQUATIONS. In this section we will only consider two linear equations each having two unknowns.

FRAME 79 METHOD OF ELIMINATION

There are a number of ways to solve simultaneous equations. In this text we will consider only one type of method of solution, that is using elimination. Let us look at a simple case:

$$2x + y = 18 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$x + y = 10 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

The difference between the L.H.S. of (i) and (ii) is that (i) has one more x. The difference between the R.H.S. of (i) and (ii) is 8 bigger. If we calculate (i) - (ii) this would give $x = 8$. We say y has been eliminated. Substituting this value of x in (i) <u>or</u> (ii) ((ii) looks simpler) gives

$$8 + y = 10$$

$$y = 2$$

The full working would be set out like this:

$$2x + y = 18 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$\underline{x + y = 10} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

$$x = 8 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i) - (ii)}$$

Substitute $x = 8$ in (ii)

$$8 + y = 10$$

$$y = 2$$

Solution: $\underline{x = 8, \quad y = 2}$

Check (i) $2 \times 8 + 2$ (ii) $8 + 2 = 10$

$= 16 + 2$

$= 18$

FRAME 80

Here are some examples:

Example 80.1

$$\begin{array}{rl} 3x + y = 9 & \ldots\ldots\ldots\ldots\ldots \quad (i) \\ \underline{x + y = 5} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \\ 2x = 4 & \ldots\ldots\ldots\ldots\ldots \quad (i) - (ii) \\ x = 2 & \end{array}$$

Substitute $x = 2$ in (ii)

$$2 + y = 5$$
$$y = 3$$

Solution: $\underline{x = 2, \quad y = 3}$

Check (i) $3 \times 2 + 3$ (ii) $2 + 3 = 5$

$= 6 + 3$

$= 9$

Example 80.2

$$\begin{array}{rl} 10x + 4y = 22 & \ldots\ldots\ldots\ldots\ldots \quad (i) \\ \underline{10x + y = 13} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \\ 3y = 9 & \ldots\ldots\ldots\ldots\ldots \quad (i) - (ii) \\ y = 3 & \end{array}$$

Substitute $y = 3$ in (ii)

$$10x + 3 = 13$$
$$10x = 10$$
$$x = 1$$

Solution: $\underline{x = 1, \quad y = 3}$

Check (i) $10 \times 1 + 4 \times 3$ (ii) $10 \times 1 + 3$

$= 10 + 12$ $\quad = 10 + 3$

$= 22$ $\quad = 13$

Example 80.3

$$3x + 5y = 29 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$6x + 5y = 38 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$
$$3x = 9 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii) - (i)}$$
$$x = 3$$

Substitute $x = 3$ in (i)

$$9 + 5y = 29$$
$$5y = 20$$
$$y = 4$$

Solution: $x = 3, \quad y = 4$

Check (i) $3 \times 3 + 5 \times 4$ $= 9 + 20$ $= 29$

(ii) $6 \times 3 + 5 \times 4$ $= 18 + 20$ $= 38$

Now try to solve these in the same way:

1. $4x + y = 9$
   $x + y = 6$

2. $5x + 3y = 54$
   $x + 3y = 30$

3. $6a + 4b = 40$
   $9a + 4b = 46$

ANSWERS 80

1. $x = 1, \quad y = 5$
2. $x = 6, \quad y = 8$
3. $a = 2, \quad b = 7$

FRAME 81

Suppose in Example 80.1 the two equations had been:

$$3x - y = 9 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$x - y = 5 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

Look carefully at the L.H.S. and see what subtraction would do:

$3x - x$ would give $2x$ as before.

$-y - (-y)$ would give $-y + y$

$= 0$ as before.

If each equation has the same number of a particular letter then that letter disappears by subtraction, as long as the letter has the same sign in each equation.

e.g.

$$3x - 2y = 8 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$\underline{5x - 2y = 12} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$
$$2x = 4 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii) - (i)}$$
$$x = 2$$

Substitute $x = 2$ in (i)

$$6 - 2y = 8$$
$$6 = 8 + 2y$$
$$-2 = 2y$$
$$-1 = y$$

Solution: $\underline{x = 2, \quad y = -1}$

Check (i) $3 \times 2 - 2(-1)$ $= 6 + 2$ $= 8$

(ii) $5 \times 2 - 2(-1)$ $= 10 + 2$ $= 12$

Solve these equations:

1. $2x - y = 8$
   $x - y = 3$

2. $3a - 2b = -8$
   $5a - 2b = -12$

ANSWERS 81

1.

$$\begin{array}{lll} 2x - y = 8 & \ldots\ldots\ldots\ldots\ldots & \text{(i)} \\ \underline{x - y = 3} & \ldots\ldots\ldots\ldots\ldots & \text{(ii)} \\ x \quad = 5 & \ldots\ldots\ldots\ldots\ldots & \text{(i) - (ii)} \end{array}$$

Substitute $x = 5$ in (ii)

$$5 - y = 3$$

$$y = 2$$

Solution: $\underline{x = 5, \quad y = 2}$

Check (i) $2 \times 5 - 2$ (ii) $5 - 2 = 3$

$= 10 - 2$

$= 8$

2.

$$\begin{array}{lll} 3a - 2b = -8 & \ldots\ldots\ldots\ldots\ldots & \text{(i)} \\ \underline{5a - 2b = -12} & \ldots\ldots\ldots\ldots\ldots & \text{(ii)} \\ 2a \quad = -4 & \ldots\ldots\ldots\ldots\ldots & \text{(ii) - (i)} \\ a \quad = -2 & & \end{array}$$

Substitute $a = -2$ in (i)

$$-6 - 2b = -8$$

$$-6 = -8 + 2b$$

$$2 = 2b$$

$$1 = b$$

Solution: $\underline{a = -2, \quad b = 1}$

Check (i) $3(-2) - 2 \times 1$ (ii) $5(-2) - 2 \times 1$

$= -6 - 2$ $= -10 - 2$

$= -8$ $= -12$

FRAME 82

So far we have only looked at equations with equal numbers of a particular letter. Of course, this rarely happens. Take for example:

$$3x + 2y = 12 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$5x + 4y = 22 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

Notice in this case subtraction will not make one of the letters disappear. But if we calculate 2 X (i) this gives:

$$6x + 4y = 24$$

Now this equation has 4y like (ii). It can be solved with (i) as described in FRAMES 79 - 81.

$$3x + 2y = 12 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$\underline{5x + 4y = 22} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

$$6x + 4y = 24 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i) X 2}$$

$$\underline{5x + 4y = 22} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(iv)}$$

$$x = 2 \qquad \text{Subtract}$$

Substitute x = 2 in (i)

$$6 + 2y = 12$$

$$2y = 6$$

$$y = 3$$

Solution: $\underline{x = 2, \quad y = 3}$

Check (i) $3 \times 2 + 2 \times 3$ (ii) $5 \times 2 + 4 \times 3$

$= 6 + 6$ $\qquad = 10 + 12$

$= 12$ $\qquad = 22$

FRAME 83

In FRAME 82 we looked at an example that required multiplying one equation. In that particular example it made sense to eliminate y as it only required an alteration to one equation. Usually it does not matter. For example:

$$2x + 5y = 29 \quad \ldots\ldots\ldots\ldots\ldots \quad (i)$$
$$3x + 2y = 16 \quad \ldots\ldots\ldots\ldots\ldots \quad (ii)$$

In this case it is no simpler to eliminate x than it is to eliminate y. Either method requires altering both equations. Let us choose to eliminate x. To do this we must make the coefficients of x the same, and then apply the methods described in FRAMES 79 - 81.

(i) X 3 makes the coefficient of x to be 6

(ii) X 2 ' ' ' ' ' ' ' ' '

The working would look like this:

$$\begin{array}{rcll} 2x + 5y & = & 29 & \ldots\ldots\ldots\ldots\ldots \quad (i) \\ \underline{3x + 2y} & \underline{=} & \underline{16} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \\ 6x + 15y & = & 87 & \ldots\ldots\ldots\ldots\ldots \quad (i) \times 3 \\ \underline{6x + 4y} & \underline{=} & \underline{32} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \times 2 \\ 11y & = & 55 & \qquad\qquad\qquad \text{Subtract} \\ y & = & 5 & \end{array}$$

Substitute $y = 5$ in (i)

$$\begin{array}{rcl} 2x + 25 & = & 29 \\ 2x & = & 4 \\ x & = & 2 \end{array}$$

Solution: $x = 2, \quad y = 5$

Check (i) $2 \times 2 + 5 \times 5$ (ii) $3 \times 2 + 2 \times 5$

$= 4 + 25$ $= 6 + 10$

$= 29$ $= 16$

FRAME 84

Here is another example:

$$3x - 8y = -4 \quad \ldots\ldots\ldots\ldots \quad (i)$$

$$7x - 3y = 69 \quad \ldots\ldots\ldots\ldots \quad (ii)$$

Which unknown would you choose to eliminate?

Suppose you choose to eliminate x. You would need to calculate:

(i) X 7 to give 21x

(ii) X 3 to give 21x

This gives:

$$21x - 56y = -28 \quad \ldots\ldots\ldots\ldots \quad (i) \times 7$$

$$21x - 9y = 207 \quad \ldots\ldots\ldots\ldots \quad (ii) \times 3$$

Then you subtract to eliminate x.

If you had chosen to eliminate y, you would need to calculate:

(i) X 3 to give -24y

(ii) X 8 to give -24y

This gives:

$$9x - 24y = -12 \quad \ldots\ldots\ldots\ldots \quad (i) \times 3$$

$$\underline{56x - 24y = 552} \quad \ldots\ldots\ldots\ldots \quad (ii) \times 8$$

Subtract $47x = 564$

$$x = 12$$

Substitute $x = 12$ in (i)

$$36 - 8y = -4$$

$$36 = 8y - 4$$

$$40 = 8y$$

$$5 = y$$

Solution: $\underline{x = 12, \quad y = 5}$

Check (i) $3 \times 12 - 8 \times 5$ (ii) $7 \times 12 - 3 \times 5$

$= 36 - 40$ $= 84 - 15$

$= -4$ $= 69$

Try the following:

1. $2x + 5y = 9$
   $x + 3y = 5$

2. $5a + 3b = -4$
   $2a + 7b = -19$

3. $4x + 5y = 7$
   $2x + 3y = 3$

4. $8m - 3n = -6$
   $5m - 7n = 27$

ANSWERS 84

1. $x = 2, \quad y = 1$
2. $a = 1, \quad b = -3$
3. $x = 3, \quad y = -1$
4. $m = -3, \quad n = -6$

FRAME 85

Now look at these simultaneous equations:

$$2x - y = 5 \quad \ldots\ldots\ldots\ldots\ldots \quad (i)$$

$$-3x + y = -8 \quad \ldots\ldots\ldots\ldots\ldots \quad (ii)$$

If you try the previous methods on this example you will see that neither x nor y can be made to disappear. This is because each unknown has a different sign in each equation:

i.e. x is positive in (i)
x is negative in (ii)
y is negative in (i)
y is positive in (ii)

Subtracting the equations will only eliminate an unknown if their coefficients are exactly the same. When there is only a sign difference between them the unknown can be eliminated by adding the equations:

e.g.

$$\begin{array}{lll} 2x - y = 5 & \ldots\ldots\ldots\ldots\ldots & (i) \\ \underline{-3x + y = -8} & \ldots\ldots\ldots\ldots\ldots & (ii) \\ -x = -3 & \ldots\ldots\ldots\ldots\ldots & (i) + (ii) \\ x = 3 & & \end{array}$$

Substitute $x = 3$ in (ii)

$$-9 + y = -8$$

$$y = 1$$

Solution: $\underline{x = 3, \quad y = 1}$

Check (i) $2 \times 3 - 1$
$= 6 - 1$
$= 5$

(ii) $-3(3) + 1$
$= -9 + 1$
$= -8$

FRAME 86

Here are some examples of eliminating an unknown with different signs:

Example 86.1

$$\begin{array}{lll} 2x + 3y = 20 & \ldots\ldots\ldots\ldots\ldots & (i) \\ \underline{5x - 3y = 22} & \ldots\ldots\ldots\ldots\ldots & (ii) \\ 7x = 42 & & \text{Add} \\ x = 6 & & \end{array}$$

Substitute $x = 6$ in (i)

$$12 + 3y = 20$$

$$3y = 8$$

$$y = 2\tfrac{2}{3}$$

Solution: $\underline{x = 6, \quad y = 2\tfrac{2}{3}}$

Check (i) $2 \times 6 + 3 \times 2\tfrac{2}{3}$
$= 12 + 8$
$= 20$

(ii) $5 \times 6 - 3 \times 2\tfrac{2}{3}$
$= 30 - 8$
$= 22$

Example 86.2

$$\begin{array}{rcll} x + 2y &=& 11 & \ldots\ldots\ldots\ldots\ldots \quad \text{(i)} \\ \underline{x - 2y} &\underline{=}& \underline{3} & \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)} \\ 2x &=& 14 & \qquad\qquad\qquad\quad \text{Add} \\ x &=& 7 & \end{array}$$

Substitute $x = 7$ in (i)

$$\begin{array}{rcl} 7 + 2y &=& 11 \\ 2y &=& 4 \\ y &=& 2 \end{array}$$

Solution: $\underline{x = 7, \quad y = 2}$

Check (i) $7 + 2 \times 2$ $\qquad$ (ii) $7 - 2 \times 2$

$= 7 + 4$ $\qquad$ $= 7 - 4$

$= 11$ $\qquad$ $= 3$

Note: Subtraction of equations would have worked as quickly to eliminate x.

Example 86.3

$$\begin{array}{rcll} 3x + 2y &=& 18 & \ldots\ldots\ldots\ldots\ldots \quad \text{(i)} \\ \underline{2x - y} &\underline{=}& \underline{6} & \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)} \\ 3x + 2y &=& 18 & \ldots\ldots\ldots\ldots\ldots \quad \text{(i)} \\ \underline{4x - 2y} &\underline{=}& \underline{12} & \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)} \\ 7x &=& 30 & \\ x &=& 4\frac{2}{7} & \end{array}$$

Substitute $x = 4\frac{2}{7}$ in (ii)

$$\begin{array}{rcl} 8\frac{4}{7} - y &=& 6 \\ 8\frac{4}{7} &=& 6 + y \\ 2\frac{4}{7} &=& y \end{array}$$

Solution: $\underline{x = 4\frac{2}{7}, \quad y = 2\frac{4}{7}}$

Check (i) $3 \times 4\frac{2}{7} + 2 \times 2\frac{4}{7}$ (ii) $2 \times 4\frac{2}{7} - 2\frac{4}{7}$

$= 12\frac{6}{7} + 5\frac{1}{7}$ $= 8\frac{4}{7} - 2\frac{4}{7}$

$= 18$ $= 6$

Note: Elimination of x here would have involved altering both equations. Therefore it was better to eliminate y.

<u>Example 86.4</u>

$$
\begin{array}{rcll}
3x + 2y & = & 1 & \ldots\ldots\ldots\ldots\ldots \quad \text{(i)} \\
\underline{5x - 3y} & \underline{=} & \underline{8} & \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)} \\
9x + 6y & = & 3 & \ldots\ldots\ldots\ldots\ldots \quad \text{(i)} \times 3 \\
\underline{10x - 6y} & \underline{=} & \underline{16} & \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)} \times 2 \\
19x & = & 19 & \qquad\qquad\qquad\qquad\quad \text{Add} \\
x & = & 1 &
\end{array}
$$

Substitute $x = 1$ in (i)

$$
\begin{array}{rcl}
3 + 2y & = & 1 \\
2y & = & -2 \\
y & = & -1
\end{array}
$$

Solution: <u>$x = 1$, $y = -1$</u>

Check (i) $3 \times 1 + 2 \times (-1)$ (ii) $5 \times 1 - 3(-1)$

$= 3 - 2$ $= 5 + 3$

$= 1$ $= 8$

Note: Elimination of x was possible by $5 \times$ (i) and $3 \times$ (ii), and then subtraction. Either way would involve as much work.

FRAME 87

When you are required to solve a pair of simultaneous equations you should choose an unknown to eliminate which will involve as few alterations as possible. Remember there is more than one way to solve a pair of simultaneous equations; it is advisable to look for the quickest.

For example:

$$x + y = 7 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

$$2x - y = 8 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

In this case the best method would be to add as this would eliminate the y straight away. To eliminate x would involve multiplying by 2 first.

Look at the following questions and write down what would be the quickest method. (It is not necessary to solve them yet):

1. $$3x - 2y = 4 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
   $$5x - 2y = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

2. $$2a + b = 8 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
   $$5a - b = 6 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

3. $$3x + 4y = 14 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
   $$3x - y = 4 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

4. $$4x - 5y = 21 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
   $$6x + 7y = -12 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

5. $$k + 3p = 2 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
   $$2k - 4p = 1 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

ANSWERS 87

1.		(ii) - (i)	this would eliminate y
	or	(i) - (ii)	
2.		(i) + (ii)	this would eliminate b
3.		(i) - (ii)	this would eliminate x
	or	(ii) - (i)	
4.	Either	(i) X 3 (ii) X 2	then subtract which would eliminate x
	or	(i) X 7 (ii) X 5	then add which would eliminate y
5.		(i) X 2	then subtract which would eliminate k

FRAME 88

Noting the answers to FRAME 87 solve the questions 1 - 5 in that FRAME fully.

ANSWERS 88

1.

$$3x - 2y = 4 \quad \ldots\ldots\ldots\ldots\ldots \quad (i)$$

$$\underline{5x - 2y = 0} \quad \ldots\ldots\ldots\ldots\ldots \quad (ii)$$

$$2x = -4 \quad \ldots\ldots\ldots\ldots\ldots \quad (ii) - (i)$$

$$x = -2$$

Substitute $x = -2$ in (i)

$$-6 - 2y = 4$$

$$-6 = 2y + 4$$

$$-10 = 2y$$

$$-5 = y$$

Solution: $\underline{x = -2, \quad y = -5}$

Check (i) $3(-2) \quad -2(-5)$

$= -6 + 10$

$= 4$

(ii) $5(-2) \quad -2(-5)$

$= -10 + 10$

$= 0$

2. $2a + b = 8$ .... (i)

$\underline{5a - b = 6}$ .... (ii)

$7a = 14$ Add

$a = 2$

Substitute $a = 2$ in (i)

$4 + b = 8$

$b = 4$

Solution: $\underline{a = 2, \quad b = 4}$

Check (i) $2 \times 2 + 4$

$= 4 + 4$

$= 8$

(ii) $5 \times 2 - 4$

$= 10 - 4$

$= 6$

3. $3x + 4y = 14$ .... (i)

$\underline{3x - y = 4}$ .... (ii)

$5y = 10$ .... (i) - (ii)

$y = 2$

Substitute $y = 2$ in (i)

$3x + 8 = 14$

$3x = 6$

$x = 2$

Solution: $\underline{x = 2, \quad y = 2}$

Check (i) $3 \times 2 + 4 \times 2$

$= 6 + 8$

$= 14$

(ii) $3 \times 2 - 2$

$= 6 - 2$

$= 4$

4.

$$\begin{array}{rcll} 4x - 5y &=& 21 & \ldots\ldots\ldots\ldots\ldots \quad (i) \\ \underline{6x + 7y} &=& \underline{-12} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \\ 12x - 15y &=& 63 & \ldots\ldots\ldots\ldots\ldots \quad (i) \times 3 \\ \underline{12x + 14y} &=& \underline{-24} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \times 2 \\ -29y &=& 87 & \ldots\ldots\ldots\ldots\ldots \quad \text{Subtract} \\ y &=& -3 & \end{array}$$

Substitute $y = -3$ in (i)

$$\begin{array}{rcl} 4x + 15 &=& 21 \\ 4x &=& 6 \\ x &=& 1\tfrac{1}{2} \end{array}$$

Solution: $\underline{x = 1\tfrac{1}{2}, \quad y = -3}$

Check (i) $4 \times 1\tfrac{1}{2} - 5(-3)$ (ii) $6 \times 1\tfrac{1}{2} + 7(-3)$

$= 6 + 15$ $= 9 - 21$

$= 21$ $= -12$

5.

$$\begin{array}{rcll} k + 3p &=& 2 & \ldots\ldots\ldots\ldots\ldots \quad (i) \\ \underline{2k - 4p} &=& \underline{1} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \\ 2k + 6p &=& 4 & \ldots\ldots\ldots\ldots\ldots \quad (i) \times 2 \\ \underline{2k - 4p} &=& \underline{1} & \ldots\ldots\ldots\ldots\ldots \quad (ii) \\ 10p &=& 3 & \qquad\qquad \text{Subtract} \\ p &=& \frac{3}{10} & \end{array}$$

Substitute $p = \frac{3}{10}$ in (i)

$$\begin{array}{rcl} k + \frac{9}{10} &=& 2 \\ k &=& 1\frac{1}{10} \end{array}$$

Solution: $\underline{k = 1\frac{1}{10}, \quad p = \frac{3}{10}}$

Check (i) $1\frac{1}{10} + \frac{9}{10}$ (ii) $2 \times 1\frac{1}{10} - 4 \times \frac{3}{10}$

$= 2$ $= 2\frac{2}{10} - 1\frac{2}{10}$

$= 1$

FRAME 89

All the simultaneous equations met so far have been written in the same way. That is, all the unknowns on one side and all the numbers on the other side. Sometimes, of course, this may not happen.

For example:

$$2a = 3b + 5 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$3a - 4 = 5b \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

We can re-arrange (i) and (ii) to make them appear the same as in the previous 12 FRAMES, e.g. :

$$2a = 3b + 5 \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$2a - 3b = 5 \quad (-3b) \ldots\ldots \quad \text{(i)}$$

$$3a - 4 = 5b \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$
$$3a = 5b + 4 \quad (+4) \ldots\ldots\ldots \quad \text{(ii)}$$
$$3a - 5b = 4 \quad (-5b) \ldots\ldots \quad \text{(ii)}$$

So:

$$\begin{matrix} 2a = 3b + 5 \\ 3a - 4 = 5b \end{matrix} \xrightarrow[\text{become}]{} \begin{matrix} 2a - 3b = 5 \\ 3a - 5b = 4 \end{matrix}$$

Then the equations can be solved as before.

Re-arrange the following simultaneous equations into the form used in previous FRAMES (There is no need to solve them).

1. $2x = 3y - 4$
   $5y = 6 - 8x$

2. $4 + 2y = 3x$
   $5x = 12y - 8$

3. $18 = 3x - 2y$

$0 = 6x + y - 10$

4. $y + 8 = 2x$

$9x = -y + 5$

5. $13 + 2y + 3x = 0$

$5y - 8 = -x$

<u>ANSWERS 89</u>

1. $2x - 3y = -4$ or $-2x + 3y = 4$

$8x + 5y = 6$ or $-8x - 5y = -6$

2. $-3x + 2y = -4$ or $3x - 2y = 4$

$5x - 12y = -8$ or $-5x + 12y = 8$

3. $3x - 2y = 18$ or $-3x + 2y = -18$

$6x + y = 10$ or $-6x - y = -10$

4. $-2x + y = -8$ or $2x - y = 8$

$9x + y = 5$ or $-9x - y = -5$

5. $3x + 2y = -13$ or $-3x - 2y = 13$

$x + 5y = 8$ or $-x - 5y = -8$

FRAME 90

Returning to the problem in FRAME 77, we will now construct the simultaneous equation from the question.

The man overheard two customers.

The first customer had bought:

2 choc-ices and 3 lollipops costing £1.34

The second customer had bought:

3 choc-ices and 1 lollipop costing £1.03

By letting the cost of a choc-ice be x pence and the cost of a lollipop be y pence, these two statements become:

$$2x + 3y = 134 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$\underline{3x + y = 103} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

$$2x + 3y = 134 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$\underline{9x + 3y = 309} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(ii) X 3}$$
$$7x = 175 \quad \text{Subtract}$$
$$x = 25$$

Substitute $x = 25$ in (i)

$$50 + 3y = 134$$
$$3y = 84$$
$$y = 28$$

Check (i) $2 \times 25 + 3 \times 28$ $= 50 + 84$ $= 134$

(ii) $3 \times 25 + 28$ $= 75 + 28$ $= 103$

Therefore, the cost of a choc-ice is 25p.

and the cost of a lollipop is 28p.

So the man who ordered one choc-ice and one lollipop would expect to pay 53p.

FRAME 91 PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS

1. Two families entering a museum pay their entrance fees. One family of 2 adults and 2 children pay £1, while the other family of 1 adult and 3 children pay 90p. Find the entrance cost for a child and an adult.

2. A record player with 12 records cost £50. The same record player with 17 similar records cost £57.50. What is the cost of the record player?

3. The sum of the ages of a father and a son is 52 years. Eight years ago, the father was eight times as old as his son. How old is the father now?

4. A 3hr. and 2hr. video tape together cost £12. The 3hr. tape is £1.50 more expensive than the 2hr. tape. Find the cost of each type of tape.

ANSWERS 91

1.	Adult:	30p.	2.	£32	
	Child:	20p.			
3.	40 years		4.	2hr.:	£5.25
				3hr.:	£6.75

FRAME 92 SUMMARY

FRAMES 77 - 78	Introduction to simultaneous equations.
FRAMES 79 - 84	Elimination by subtraction.
FRAMES 85 - 86	Elimination by addition.
FRAME 87	Looking for the shortest method.
FRAME 88	Examples.
FRAME 89	Re-arrangement of equations.
FRAMES 90 - 91	Problems leading to simultaneous equations.

CHAPTER 6

TERMINAL TESTS AND RESULTS

6.1 Target Population

The programmed sections of Chapter 5 were tested on four different groups at the Lilley and Stone Upper School, Newark, during the Autumn Term, 1983. The groups within this school are set by ability as recommended by our feeder school, Sconce Hills High School. Each year contains two sets following an O-level course, four sets following a CSE course (Syllabus 1) and three sets following a limited Grade CSE (Syllabus II). These CSE groups are themselves set by ability to form one group containing the most able.

The algebra contained within the programmed text covers those relevant parts of the J.M.B. (Syllabus C) and E.M.R.E.B. It was decided to use one O-level group and the most able CSE group from each year.

The algebra contained within the text was of an O-level standard and it was thought unwise to use it on a CSE class unless there was a fair proportion of possible Grade 1 candidates.

The groups are distinguished in the table of results as 4-O, 4-C, 5-O and 5-C :

4-O	4th year O-level
4-C	4th year CSE-level
5-O	5th year O-level
5-C	5th year CSE-level

Mathematics has 5 out of 40 periods in a week. The text was used as classwork and homework over a period of one to two weeks for each section. The sections 1 and 2 were used on 4-O and 4-C while the sections 3 and 4 were used on 5-O and 5-C. In the event, because of mock examinations and general shortage of time, section 4 was never actually tested.

After much consideration it was decided not to test this text in another school at this early stage because, among other things, it would be impossible to administer the test consistently in another school and with completely comparable classes. The text was presented to the pupils as booklets which they kept. It was the only teaching material used on those subjects, but some of the students had met some of the items before. The teacher helped where necessary and issued some guidance on the pace at which they should be working.

6.2 Attitude Questionnaire

The opinion of the pupils was sought immediately after completing the programmed text. As the success of a programmed text depends so heavily on self-motivation, it was thought some measure of this opinion was necessary. The results to Q1 - Q4 are expressed as percentages of the total returns in 6.4. Answers to Q5 are considered in the comments on the results in Chapter 7.

ATTITUDE QUESTIONNAIRE

This is an opportunity for you to express your views on the booklets you have just completed. Please give serious thought to your answers as your opinions will be valued and possibly acted upon.

Q1. How did you prefer using these booklets compared with normal classwork?	**Better**	**No Preference**	**Worse**
Q2. How did having answers immediately available make you feel?	**Better**	**No Preference**	**Worse**
Q3. How did you like being allowed to work at your own pace?	**Better**	**No Preference**	**Worse**
Q4. How do you rate your own understanding of these topics in comparison with your normal classwork?	**Better**	**No Preference**	**Worse**

Q5. Were there any FRAMES that you found confusing or particularly difficult to understand? Try to give, if possible, reasons why.

6.3 THE TERMINAL TESTS AND MARKING SCHEMES.

TERMINAL TEST I

FACTORS

1. Find the set of factors of the following numbers:

   (a) 15 (b) 24 (c) 13

2. Write the following numbers in prime factor form:

   (a) 10 (b) 12 (c) 30

3. Factorise fully:

   (a) x^2y (b) $4xy^2$

4. Look at $3x^2y + 12xy^2$

   (a) What are the common factors?

   (b) Factorise the expression $3x^2y + 12xy^2$ fully.

5. Factorise the following fully:

   (a) $3a^2b - 3ab^2$

   (b) $10x^2 - 6xy + 8x$

   (c) $a^2b - ab + ab^2c$

6. Factorise:

   (a) $2ax - ay + 2bx - by$

   (b) $6mp + 9mq - 2np - 3nq$

   (c) $x^2y - ax^2 - 3y + 3a$

7. Factorise the following quadratic expressions:

   (a) $x^2 + 12x + 32$ (b) $x^2 + 5x - 14$

   (c) $a^2 - 14a + 33$ (d) $10b^2 + 3b - 4$

   (e) $6x^2 - 11x - 10$

8. Look at the following expressions:

(i) $a^2 - b^2$

(ii) $4x^2 - 9y^2$

(iii) $25a^2m^2 - 64p^2$

(a) What is the name given to expressions like (i) - (iii)?

(b) Factorise (i) - (iii) fully.

TERMINAL TEST I

ANSWERS AND MARKING SCHEME

			MARK
1.	(a)	1, 3, 5, 15	1
	(b)	1, 2, 3, 4, 6, 8, 12, 24	1
	(c)	1, 13	1
2.	(a)	2 X 5	1
	(b)	2 X 2 X 3	1
	(c)	2 X 3 X 5	1
3.	(a)	x X x X y	1
	(b)	2 X 2 X x X y X y	1
4.	(a)	3, x, y	1
	(b)	3xy(x + 4y) or 3 X x X y X (x + 4y)	1
5.	(a)	3ab(a - b)	1
	(b)	2x(5x - 3y + 4)	1
	(c)	ab(a - 1 + bc)	1
6.	(a)	(2x - y)(a + b)	1
	(b)	(2p + 3q)(3m - n)	1
	(c)	$(y - a)(x^2 - 3)$	1

			MARK
7.	(a)	(x + 4)(x + 8)	1
	(b)	(x - 2)(x + 7)	1
	(c)	(a - 3)(a - 11)	1
	(d)	(5b + 4)(2b - 1)	1
	(e)	(2x - 5)(3x + 2)	1
8.	(a)	Difference of two squares	1
	(b)	(i) (a + b)(a - b)	1
		(ii) (2x + 3y)(2x - 3y)	1
		(iii) (5am + 8p)(5am - 8p)	1
		TOTAL	25

TERMINAL TEST II

SIMPLE EQUATIONS

1. (a) From the following equations state which are in solution form:

(i) $3 + x = 2$ (ii) $4 = x - 1$

(iii) $x = 8$ (iv) $5 - x = 2x + 10$

(v) $5 = x$ (vi) $x + x = 10$

(b) Which of the options (i) - (vi) are said to be equivalent.

2. Solve the following equations:

(a) $x - 8 = 10$

(b) $13 = x + 5$

(c) $4x = 20$

(d) $\frac{x}{5} = 10$

(e) $2x + 3 = 5x - 60$

3. Solve:

(a) $12(x - 4) = 7(x + 6)$

(b) $3(2x - 5) - 2(x - 3) = 3$

4. Solve:

(a) $\frac{x}{6} - \frac{2x}{15} = \frac{1}{3}$

(b) $\frac{2(x + 1)}{3} - 4 = \frac{(2x - 15)}{2}$

5. Check whether $z = 7$ is the solution to the equation

$$\frac{2(z - 1)}{3} - \frac{z}{7} = 4z - 25$$

6. A man and his wife went to the theatre. He found that he spent £7.10 on two circle seats and a programme. The cost of a programme was 50p.

(a) By letting x represent the cost of one circle seat in pounds, form an equation in x.

(b) Solve the equation and hence state the cost of one circle seat.

7. Solve:

$$\frac{1}{x} + \frac{7}{x-1} = \frac{8}{x-2}$$

TERMINAL TEST II

ANSWERS AND MARKING SCHEME

	MARK
1. (a) (ii) and (v)	1
(b) (ii), (v) & (vi)	2 (1 for only 2)
2. (a) $x = 18$	1
(b) $x = 8$	1
(c) $x = 5$	1
(d) $x = 50$	1
(e) $x = 21$	1
3. (a) $12x - 48 = 7x + 42$	1
$x = 18$	1
(b) $6x - 15 - 2x + 6 = 3$	1
$x = 3$	1
4. (a) $5x - 4x = 10$ (Multiplication by 30)	1
$x = 10$	1
(b) $4(x + 1) - 24 = 3(2x - 15)$ (Multiplication by 6)	1
$4x + 4 - 24 = 6x - 45$	1
$x = 12\frac{1}{2}$	1
5. Correct substitution is L.H.S. giving 3	1
Correct substitution is R.H.S. giving 3	1

	MARK
6. (a) $2x + 0.5 = 7.1$	1
(b) $x = 3.30$	1
Cost of one seat £3.30	1
7. $(x - 1)(x - 2) + 7x(x - 2) = 8x(x - 1)$	1
$2 - 17x = -8x$	1
$x = \frac{2}{9}$	1
TOTAL	25

TERMINAL TEST III

QUADRATIC EQUATIONS

1. Look at the following equations:

(i) $x^2 + 2x + 4 = 0$

(ii) $x^2 = 3x - 2$

(iii) $2x^2 - 2 = 5x$

(iv) $3x^2 + 4 = 2x^2 - 2x$

(v) $x^2 + 2x = 8 + x^2$

(vi) $2x + 4 = -x^2$

(a) State which of the equations are quadratic equations.

(b) Which of the equations are equivalent to one another?

2. By using FACTORS solve the following quadratic equations:

(a) $x^2 - 5x - 24 = 0$

(b) $3x^2 - 7x + 2 = 0$

(c) $18x^2 + 25x - 3 = 0$

3. State the formula that solves the quadratic equation:

$ax^2 + bx + c = 0$

4. By using the formula from question 3 solve the quadratic equation:

$2x^2 - 8x + 1 = 0$, giving your answers correct to one decimal place.

5. The length of a rectangle is 6m. greater than its width. The area of the rectangle is $520m.^2$ Let the width be represented by x.

   (a) Form an equation in x

   (b) Solve the equation

   (c) Find the length and the width of the rectangle.

6. Look at the equations:

   (i) $x^2 + 5x + 1 = 0$

   (ii) $2x^2 + x + 1 = 0$

   (iii) $3x^2 = x - 2$

   (iv) $x^2 - 6x + 9 = 0$

   (v) $4x = x^2 - 8$

   Which of the equations given here have:

   (a) no real solutions

   (b) one real solution

   (c) two real solutions?

   NOTE It is not necessary to give the solutions.

TERMINAL TEST III

ANSWERS AND MARKING SCHEME

	MARK
1. (a) (i), (ii), (iii), (iv), (vi)	1
(b) (i), (iv), (vi)	2 (1 for one missing
2. (a) $(x + 3)(x - 8) = 0$	1
$x = -3$ or 8	1
(b) $(3x - 1)(x - 2) = 0$	1
$x = \frac{1}{3}$ or 2	1
(c) $(9x - 1)(2x + 3) = 0$	1
$x = \frac{1}{9}$ or $\frac{-3}{2}$	1
3. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	2
4. $x = \frac{8 \pm \sqrt{(-8)^2 - 4(2)(1)}}{4}$	1
$= \frac{8 \pm \sqrt{56}}{4}$	1
$= 3.9$ or 0.1	2
5. (a) $x(x + 6) = 520$	1
(b) $x^2 + 6x - 520 = 0$	1
$x = 20$ or -26	1
(c) Width: 20m.	1
Length: 26m.	1

			MARKS
6.	(a)	(ii), (iii)	2
	(b)	(iv)	1
	(c)	(i), (v)	2
		TOTAL	25

TERMINAL TEST IV

SIMULTANEOUS EQUATIONS

1. Solve the following simultaneous equations:

(a)
$$2x + 3y = 19$$
$$2x + y = 9$$

(b)
$$5x + 3y = 11$$
$$10x + 2y = 14$$

(c)
$$7x + 2y = 26$$
$$3x - 2y = 14$$

(d)
$$4p + 3q = 22$$
$$5p - 4q = 43$$

2. State what you would do to solve these simultaneous equations by the <u>quickest</u> method (you need not actually solve them):

(a) $x + 5y = 26$ .... (i)

$x + 2y = 14$ .... (ii)

(b) $3x - 2y = 10$ .... (i)

$x + 2y = 4$ .... (ii)

(c) $3x + 2y = 10$ .... (i)

$6x + 3y = 12$ .... (ii)

3. Re-arrange the following simultaneous equations and then solve them:

$$5x = 2y + 1$$
$$y = 3x - 2$$

4. Two adults' tickets and five children's tickets at the circus cost £3.50, whereas three adults' tickets and four children's tickets cost £3.85. What would be the cost for four adults and five children?

TERMINAL TEST IV

ANSWERS AND MARKING SCHEME

	MARKS
1. (a) $x = 2, \quad y = 5$	2
(b) $x = 1, \quad y = 2$	2
(c) $x = 4, \quad y = -1$	2
(d) $p = 7, \quad q = -2$	2
2. (a) (i) - (ii) or (ii) - (i)	1
(b) (i) + (ii)	1
(c) (i) X 2 - (ii) or (ii) - 2 X (i)	2
3. $5x - 2y = 1$ or $-5x + 2y = -1$	
$-3x + y = -2$ or $3x - y = 2$	2
$x = 3$	
$y = 7$	2
4. Construction of equations	1
Solution: Adult 75p	
Child 40p	2
4 Adults and 5 Children £5.00	1
TOTAL	20

6.4 RESULTS OF TERMINAL TESTS AND ATTITUDE QUESTIONNAIRE.

TABLE 1

4th Form C.S.E. Pupils   Terminal Test 1 - Factors

Pupil	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	T	%
C1	2	2	1	2	3	2	3	2	17	64
C2	0	2	1	2	3	2	3	2	15	60
C3			A B S	E N	T					
C4	1	2	1	0	2	2	2	2	12	48
C5	2	2	1	0	2	2	3	2	14	56
C6	3	3	2	2	3	3	4	3	23	92
C7	3	2	1	0	2	2	1	2	13	52
C8	2	2	1	0	1	1	0	1	8	32
C9	1	2	2	0	1	2	1	2	11	44
C10	2	2	2	2	1	1	0	2	12	48
C11	3	2	2	2	3	2	3	2	19	76
C12	2	1	1	2	3	2	3	2	16	64
C13	2	1	1	2	3	2	3	2	16	64
C14	0	0	1	1	0	0	1	1	4	16
C15	2	2	1	0	1	2	3	2	13	52
C16	2	3	2	0	2	2	3	2	16	64
C17	3	2	1	0	2	2	3	2	15	60
C18	2	1	1	0	2	1	0	1	8	32
C19	3	3	2	2	1	2	4	4	21	84
C20	3	2	1	2	2	2	2	2	16	64
C21	1	0	0	1	0	0	0	1	3	12
C22	1	1	0	0	2	0	0	1	5	20
C23	2	3	1	0	1	2	3	2	14	56
C24	3	3	2	2	2	2	4	4	22	88
C25	2	2	1	0	2	2	3	2	14	56
T	47	45	29	22	44	40	52	48		
%	65.3	62.5	60.4	45.8	61.1	55.6	43.3	50		

TABLE 2

4th Form 'O' - Level Pupils Terminal Test 1 - Factors

Pupil	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	T	%
01	2	3	2	2	2	2	3	3	19	76
02	3	3	2	2	3	2	3	3	21	84
03	3	3	2	2	2	2	3	2	19	76
04	3	3	1	2	2	3	4	2	20	80
05	3	2	2	1	2	2	2	2	16	64
06	2	3	2	2	1	2	3	3	18	72
07	3	3	2	2	3	3	5	4	25	100
08	3	3	1	0	2	3	2	4	18	72
09	1	3	1	2	3	3	5	4	22	88
010	3	3	2	2	3	0	1	1	15	60
011	3	3	2	2	2	1	2	4	19	76
012	1	3	2	2	2	1	2	4	17	68
013	3	3	2	2	2	3	3	2	20	80
014			A B	S E N	T					
015	2	3	1	0	1	2	3	4	16	64
016	2	3	2	2	3	2	2	4	20	80
017	3	3	2	1	3	1	2	4	19	76
018	3	2	2	1	2	3	4	3	20	80
019	1	3	2	2	3	0	1	4	16	64
020	3	3	1	2	3	3	5	1	21	84
021	0	2	2	2	3	2	5	3	19	76
022	3	3	2	2	2	0	2	3	17	68
023	3	3	2	2	2	3	3	3	21	84
024			A B	S E N	T					
025	3	3	1	1	2	1	3	3	17	68
026	2	3	1	2	2	1	3	3	17	68
027	2	3	2	2	3	3	4	3	22	88
028	3	3	2	2	1	0	2	3	16	64
T	63	75	45	44	59	48	77	79		
%	80.8	96.2	86.6	84.6	75.6	61.5	59.2	76.0		

TABLE 3

4th Form C.S.E. Pupils Terminal Test 2 - Simple Equations

Pupil	Q1	Q2	Q3	Q4	Q5	Q6	Q7		T	%
C1	2	5	3	2	2	2	0		16	64
C2	2	4	3	2	2	2	0		15	60
C3	1	2	2	2	2	0	0		9	36
C4	1	3	1	0	2	0	0		7	28
C5	2	4	3	2	2	2	0		15	60
C6	3	5	4	5	2	3	0		22	88
C7	2	4	3	2	2	2	0		15	60
C8	1	3	0	0	2	0	0		6	24
C9	2	4	3	2	2	2	0		15	60
C10				A B S	E N	T				
C11	2	4	2	2	2	0	1		13	52
C12	2	4	2	2	2	0	0		12	48
C13	2	4	3	2	2	2	0		15	60
C14	1	3	1	2	2	0	0		9	36
C15	2	4	3	2	2	2	1		16	64
C16	2	5	3	2	2	2	0		16	64
C17	2	5	3	2	2	2	0		16	64
C18	0	3	1	0	0	0	0		4	16
C19	3	5	4	5	2	3	2		24	96
C20	2	4	3	2	2	2	0		15	60
C21	0	2	1	0	0	0	0		3	12
C22	2	5	3	2	2	2	0		16	64
C23				A B S	E N	T				
C24	3	5	3	5	2	3	0		21	84
C25	2	4	3	2	2	2	0		15	60
T	41	91	57	47	42	33	4			
%	59.4	79.1	62.0	40.9	91.3	47.8	5.8			

TABLE 4

4th Form '0' - Level Pupils Terminal Test 2 - Simple Equations

Pupil	Q1	Q2	Q3	Q4	Q5	Q6	Q7		T	%
01	2	5	4	2	2	3	3		21	84
02	3	5	4	5	2	3	0		22	88
03	2	5	3	2	2	3	3		20	80
04	3	5	3	2	2	3	0		18	72
05	2	5	4	2	2	3	0		18	72
06	2	5	4	2	2	3	3		21	84
07	3	5	4	5	2	3	3		25	100
08	2	5	4	2	2	3	3		21	84
09	3	5	4	2	2	3	1		20	80
010	1	4	1	0	1	0	0		7	28
011	2	5	4	5	2	3	3		24	96
012	2	5	4	5	2	3	3		24	96
013	3	5	4	5	2	3	3		25	100
014	1	4	1	0	2	0	0		8	32
015	2	5	4	2	2	3	3		21	84
016	3	5	4	2	2	3	3		22	88
017	2	5	4	2	2	3	3		21	84
018	3	5	4	2	2	3	3		22	88
019	1	5	1	0	1	0	0		8	32
020	3	5	4	2	2	3	3		22	88
021	2	5	3	5	2	3	0		20	80
022	2	5	4	2	2	3	0		18	72
023	3	5	4	2	2	3	0		19	76
024	2	5	4	2	2	3	0		18	72
025	2	5	4	2	2	0	3		18	72
026	2	5	4	2	2	0	3		18	72
027	3	5	4	5	2	3	1		23	92
028	1	4	2	0	2	0	0		9	36
T	62	137	98	69	54	66	47			
%	73.8	97.9	87.5	49.3	96.4	78.6	56.0			

TABLE 5

5th Form C.S.E. Pupils Terminal Test 3 - Quadratic Equations

Pupil	Q1	Q2	Q3	Q4	Q5	Q6			T	%
C1	1	2	2	3	4	1			13	52
C2	2	3	2	3	4	1			15	60
C3	3	6	2	4	5	5			25	100
C4				A B E S E N T						
C5	2	3	2	2	4	1			14	56
C6	0	1	2	2	0	3			8	32
C7	2	4	2	1	4	5			18	72
C8	0	1	0	0	2	0			3	12
C9	2	3	2	1	2	5			15	60
C10	2	4	2	3	2	5			18	72
C11	2	3	2	1	4	1			13	52
C12	3	5	2	4	5	5			24	96
C13	3	6	2	1	5	5			22	88
C14	2	3	2	1	3	1			12	48
C15	1	2	2	1	3	1			10	40
C16	2	3	2	4	0	3			14	56
C17	1	2	2	4	5	3			17	68
C18	2	2	2	4	3	3			16	64
C19	1	2	2	1	5	3			14	56
C20	1	4	2	1	0	1			9	36
C21	2	3	2	2	3	5			17	68
C22	2	3	2	2	3	5			17	68
C23	0	1	2	2	0	1			6	24
C24	2	3	2	2	3	3			15	60
T	38	69	44	49	69	66				
%	55.1	50.0	95.7	53.3	60.0	57.4				

TABLE 6

5th Form 'O' - Level Pupils Terminal Test 3 - Quadratic Equations

Pupil	Q1	Q2	Q3	Q4	Q5	Q6			T	%
01	2	4	2	4	2	3			17	68
02	3	6	2	2	5	5			23	92
03	2	4	2	2	2	3			15	60
04	2	3	0	0	1	1			7	28
05	1	6	2	1	1	1			12	48
06	2	6	2	4	2	5			21	84
07	3	1	2	4	2	3			15	60
08	3	4	2	4	2	3			18	72
09	2	6	2	4	2	3			19	76
010	1	4	0	0	1	1			7	28
011	2	4	0	0	4	3			13	52
012	2	4	2	4	2	3			17	64
013	3	6	2	4	5	5			25	100
014	3	6	2	2	2	5			20	80
015	2	4	2	2	2	3			15	60
016	3	5	2	4	5	5			24	96
017	2	4	2	4	2	5			19	76
018			A B S	E N	T					
019	2	4	2	1	2	3			14	56
020	1	2	0	0	1	1			5	20
021	2	4	2	2	4	3			17	68
022	3	6	2	4	2	5			22	88
023	2	4	2	4	2	3			17	68
024	3	6	2	2	5	3			21	84
025			A B	S E N	T					
026			A B	S E N	T					
027	1	3	0	0	0	1			5	20
T	52	106	38	58	58	76				
%	72.2	73.6	79.2	60.4	48.3	63.3				

TABLE 7

SUMMARY:- Terminal Test 1 - Factors

Marks %	4-C %	4-O %	Total %
$\geqslant$ 90	4.2	3.8	4
$\geqslant$ 80	12.5	38.5	26
$\geqslant$ 70	16.7	65.4	42
$\geqslant$ 60	45.8	100	74
$\geqslant$ 50	66.7		84
$\geqslant$ 40	79.2		90
$\geqslant$ 30	87.5		94
$\geqslant$ 20	91.7		96
$\geqslant$ 10	100		100
$\geqslant$ 0			
$\bar{x}$	54.2	75.4	65.2
σ	20.8	9.3	19.1

TABLE 8

SUMMARY:- Terminal Test 2 - Simple Equations

Marks %	4-C %	4-O %	Total %
$\geqslant$ 90	4.3	17.9	11.8
$\geqslant$ 80	13.0	60.7	39.2
$\geqslant$ 70	13.0	85.7	52.9
$\geqslant$ 60	65.2	85.7	76.5
$\geqslant$ 50	69.6	85.7	78.4
$\geqslant$ 40	73.9	85.7	80.4
$\geqslant$ 30	82.6	96.4	90.2
$\geqslant$ 20	91.3	100	96.1
$\geqslant$ 10	100		100
$\geqslant$ 0			
$\bar{x}$	54.8	76.1	66.7
σ	20.9	19.8	22.9

TABLE 9

SUMMARY:- Terminal Test 3 - Quadratic Equations

Marks %	5-C %	5-0 %	Total %
$\geqslant 90$	8.7	12.5	10.6
$\geqslant 80$	13.0	29.2	21.3
$\geqslant 70$	21.7	41.7	31.9
$\geqslant 60$	52.2	70.8	61.7
$\geqslant 50$	73.9	79.2	76.6
$\geqslant 40$	82.6	83.3	83.0
$\geqslant 30$	91.3	83.3	87.2
$\geqslant 20$	95.7	100	97.9
$\geqslant 10$	100		100
$\geqslant 0$			
$\bar{x}$	58.2	64.5	61.4
σ	20.6	22.5	21.8

TABLE 10 ATTITUDE SURVEY - PERCENTAGE OF COMPLETED RETURNS

	Better	No Preference	Worse
Q1.	53	28	19
Q2.	73	15	12
Q3.	41	32	27
Q4	46	36	18

CHAPTER 7

DISCUSSION OF RESULTS AND REVISIONS

7.1 Introduction

The results of each terminal test used are shown in Chapter 6, Tables 1 - 9 inclusive. Table 10 shows the results of the Attitude Questionnaire.

Tables 1 - 6 show the individual scores of the pupils followed by their percentage mark. At the foot of each question column the total marks scored for that question are recorded. This is followed by a percentage that shows the ratio of scored marks to the total available for each individual question. It will be considered that a ratio of less than 50% indicates a possibly too difficult question.

Tables 7 - 9 summarise the earlier tables and show the cumulative percentages of pupils who score over a particular mark. Thus, it is these tables that must be considered when deciding whether or not the criterion behaviour has been achieved. Firstly at the foot of these tables are recorded the Mean ($\bar{x}$) and Standard Deviation (σ) for each group and the year considered as a whole.

In consideration of these results and professional criticism, there now follow some recommended alterations thought necessary.

7.2 Factors

FRAME 4

More explanation should be given to the method of finding a prime number. In particular, the number 2 could have been pointed out as the only even prime number. This would have speeded up the process of finding prime numbers as some students were checking far too many numbers.

FRAME 5

It was felt more advisable to letter the equations as opposed to numbering them. This could have caused some confusion in reading.

FRAME 7

Some confirmation of the cummutative law should be included here rather than later.

FRAMES 8 and 9

In retrospect it was felt that these FRAMES could have been expanded to three. A frame where the student was simply finding a common factor and not writing in factorised form may have been more worthwhile.

FRAME 15

This FRAME caused some difficulty. It is possibly too long, containing too much information and technique. Instead of noting that there is more than one way to group, this FRAME could be split into two, one describing the skill of grouping, i.e.

$$ax + bx + ay + by$$
$$x(a + b) + y(a + b)$$

or

$$ax + bx + ay + by$$
$$ax + ay + bx + by$$
$$a(x + y) + b(x + y)$$

and the other, taking the final step to factorise.

FRAME 25

This FRAME somewhat takes for granted that the student has noticed the required pattern. This was to prove not the case and the FRAME should have been more explicit in pointing out why it is possible to pick out the factors in one step.

FRAMES 26 and 27

It is thought better to reverse the examples used in these FRAMES. $6x^2 + 17x + 5$ would have been a simpler example to use to illustrate a new idea as opposed to $2x^2 - 5x - 3$.

Terminal Test 1

This test went quite well and achieved the criterion behaviour of 90/40, i.e. 90% of candidates achieving 40% or over (See Table 7). The O-level pupils achieved good results in this test and possibly found this early stage of the text quite easy. Question 4 proved difficult to the CSE group as many confused (a) and (b) and could not distinguish between them. The other question that proved difficult was Number 7 where few of the CSE group managed (d) and (e) correctly.

Overall Results

This section was undoubtably the best of the three tested. All the O-level set achieved marks over 60% and the majority of the CSE group achieved good results. The criterion behaviour stated in Chapter 4 was 90/40 and possibly in retrospect over optimistic.

The results of the CSE set were 79.2/40

The results of the O-level set were 100/40

And overall were 90/40.

7.3 Simple Equations

FRAME 33

At this stage simple equations are being considered at a 'concrete level' and it is now thought wrong to introduce the idea of adding -1 to either side in this FRAME. An extra FRAME making this step should perhaps be added after FRAME 34.

FRAME 44

It was perhaps a mistake to include Example 44.3 at this stage. An extra FRAME at least is necessary to include examples involving negative coefficients of the unknown.

FRAME 45

The alternative way of solution using factors claims to make numerical work easier. The examples in this FRAME do not show this and no response was required by the student. It was felt that either the FRAME should be missed out altogether or more detail given as to its use in the technique described in FRAME 46.

FRAME 46

It was at this stage of the text that certain lines of working were missed out. Most pupils by this stage were wanting to make this step and many had already done so. In retrospect it is felt that the insistence of all working being shown to this stage was perhaps unproductive, and the step of missing out the lines of working should have been made earlier.

FRAME 48

The questions asked of the student in this FRAME proved difficult and worked solutions are thought necessary.

FRAME 57

This FRAME proved difficult to understand, especially among the pupils of a more modest ability. The question on this kind of equation in the Terminal Test proved to be a disaster in the CSE set. Firstly, it is thought not enough to say: "To remove denominators in this case we multiply by $t(t + 1)$".

A parallel situation using numerical denominators would be useful and the skill of recognising the lowest common multiple of the denominators should be covered in a FRAME on its own. To continue to solve the equation in one FRAME was perhaps too much. It has been pointed out that no use of factors was made and skills learnt in Section 1 could have been quoted.

Terminal Test 2

Question 7 produced very poor results in the CSE set but acceptable results in the O-level set. The CSE set also found the problem question difficult although that perhaps is to be accepted given their overall ability. The surprising result was Question 4 where non-acceptable results (40.9% and 49.3%) were achieved by both sets. 4(b) perhaps needs revising as many candidates scored 0 after omitting to multiply -4 by 6.

Overall Results

Neither group achieved the criterion behaviour (See Table 8).

The CSE set achieved 73.9/40

The O-level set achieved 85.7/40

Giving the overall 80.4/40

The text did extend many of the students (particularly in the CSE set) beyond the level most appropriate to the majority and brings into question whether objectives of the calibre of question 7 should have been included.

7.4 Quadratic Equations

FRAME 66

Extra time should be spent here on the re-arrangement of quadratic equations into the standard way. An extra FRAME put in before this one would do this, using the same examples that need to be factorised.

FRAME 68

It was originally thought too much for this text to include 'completion of square'. The inclusion of this FRAME was inevitable if the solution by formula of quadratics was to be included. The feeling of many students and colleagues was that this FRAME was out of context and more explanation was needed.

FRAME 74

The remark: "Certainly it is not easily factorised" was misleading and lead some students to believe that it was possible to find real factors. This sentence should be missed out and a statement included: "It is not possible to factorise Eq. 74.1 into real factors so we will try the formula".

Terminal Test 3

The results of this test were very pleasing, in particular the CSE set. The only question failing to reach an acceptable result was surprisingly Question 5 in the O-level set. There does not seem to be an explanation for this as the errors were not consistent. The other surprising result was Question 3 where only memory was being tested and yet the CSE set received a 95.7% mark on this question whereas the O-level set only managed 79.2% It is thought now, perhaps, unwise to include Question 3 as an error here would mean a score of 0 on Question 4 as well.

Overall Results

Again neither group achieved the criterion behaviour but only just fell short.

The CSE set achieved 82.6/40

The O-level set achieved 83.3/40

Giving the overall 83/40.

The most surprising observation was the similarity of the results between the two sets in the 5th year, as this was not expected when their overall abilities were taken into account.

7.5 Attitude Questionnaire

It was decided not to attempt any numerical measurement of attitudes and the results shown in Table 10 are a crude measurement of subjective responses to Questions 1 - 4 on the Attitude Questionnaire. The replies of the final question were considered when writing this Chapter and varied from being helpful to being left blank. The fact that the pupils were asked to submit their opinions and were told this from the beginning encouraged many of them to have a positive attitude to this experiment.

7.6 General Conclusions

It was evident from the start that the 5th year sets would be hampered by time. Their 'mock' examinations were looming on the horizon and in the end they only had time to complete Section 3 satisfactorily enough to be tested. In addition this project was designed to be a complete study although the validation testing had to be spread over two year groups if the testing was to be completed in one term. This caused two major difficulties. Firstly, although the groups were theoretically the same ability in each year it made overall comparison between the first two Sections and Section 3 invalid. Secondly, the 5th year groups working through QUADRATIC EQUATIONS continually read references back to earlier FRAMES. Although they had experienced the subject matter of these earlier FRAMES in their 4th year, it was unhelpful that the referencing made them appear to have 'missed out' on something. However the text would normally be used throughout the 4th and 5th year and continuity between the Sections was seen as a major feature of the project.

The pupils used in the validation testing must not be seen as necessarily representative of the top 20 - 30% of the school population. No far-reaching conclusions can be made and it is not within the scope of this project to attempt to do so. A programmed text of this sort is still 'printed matter' and some students have shown an inability to succeed by using it.

The extra commitment necessary by the pupil in an exercise of this kind should also not be under-estimated. These sets saw this text as an innovation and for the most part were trying to please by being successful. The absence of the novelty of the situation could result in lowering of self-motivation so necessary in a successful application of a programmed text.

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