A Lyapunov-based approach for recursive continuous higher order nonsingular terminal sliding-mode control

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A Lyapunov-based approach for recursive continuous higher-order nonsingular terminal sliding mode control

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***Abstract*—A recursive continuous higher-order nonsingular terminal-sliding-mode (TSM) controller is proposed in this paper for nonlinear systems. A new integral TSM manifold is constructed in a recursive manner by modifying the tool of adding a power of integrator instead of exploring nonrecursive design directly. A super-twisting like reaching law is designed to achieve continuous control action without sacrificing disturbance rejection specification as that in boundary-layer approaches. By the new Lyapunov-based design, the proposed control method admits the following new features: 1) rather than imposing some existence condition for nonrecursive design, the proposed method admits the certainty for chosen fractional power to guarantee the finite-time stability of the closed-loop system; 2) an explicit Lyapunov function approach is proposed to establish finite-time stability of the closed-loop system; and 3) The proposed method is shown to be tunable to exhibit desired transient performance and control energy restriction.**

***Index Terms*—Recursive higher-order nonsingular terminal sliding mode; continuous finite-time control; Lyapunov stability; nonlinear systems.**

I. Introduction

Sliding-mode control (SMC) is recognized as one of the most efficient nonlinear robust control approaches in control systems for uncertain nonlinear systems subject to external disturbances [1–6]. Among the SMC methods, the terminal sliding mode (TSM) control has attracted a great deal of attention due to the prosperous property of finite-time convergence in the sliding phase, which brings about many advantages such as smaller steady-state tracking error and faster convergence rate [7]. A key drawback of TSM control is the singularity problem of the control law [8], which can be addressed by the so-called nonsingular TSM control approach [8–13].

It has been noticed that most existing nonsingulrar TSM control approaches can only address the control design problems of a class of second-order uncertain nonlinear systems [8, 11]. As such, much research has been concentrated on extending the second-order nonsingulrar TSM control to higher-order ones [9, 12, 13]. One most effective approach to realize higher-order nonsingulrar TSM control [9, 13] is the utilization of nonrecursive design method presented in [14]. The structure therein is concise due to the nonrecursive design and the controller gains can even be designed straightforwardly by following the pole placement method in linear control theory. However, it should be pointed out that the selection of the fractional power for this kind of higher-order nonsingular TSM is actually constrained to become an existence condition. That is, one can solely obtain the conclusion that there exists a parameter $\epsilon \in (0,1)$ such that the nonrecursive TSM can achieve the goal of finite-time control. Nevertheless, for any given $\epsilon \in (0,1)$, no result can show that the closed-loop system under the proposed controller is stable or not. As such, it is imperative to develop new solutions on TSM control design and analysis to overcome this drawback.

A new recursive continuous higher-order nonsingular TSM control approach is developed in this paper. The new sliding manifold and controller are constructed by means of modifying the tool of adding a power integrator [15] instead of utilizing the nonrecursive design directly. This admits the certainty for choosing fractional power that guarantees the finite-time stability of the closed-loop system. An explicit Lyapunov function is constructed for the closed-loop error system, which can be used to evaluate the upper bounds of the system states.

In addition, to reduce the well-known chattering effects [1, 16, 17] of nonsingular TSM control, a recursive continuous higher-order nonsingular TSM control is proposed by exploiting the super-twisting algorithm as the reaching law. The resultant reaching law admits a continuous control action that substantially alleviate the chattering influence. The dynamics of both the reaching and sliding phases are combined together and a composite Lyapunov function is constructed for stability analysis. By virtue of the Lyapunov-based design and analysis, the transient system states of the presented control approach can be evaluated in terms of initial states and control parameters.

To conclude, the main contributions and merits of the proposed work are summarized as follows:

1) To overcome the existence condition on power selection in existing nonsingular TSM manifolds [9, 13], a novel higher-order nonsingular TSM manifold is proposed by using a new principle for fractional power design, which

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provides sufficient conditions on finite-time stability with given power factor.

2) Due to the structural difference of the manifold design, the existing geometric homogeneity approach [14] for stability analysis is not applicable for the proposed method. Thus a well explicit Lyapunov function approach is developed to prove the finite-time stability of the resultant closed-loop system under the proposed approach.
3) As a byproduct, the new structure of proposed higher-order nonsingular TSM controller provides an additional design of freedom for parameter tuning. It is shown to be tunable to exhibit desired transient performance and control energy restriction.

II. Preliminaries and motivation

A. Definitions and useful lemmas

To begin with, we present some useful notations and definitions as follows.

(a) Let $\mathbb{N}_{j:k} := \{j, j+1, \dots, k\}$ be a set of nonnegative integers with j and k are integers satisfying $0 \le j \le k$. The symbol $\mathcal{L}_\infty$ denotes the set of all signals with bounded infinity-norms. We further define $\lfloor z \rceil^\alpha := \mathrm{sgn}(z)|z|^\alpha$, $\alpha > 0$, $\forall z \in \mathbb{R}$, where $\mathrm{sgn}(\cdot)$ is the standard signum function.
(b) The homogeneity properties [14] are provided below for convenience of the reader. For real numbers $r_i > 0$ for $i \in \mathbb{N}_{1:n}$ and a fixed choice of coordinates $\boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$. The dilation $\Delta^r : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is defined by $\Delta^r(\varepsilon, \boldsymbol{x}) := (\varepsilon^{r_1} x_1, \dots, \varepsilon^{r_n} x_n)$, $\forall \varepsilon > 0$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be of Δ^r-homogeneous of degree τ, denoted by $V \in \mathcal{H}^\tau_{\Delta^r}$ if $V(\Delta^r(\varepsilon, \boldsymbol{x})) = \varepsilon^\tau V(\boldsymbol{x})$. A Δ^r-homogenous 2-norm is defined as $\|\boldsymbol{x}\|_{\Delta^r} := \left(\sum_{i=1}^n |x_n|^{2/r_i}\right)^{1/2}$.

The inequalities presented by the following lemma is crucial for the derivation of the main result of the paper.

Lemma 1: [18] For $x \in \mathbb{R}, y \in \mathbb{R}$, and $\ell \ge 1$ is a constant, the following inequalities hold

$$
\begin{aligned}
|\lfloor x \rceil - \lfloor y \rceil|^\ell &\le 2^{\ell-1} \left| \lfloor x \rceil^\ell - \lfloor y \rceil^\ell \right|, \\
\left| \lfloor x \rceil^{\frac{1}{\ell}} - \lfloor y \rceil^{\frac{1}{\ell}} \right| &\le 2^{\frac{\ell-1}{\ell}} |x - y|^{\frac{1}{\ell}}, \\
|x + y|^\ell &\le 2^{\ell-1} |x^\ell + y^\ell|, \\
(|x| + |y|)^{\frac{1}{\ell}} \le |x|^{\frac{1}{\ell}} + |y|^{\frac{1}{\ell}} &\le 2^{\frac{\ell-1}{\ell}} (|x| + |y|)^{\frac{1}{\ell}}.
\end{aligned}
$$

Lemma 2: [19] If $p_1 > 0$ and $p_2 \ge 1$, then for $\forall x, y \in \mathbf{R}$, we have $\left| \lfloor x \rceil^{p_1/p_2} - \lfloor y \rceil^{p_1/p_2} \right| \le 2^{1-1/p_2} \left| \lfloor x \rceil^{p_1} - \lfloor y \rceil^{p_1} \right|^{1/p_2}$.

B. Motivation

We consider the following class of full-order nonlinear systems

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t), \\
\dot{x}_n(t) &= b(t, \boldsymbol{x}) u(t) + f(t, \boldsymbol{x}) + d(t, \boldsymbol{x}),
\end{aligned}
\tag{1}
$$

where $\boldsymbol{x} := [x_1, x_2, \cdots, x_n]^\top \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ are system state and control input, respectively. $f(t, \boldsymbol{x})$ and $b(t, \boldsymbol{x}) \neq 0$ are functions in terms of system state $\boldsymbol{x}$. The function $d(t, \boldsymbol{x})$ denotes the parameter uncertainties and external disturbances. Here it is supposed that the function $d(t, \boldsymbol{x})$ is unknown but its amplitude and its derivative are bounded such that $d(t, \boldsymbol{x}) < \delta_0$ and $\dot{d}(t, \boldsymbol{x}) < \delta$, where δ_0 and δ are positive constants. Without possible confusion, we will use $d(t)$ and $\dot{d}(t)$ subsequently instead of $d(t, \boldsymbol{x})$ and $\dot{d}(t, \boldsymbol{x})$ respectively for neatness of expression.

By virtue of nonrecursive finite-time design [14], the nonsingular TSM manifold [9] for system (1) is generally designed as follows

$$
s(t) = x_n(t) - x_n(0) + \int_0^t \sum_{i=1}^n \beta_i \lfloor x_i(t) \rceil^{\alpha_i} d\tau, \tag{2}
$$

where $\alpha_{i-1} = \alpha_i \alpha_{i+1} / (2\alpha_{i+1} - \alpha_i)$, $\alpha_{n+1} = 1$, $\alpha_n = \alpha_0 \in (1 - \epsilon, 1)$, and $\beta_i > 0$ should be assigned such that polynomial $\lambda^n + \beta_n \lambda^{n-1} + \cdots + \beta_2 \lambda + \beta_1$ is Hurwitz. It has been shown that there always exists a $\epsilon \in (0, 1)$ such that once the sliding mode $s(t) = \dot{s}(t) = 0$ achieves, the system dynamics will converge to their desired equilibrium in finite time. However, for any predesigned ϵ, no controller can theoretically rigorously ensure the desired stability result as stated above.

Consequently, we intend to develop a new recursive continuous higher-order nonsingular TSM controller whose power can be explicitly assigned to guarantee finite-time stability of the closed-loop system.

III. Recursive continuous full-order nonsingular TSM control design

Let the homogeneous degree τ be a negative real number, i.e., $\tau \in (-\infty, 0)$. Define the weighting element r_i satisfying $r_i = r_{i-1} + \tau$ and $r_i > 0$ for $i \in \mathbb{N}_{1:n}$. Let ρ satisfy $\rho \ge \max_{i \in \mathbb{N}_{1:n}} \{r_i\}$. A new recursive nonsingular TSM manifold is constructed as follows

$$
s(t) = x_n(t) - x_n(0) + \int_0^t \lambda_n \lfloor \sigma(\tau) \rceil^{\frac{r_{n+1}}{\rho}} d\tau, \tag{3}
$$

with

$$
\sigma(t) = \sum_{i=1}^n \beta_i \lfloor x_i(t) \rceil^{\frac{\rho}{r_i}}, \tag{4}
$$

where β_i for $i \in \mathbb{N}_{1:n}$ are coefficients to be *recursively* assigned as

$$
\beta_i = \prod_{k=i}^{n-1} (\lambda_k)^{\frac{\rho}{r_{k+1}}}, \ \beta_n = 1, \tag{5}
$$

for $i \in \mathbb{N}_{1:n-1}$, where λ_i is selected such that

$$
\lambda_i \ge \frac{L}{2^{i-1}} + \bar{c}_i + \hat{c}_i, \tag{6}
$$

with L being a constant gain to be assigned and

$$
\begin{aligned}
\bar{c}_i &= \frac{r_i}{\mu 2^{r_i/\rho}} \left(\frac{(2\mu - r_i) 2^{i - r_i/\rho}}{\mu L} \right)^{\frac{2\mu(2\mu - r_i)}{\rho r_i}}, \\
\hat{c}_i &= \frac{L}{2^{i+1}} + \sum_{k=1}^{i-1} \psi_{ik},
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
\phi_{ik} =& \frac{2^{1-r_i/\rho}(2\mu - r_i - \tau)}{r_k}\left(\prod_{l=k}^{i-1}(\lambda_l)^{\frac{\rho}{r_{l+1}}}\right)(\lambda_{k-1})^{\frac{\rho - r_k}{r_k}}\lambda_k,\\
\psi_{ik} =& \frac{2\phi_{ik}(2\mu - \tau - \rho)}{2\mu}\left(\frac{2^{i+1}\phi_{ik}(\tau+\rho)}{\mu L}\right)^{\frac{\tau+\rho}{2\mu-\tau-\rho}}.
\end{aligned} \tag{8}
$$

With the above new nonsingular TSM manifold (3), a recursive nonsingular TSM controller is designed as follows

$$
u(t) = b^{-1}(t, \boldsymbol{x})\left(u_{eq}(t) + u_r(t)\right), \tag{9}
$$

with the equivalent control law designed as

$$
u_{eq}(t) = -f(t, \boldsymbol{x}) - \lambda_n \lfloor \sigma(t) \rceil^{\frac{r_{n+1}}{\rho}}, \tag{10}
$$

and the reaching law constructed as

$$
u_r(t) = -k_1 \lfloor s(t) \rceil^{\frac{1}{2}} - k_2 \int_0^t \lfloor s(\tau) \rceil^0 d\tau. \tag{11}
$$

It should be noticed that the designed continuous nonsingular TSM manifold is different from those given in [9]. For nonrecursive TSM manifold (2), the gains β_i can be predesigned such that $\lambda^n + \beta_n\lambda^{n-1} + \cdots + \beta_2\lambda + \beta_1$ is Hurwitz, however there is no guidance for assignment of power factor ϵ guaranteeing stability. In the new manifold (3), the power factors ρ and r_i can be predesigned, and the finite-time stability of the closed-loop system can be guaranteed if the gains β_i can be assigned recursively based on (5)-(8). In addition, a super-twisting like reaching law (11) is constructed to generate a continuous control action that considerably reduces the chattering effects.

IV. Performance and Stability Analysis

The performance and stability analysis is Lyapunov-based here. To begin with, we give a group of variable definitions as follows. Let $\mu \geq \rho$ be a positive constant. Define $\vec{x}_i = (x_1, \ldots, x_i)^\top, i \in \mathbb{N}_{1:n}$ and the following virtual control laws

$$
\begin{aligned}
\xi_0 =& 0,\\
\alpha_i =& -\lambda_{i-1}\lfloor \xi_{i-1} \rceil^{\frac{r_i}{\rho}},\\
\xi_i =& \lfloor x_i \rceil^{\frac{\rho}{r_i}} - \lfloor \alpha_i \rceil^{\frac{\rho}{r_i}},
\end{aligned} \tag{12}
$$

for $i \in \mathbb{N}_{1:n-1}$, where λ_i are gains satisfying (6). We further define a positive definite Lyapunov function as

$$
V_i(\vec{x}_i) = \sum_{k=1}^{i} W_k(\vec{x}_k), \tag{13}
$$

with

$$
W_k(\vec{x}_k) = \int_{\alpha_k}^{x_k}\left\lfloor \lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} \right\rceil^{\frac{2\mu - r_k - \tau}{\rho}} d\eta. \tag{14}
$$

The proof of positive definiteness for $W_k(\vec{x}_k)$ is shown in appendix. With the above defined variables and Lyapunov functions, we have the following properties whose proofs are given in the appendix.

Proposition 1: By selecting $\bar{c}_i$ and $\hat{c}_i$ as shown in (7) with ϕ_{ik} and ψ_{ik} defined in (8), one can derive that

$$
\sum_{k=1}^{i-1}\frac{\partial W_i}{\partial x_k}\dot{x}_k \leq \frac{L}{2^{i-1}}\sum_{k=1}^{i-2}\xi_k^{\frac{2\mu}{\rho}} + \frac{L}{2^i}\xi_{i-1}^{\frac{2\mu}{\rho}} + \hat{c}_i\xi_i^{\frac{2\mu}{\rho}}. \tag{15}
$$

Proposition 2: Suppose that the gains λ_i are selected such that $\lambda_1 \geq L$ and $\lambda_i \geq \frac{L}{2^{i-1}} + \bar{c}_i + \hat{c}_i$ for $i \in \mathbb{N}_{1:n-1}$ (as shown in (6)) are satisfied. The defined Lyapunov function $V_i(\vec{x}_i)$ has the following property

$$
\begin{aligned}
\dot{V}_i(\vec{x}_i) \leq& -\frac{L}{2^{i-1}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_i^{\frac{2\mu}{\rho}}\right)\\
&+ \lfloor \xi_i \rceil^{\frac{2\mu - r_i - \tau}{\rho}}(x_{i+1} - \alpha_{i+1}),
\end{aligned} \tag{16}
$$

for $i \in \mathbb{N}_{1:n-1}$.

With the help of the above two Propositions, the main result of the paper can be derived and summarized in the following theorem.

Theorem 1: Suppose that the control parameters in the reaching law are designed such that $k_1 > 0$ and $k_2 > 3\delta + 2(\delta/k_1)^2$. For the full-order dynamic system (1), the new recursive nonsingular TSM controller (9)-(10)-(11) with sliding manifold designed by (3)-(4) renders the following properties:

- the resultant closed-loop system is globally stable in the sense that $x_i \in \mathcal{L}_\infty$, $u \in \mathcal{L}_\infty$ and $s \in \mathcal{L}_\infty$; furthermore, the state $x_i(t)$ is upper bounded by
$$
|x_i(t)| \leq \delta_i(\vec{x}(0), \tau, \vec{r}, \vec{\beta}), \tag{17}
$$
with $\delta_i(\vec{x}(0), \tau, \vec{r}, \vec{\beta})$ being a constant in terms of initial states and controller parameters;
- the finite-time stabilization of system states is guaranteed, i.e., $x_i \to 0$ as $t \to T$ with T being a finite time.

Proof. The proof of the theorem is divided into two parts. The first part will provide the bounded stability of the proposed nonsingular TSM controller. In the second part, the finite-time stability is further proved.

Part I: Define an auxiliary variable

$$
v(t) = d(t) - k_2\int_0^t \lfloor s(\tau) \rceil^0 d\tau.
$$

Taking derivative of the sliding manifold (3) along system dynamics (1) gives

$$
\begin{aligned}
\dot{s}(t) =& -k_1\lfloor s(t) \rceil^{\frac{1}{2}} + v(t),\\
\dot{v}(t) =& -k_2\lfloor s(t) \rceil^0 + \phi(t),
\end{aligned} \tag{18}
$$

where $\phi(t) = \dot{d}(t)$. Suppose that $\phi(t)$ satisfies $\phi(t) \leq \delta$. It follows from [20] that for a quadratic Lyapunov function $V_s(s, v) = \zeta^T P\zeta$ with $\zeta = \left[\lfloor s \rceil^{1/2}, v\right]^T$ and

$$
P = \begin{bmatrix} 4k_2 + k_1^2 & -k_1\\ -k_1 & 2 \end{bmatrix},
$$

the derivative of $V_s(s, v)$ satisfy the following

$$
\dot{V}_s(s, v) \leq -\frac{1}{|s|^{1/2}}\zeta^T Q\zeta \leq -\gamma_s V_s^{1/2}(s, v), \tag{19}
$$

where

$$Q = \frac{k_1}{2}\begin{bmatrix} 2k_2 + k_1^2 - 2\delta & -(k_1 + 2\delta/k_1) \\ -(k_1 + 2\delta/k_1) & 1 \end{bmatrix},$$

and $\gamma_s = \lambda_{\min}^{1/2}\{P\}\lambda_{\min}\{Q\}/\lambda_{\max}\{P\}$. By using Bihari's inequality [21] gives

$$||\zeta(t)|| \leq V_s^{\frac{1}{2}}(t)/\lambda_{\min}\{P\} \leq \zeta_{\max}, \tag{20}$$

where $\zeta_{\max} = V_s^{\frac{1}{2}}(0)/\lambda_{\min}\{P\}$. It should be noted that the controller proposed in this paper is a kind of second-order sliding mode controller using super-twisting algorithm [3]. Different from the traditional first-order sliding mode controller, s may reach zero non-monotonically ensuring the existence of the sliding mode.

By means of Lemmas 1, it can be calculated that the derivative of $V_n(\vec{x}_n)$ satisfies

$$\begin{aligned}\dot{V}_n(\vec{x}_n) \leq& -\frac{L}{2^{n-1}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_{n-1}^{\frac{2\mu}{\rho}}\right) + (\bar{c}_n + \hat{c}_n)\xi_n^{\frac{2\mu}{\rho}} \\ &+ \lfloor \xi_n \rceil^{\frac{2\mu - r_n - \tau}{\rho}}\left(b(t,\boldsymbol{x})u(t) + f(t,\boldsymbol{x}) + d(t,\boldsymbol{x})\right).\end{aligned} \tag{21}$$

By definitions (5) and (12), it can be derived that

$$\xi_n = \lfloor x_n \rceil^{\frac{\rho}{r_n}} + \sum_{j=1}^{n-1}\left(\beta_j \lfloor x_j \rceil^{\frac{\rho}{r_j}}\right).$$

Note that we further have $\sigma(t) = \xi_n$ by definition (4). With this in mind, substituting the nonsingular TSM control law (9) into the (21) gives

$$\begin{aligned}\dot{V}_n(\vec{x}_n) \leq& -\frac{L}{2^{n-1}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_{n-1}^{\frac{2\mu}{\rho}}\right) + (\bar{c}_n + \hat{c}_n)\xi_n^{\frac{2\mu}{\rho}} \\ &+ \lfloor \xi_n \rceil^{\frac{2\mu - r_n - \tau}{\rho}}\left(-\lambda_n \lfloor \xi_n \rceil^{\frac{r_{n+1}}{\rho}} + d + u_r\right).\end{aligned} \tag{22}$$

It can be derived from (20) that

$$\begin{aligned}|d + u_r| =& |v - k_1 \lfloor s \rceil^{\frac{1}{2}}| \\ \leq& \|\zeta\|\sqrt{2\max\{k_1^2, 1\}} \\ \leq& \zeta_{\max}\sqrt{2\max\{k_1^2, 1\}},\end{aligned} \tag{23}$$

By Young's inequality and (23), we have

$$\lfloor \xi_n \rceil^{\frac{2\mu - r_n - \tau}{\rho}}(d + u_r) \leq \tilde{c}_n \xi_n^{\frac{2\mu}{\rho}} + m, \tag{24}$$

where $\tilde{c}_n$ is any small positive constant and $m = \frac{r_n+\tau}{2\mu}\left(\frac{2\mu - r_n - \tau}{2\mu\tilde{c}_n}\right)^{\frac{2\mu(2\mu - r_n - \tau)}{\rho(r_n+\tau)}}\left(\zeta_{\max}\sqrt{2\max\{k_1^2, 1\}}\right)^{\frac{2\mu}{r_n+\tau}}$. Inserting the above inequality into (22) gives

$$\dot{V}_n(\vec{x}_n) \leq -\frac{L}{2^{n-1}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_n^{\frac{2\mu}{\rho}}\right) + m, \tag{25}$$

where λ_n is chosen such that $\lambda_n \geq L/2^{n-1} + \bar{c}_n + \hat{c}_n + \tilde{c}_n$.

Define a composite candidate Lyapunov function for the closed loop systems as follows

$$U(\vec{x}_n, s, v) = V_n(\vec{x}_n) + \bar{V}_s(s, v), \tag{26}$$

with $\bar{V}_s(s,v) = \nu V_s(s,v)^{(2\mu-\tau)/(-2\tau)}$, where ν is a positive constant given by

$$\nu = \left(\frac{\alpha_0 L}{2^{n-2}\gamma_s(1 - 2\mu/\tau)}\right)^{\frac{2\mu-\tau}{-\tau}}.$$

The function $U(\vec{x}_n, s, v)$ is positive definite and proper since $\tau < 0$. Taking into account inequality (19) and (25), one obtains that the derivative of $U(\vec{x}_n, s, v)$ satisfies

$$\begin{aligned}\dot{U}(\vec{x}_n, s, v) \leq& -\frac{L}{2^{n-1}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_n^{\frac{2\mu}{\rho}}\right) \\ &- \nu\gamma_s\left(\frac{1}{2} - \frac{\mu}{\tau}\right)V_s(s,v)^{\frac{\mu}{-\tau}} + m.\end{aligned} \tag{27}$$

By definition of $V_n(\vec{x}_n)$, it follows from the properties of homogeneity that

$$\alpha_0 V_n^{\frac{2\mu}{2\mu-\tau}}(\vec{x}_n) \leq \left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_n^{\frac{2\mu}{\rho}}\right), \tag{28}$$

where $\alpha_0 = (r_1/\rho - 1)^{2\mu/(2\mu-\tau)}$. Collecting the above facts in (27) and (28), we have

$$\begin{aligned}\dot{U}(\vec{x}_n, s, v) \leq& -\frac{L}{2^{n-1}}\alpha_0\left(V_n^{\frac{2\mu}{2\mu-\tau}}(\vec{x}_n) + \bar{V}_s^{\frac{2\mu}{2\mu-\tau}}(s,v)\right) + m, \\ \leq& -cU^{\alpha}(\vec{x}_n, s, v) + m,\end{aligned} \tag{29}$$

where $c := \frac{L\alpha_0}{2^{n-1}} \in (0, +\infty)$ and $\alpha := \frac{2\mu}{2\mu-\tau} \in (0,1)$. For neatness of presentation, we use $U(t)$ denoting the solution of $U(\vec{x}_n, s, v)$ along the closed-loop system dynamics. By using comparison lemma [22], it can be obtained that the solution $U(t)$ satisfies

$$U(t) \leq \begin{cases} \left[U_0^{1-\alpha} - c(1-\alpha)t\right]^{\frac{1}{1-\alpha}} + U_{\mathrm{u}}, & 0 \leq t < T_{\mathrm{u}} \\ U_{\mathrm{u}}, & t \geq T_{\mathrm{u}} \end{cases} \tag{30}$$

where $U_0 := U(\vec{x}_n(0), s(0), v(0))$ denotes the initial value of $U(\vec{x}_n, s, v)$, and U_{u} and T_u represent the ultimate bound and convergence time of $U(t)$, given by

$$U_{\mathrm{u}} = \left(\frac{m2^{n-1}}{L\alpha_0}\right)^{\frac{2\mu-\tau}{2\mu}}, \tag{31}$$

and

$$T_{\mathrm{u}} = \frac{2^{n-1}(2\mu - \tau)U_0^{\frac{-\tau}{2\mu-\tau}}}{-L\alpha_0\tau}. \tag{32}$$

Consequently, we can conclude from (29)-(32) that the system states and control input are ultimate bounded, i.e., $x_i \in \mathcal{L}_\infty$, $u \in \mathcal{L}_\infty$ and $s \in \mathcal{L}_\infty$.

By the definition of homogeneity, it can be verified that the homogeneous degree of $V_n(\vec{x}_n)$ is $\frac{2\mu-\tau}{2}$, i.e., $V_n(\vec{x}_n) \in \mathcal{H}_{\Delta^r}^{(2\mu-\tau)/2}$. With this in mind, it follows from homogeneity property that there exists a positive constant $\underline{c}$ such that

$$\underline{c}\|x\|_{\Delta^r}^{\frac{2\mu-\tau}{2}} \leq V_n(\vec{x}_n) \leq U(\vec{x}_n, s). \tag{33}$$

The system state is therefore upper bounded by

$$|x_i(t)| \leq \|x\|_{\Delta^r}^{r_i} \leq \delta_i(\vec{x}(0), \tau, \vec{r}, \vec{\beta}), \tag{34}$$

where

$$\delta_i(\vec{x}(0),\tau,\vec{r},\vec{\beta}) := \left(\frac{U_0+U_{\mathrm{u}}}{\underline{c}}\right)^{\frac{2r_i}{2\mu-\tau}}.$$

Part II: Since the boundedness property of system states and control input has been guaranteed as illustrated in Part I, we will establish the finite-time stability of the closed-loop system as follows.

The finite-time convergence of the reaching phase directly follows from the dynamics of sliding manifold as shown in (19). It can be obtained from (19) that there exists a finite time $T_r = 2V_{s0}^{1/2}/\gamma_s$ with $V_{s0} = V_s(s(0), v(0))$ such that $V_s(s(t), v(t)) \to 0$, $s(t) \to 0$ and $v(t) \to 0$ as $t \to T_r$. Once the sliding manifold is reached, we have $s(t) = 0$ and $v(t) \to 0$ for $t \geq T_r$. It can be obtained from (18) that $s(t) = \dot{s}(t) = 0$ and hence $d(t) + u_r(t) = 0$ for $t \geq T_r$.

Since we have assigned that $\lambda_n \geq L/2^{n-1} + \bar{c}_n + \hat{c}_n + \tilde{c}_n$, it further follows from (22) that

$$\begin{aligned}\dot{V}_n(\vec{x}_n) \leq& -\frac{L}{2^{n-1}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_n^{\frac{2\mu}{\rho}}\right)\\ \leq& -cV_n^{\frac{2\mu}{2\mu-\tau}}(\vec{x}_n),\end{aligned} \tag{35}$$

for $t \geq T_r$. Finally, it can be obtained from (35) that $V_n(\vec{x}_n(t)) \to 0$ and hence $x_i(t) \to 0$ as $t \to T$ where $T = T_r + T_s$ with $T_s = \frac{2^{n-1}(2\mu-\tau)V_{n0}^{\frac{-\tau}{2\mu-\tau}}}{-L\alpha_0\tau}$ and $V_{n0} = V_n(\vec{x}_n(0))$.

V. DISCUSSION

In this section, we will further discuss the differences between the proposed one and the existing ones. The TSM manifold proposed in this paper (3) shares some similarities with the TSM manifold (2) proposed by [9, 13], but the design principles of the two TSM manifolds are quite different, which are reflected in the following aspects.

1) *Structure Difference*: The term in the integral part of the TSM manifold in (2) is a sum of several fractional power items, i.e., $\sum_{i=1}^{n}\beta_i\lfloor x_i(t)\rceil^{\alpha_i}$, while the corresponding one in the proposed manifold is a fractional power of a sum of several items, i.e., $\lambda_n\lfloor\sigma(\tau)\rceil^{\frac{r_{n+1}}{\rho}}$ with $\sigma(t) = \sum_{i=1}^{n}\beta_i\lfloor x_i(t)\rceil^{\frac{\rho}{r_i}}$.

2) *Fractional Power and Gain Design*: The fractional powers α_i and gains β_i of TSM manifold in (2) are determined by following a non-recursive manner. The designed powers and gains provide an existence condition to guarantee finite-time stability. However, the proposed TSM manifold in the paper gives a systematic approach to design fractional powers and gains following a recursive design manner, which are sufficient to guarantee finite-time stability.

For an intuitive and clear explanation, we revisit the design processes of the two kinds of TSMs for a concrete numerical example. Consider a third-order nonlinear system as follows

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= 100u + x_1^2x_2 + x_3^3 + d(t),\end{aligned} \tag{36}$$

where $d(t)$ is the external disturbance.

The proposed TSM manifold is designed as

$$s(t) = x_3(t) - x_3(0) + \int_0^t \lambda_3\lfloor\sigma(\tau)\rceil^{\frac{r_4}{\rho}}d\tau, \tag{37}$$

where $\sigma(t) = \beta_1\lfloor x_1\rceil^{\frac{\rho}{r_1}} + \beta_2\lfloor x_2\rceil^{\frac{\rho}{r_2}} + \lfloor x_3\rceil^{\frac{\rho}{r_3}}$. The proposed continuous nonsingular TSM controller is designed as

$$\begin{aligned}u_{eq} &= x_1^2x_2 + x_3^3 + \lambda_3\lfloor\sigma(t)\rceil^{\frac{r_4}{\rho}},\\ u_r &= k_1\lfloor s\rceil^{\frac{1}{2}} + k_2\int_0^t\lfloor s(\tau)\rceil^0 d\tau,\\ u &= -\frac{1}{100}(u_{eq}+u_r).\end{aligned} \tag{38}$$

Following the nonsingular TSM manifold (2), one has

$$s(t) = x_3(t) - x_3(0) + \int_0^t \beta_1\lfloor x_1\rceil^{\alpha_1} + \beta_2\lfloor x_2\rceil^{\alpha_2} + \beta_3\lfloor x_3\rceil^{\alpha_3}d\tau, \tag{39}$$

and the controller is

$$\begin{aligned}u_{eq} &= x_1^2x_2 + x_3^3 + \beta_1\lfloor x_1\rceil^{\alpha_1} + \beta_2\lfloor x_2\rceil^{\alpha_2} + \beta_3\lfloor x_3\rceil^{\alpha_3},\\ u_r &= k_1\lfloor s\rceil^{\frac{1}{2}} + k_2\int_0^t\lfloor s(\tau)\rceil^0 d\tau,\\ u &= -\frac{1}{100}(u_{eq}+u_r),\end{aligned} \tag{40}$$

with $\alpha_{i-1} = \alpha_i\alpha_{i+1}/(2\alpha_{i+1} - \alpha_i)$, $\alpha_4 = 1$, $\alpha_3 = \alpha_0$. It is clear that both the controllers and sliding manifolds are quite different from structure, fractional power selection and gain design, which can support the statements about the contributions in the Introduction.

Owing to the above mentioned difference of the sliding manifold design, the proposed TSM manifold in the paper shows some interesting properties over the existing ones [9,13], which are summarized below.

- First, for the existing TSM manifold (2), the parameter chosen following the non-recursive manner can only admit an existence condition for finite-time stability. That is, with given fractional powers and gains following the design guidelines, the closed-loop system could be finite-time stable, but there is a lack of rigorous theoretical proof for that. However, in the proposed TSM manifold (3), for any $\tau \in (-\infty, 0)$, we establish rigorous finite-time stability theoretically.
- Second, the stability of the new result is established in the context of Lyapunov theory rather than geometric theory. That is, instead of using the geometric approach to establish the stability in existing TSM manifold (2), some new tools, in particular an explicit Lyapunov function approach, are proposed to establish finite-time stability of the closed-loop system under the proposed control approach.
- Last, it can be observed from (37) and (39) that the main difference between the proposed TSM manifold and the existing one lies in that the parameter ρ provides an additional degree of freedom on parameter tuning. This parameter admits the possibility for the proposed method to obtain a better trade-off between various conflict performance specifications.

Remark 1. In practical implementation, the requirement of full-order states will possibly meet some difficulties as full state measurements are expensive and even impossible in some cases. Actually, similar to many existing output feedback sliding mode control approaches [23] [24], it is straightforward to combine the proposed algorithm with the well-known higher-order sliding mode observer to form an output feedback control algorithm. Both the design and analysis of the resultant output feedback control system is relatively simple by utilizing the finite-time separation principle of observer and controller design [3] [17], which will not be given the details due to space limitation.

VI. Simulations

In simulation, the numerical example in (36) is taken for performance analysis and comparison. The sliding manifold and the controller are designed as (37) and (39), respectively.

In (37), the parameters are set as $\tau = -1/6, r_1 = 1$, $r_i = r_{i-1} + \tau$ for $i \in \mathbb{N}_{1:4}$, $\rho = 2$, $\beta_1 = 100$, $\beta_2 = 10$ and $\lambda_3 = 100$. In (38), we choose $k_1 = 500$, $k_2 = 500$. For the existing TSM controller (39) and (40), we set $\alpha_0 = 0.75$, $\beta_1 = 50$, $\beta_2 = 30$, $\beta_3 = 30$, $k_1 = 20000$, $k_2 = 20000$. The initial values of the system are set as $[x_1(0), x_2(0), x_3(0)]^T = [1, 1, 1]^T$. Response curves of states and the control input for system (36) in the absence of external disturbance are shown in Fig. 1. The corresponding phase trajectory of the closed-loop system is shown in Fig. 2.

We further investigate how the parameter ρ that provides an additional degree of freedom parameter tuning can affect the control performance. To do so, different cases for $\rho = 2,\ 5,\ 10$ and 15 are carried out, while the simulation results are shown in Figs. 1 and 2. As shown by Fig. 1, it can be observed that smaller ρ will result in faster dynamic responses, but demand larger control energies. Therefore, the parameter ρ can be served as an adjustment to balance the convergent rate and the control energy for the system.

The following part will focus on disturbance rejection performance. Toward that end, the external disturbance $d(t) = 100 + 30sin(2\pi t)$ is taken into account, which is assumed to impose on the system at $t = 15$ sec. Response curves of states and the control input for system (36) under the two controllers are shown in Fig. 3. As shown by Fig. 3(a)-Fig. 3(c), the states of the system are driven to the desired equilibrium under the proposed controller with lower drop/rise and shorter settling time than the existing TSM control strategy. As such, the external disturbance can be attenuated effectively by the proposed control law, which maintains the nominal control performance of the system and provides a better disturbance rejection performance.

Remark 2. The parameter ρ provides an additional freedom for parameter tuning for the proposed algorithm. Actually, this parameter has great impacts on control performance. As shown by the simulation results in Fig. 1, larger ρ will lead to larger overshoot and longer settling time, but the transient control effort is relatively mild especially at the beginning of setpoint changes. In practice, it is suggested to select appropriate parameter ρ to balance the transient performance specifications like overshoot, settling time and control energy constraint.

VII. Conclusion

The full-order finite-time sliding mode control problem has been addressed by designing a recursive continuous higher-order nonsingular TSM controller. It has been shown that the proposed approach admits the certainty of given power factor to ensure finite-time stability of the closed-loop system. This has overcome the theoretical limitation of existence condition of existing higher-order nonsingular TSM control

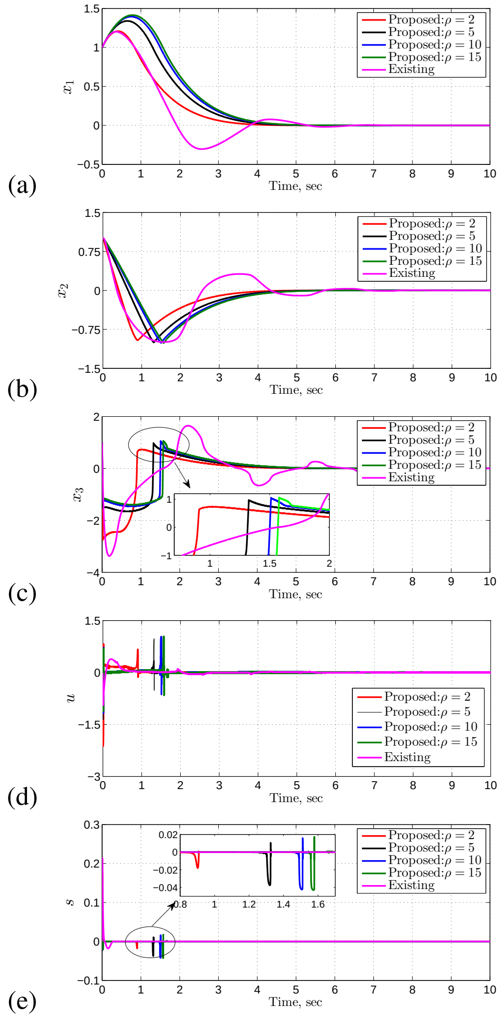

Fig. 1. Response curves in the absence of disturbances. (a) system state x_1; (b) system state x_2; (c) system state x_3; (d) control input u; (e) sliding manifold s.

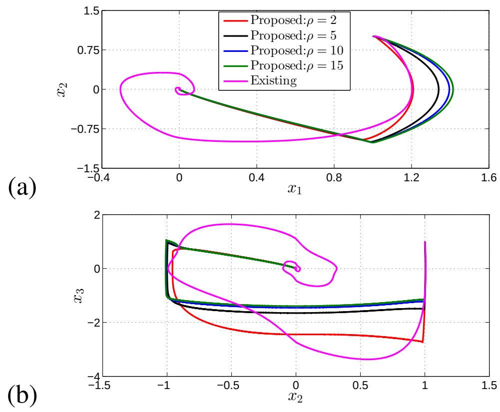

Fig. 2. Phase portrait of the closed-loop system in the absence of disturbance under the proposed approach: (a) $2-D$ of x_1 and x_2; (b) $2-D$ of x_2 and x_3.

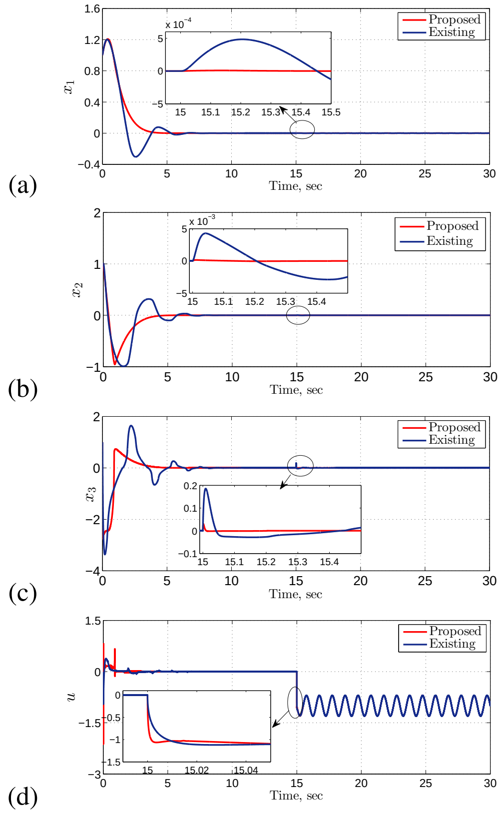

Fig. 3. Response curves in the presence of disturbances. (a) system state x_1; (b) system state x_2; (c) system state x_3; (d) control input u.

on the basis of nonrecursive design. The new controller has also brought about several new features such as possibility to assess transient performance specification and continuous control action to alleviate chattering effects.

Appendix

A. Proof of positive definiteness for $W_k(\vec{x}_k)$

The expression of W_k is rewritten as follows:

$$W_k(\vec{x}_k) = \int_{\alpha_k}^{x_k} \left\lfloor \lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} \right\rceil^{\frac{2\mu - r_k - \tau}{\rho}} d\eta. \tag{41}$$

Next, we will consider the following two cases.

Case 1. $x_k > \alpha_k$, then $\eta \in (\alpha_k, x_k)$.

Under this condition, we have $\lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} \geq 0$. From this, we obtain

$$\begin{aligned} W_k(\vec{x}_k) &= \int_{\alpha_k}^{x_k} \left(\lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} \right)^{\frac{2\mu - r_k - \tau}{\rho}} d\eta \\ &= \int_{\alpha_k}^{x_k} \left[\left(\lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} \right)^{\frac{r_k}{\rho}} \right]^{\frac{2\mu - r_k - \tau}{r_k}} d\eta. \end{aligned} \tag{42}$$

By Lemma 2, it obtains

$$(\eta - \alpha_k) \leq 2^{1 - \frac{r_k}{\rho}} \left(\lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} \right)^{\frac{r_k}{\rho}}. \tag{43}$$

Combining (42) and (43), we have

$$\begin{aligned} W_k(\vec{x}_k) &\geq \int_{\alpha_k}^{x_k} 2^{\frac{r_k}{\rho} - 1} (\eta - \alpha_k)^{\frac{2\mu - r_k - \tau}{r_k}} d\eta \\ &= 2^{\frac{(r_k - \rho)(2\mu - r_k - \tau)}{\rho r_k}} \frac{r_k}{2\mu - \tau} \left(x_k - \alpha_k \right)^{\frac{2\mu - \tau}{r_k}}. \end{aligned} \tag{44}$$

Case 2. $x_k < \alpha_k$, then $\eta \in (x_k, \alpha_k)$.

Defining $f(\eta) = \lfloor \eta \rceil^{\frac{\rho}{r_k}} - \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}}$, it can be obtained

$$\begin{aligned} W_k(\vec{x}_k) &= -\int_{x_k}^{\alpha_k} |f(\eta)|^{\frac{2\mu - r_k - \tau}{\rho}} sign(f(\eta)) d\eta \\ &= \int_{x_k}^{\alpha_k} |f(\eta)|^{\frac{2\mu - r_k - \tau}{\rho}} sign(-f(\eta)) d\eta \\ &= \int_{x_k}^{\alpha_k} \lfloor -f(\eta) \rceil^{\frac{2\mu - r_k - \tau}{\rho}} d\eta \\ &= \int_{x_k}^{\alpha_k} \left\lfloor \lfloor \alpha_k \rceil^{\frac{\rho}{r_k}} - \lfloor \eta \rceil^{\frac{\rho}{r_k}} \right\rceil^{\frac{2\mu - r_k - \tau}{\rho}} d\eta \end{aligned} \tag{45}$$

After the similar analysis with *Case 1*, we have

$$\begin{aligned} W_k(\vec{x}_k) &\geq \int_{\alpha_k}^{x_k} 2^{\frac{r_k}{\rho} - 1} (\eta - \alpha_k)^{\frac{2\mu - r_k - \tau}{r_k}} d\eta \\ &= 2^{\frac{(r_k - \rho)(2\mu - r_k - \tau)}{\rho r_k}} \frac{r_k}{2\mu - \tau} \left(\alpha_k - x_k \right)^{\frac{2\mu - \tau}{r_k}}. \end{aligned} \tag{46}$$

Therefore, we can get that $W_k(\vec{x}_k)$ is positive definite.

B. Proof of Proposition 1

Keeping in mind the definitions of $W_i(\vec{x}_i)$, ϕ_{ik} and ψ_{ik} in (14) and (8), by utilizing Lemma 1 and Young's inequality, we have the following derivations

$$\begin{aligned} \frac{\partial W_i}{\partial x_k} \dot{x}_k \leq & 2^{1 - r_i/\rho} \frac{2\mu - r_i - \tau}{\rho} |\xi_i|^{\frac{2\mu - \tau - \rho}{\rho}} \left| \frac{\partial \lfloor \alpha_i \rceil^{\frac{\rho}{r_i}}}{\partial x_k} \dot{x}_k \right| \\ \leq & \phi_{ik} |\xi_i|^{\frac{2\mu - \tau - \rho}{\rho}} \left(|\xi_k|^{\frac{\rho - r_k}{\rho}} + |\xi_{k-1}|^{\frac{\rho - r_k}{\rho}} \right) \\ & \times \left(|\xi_{k+1}|^{\frac{r_{k+1}}{\rho}} + |\xi_k|^{\frac{r_{k+1}}{\rho}} \right) \\ \leq & \frac{L}{2^{i+1}} \left(|\xi_{k-1}|^{\frac{2\mu}{\rho}} + |\xi_k|^{\frac{2\mu}{\rho}} + |\xi_{k+1}|^{\frac{2\mu}{\rho}} \right) + \psi_{ik} \xi_i^{\frac{2\mu}{\rho}}. \end{aligned} \tag{47}$$

The conclusion of the proposition can be derived by taking sum of the inequalities (47) in terms of index k.

C. Proof of Proposition 2

With the definitions of α_1, ξ_1 and $V_1(x_1)$ given by (12) and (13) in mind, differentiating $V_1(x_1)$ along system dynamics (1) gives

$$\dot{V}_1(x_1) = \lfloor \xi_1 \rceil^{\frac{2\mu - \tau - r_1}{\rho}} (x_2 - \alpha_2) + \lfloor \xi_1 \rceil^{\frac{2\mu - \tau - r_1}{\rho}} \alpha_2, \tag{48}$$

where α_2 is a virtual control law designed as (12). Since the parameter λ_1 has been selected such that $\lambda_1 \geq L$, it then follows from (48) that

$$\dot{V}_1(x_1) \leq -L \xi_1^{\frac{2\mu}{\rho}} + \lfloor \xi_1 \rceil^{\frac{2\mu - \tau - r_1}{\rho}} (x_2 - \alpha_2). \tag{49}$$

The proof in the following is presented in an inductive way. Consequently, we suppose that at step $i-1$, the candidate Lyapunov function $V_{i-1}(\vec{x}_{i-1})$ satisfies

$$\begin{aligned}\dot{V}_{i-1}(\vec{x}_{i-1}) \leq &- \frac{L}{2^{i-2}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_{i-1}^{\frac{2\mu}{\rho}}\right) \\ &+ \lfloor \xi_{i-1} \rceil^{\frac{2\mu - r_{i-1} - \tau}{\rho}} (x_i - \alpha_i).\end{aligned} \tag{50}$$

It is clearly shown by (49) that the inequality (50) holds for the case when $i = 2$. By definitions of $V_i(\vec{x}_i)$ and $W_k(\vec{x}_k)$ given in (13) and (14) respectively, we have $V_i(\vec{x}_i) = V_{i-1}(\vec{x}_{i-1}) + W_i(\vec{x}_i)$. As such, the derivative of $V_i(\vec{x}_i)$ can be calculated, which is estimated as follows

$$\begin{aligned}\dot{V}_i(\vec{x}_i) \leq &- \frac{L}{2^{i-2}}\left(\xi_1^{\frac{2\mu}{\rho}} + \cdots + \xi_{i-1}^{\frac{2\mu}{\rho}}\right) + \sum_{k=1}^{i-1} \frac{\partial W_i}{\partial x_k}\dot{x}_k \\ &+ \lfloor \xi_{i-1} \rceil^{\frac{2\mu - r_{i-1} - \tau}{\rho}} (x_i - \alpha_i) + \frac{\partial W_i}{\partial x_i} x_{i+1}.\end{aligned} \tag{51}$$

To derive the result in the proposition, we have to estimate the rest terms in the above inequality. Firstly, by Lemma 1, we have

$$\lfloor \xi_{i-1} \rceil^{\frac{2\mu - r_{i-1} - \tau}{\rho}} (x_i - \alpha_i) \leq \frac{L}{2^i}\xi_{i-1}^{\frac{2\mu}{\rho}} + \bar{c}_i \xi_i^{\frac{2\mu}{\rho}}, \tag{52}$$

where $\bar{c}_i$ has been defined in (7). As λ_i has been assigned to satisfy inequality (6), by inserting the estimations in (52) and (15) into (51), we can derive the inequality shown in (16).

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