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Synchronization of Generally Uncertain Markovian Inertial Neural Networks with Random Connection Weight Strengths and Image Encryption Application

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***Abstract*—This paper focuses on the synchronization problem of delayed inertial neural networks (INNs) with generally uncertain Markovian jumping and their applications in image encryption. The random connection weight strengths and generally uncertain Markovian are discussed in INNs model. Compared with most existing INNs models that have constant connection weight strengths, our model is more practical because connection weight strengths of INNs may randomly vary owing to the external and internal environment and human factor. The delay-range-dependent synchronization conditions (DRDSC) could be obtained by adopting the delay-product-term Lyapunov-Krasovskii functional (DPTLKF) and higher order polynomial based relaxed inequality (HOPRII). In addition, the desired controllers are obtained by solving a set of linear matrix inequalities. Finally, two examples are shown to demonstrate the effectiveness of the proposed results.**

***Index Terms*—Inertial neural networks, random connection weight strengths, delay-product-term Lyapunov-Krasovskii functional, higher order polynomial based relaxed inequality**

I. Introduction

In recent years, neural networks have received considerable attention because of their broad applications in pattern recognition, associative memories, signal processing and secure communication, etc [1]–[8]. Due to the finite switching speed of the amplifier, time delays often occur between neurons and may result in complicated chaotic dynamics [3]–[12].

There are many papers devoting to Markovian jumping systems (MJSs) because many practical systems experience random changes in their parameters and structures [13]–[27]. In [28], the stability problem of discrete-time linear systems with random jumping parameters was investigated, and necessary and sufficient conditions of mean square stability were obtained. The global exponential stability problem was addressed for delayed recurrent neural networks with Markovian jumping parameters in [29]. For relaxing the traditional assumption in MJSs, the stability analysis and stabilization problems of discrete-time Markovian linear systems with partially known transition probabilities were investigated in [30]. After that, the stability and synchronization of discrete-time neural networks were investigated based on mixed mode-dependent time delays and Markovian jumping in [31]. By considering the partially known transition rates, the stability and synchronization problems of Markovian jumping neural networks with time-varying delays were investigated in [20]. Based on [20] and [30], the finite-time stochastically stability problem of Markovian jumping memristive neural networks with partly unknown transition rates was concerned by introducing the Markovian switching Lyapunov functional. The delay-dependent robust fault detection problem of MJSs with partly unknown transition rates was addressed in [32]. Different from partly unknown transition rates, the passivity problem of Markovian jumping neural networks with piecewise-constant transition rates was studied in [33]. By fully considering the property of transition rates and the characteristic of uncertain domains, the stability problems of Markovian jumping linear systems with uncertain transition rates were proposed [34]. Based on [34] and [35], Markovian nonlinearly coupled neural networks with generally uncertain transition rates were investigated in [27], and the corresponding local synchronization conditions were proposed.

Inertial neural networks (INNs), as a kind of special neural networks, were described by the second-order differential equation [36]. In recent years, the INNs have attracted much attention [37]–[42]. The second-order term is named inertial term in INNs and could produce more complex dynamic behavior than normal neural networks with first-order term. Based on integral inequality method, the finite-time synchronization conditions of INNs were proposed in [37]. The fixed-time synchronization of inertial memristor-based neural networks was investigated, and four different feedback controllers were designed in [40]. Because of the chaotic characteristics of INNs, they were utilized in image encryption/decryption [43] and [44]. The synchronization of Markovian INNs was investigated in [43], and the obtained results were applied in image encryption. Based on [43], the synchronization conditions of delayed INNs with generally Markovian jumping were proposed in [45]. In [45], the uncertain and unknown elements in the transition rates matrix are solved based on Schur complement and matrix inequality Lemma, which increases the dimension and computational burden of INNs synchronization conditions. In addition, the

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connection weight strengths of INNs may randomly vary because of environmental and artificial factors. It is necessary to further consider the synchronization and image encryption problems of INNS with random connection weight strengths.

Motivated by the foregoing discussions, this paper explores the synchronization of generally uncertain Markovian INNs with random connection weight strengths and their applications in image encryption. Based on delay-product-term Lyapunov - Krasovskii functional (DPTLKF) and higher order polynomial-based relaxed inequality (HOPRII) containing more information of upper and low bounds of time delay and time delay derivative, the new synchronization conditions and desired controllers are proposed to ensure drive system and response system synchronized in mean square. The contributions of this paper are summarized as follows.

1) Compared with most existing INNs with constant connection weight strengths [37]–[39], the INNs in this paper is more practical because connection weight strengths of INNs may randomly vary owing to the influence of device performances and the uncertainties of external environment.
2) The DPTLKF is adopted in this paper to deal with the synchronization of generally uncertain Markovian INNS with random connection weight strengths, which makes the synchronization conditions include more information about time-varying delay and its derivative.
3) Different from the existing generally uncertain Markovian NNs [45]–[47], Schur complement and matrix inequality Lemma are not adopted in the process of handling generally uncertain Markovian, and only one set of relaxation variables are utilized to deal with generally uncertain transition rates in this paper, which reduces the dimension and computational complexity of synchronization conditions. By in-depth exploiting relationships among the generally uncertain transition rates, the new synchronization criteria are derived for the INNs.

The rest of this paper is arranged as follows. In Section II, some preliminaries and generally uncertain INNs are introduced. In Section III, the synchronization criteria and controllers of INNs are proposed. In Section IV, two examples are given to demonstrate the validity of the proposed results. Finally, conclusions are drawn in Section V.

II. Problem Statement and Preliminaries

A. Notation

Notation: Throughout the article, $\mathcal{R}^n$ and $\mathcal{R}^{n\times n}$ denote n-dimensional Euclidean space and $n\times n$ real matrices, respectively. $\|\cdot\|$ stands for Euclidean vector norm. A^T denotes the transpose of matrix A. When $A>0$, A means a symmetric positive definite matrix. $\mathrm{Sym}\{A\}=A+A^T$. E_n denotes the n-dimensional identity matrix. $\begin{pmatrix} A & B \\ * & C \end{pmatrix}$ stands for $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$. $\mathcal{E}$ denotes the mathematical expectation. $e_x=[0_{n\times(x-1)n}, E_n, 0_{n\times(17-x)n}]$ $(x=1,2,\cdots,17)$.

B. Problem Formulation

$\{\imath_t,\ t\geq 0\}$ is a right-continuous Markovian process on the probability space, which takes values in a finite state space $\mathcal{C}=\{1,2,\cdots,\mathcal{M}\}$ with generator $\Omega=(\lambda_{ab}),(a,b\in\mathcal{C})$ given by

$$Pr\{\imath_{t+\Delta t}=b\mid \imath_t=a\}=\begin{cases}\lambda_{ab}\Delta t+o(\Delta t), a\neq b,\\ 1+\lambda_{aa}\Delta t+o(\Delta t), a=b,\end{cases}$$

where $\Delta t>0$ and $\lim\limits_{\Delta t\to 0}(o(\Delta t)/\Delta t)=0$. $\lambda_{ab}\geq 0$ $(a\neq b)$ is the transition rate from mode a at time t to mode b at time $t+\Delta t$, and $\lambda_{aa}=-\sum\limits_{b=1,b\neq a}^{\mathcal{M}}\lambda_{ab}$.

In this paper, transition rates of the jumping process are generally uncertain. For instance, transition rate matrix (TRM) Ω with $\mathcal{M}$ operation modes may be expressed as

$$\Omega=\begin{pmatrix} \tilde{\lambda}_{11}+\triangle_{11} & ? & \tilde{\lambda}_{13}+\triangle_{13} & \cdots & \tilde{\lambda}_{1\mathcal{M}}+\triangle_{1\mathcal{M}} \\ ? & ? & \tilde{\lambda}_{23}+\triangle_{23} & \cdots & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\lambda}_{\mathcal{M}1}+\triangle_{\mathcal{M}1} & ? & ? & \cdots & \tilde{\lambda}_{\mathcal{M}\mathcal{M}}+\triangle_{\mathcal{M}\mathcal{M}} \end{pmatrix}, \tag{1}$$

where "?" denotes the completely unknown transition rate. $\tilde{\lambda}_{ab}$ and $\triangle_{ab}$ denote the estimate value and estimate error of the uncertain transition rate λ_{ab}, respectively. $\|\triangle_{ab}\|\leq\varpi_{ab}$ and $\varpi_{ab}\geq 0$. $\tilde{\lambda}_{ab}$, ϖ_{ab} are known. For the clarity of the notation, $\forall\ a\in\mathcal{C}$, we denote $\mathcal{C}^a=\mathcal{C}^a_k\bigcup\mathcal{C}^a_{uk}$, where $\mathcal{C}^a_k=\{b:$ The estimate value of λ_{ab} is known for $b\in\mathcal{C}\}$ and $\mathcal{C}^a_{uk}=\{b:$ The estimate value of λ_{ab} is unknown for $b\in\mathcal{C}\}$. On the basis of the characteristics of the transition rates, the following cases are assumed.
If $\mathcal{C}^a_k\neq\mathcal{C}$, and $a\notin\mathcal{C}^a_k$, then $\tilde{\lambda}_{ab}-\varpi_{ab}\geq 0,(\forall b\in\mathcal{C}^a_k)$.
If $\mathcal{C}^a_k\neq\mathcal{C}$, and $a\in\mathcal{C}^a_k$, then $\tilde{\lambda}_{ab}-\varpi_{ab}\geq 0,(\forall b\in\mathcal{C}^a_k, b\neq a)$, $\tilde{\lambda}_{aa}+\varpi_{aa}\leq 0$, and $\sum\limits_{b\in\mathcal{C}^a_k}\tilde{\lambda}_{ab}\leq 0$.
If $\mathcal{C}^a_k=\mathcal{C}$, then $\tilde{\lambda}_{ab}-\varpi_{ab}\geq 0,(\forall b\in\mathcal{C}, b\neq a),\tilde{\lambda}_{aa}=-\sum\limits_{b=1,b\neq a}^{\mathcal{M}}\tilde{\lambda}_{ab}\leq 0$, and $\varpi_{aa}=\sum\limits_{b=1,b\neq a}^{\mathcal{M}}\varpi_{ab}\geq 0$.

Remark 1: Different from most existing TRM, the generally uncertain TRM (1) is considered for INNs in this paper, which is more general and applicable. When $\triangle_{ab}=0$, generally uncertain TRM (1) is reduced to partially unknown TRM. When $\mathcal{C}^a_k=\mathcal{C}$ $(\forall b\in\mathcal{C})$, generally uncertain TRM (1) is reduced to bounded uncertain TRM.

Consider the INNs as follows

$$\frac{d^2u_k(t)}{dt^2}=-a_k(\imath_t)\frac{du_k(t)}{dt}-b_k(\imath_t)u_k(t)+\alpha(t)\sum_{l=1}^{n}w^1_{kl}(\imath_t)$$

$$f_l(u_l(t))+\beta(t)\sum_{l=1}^{n}w^2_{kl}(\imath_t)f_l(u_l(t-\tau(t)))+T_k \tag{2}$$

where $k=1,2,\cdots,n$, $u_k(t)$ denotes the state of the kth neuron at time t. $a_k(\imath_t)>0$, $b_k(\imath_t)>0$ are constants. $w^1_{kl}(\imath_t)$ and $w^2_{kl}(\imath_t)$ are connection weights related to the neurons. $f_l(.)$ is the activation function of lth neurons with $f_l(0)=0$. T_k is the external input of the kth neuron at time t. The time-varying delay $\tau(t)$ satisfies $0\leq\tau(t)\leq\tau$, $\mu_1\leq\dot{\tau}(t)\leq\mu_2$, where τ, μ_1 and μ_2 are constants. $\alpha(t)$, $\beta(t)$ denote the random connection weight strengths. The initial condition associated with (2) is given as follows: $u_k(s)=\phi_k(s)$, $\frac{du_k(s)}{ds}=\tilde{\phi}_k(s)$, $s\in([-\tau,0])$ and $\phi_k(t)$, $\tilde{\phi}_k(t)\in\mathcal{C}([-\tau,0],\mathcal{R})$. The mathematical expectations and variances of $\alpha(t)$ and $\beta(t)$ are $\mathcal{E}\{\alpha(t)\}=\bar{\alpha}$, $\mathcal{E}\{(\alpha(t)-\bar{\alpha})^2\}=\nu_\alpha$, $\mathcal{E}\{\beta(t)\}=\bar{\beta}$ and $\mathcal{E}\{(\beta(t)-\bar{\beta})^2\}=\nu_\beta$.

Remark 2: Different from most existing INNs [37], [42]–[45] with constant connection weight strengths, the random connection weight strengths are considered for INNS in this paper, which is more practical because connection weight strengths of INNs may randomly vary owing to environmental and artificial interferences.

Assumption 1 [48] For any $u_1,u_2\in\mathcal{R}$, there are constants l_k^-, l_k^+, such that

$$l_k^-\le\frac{f_k(u_1)-f_k(u_2)}{u_1-u_2}\le l_k^+, k=1,2,\cdots,n.$$

We denote

$$L_1=\mathrm{diag}\{l_1^+l_1^-,\cdots,l_n^+l_n^-\},\ L_2=\mathrm{diag}\{\frac{l_1^++l_1^-}{2},\cdots,\frac{l_n^++l_n^-}{2}\}.$$

For constant $\mho_k$, the following transformation is employed

$$\flat_k(t)=\frac{du_k(t)}{dt}+\mho_k u_k(t), k=1,2,\cdots,n. \quad (3)$$

Then, system (2) is rewritten as the following form

$$\begin{aligned}\frac{du_k(t)}{dt}=&-\mho_k u_k(t)+\flat_k(t)\\ \frac{d\flat_k(t)}{dt}=&-[b_k(\imath_t)+\mho_k(\mho_k-a_k(\imath_t))]u_k(t)-(a_k(\imath_t)-\mho_k)\\ &\times\flat_k(t)+\alpha(t)\sum_{l=1}^n w_{kl}^1(\imath_t)f_l(u_l(t))\\ &+\beta(t)\sum_{l=1}^n w_{kl}^2(\imath_t)f_l(u_l(t-\tau(t)))+T_k\\ =&-\tilde{a}_k(\imath_t)u_k(t)-\tilde{b}_k(\imath_t)\flat_k(t)+\alpha(t)\sum_{l=1}^n w_{kl}^1(\imath_t)\\ &\times f_l(u_l(t))+\beta(t)\sum_{l=1}^n w_{kl}^2(\imath_t)f_l(u_l(t-\tau(t)))+T_k,\end{aligned} \quad (4)$$

where $\tilde{a}_k(\imath_t)=b_k(\imath_t)+\mho_k(\mho_k-a_k(\imath_t))$, $\tilde{b}_k(\imath_t)=a_k(\imath_t)-\mho_k$. The initial condition associated with (4) is considered as follows $u_k(s)=\phi_k(s)$, $\frac{du_k(s)}{ds}=\tilde{\phi}_k(s)$, $\flat_k(s)=\tilde{\phi}_k(s)+\mho_k\phi_k(s)$, and $s\in([-\tau,0])$.

Now, system (4) is rewritten as the following form

$$\begin{aligned}\frac{du(t)}{dt}=&-Au(t)+\flat(t),\\ \frac{d\flat(t)}{dt}=&-B(\imath_t)\flat(t)-C(\imath_t)u(t)+\alpha(t)W^1(\imath_t)f(u(t))\\ &+\beta(t)W^2(\imath_t)f(u(t-\tau(t)))+T,\end{aligned} \quad (5)$$

where $u(t)=[u_1(t),u_2(t),\ldots,u_n(t)]^T$, $\flat(t)=[\flat_1(t),\flat_2(t),\ldots,\flat_n(t)]^T$, $A=\mathrm{diag}\{\mho_1,\cdots,\mho_n\}$, $B(\imath_t)=\mathrm{diag}\{a_1(\imath_t)-\mho_1,\cdots,a_n(\imath_t)-\mho_n\}$, $C(\imath_t)=\mathrm{diag}\{b_1(\imath_t)+\mho_1(\mho_1-a_1(\imath_t)),\cdots,b_n(\imath_t)+\mho_n(\mho_n-a_n(\imath_t))\}$, $W^1(\imath_t)=(w_{kl}^1(\imath_t))_{n\times n}$, $W^2(\imath_t)=(w_{kl}^2(\imath_t))_{n\times n}$, $T=[T_1,\cdots,T_n]$.

To be more convenient, each possible value of $\imath_t$ is denoted by a ($a\in\mathcal{C}$). According to (5), one gets

$$\begin{aligned}\frac{du(t)}{dt}=&-Au(t)+\flat(t),\\ \frac{d\flat(t)}{dt}=&-B_a\flat(t)-C_a u(t)+\alpha(t)W_a^1 f(u(t))\\ &+\beta(t)W_a^2 f(u(t-\tau(t)))+T\end{aligned} \quad (6)$$

For the drive system (6), the response system is considered as

$$\begin{aligned}\frac{d\hat{u}(t)}{dt}=&-A\hat{u}(t)+\hat{\flat}(t)+v_1(t),\\ \frac{d\hat{\flat}(t)}{dt}=&-B_a\hat{\flat}(t)-C_a\hat{u}(t)+\alpha(t)W_a^1\hat{f}(\hat{u}(t))\\ &+\beta(t)W_a^2\hat{f}(\hat{u}(t-\tau(t)))+T+v_2(t),\end{aligned} \quad (7)$$

where $\hat{u}(t)=[\hat{u}_1(t),\hat{u}_2(t),\ldots,\hat{u}_n(t)]^T$, $\hat{\flat}(t)=[\hat{\flat}_1(t),\hat{\flat}_2(t),\ldots,\hat{\flat}_n(t)]^T$ are state responses of controlled system. $v_1(t)\in\mathcal{R}^n$, $v_2(t)\in\mathcal{R}^n$ are control inputs. The errors are $e_1(t)=\hat{u}(t)-u(t)$, $e_2(t)=\hat{\flat}(t)-\flat(t)$, and $g(e_1(t))=\hat{f}(\hat{u}(t))-f(u(t))$. Then, the error dynamic system is given as follows

$$\begin{aligned}\frac{de_1(t)}{dt}=&-Ae_1(t)+e_2(t)+v_1(t),\\ \frac{de_2(t)}{dt}=&-B_a e_2(t)-C_a e_1(t)+\alpha(t)W_a^1 g(e_1(t))\\ &+\beta(t)W_a^2 g(e_1(t-\tau(t)))+v_2(t).\end{aligned} \quad (8)$$

The control inputs of the error dynamic system (8) are given as follows

$$\begin{aligned}v_1(t)&=L_{1a}e_1(t),\\ v_2(t)&=L_{2a}e_2(t).\end{aligned} \quad (9)$$

Then, system (8) is rewritten as the following form

$$\begin{aligned}\frac{de_1(t)}{dt}=&-(A-L_{1a})e_1(t)+e_2(t),\\ \frac{de_2(t)}{dt}=&-(B_a-L_{2a})e_2(t)-C_a e_1(t)+\alpha(t)W_a^1 g(e_1(t))\\ &+\beta(t)W_a^2 g(e_1(t-\tau(t))).\end{aligned} \quad (10)$$

Lemma 1: ([49]). For the real scalar $\alpha\in(0,1)$, vectors ω_1, ω_2, given symmetric matrices $W_1>0$, $W_2>0$, if a matrix $X\in\mathcal{R}^{n\times n}$ satisfies $\begin{bmatrix}W_1 & X\\ * & W_2\end{bmatrix}\ge 0$, then the following inequality holds

$$\begin{aligned}&\frac{1}{\alpha}\omega_1^T W_1\omega_1+\frac{1}{1-\alpha}\omega_2^T W_2\omega_2\\ &\ge\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}^T\begin{bmatrix}W_1 & X\\ * & W_2\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}.\end{aligned} \quad (11)$$

Lemma 2: (HOPRII, [50]). $x(t)$ is a differentiable function in $[a_1,a_2]\to\mathcal{R}^n$ for a time-varying scalar $a(t)\in[a_1,a_2]$. For symmetric matrices $Z_l=Z_l^T>0$, $(l=1,2)$, and any matrices N_1, N_2, the following inequality holds

$$a_{12}\int_{a_1}^{a(t)}\dot{x}^T(s)Z_1\dot{x}(s)ds+a_{12}\int_{a(t)}^{a_2}\dot{x}^T(s)Z_2\dot{x}(s)ds\ge$$

$$\varsigma^T(t)\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}^T\Delta\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\varsigma(t), \tag{12}$$

where

$$\Delta=\begin{bmatrix}\widetilde{Z_1}+(1-\gamma)\widetilde{\mho}_1 & (1-\gamma)N_1+\gamma N_2\\ * & \widetilde{Z_2}+\gamma\widetilde{\mho}_2\end{bmatrix},$$

$$\gamma=\frac{a(t)-a_1}{a_{12}}, a_{12}=a_2-a_1,$$

$$\varsigma(t)=[x^T(a_2),\ x^T(a(t)),\ x^T(a_1),\ \varsigma_1^T(t),\ \varsigma_2^T(t),\ \varsigma_3^T(t),\ \varsigma_4^T(t),\ \varsigma_5^T(t),\ \varsigma_6^T(t)]^T,$$

$$\varsigma_1(t)=\frac{1}{a(t)-a_1}\int_{a_1}^{a(t)}x(s)ds,$$

$$\varsigma_2(t)=\frac{1}{a_2-a(t)}\int_{a(t)}^{a_2}x(s)ds,$$

$$\varsigma_3(t)=\frac{2}{(a(t)-a_1)^2}\int_{a_1}^{a(t)}\int_u^{a(t)}x(s)dsdu,$$

$$\varsigma_4(t)=\frac{2}{(a_2-a(t))^2}\int_{a(t)}^{a_2}\int_u^{a_2}x(s)dsdu,$$

$$\varsigma_5(t)=\frac{6}{(a(t)-a_1)^3}\int_{a_1}^{a(t)}\int_u^{a(t)}\int_v^{a(t)}x(s)dsdvdu,$$

$$\varsigma_6(t)=\frac{6}{(a_2-a(t))^3}\int_{a(t)}^{a_2}\int_u^{a_2}\int_v^{a_2}x(s)dsdvdu,$$

$$\widetilde{Z}_1=\mathrm{diag}\{Z_1,3Z_1,5Z_1,7Z_1\},\ \widetilde{Z}_2=\mathrm{diag}\{Z_2,3Z_2,5Z_2,7Z_2\},$$

$$\widetilde{\mho}_1=\widetilde{Z}_1-N_2\widetilde{Z}_2^{-1}N_2^T,\ \widetilde{\mho}_2=\widetilde{Z}_2-N_1^T\widetilde{Z}_1^{-1}N_1,$$

$$\omega_1=[(\widetilde{e}_2-\widetilde{e}_3)^T,(\widetilde{e}_2+\widetilde{e}_3-2\widetilde{e}_4)^T,(\widetilde{e}_2-\widetilde{e}_3+6\widetilde{e}_4-6\widetilde{e}_6)^T,$$
$$(\widetilde{e}_2+\widetilde{e}_3-12\widetilde{e}_4+30\widetilde{e}_6-20\widetilde{e}_8)^T]^T,$$

$$\omega_2=[(\widetilde{e}_1-\widetilde{e}_2)^T,(\widetilde{e}_1+\widetilde{e}_2-2\widetilde{e}_5)^T,(\widetilde{e}_1-\widetilde{e}_2+6\widetilde{e}_5-6\widetilde{e}_7)^T,$$
$$(\widetilde{e}_1+\widetilde{e}_2-12\widetilde{e}_5+30\widetilde{e}_7-20\widetilde{e}_9)^T]^T,$$

$$\widetilde{e}_k=[0_{n\times(k-1)n}\ I_{n\times n}\ 0_{n\times(9-k)n}],(k=1,2,\cdots,9).$$

Remark 3: According to [50], the matrix Δ in (12) could provide extra freedom to increase the accuracy of estimating the integral terms.

Lemma 3: ([51]). For the matrix $S>0$, the following inequality holds

$$\int_{l_1}^{l_2}v^T(s)Sv(s)ds\geq\frac{1}{l_2-l_1}(\int_{l_1}^{l_2}v(s)ds)^TS(\int_{l_1}^{l_2}v(s)ds)$$
$$+\frac{3}{l_2-l_1}\Xi_1^TS\Xi_1+\frac{5}{l_2-l_1}\Xi_2^TS\Xi_2, \tag{13}$$

where

$$\Xi_1=\int_{l_1}^{l_2}v(s)ds-\frac{2}{l_2-l_1}\int_{l_1}^{l_2}\int_u^{l_2}v(s)dsdu,$$

$$\Xi_2=\int_{l_1}^{l_2}v(s)ds-\frac{6}{l_2-l_1}\int_{l_1}^{l_2}\int_u^{l_2}v(s)dsdu$$
$$+\frac{12}{(l_2-l_1)^2}\int_{l_1}^{l_2}\int_u^{l_2}\int_v^{l_2}v(s)dsdvdu$$

.

III. Main results

In this section, the synchronization conditions of generally uncertain Markovian INNs are obtained based on DPTLKF and HOPRII.

Theorem 1. Under Assumption 1, the drive system (6) and response system (7) are synchronous if there are any matrices $X_1\in\mathcal{R}^{2n\times 2n}$, $X_2\in\mathcal{R}^{6n\times 6n}$, $Y_l\in\mathcal{R}^{4n\times 4n}$ $(l=1,2)$, invertible matrices $\hat{F}_l,\ F_l\in\mathcal{R}^{n\times n}$ $(l=1,2)$, symmetric matrices $P_1\in\mathcal{R}^{4n\times 4n}$, $P_v\in\mathcal{R}^{2n\times 2n}, P_v>0$ $(v=2,3)$, Q_{kp}, $Q_{kq}\in\mathcal{R}^{2n\times 2n}$ $(k=3,4)$, Z_l $(l=1,2)$, symmetric positive definite matrices H_a, K_a, V_{lab} $(l=1,2)$, R, $M_1\in\mathcal{R}^{2n\times 2n}$, $M_2\in\mathcal{R}^{n\times n}$ and $Q_l\in\mathcal{R}^{2n\times 2n}$ $(l=1,2)$, positive definite diagonal matrices R_1, R_2, R_3, such that for any $a\in\mathcal{C}$, the succeeding matrix inequalities are satisfied.

If $a\notin\mathcal{C}_k^a$,

$$H_b-H_a-V_{1ab}\leq 0,\forall b\in\mathcal{C}_k^a, \tag{14}$$
$$K_b-K_a-V_{2ab}\leq 0,\forall b\in\mathcal{C}_k^a, \tag{15}$$
$$H_b-H_a\leq 0,\forall b\in\mathcal{C}_{uk}^a,b\neq a, \tag{16}$$
$$K_b-K_a\leq 0,\forall b\in\mathcal{C}_{uk}^a,b\neq a, \tag{17}$$

$$\begin{bmatrix}\Xi(\mu_1,0) & \Sigma_1^TY_2\\ * & -\widehat{M}_2\end{bmatrix}<0, \tag{18}$$

$$\begin{bmatrix}\Xi(\mu_1,\tau) & \Sigma_2^TY_1^T\\ * & -\widehat{M}_2\end{bmatrix}<0, \tag{19}$$

$$\begin{bmatrix}\Xi(\mu_2,0) & \Sigma_1^TY_2\\ * & -\widehat{M}_2\end{bmatrix}<0, \tag{20}$$

$$\begin{bmatrix}\Xi(\mu_2,\tau) & \Sigma_2^TY_1^T\\ * & -\widehat{M}_2\end{bmatrix}<0, \tag{21}$$

$$\begin{bmatrix}P_2 & X_1\\ * & P_3\end{bmatrix}\geq 0,\ \begin{bmatrix}\hat{Q}_{3q} & X_2\\ * & \hat{Q}_{4q}\end{bmatrix}\geq 0, \tag{22}$$

$$\vartheta_0>0,\ \vartheta_1>0,\ Q_{k,l}>0,\ F_{\sigma,l}>0,(k=3,4,\sigma,l=1,2), \tag{23}$$

where

$\Xi(\mu_1,0)=\Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=0}+\Gamma_t+\Gamma_{aa}$,
$\Xi(\mu_2,0)=\Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=0}+\Gamma_t+\Gamma_{aa}$,
$\Xi(\mu_1,\tau)=\Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=\tau}+\Gamma_t+\Gamma_{aa}$,
$\Xi(\mu_2,\tau)=\Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=\tau}+\Gamma_t+\Gamma_{aa}$,
$\Gamma(\dot{\tau}(t),\tau(t))=\dot{\tau}(t)\phi_2^TQ_1\phi_2+\mathrm{Sym}\{[\tau(t)e_7^T+(\tau-\tau(t))e_8^T]R(e_1-e_3)\}+\dot{\tau}(t)[\Pi_3^TP_2\Pi_3-\Pi_4^TP_3\Pi_4]+\mathrm{Sym}\{\Pi_1^TP_1\Pi_2(t)\}+\mathrm{Sym}\{\Pi_3^TP_2\Pi_5\}+\mathrm{Sym}\{\Pi_4^TP_3\Pi_6\}+\Pi_{11}^T(Q_{3p}-\tau(t)Q_{3q})\Pi_{11}-(1-\dot{\tau}(t))\Pi_{12}^T(Q_{3p}-\tau(t)Q_{3q})\Pi_{12}+(1-\dot{\tau}(t))\Pi_{12}^T(Q_{4p}+(\tau-\tau(t))Q_{4q})\Pi_{12}-\Pi_{13}^T(Q_{4p}+(\tau-\tau(t))Q_{4q})\Pi_{13}-\mathrm{Sym}\{\Pi_7^T\hat{Q}_{3q}\Pi_8+\Pi_9^T\hat{Q}_{4q}\Pi_{10}\}-\varrho\tau\Pi_8^T\hat{Q}_{3q}\Pi_8-(1-\varrho)\tau\Pi_{10}^T\hat{Q}_{4q}\Pi_{10}-\frac{1}{\tau}\begin{bmatrix}\Pi_7\\ \Pi_9\end{bmatrix}^T\begin{bmatrix}\hat{Q}_{3q} & X_2\\ * & \hat{Q}_{4q}\end{bmatrix}\begin{bmatrix}\Pi_7\\ \Pi_9\end{bmatrix}-\begin{bmatrix}\Sigma_1\\ \Sigma_2\end{bmatrix}^T\hat{\psi}_2(\varrho,1-\varrho)\begin{bmatrix}\Sigma_1\\ \Sigma_2\end{bmatrix}$,

$\Gamma_t=\mathrm{Sym}\{e_1^TH_ae_{14}+e_{13}^TK_ae_{15}\}+\phi_1^TQ_1\phi_1+e_1^TZ_1e_1$
$-e_2^TZ_1e_2+e_2^TZ_2e_2-e_3^TZ_2e_3+\tau^2e_{14}^TM_2e_{14}-\phi_2^TQ_1\phi_2$
$+\phi_1^TQ_2\phi_1-\phi_3^TQ_2\phi_3+\mathrm{Sym}\{e_1^TF_1e_{13}-e_1^TF_1(A-L_{1a})e_1$

$- e_1^T F_1 e_{14} - e_{14}^T F_2 (A - L_{1a}) e_1 + e_{14}^T F_2 e_{13} - e_{14}^T F_2 e_{14}$
$- e_{13}^T \hat{F}_1 (B_a - L_{2a}) e_{13} + e_{13}^T \hat{F}_1 \bar{\alpha} W_a^1 e_4 + e_{13}^T \hat{F}_1 \bar{\beta} W_a^2 e_5$
$- e_{13}^T \hat{F}_1 e_{15} - e_{13}^T \hat{F}_1 C_a e_1 - e_{15}^T \hat{F}_2 (B_a - L_{2a}) e_{13} + e_{15}^T \hat{F}_2 \bar{\alpha} W_a^1 e_4$
$+ e_{15}^T \hat{F}_2 \bar{\beta} W_a^2 e_5 - e_{15}^T \hat{F}_2 e_{15} - e_{15}^T \hat{F}_2 C_a e_1\} + \tau \Pi_{11}^T M_1 \Pi_{11}$
$+ \phi_1^T \Gamma_1 \phi_1 + \phi_2^T \Gamma_2 \phi_2 + \phi_3^T \Gamma_3 \phi_3$,
$\Gamma_{aa} = e_1^T \sum_{b \in \mathcal{C}_k^a} [\underline{\lambda}_{ab} (H_b - H_a) + 2\varpi_{ab} V_{1ab}] e_1$
$+ e_{13} \sum_{b \in \mathcal{C}_k^a} [\underline{\lambda}_{ab} (K_b - K_a) + 2\varpi_{ab} V_{2ab}] e_{13}$,
$\vartheta_\varrho = P_1 + \tau \Phi_1^T (\varrho P_2 + (1-\varrho) P_3) \Phi_1 + \mathrm{Sym}\{\Phi_1^T P_2 \Phi_2 + \Phi_1^T P_3 \Phi_3\} + \frac{1}{\tau} \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}^T \begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}$,
$\vartheta_0 = \vartheta_\varrho|_{\tau(t)=0}$, $\vartheta_1 = \vartheta_\varrho|_{\tau(t)=\tau}$, $\varrho = \frac{\tau(t)}{\tau}$,
$\Phi_1 = [\hat{e}_1^T \ 0]^T$, $\Phi_2 = [0 \ \hat{e}_3^T]^T$, $\Phi_3 = [0 \ \hat{e}_4^T]^T$,
$\hat{e}_m = [0_{n\times(m-1)n} \ I_{n\times n} \ 0_{n\times(4-m)n}]$, $m = 1, 2, 3, 4$.
$\phi_1 = [e_1^T, e_4^T]$, $\phi_2 = [e_2^T, e_5^T]$, $\phi_3 = [e_3^T, e_6^T]$,
$\hat{\psi}_2(\varrho, 1-\varrho) = \begin{bmatrix} (2-\varrho)\widehat{M}_2 & (1-\varrho)Y_1 + \varrho Y_2 \\ * & (1+\varrho)\widehat{M}_2 \end{bmatrix}$,
$\widehat{M}_2 = \mathrm{diag}\{M_2, 3M_2, 5M_2, 7M_2\}$,
$\hat{\Lambda}_a = \varpi_{aa} - \sum_{b \in \mathcal{C}_k^a, b \neq a} \varpi_{ab}$,
$\underline{\lambda}_a = -\bar{\lambda}_{aa} - \sum_{b \in \mathcal{C}_k^a, b \neq a} \underline{\lambda}_{ab}$,
$\underline{\lambda}_{ab} = \tilde{\lambda}_{ab} - \varpi_{ab}$, $\bar{\lambda}_{ab} = \tilde{\lambda}_{ab} + \varpi_{ab}$,
$\Pi_1 = [e_1^T, e_2^T, \tau(t) e_7^T, (\tau - \tau(t)) e_8^T]^T$,
$\Pi_2(t) = [e_{14}^T, (1-\dot{\tau}(t)) e_{16}^T, e_1^T - (1-\dot{\tau}(t)) e_2^T, (1-\dot{\tau}(t)) e_2^T - e_3^T]^T$,
$\Pi_3 = [e_1^T, e_7^T]^T$, $\Pi_4 = [e_1^T, e_8^T]^T$,
$\Pi_5 = [\varrho \tau e_{14}^T, e_1^T - (1-\dot{\tau}(t)) e_2^T - \dot{\tau}(t) e_7^T]^T$,
$\Pi_6 = [(1-\varrho)\tau e_{14}^T, (1-\dot{\tau}(t)) e_2^T - e_3^T + \dot{\tau}(t) e_8^T]^T$,
$\Pi_7 = [0, e_1^T - e_2^T, 0, -e_1^T - e_2^T + 2e_7^T, 0, e_1^T - e_2^T + 6e_7^T - 6e_9^T]^T$,
$\Pi_8 = [e_7^T, 0, e_7^T - e_9^T, 0, e_7^T - 3e_9^T + 2e_{11}^T, 0]^T$,
$\Pi_9 = [0, e_2^T - e_3^T, 0, -e_2^T - e_3^T + 2e_8^T, 0, e_2^T - e_3^T + 6e_8^T - 6e_{10}^T]^T$,
$\Pi_{10} = [e_8^T, 0, e_8^T - e_{10}^T, 0, e_8^T - 3e_{10}^T + 2e_{12}^T, 0]^T$,
$\Pi_{11} = [e_1^T, e_{14}^T]^T$, $\Pi_{12} = [e_2^T, e_{16}^T]^T$, $\Pi_{13} = [e_3^T, e_{17}^T]^T$,
$\Sigma_1 = [(e_1 - e_2)^T, (e_1 + e_2 - 2e_7)^T, (e_1 - e_2 + 6e_7 - 6e_9)^T, (e_1 + e_2 - 12e_7 + 30e_9 - 20e_{11})^T]^T$,
$\Sigma_2 = [(e_2 - e_3)^T, (e_2 + e_3 - 2e_8)^T, (e_2 - e_3 + 6e_8 - 6e_{10})^T, (e_2 + e_3 - 12e_8 + 30e_{10} - 20e_{12})^T]^T$,
$Q_{31} = Q_{3p}$, $Q_{41} = Q_{4p}$, $Q_{32} = Q_{3p} - \tau Q_{3q}$,
$Q_{42} = Q_{4p} + \tau Q_{4q}$,
$\digamma_{1,1} = \digamma_1|_{\dot{\tau}(t)=\mu_1}$, $\digamma_{1,2} = \digamma_1|_{\dot{\tau}(t)=\mu_2}$,
$\digamma_{2,1} = \digamma_2|_{\dot{\tau}(t)=\mu_1}$, $\digamma_{2,2} = \digamma_2|_{\dot{\tau}(t)=\mu_2}$,
$\digamma_1 = M_1 + \dot{\tau}(t) Q_{3q} + \begin{bmatrix} 0 & Z_1 \\ Z_1 & 0 \end{bmatrix}$,
$\digamma_2 = M_1 + \dot{\tau}(t) Q_{4q} + \begin{bmatrix} 0 & Z_2 \\ Z_2 & 0 \end{bmatrix}$,
$\hat{Q}_{3q} = \mathrm{diag}\{\digamma_1, 3\digamma_1, 5\digamma_1\}$,
$\hat{Q}_{4q} = \mathrm{diag}\{\digamma_2, 3\digamma_2, 5\digamma_2\}$,
$\Gamma_{\alpha_1} = \begin{bmatrix} -R_{\alpha_1} E_1 & R_{\alpha_1} E_2 \\ * & -R_{\alpha_1} \end{bmatrix}$ $(\alpha_1 = 1, 2, 3)$.

If $a \in \mathcal{C}_k^a$,

$$H_b - H_a - V_{1ab} \leq 0, \forall b \in \mathcal{C}, b \neq a, \tag{24}$$
$$K_b - K_a - V_{2ab} \leq 0, \forall b \in \mathcal{C}, b \neq a, \tag{25}$$

$$\begin{bmatrix} \hat{\Xi}(\mu_1, 0) & \Sigma_1^T Y_2 \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{26}$$
$$\begin{bmatrix} \hat{\Xi}(\mu_1, \tau) & \Sigma_2^T Y_1^T \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{27}$$
$$\begin{bmatrix} \hat{\Xi}(\mu_2, 0) & \Sigma_1^T Y_2 \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{28}$$
$$\begin{bmatrix} \hat{\Xi}(\mu_2, \tau) & \Sigma_2^T Y_1^T \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{29}$$
$$\begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \geq 0, \ \begin{bmatrix} \hat{Q}_{3q} & X_2 \\ * & \hat{Q}_{4q} \end{bmatrix} \geq 0 \tag{30}$$
$$\vartheta_0 > 0, \ \vartheta_1 > 0, \ Q_{k,l} > 0, \ \digamma_{\sigma,l} > 0, (k = 3, 4, \sigma, l = 1, 2), \tag{31}$$

where
$\hat{\Xi}(\mu_1, 0) = \Gamma(\dot{\tau}(t), \tau(t))|_{\dot{\tau}(t)=\mu_1, \tau(t)=0} + \Gamma_t + \hat{\Gamma}_{aa}$,
$\hat{\Xi}(\mu_2, 0) = \Gamma(\dot{\tau}(t), \tau(t))|_{\dot{\tau}(t)=\mu_2, \tau(t)=0} + \Gamma_t + \hat{\Gamma}_{aa}$,
$\hat{\Xi}(\mu_1, \tau) = \Gamma(\dot{\tau}(t), \tau(t))|_{\dot{\tau}(t)=\mu_1, \tau(t)=\tau} + \Gamma_t + \hat{\Gamma}_{aa}$,
$\hat{\Xi}(\mu_2, \tau) = \Gamma(\dot{\tau}(t), \tau(t))|_{\dot{\tau}(t)=\mu_2, \tau(t)=\tau} + \Gamma_t + \hat{\Gamma}_{aa}$,
In matrices $\hat{\Xi}(\mu_1, 0)$, $\hat{\Xi}(\mu_2, 0)$, $\hat{\Xi}(\mu_1, \tau)$ and $\hat{\Xi}(\mu_2, \tau)$, only the element $\hat{\Gamma}_{aa}$ is different from Γ_{aa}. The other elements are the same as the elements in $\Xi(\mu_1, 0)$, $\Xi(\mu_2, 0)$, $\Xi(\mu_1, \tau)$ and $\Xi(\mu_2, \tau)$.
$\hat{\Gamma}_{aa} = e_1^T [\sum_{b \in \mathcal{C}_k^a, b \neq a}^{\mathcal{M}} [\underline{\lambda}_{ab} (H_b - H_a) + 2\varpi_{ab} V_{1ab}]$
$+ \underline{\lambda}_a (H_l - H_a) + 2\hat{\Lambda}_a V_{1al}] e_1$
$+ e_{13}^T [\sum_{b \in \mathcal{C}_k^a, b \neq a}^{\mathcal{M}} [\underline{\lambda}_{ab} (K_b - K_a) + 2\varpi_{ab} V_{2ab}]$
$+ \underline{\lambda}_a (K_l - K_a) + 2\hat{\Lambda}_a V_{2al}] e_{13}$, $(l \in \mathcal{C}_{uk}^a)$.

Proof Consider the DPTLKF of generally uncertain Markovian error INNs as follows
$V(t, a) = \sum_{r=1}^{7} V_r(t, a)$,

$$V_1(t, a) = e_1^T(t) H_a e_1(t) + e_2^T(t) K_a e_2(t), \tag{32}$$
$$V_2(t, a) = \int_{t-\tau(t)}^{t} \phi^T(s) Q_1 \phi(s) ds + \int_{t-\tau}^{t} \phi^T(s) Q_2 \phi(s) ds, \tag{33}$$
$$V_3(t, a) = \int_{t-\tau}^{t} e_1^T(s) ds R \int_{t-\tau}^{t} e_1(s) ds, \tag{34}$$
$$V_4(t, a) = \zeta_1^T(t) P_1 \zeta_1(t) + \tau(t) \zeta_2^T(t) P_2 \zeta_2(t) + (\tau - \tau(t)) \zeta_3^T(t) P_3 \zeta_3(t), \tag{35}$$
$$V_5(t, a) = \int_{t-\tau(t)}^{t} \zeta_4^T(s) Q_3(\tau(t)) \zeta_4(s) ds + \int_{t-\tau}^{t-\tau(t)} \zeta_4^T(s) Q_4(\tau(t)) \zeta_4(s) ds, \tag{36}$$
$$V_6(t, a) = \int_{-\tau}^{0} \int_{t+\theta}^{t} \zeta_4^T(s) M_1 \zeta_4(s) ds d\theta, \tag{37}$$
$$V_7(t, a) = \tau \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{e}_1^T(s) M_2 \dot{e}_1(s) ds d\theta, \tag{38}$$

where
$\phi(t) = [e_1^T(t), g^T(e_1(t))]^T$,
$\zeta_1(t) = [e_1^T(t), e_1^T(t-\tau(t)), \int_{t-\tau(t)}^{t} e_1^T(s)ds,$
$\int_{t-\tau}^{t-\tau(t)} e_1^T(s)ds]^T$,
$\zeta_2(t) = [e_1^T(t), \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} e_1^T(s)ds]^T$,
$\zeta_3(t) = [e_1^T(t), \frac{1}{\tau-\tau(t)}\int_{t-\tau}^{t-\tau(t)} e_1^T(s)ds]^T$,
$\zeta_4(t) = [e_1^T(t), \dot{e}_1^T(t)]^T$,
$Q_3(\tau(t)) = Q_{3p} - \tau(t)Q_{3q}$, $Q_4(\tau(t)) = Q_{4p} + (\tau-\tau(t))Q_{4q}$.

In order to prove the positive definiteness of the LKF $V(t,a)$, $V_4(t,a)$ is rewritten as

$$\begin{aligned}V_4(t,a) =&\zeta_1^T(t)\{P_1 + \tau(t)(\Phi_1 + \frac{1}{\tau(t)}\Phi_2)^T P_2(\Phi_1 + \frac{1}{\tau(t)}\Phi_2)\\&+(\tau-\tau(t))(\Phi_1 + \frac{1}{\tau-\tau(t)}\Phi_3)^T P_3(\Phi_1\\&+\frac{1}{\tau-\tau(t)}\Phi_3)\}\zeta_1(t)\\=&\zeta_1^T(t)[P_1 + \tau\Phi_1^T(\varrho P_2 + (1-\varrho)P_3)\Phi_1\\&+\mathrm{Sym}\{\Phi_1^T P_2\Phi_2 + \Phi_1^T P_3\Phi_3\}\\&+\frac{1}{\tau}(\frac{1}{\varrho}\Phi_2^T P_2\Phi_2 + \frac{1}{1-\varrho}\Phi_3^T P_3\Phi_3)]\zeta_1(t),\end{aligned} \tag{39}$$

where
$\Phi_1 = [\hat{e}_1^T\ 0]^T$, $\Phi_2 = [0\ \hat{e}_3^T]^T$, $\Phi_3 = [0\ \hat{e}_4^T]^T$, $\hat{e}_m = [0_{n\times(m-1)n}\ I_{n\times n}\ 0_{n\times(4-m)n}]$, $m = 1,2,3,4$.

$P_2 > 0$ and $P_3 > 0$, then by utilizing Lemma 1, for any matrix X_1 and $\begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \geq 0$, one has

$$\frac{1}{\varrho}\Phi_2^T P_2\Phi_2 + \frac{1}{1-\varrho}\Phi_3^T P_3\Phi_3 \geq \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}^T \begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}. \tag{40}$$

Thus we have

$$V_4(t,a) \geq \zeta_1^T(t)\vartheta_\varrho\zeta_1(t), \tag{41}$$

where

$$\begin{aligned}\vartheta_\varrho =& P_1 + \tau\Phi_1^T(\varrho P_2 + (1-\varrho)P_3)\Phi_1 + \mathrm{Sym}\{\Phi_1^T P_2\Phi_2\\&+\Phi_1^T P_3\Phi_3\} + \frac{1}{\tau}\begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}^T \begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix}.\end{aligned} \tag{42}$$

From $\vartheta_0 > 0$ and $\vartheta_1 > 0$ in (23), we get $V_4(t,a) > 0$. From $Q_{k,l} > 0\,(k = 3,4, l = 1,2)$, we get $V_5(t,a) > 0$. $\mathcal{L}$ is defined as the weak infinitesimal operator. Then, calculating the time derivative of $V(t,a)$, one has

$$\begin{aligned}&\mathcal{L}V_1(t,a)\\&= 2e_1^T(t)H_a\dot{e}_1(t) + 2e_2^T(t)K_a\dot{e}_2(t)\\&+\sum_{b=1}^{\mathcal{M}}\lambda_{ab}e_1^T(t)H_b e_1(t) + \sum_{b=1}^{\mathcal{M}}\lambda_{ab}e_2^T(t)K_b e_2(t),\end{aligned} \tag{43}$$

$$\begin{aligned}&\mathcal{L}V_2(t,a)\\&= \phi^T(t)Q_1\phi(t) - (1-\dot{\tau}(t))\phi^T(t-\tau(t))Q_1\\&\phi(t-\tau(t)) + \phi^T(t)Q_2\phi(t) - \phi^T(t-\tau)Q_2\phi(t-\tau),\end{aligned} \tag{44}$$

$$\begin{aligned}&\mathcal{L}V_3(t,a)\\&= 2[\tau(t)\times\frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} e_1^T(s)ds + (\tau-\tau(t))\\&\times\frac{1}{\tau-\tau(t)}\int_{t-\tau}^{t-\tau(t)} e_1^T(s)ds]R(e_1(t) - e_1(t-\tau)),\end{aligned} \tag{45}$$

$$\begin{aligned}&\mathcal{L}V_4(t,a)\\&= 2\zeta_1^T(t)P_1\dot{\zeta}_1(t) + \dot{\tau}(t)\zeta_2^T(t)P_2\zeta_2(t) + 2\tau(t)\zeta_2^T(t)P_2\\&\times\dot{\zeta}_2(t) - \dot{\tau}(t)\zeta_3^T(t)P_3\zeta_3(t) + 2(\tau-\tau(t))\zeta_3^T(t)P_3\dot{\zeta}_3(t),\end{aligned} \tag{46}$$

$$\begin{aligned}&\mathcal{L}V_5(t,a)\\&= \zeta_4^T(t)(Q_{3p} - \tau(t)Q_{3q})\zeta_4(t) - (1-\dot{\tau}(t))\zeta_4^T(t-\tau(t))(Q_{3p}\\&-\tau(t)Q_{3q})\zeta_4(t-\tau(t)) + (1-\dot{\tau}(t))\zeta_4^T(t-\tau(t))(Q_{4p}\\&+(\tau-\tau(t))Q_{4q})\zeta_4(t-\tau(t)) - \zeta_4^T(t-\tau)(Q_{4p} + (\tau\\&-\tau(t))Q_{4q})\zeta_4(t-\tau) - \dot{\tau}(t)\int_{t-\tau(t)}^{t}\zeta_4^T(s)Q_{3q}\zeta_4(s)ds\\&-\dot{\tau}(t)\int_{t-\tau}^{t-\tau(t)}\zeta_4^T(s)Q_{4q}\zeta_4(s)ds,\end{aligned} \tag{47}$$

$$\begin{aligned}&\mathcal{L}V_6(t,a)\\&= \tau\zeta_4^T(t)M_1\zeta_4(t) - \int_{t-\tau}^{t}\zeta_4^T(s)M_1\zeta_4(s)ds,\end{aligned} \tag{48}$$

$$\begin{aligned}&\mathcal{L}V_7(t,a)\\&= \tau^2\dot{e}_1^T(t)M_2\dot{e}_1(t) - \tau\int_{t-\tau}^{t}\dot{e}_1^T(s)M_2\dot{e}_1(s)ds,\end{aligned} \tag{49}$$

where
$\dot{\zeta}_1(t) = [\dot{e}_1^T(t),\ (1-\dot{\tau}(t))\dot{e}_1^T(t-\tau(t)),\ e_1^T(t) - (1-\dot{\tau}(t))e_1^T(t-\tau(t)), (1-\dot{\tau}(t))e_1^T(t-\tau(t)) - e_1^T(t-\tau)]^T$,
$\dot{\zeta}_2(t) = [\dot{e}_1^T(t),\ \frac{1}{\tau(t)}e_1^T(t) - \frac{(1-\dot{\tau}(t))}{\tau(t)}e_1^T(t-\tau(t)) - \frac{\dot{\tau}(t)}{\tau^2(t)}\int_{t-\tau(t)}^{t} e_1^T(s)ds]^T$,
$\dot{\zeta}_3(t) = [\dot{e}_1^T(t),\ \frac{(1-\dot{\tau}(t))}{\tau-\tau(t)}e_1^T(t-\tau(t)) - \frac{1}{\tau-\tau(t)}e_1^T(t-\tau) + \frac{\dot{\tau}(t)}{(\tau-\tau(t))^2}\int_{t-\tau}^{t-\tau(t)} e_1^T(s)ds]^T$.

By using Lemma 2, one gets

$$\begin{aligned}&-\tau\int_{t-\tau}^{t}\dot{e}_1^T(s)M_2\dot{e}_1(s)ds\\&= -\tau\int_{t-\tau(t)}^{t}\dot{e}_1^T(s)M_2\dot{e}_1(s)ds - \tau\int_{t-\tau}^{t-\tau(t)}\dot{e}_1^T(s)M_2\dot{e}_1(s)ds\\&\leq -\chi^T(t)\left\{\begin{bmatrix}\Sigma_1\\\Sigma_2\end{bmatrix}^T\hat{\psi}_1(\varrho,1-\varrho)\begin{bmatrix}\Sigma_1\\\Sigma_2\end{bmatrix}\right\}\chi(t)\\&\leq -\chi^T(t)\left\{\begin{bmatrix}\Sigma_1\\\Sigma_2\end{bmatrix}^T\hat{\psi}_2(\varrho,1-\varrho)\begin{bmatrix}\Sigma_1\\\Sigma_2\end{bmatrix}\right\}\chi(t)\\&+\chi^T(t)[\varrho\Sigma_2^T Y_1^T\widehat{M}_2^{-1}Y_1\Sigma_2\\&+(1-\varrho)\Sigma_1^T Y_2\widehat{M}_2^{-1}Y_2^T\Sigma_1]\chi(t),\end{aligned} \tag{50}$$

where
$\hat{\psi}_1(\varrho,1-\varrho) = \begin{bmatrix}\widehat{M}_2 + (1-\varrho)\hat{\Pi}_1 & (1-\varrho)Y_1 + \varrho Y_2\\ * & \widehat{M}_2 + \varrho\hat{\Pi}_2\end{bmatrix}$,
$\hat{\Pi}_1 = \widehat{M}_2 - Y_2\widehat{M}_2^{-1}Y_2^T$, $\hat{\Pi}_2 = \widehat{M}_2 - Y_1^T\widehat{M}_2^{-1}Y_1$,
$\widehat{M}_2$, $\hat{\psi}_2(\varrho,1-\varrho)$, ϱ, Σ_1, Σ_2 are given in Theorem 1.

By using the integration formula, the zero equalities with any symmetric matrices Z_l $(l=1,2)$ hold

$$0 = e_1^T(t)Z_1e_1(t) - e_1^T(t-\tau(t))Z_1e_1(t-\tau(t)) - 2\int_{t-\tau(t)}^{t} e_1^T(s)Z_1\dot{e}_1(s)ds, \tag{51}$$

$$0 = e_1^T(t-\tau(t))Z_2e_1(t-\tau(t)) - e_1^T(t-\tau)Z_2e_1(t-\tau) - 2\int_{t-\tau}^{t-\tau(t)} e_1^T(s)Z_2\dot{e}_1(s)ds. \tag{52}$$

From (48), one gets

$$-\int_{t-\tau}^{t}\zeta_4^T(s)M_1\zeta_4(s)ds = -\int_{t-\tau}^{t-\tau(t)}\zeta_4^T(s)M_1\zeta_4(s)ds - \int_{t-\tau(t)}^{t}\zeta_4^T(s)M_1\zeta_4(s)ds. \tag{53}$$

Considering single integral terms in (47), (48) and (53), we get

$$\Delta_1 = -\int_{t-\tau(t)}^{t}\zeta_4^T(s)(M_1 + \dot{\tau}(t)Q_{3q} + \begin{bmatrix} 0 & Z_1 \\ Z_1 & 0 \end{bmatrix})\zeta_4(s)ds, \tag{54}$$

$$\Delta_2 = -\int_{t-\tau}^{t-\tau(t)}\zeta_4^T(s)(M_1 + \dot{\tau}(t)Q_{4q} + \begin{bmatrix} 0 & Z_2 \\ Z_2 & 0 \end{bmatrix})\zeta_4(s)ds. \tag{55}$$

By using Lemma 3, the upper bounds of (54) and (55) are obtained as follows

$$\begin{aligned}
&\Delta_1 + \Delta_2 \\
&\le -\chi^T(t)\{\frac{1}{\tau(t)}(\Pi_7 + \tau(t)\Pi_8)^T\hat{Q}_{3q}(\Pi_7 + \tau(t)\Pi_8) \\
&+ \frac{1}{\tau-\tau(t)}(\Pi_9 + (\tau-\tau(t))\Pi_{10})^T\hat{Q}_{4q}(\Pi_9 \\
&+ (\tau-\tau(t))\Pi_{10})\}\chi(t) \\
&= -\chi^T(t)\{\mathrm{Sym}\{\Pi_7^T\hat{Q}_{3q}\Pi_8 + \Pi_9^T\hat{Q}_{4q}\Pi_{10}\} + \varrho\tau\Pi_8^T\hat{Q}_{3q}\Pi_8 \\
&+ (1-\varrho)\tau\Pi_{10}^T\hat{Q}_{4q}\Pi_{10} + \frac{1}{\tau}(\frac{1}{\varrho}\Pi_7^T\hat{Q}_{3q}\Pi_7 \\
&+ \frac{1}{1-\varrho}\Pi_9^T\hat{Q}_{4q}\Pi_9)\}\chi(t)
\end{aligned} \tag{56}$$

By using Lemma 1 and (56), the following inequality is obtained

$$\begin{aligned}
&\Delta_1 + \Delta_2 \\
&\le -\chi^T(t)\{\mathrm{Sym}\{\Pi_7^T\hat{Q}_{3q}\Pi_8 + \Pi_9^T\hat{Q}_{4q}\Pi_{10}\} + \varrho\tau\Pi_8^T\hat{Q}_{3q}\Pi_8 \\
&+ (1-\varrho)\tau\Pi_{10}^T\hat{Q}_{4q}\Pi_{10} + \frac{1}{\tau}\begin{bmatrix} \Pi_7 \\ \Pi_9 \end{bmatrix}^T \begin{bmatrix} \hat{Q}_{3q} & X_2 \\ * & \hat{Q}_{4q} \end{bmatrix} \\
&\times \begin{bmatrix} \Pi_7 \\ \Pi_9 \end{bmatrix}\}\chi(t).
\end{aligned} \tag{57}$$

From Assumption 1, one gets

$$\phi^T(t)\begin{bmatrix} -R_1E_1 & R_1E_2 \\ * & -R_1 \end{bmatrix}\phi(t) \ge 0, \tag{58}$$

$$\phi^T(t-\tau(t))\begin{bmatrix} -R_2E_1 & R_2E_2 \\ * & -R_2 \end{bmatrix}\phi(t-\tau(t)) \ge 0, \tag{59}$$

$$\phi^T(t-\tau)\begin{bmatrix} -R_3E_1 & R_3E_2 \\ * & -R_3 \end{bmatrix}\phi(t-\tau) \ge 0. \tag{60}$$

From (10), one gets

$$0 = 2\mathcal{E}\{[e_1^T(t)F_1 + \dot{e}_1^T(t)F_2][-(A-L_{1a})e_1(t) + e_2(t) - \dot{e}_1^T(t)]\}, \tag{61}$$

$$0 = 2\mathcal{E}\{[e_2^T(t)\hat{F}_1 + \dot{e}_2^T(t)\hat{F}_2][-(B_a-L_{2a})e_2(t) - C_ae_1(t) + \alpha(t)W_a^1g(e_1(t)) + \beta(t)W_a^2g(e_1(t-\tau(t))) - \dot{e}_2^T(t)]\}. \tag{62}$$

Then combining (32)-(62), we get $\mathcal{E}\{\mathcal{L}V(t,a)\} \le \mathcal{E}\{\chi^T(t)\wp(t)\chi(t)\} < 0$, where

$$\begin{aligned}
\wp(s) &= \bar{\psi}(\dot{\tau}(t),\tau(t)) + \varrho\Sigma_2^TY_1^T\widehat{M}_2^{-1}Y_1\Sigma_2 \\
&+ (1-\varrho)\Sigma_1^TY_2\widehat{M}_2^{-1}Y_2^T\Sigma_1,
\end{aligned} \tag{63}$$

$\bar{\psi}(\dot{\tau}(t),\tau(t))$
$= \Gamma_t + \Gamma(\dot{\tau}(t),\tau(t)) + e_1^T\sum_{b=1}^{\mathcal{M}}\lambda_{ab}H_be_1 + e_{13}^T\sum_{b=1}^{\mathcal{M}}\lambda_{ab}K_be_{13}$,
$\chi(t) = [e_1^T(t), e_1^T(t-\tau(t)), e_1^T(t-\tau), g^T(e_1(t)),$
$g^T(e_1(t-\tau(t))), g^T(e_1(t-\tau)), \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}e_1^T(s)ds,$
$\frac{1}{\tau-\tau(t)}\int_{t-\tau}^{t-\tau(t)}e_1^T(s)ds, \frac{2}{(\tau(t))^2}\int_{t-\tau(t)}^{t}\int_{\theta}^{t}e_1^T(s)dsd\theta,$
$\frac{2}{(\tau-\tau(t))^2}\int_{t-\tau}^{t-\tau(t)}\int_{\theta}^{t-\tau(t)}e_1^T(s)dsd\theta,$
$\frac{6}{(\tau(t))^3}\int_{t-\tau(t)}^{t}\int_{\theta}^{t}\int_{\alpha}^{t}e_1^T(s)dsd\alpha d\theta,$
$\frac{6}{(\tau-\tau(t))^3}\int_{t-\tau}^{t-\tau(t)}\int_{\theta}^{t-\tau(t)}\int_{\alpha}^{t-\tau(t)}e_1^T(s)dsd\alpha d\theta,$
$e_2^T(t), \dot{e}_1^T(t), \dot{e}_2^T(t), \dot{e}_1^T(t-\tau(t)), \dot{e}_1^T(t-\tau)]^T$.

According to the convex theory, $\mathcal{E}\{\chi^T(t)\wp(t)\chi(t)\} < 0$ could be satisfied for all $(\dot{\tau}(t),\tau(t)) \in [\mu_1,\mu_2]\times[0,\tau]$ if it is satisfied at the vertices of the interval $[\mu_1,\mu_2]\times[0,\tau]$.

According to Schur complement, if one gets $\mathcal{E}\{\bar{\psi}^\gamma\} < 0$ $(\gamma = 1,2,3,4)$, then one gets $\mathcal{E}\{\chi^T(t)\wp(t)\chi(t)\} < 0$, where,

$$\begin{aligned}
\bar{\psi}^1 &= \begin{bmatrix} \bar{\psi}(\mu_1,0) & \Sigma_1^TY_2 \\ * & -\widehat{M}_2 \end{bmatrix}, \bar{\psi}^2 = \begin{bmatrix} \bar{\psi}(\mu_1,\tau) & \Sigma_2^TY_1^T \\ * & -\widehat{M}_2 \end{bmatrix}, \\
\bar{\psi}^3 &= \begin{bmatrix} \bar{\psi}(\mu_2,0) & \Sigma_1^TY_2 \\ * & -\widehat{M}_2 \end{bmatrix}, \bar{\psi}^4 = \begin{bmatrix} \bar{\psi}(\mu_2,\tau) & \Sigma_2^TY_1^T \\ * & -\widehat{M}_2 \end{bmatrix}.
\end{aligned} \tag{64}$$

$\bar{\psi}(\mu_1,0) = \bar{\psi}(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=0}$,
$\bar{\psi}(\mu_2,0) = \bar{\psi}(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=0}$,
$\bar{\psi}(\mu_1,\tau) = \bar{\psi}(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=\tau}$,
$\bar{\psi}(\mu_2,\tau) = \bar{\psi}(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=\tau}$.

Now, we prove that $\mathcal{E}\{\bar{\psi}^\gamma\} < 0$ $(\gamma = 1,2,3,4)$ hold if (14)-(23) are satisfied. Hence, if $\mathcal{E}\{\bar{\psi}^\gamma\} < 0$ $(\gamma = 1,2,3,4)$ hold, $\mathcal{E}\{\chi^T(t)\wp(t)\chi(t)\} < 0$ is satisfied. Then the error system (10) is globally asymptotically stable in the mean square, and the generally Markovian INNs drive system (6) and response system (7) are synchronous.

Let $\hat{\Upsilon}_1^\gamma = \hat{\Theta}_1^\gamma + \sum_{b=1}^{\mathcal{M}}\lambda_{ab}H_b$, $\hat{\Upsilon}_2^\gamma = \hat{\Theta}_2^\gamma + \sum_{b=1}^{\mathcal{M}}\lambda_{ab}K_b$,
$\hat{\Theta}_1^\gamma = e_1(\Gamma_t + \bar{\Gamma}^\gamma)e_1^T$, $\hat{\Theta}_2^\gamma = e_{13}(\Gamma_t + \bar{\Gamma}^\gamma)e_{13}^T$, $(\gamma = 1,2,3,4)$.
$\bar{\Gamma}^1 = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=0}$,
$\bar{\Gamma}^2 = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=\tau}$,

$\bar{\Gamma}^3 = \Gamma(\dot{\tau}(t), \tau(t))|_{\dot{\tau}(t)=\mu_2, \tau(t)=0}$,
$\bar{\Gamma}^4 = \Gamma(\dot{\tau}(t), \tau(t))|_{\dot{\tau}(t)=\mu_2, \tau(t)=\tau}$.

If $a \notin \mathcal{C}_k^a$, one gets $\sum\limits_{b=1}^{\mathcal{M}} \lambda_{ab} H_b = \sum\limits_{b\in\mathcal{C}_k^a} \lambda_{ab} H_b + \lambda_{aa} H_a + \sum\limits_{b\in\mathcal{C}_{uk}^a, b\neq a} \lambda_{ab} H_b$, where $\lambda_{ab} \geq 0, b \neq a$ and $0 \leq \sum\limits_{b\in\mathcal{C}_{uk}^a, b\neq a} \lambda_{ab} = -\lambda_{aa} - \sum\limits_{b\in\mathcal{C}_k^a} \lambda_{ab}$.

From $H_b - H_a - V_{1ab} \leq 0$ $(\forall b \in \mathcal{C}_k^a)$ in (14) and $H_b - H_a \leq 0$ $(\forall b \in \mathcal{C}_{uk}^a, b \neq a)$ in (16), one has

$$\begin{aligned}
&\sum_{b\in\mathcal{C}_k^a} \lambda_{ab} H_b + \lambda_{aa} H_a + \sum_{b\in\mathcal{C}_{uk}^a, b\neq a} \lambda_{ab} H_b \\
&\leq \sum_{b\in\mathcal{C}_k^a} \lambda_{ab} H_b + \lambda_{aa} H_a + \sum_{b\in\mathcal{C}_{uk}^a, b\neq a} \lambda_{ab} H_a \\
&= \sum_{b\in\mathcal{C}_k^a} \lambda_{ab} H_b + \lambda_{aa} H_a + (-\lambda_{aa} - \sum_{b\in\mathcal{C}_k^a} \lambda_{ab}) H_a \\
&= \sum_{b\in\mathcal{C}_k^a} \lambda_{ab} (H_b - H_a) \\
&= \sum_{b\in\mathcal{C}_k^a} (\underline{\lambda}_{ab} + \varpi_{ab} + \triangle_{ab})(H_b - H_a) \\
&= \sum_{b\in\mathcal{C}_k^a} [\underline{\lambda}_{ab}(H_b - H_a) + (\varpi_{ab} + \triangle_{ab})(H_b - H_a)]. \qquad (65)
\end{aligned}$$

Noting that $V_{1ab} \geq 0$, $\triangle_{ab} \in [-\varpi_{ab}, \varpi_{ab}]$, $\forall b \in \mathcal{C}_k^a$, and $H_b - H_a \leq 0$, $\forall b \in \mathcal{C}_{uk}^a, b \neq a$, one has

$$(\varpi_{ab} + \triangle_{ab})(H_b - H_a) \leq (\varpi_{ab} + \triangle_{ab}) V_{1ab} \leq 2\varpi_{ab} V_{1ab}, \forall b \in \mathcal{C}_k^a. \qquad (66)$$

Hence, one gets

$$\hat{\Upsilon}_1^\gamma \leq \hat{\Theta}_1^\gamma + \sum_{b\in\mathcal{C}_k^a} (\underline{\lambda}_{ab}(H_b - H_a) + 2\varpi_{ab} V_{1ab}). \qquad (67)$$

Similarly, for $K_b - K_a - V_{2ab} \leq 0$ $(\forall b \in \mathcal{C}_k^a)$ in (15) and $K_b - K_a \leq 0$ $(\forall b \in \mathcal{C}_{uk}^a, b \neq a)$ in (17), one has

$$\hat{\Upsilon}_2^\gamma \leq \hat{\Theta}_2^\gamma + \sum_{b\in\mathcal{C}_k^a} (\underline{\lambda}_{ab}(K_b - K_a) + 2\varpi_{ab} V_{2ab}). \qquad (68)$$

Hence, from (14)-(23), one gets $\mathcal{E}\{\bar{\psi}^\gamma\} < 0, (\gamma = 1, 2, 3, 4)$.

If $a \in \mathcal{C}_k^a$, according to $\lambda_{aa} = -\sum\limits_{b\in\mathcal{C}, b\neq a} \lambda_{ab}$, there is $H_b \leq H_l$, for $b, l \in \mathcal{C}_{uk}^a$, one has

$$\begin{aligned}
&\sum_{b=1}^{\mathcal{M}} \lambda_{ab} H_b \\
&= \sum_{b\in\mathcal{C}_k^a, b\neq a} \lambda_{ab} H_b + \sum_{b\in\mathcal{C}_{uk}^a} \lambda_{ab} H_b + \lambda_{aa} H_a \\
&= \sum_{b\in\mathcal{C}_k^a, b\neq a} \lambda_{ab}(H_b - H_a) + \sum_{b\in\mathcal{C}_{uk}^a} \lambda_{ab}(H_b - H_a) \\
&\leq \sum_{b\in\mathcal{C}_k^a, b\neq a} \lambda_{ab}(H_b - H_a) + \sum_{b\in\mathcal{C}_{uk}^a} \lambda_{ab}(H_l - H_a), \ (l \in \mathcal{C}_{uk}^a). \qquad (69)
\end{aligned}$$

From $H_b - H_a - V_{1ab} \leq 0$ $(\forall b \in \mathcal{C}, b \neq a)$ in (24), one has

$$\begin{aligned}
&\sum_{b\in\mathcal{C}_k^a, b\neq a} \lambda_{ab}(H_b - H_a) \\
&\leq \sum_{b\in\mathcal{C}_k^a, b\neq a} [\underline{\lambda}_{ab}(H_b - H_a) + 2\varpi_{ab} V_{1ab}], \qquad (70) \\
&\sum_{b\in\mathcal{C}_{uk}^a} \lambda_{ab}(H_l - H_a) \\
&\leq \sum_{b\in\mathcal{C}_{uk}^a} [\underline{\lambda}_{ab}(H_l - H_a) + 2\varpi_{ab} V_{1al}]. \qquad (71)
\end{aligned}$$

From $\lambda_{aa} = -\sum\limits_{b\in\mathcal{C}_k^a, b\neq a} \lambda_{ab} - \sum\limits_{b\in\mathcal{C}_{uk}^a} \lambda_{ab}$, one gets

$$\sum_{b\in\mathcal{C}_{uk}^a} \underline{\lambda}_{ab} = -\bar{\lambda}_{aa} - \sum_{b\in\mathcal{C}_k^a, b\neq a} \underline{\lambda}_{ab} = \underline{\lambda}_a, \qquad (72)$$

$$\sum_{b\in\mathcal{C}_{uk}^a} \bar{\lambda}_{ab} = -\underline{\lambda}_{aa} - \sum_{b\in\mathcal{C}_k^a, b\neq a} \bar{\lambda}_{ab}. \qquad (73)$$

Then, we have

$$\begin{aligned}
&\sum_{b\in\mathcal{C}_{uk}^a} \varpi_{ab} \\
&= \frac{1}{2}(-\sum_{b\in\mathcal{C}_{uk}^a} \underline{\lambda}_{ab} + \sum_{b\in\mathcal{C}_{uk}^a} \bar{\lambda}_{ab}) \\
&= \frac{1}{2}(\bar{\lambda}_{aa} + \sum_{b\in\mathcal{C}_k^a, b\neq a} \underline{\lambda}_{ab} - \underline{\lambda}_{aa} - \sum_{b\in\mathcal{C}_k^a, b\neq a} \bar{\lambda}_{ab}) \\
&= \varpi_{aa} - \sum_{b\in\mathcal{C}_k^a, b\neq a} \varpi_{ab} \\
&= \hat{\Lambda}_a. \qquad (74)
\end{aligned}$$

Thus, we get

$$\sum_{b\in\mathcal{C}_{uk}^a} \lambda_{ab}(H_b - H_a) \leq \underline{\lambda}_a(H_l - H_a) + 2\hat{\Lambda}_a V_{1al}. \qquad (75)$$

Hence, one gets

$$\begin{aligned}
\hat{\Upsilon}_1^\gamma \leq\ & \hat{\Theta}_1^\gamma + \underline{\lambda}_a(H_l - H_a) + 2\hat{\Lambda}_a V_{1al} \\
&+ \sum_{b\in\mathcal{C}_k^a, b\neq a} (\underline{\lambda}_{ab}(H_b - H_a) + 2\varpi_{ab} V_{1ab}). \qquad (76)
\end{aligned}$$

Similarly, for $K_b - K_a - V_{2ab} \leq 0$ $(\forall b \in \mathcal{C}, b \neq a)$ in (25), one has

$$\begin{aligned}
\hat{\Upsilon}_2^\gamma \leq\ & \hat{\Theta}_2^\gamma + \underline{\lambda}_a(K_l - K_a) + 2\hat{\Lambda}_a V_{2al} \\
&+ \sum_{b\in\mathcal{C}_k^a, b\neq a} (\underline{\lambda}_{ab}(K_b - K_a) + 2\varpi_{ab} V_{2ab}). \qquad (77)
\end{aligned}$$

Hence, from (24) to (31), one gets $\mathcal{E}\{\bar{\psi}^\gamma\} < 0$ $(\gamma = 1, 2, 3, 4)$.

Remark 4: In most existing generally uncertain Markovian neural networks [27], [45], [46], Schur complement and matrix inequality of Lemma 2 in [52] are adopted to deal with the estimate error $\triangle_{ab}$ and the completely unknown transition rate "?" in the generally uncertain TRM. Different from these papers, only one set of relaxation variables are adopted in this paper to deal with generally uncertain rates of INNS, which reduces the dimension and computational complexity of synchronization conditions.

Remark 5: The DPTLKF in (32) and HOPRII are applied in this paper to obtain the new synchronization of generally uncertain INNS, and the new delay-range-dependent synchronization conditions including more information about time-varying delay and its derivative are proposed in Theorem 1.

Remark 6: From the viewpoint of practical application, the Lyapunov functions contain more information about INNs system, which could reduce the conservativeness of the synchronization conditions and obtain better controllers of the system.

In Theorem 1, the synchronization problem of Markovian inertial neural networks is investigated. Since matrices F_l, $\hat{F}_l$, and L_{la} $(l=1,2)$ are not given, the matrix inequalities are nonlinear. Hence, the desired controllers could not be directly solved. According to [26], [53], the nonlinear matrix inequalities are converted into linear matrix inequalities in Theorem 2, and the corresponding controllers could be obtained.

Theorem 2. Under Assumption 1, the given scalars ρ_1, ρ_2, the drive system (6) and response system (7) are synchronous if there are any matrices $X_1 \in \mathcal{R}^{2n\times 2n}$, $X_2 \in \mathcal{R}^{6n\times 6n}$, $Y_l \in \mathcal{R}^{4n\times 4n}$ $(l=1,2)$, $J_{la} \in \mathcal{R}^{n\times n}$ $(l=1,2)$, any invertible matrices $\mathcal{F}$, $\hat{\mathcal{F}}$, symmetric matrices $P_1 \in \mathcal{R}^{4n\times 4n}$, $P_v \in \mathcal{R}^{2n\times 2n}, P_v > 0$ $(v=2,3)$, Q_{kp}, $Q_{kq} \in \mathcal{R}^{2n\times 2n}$ $(k=3,4)$, Z_l $(l=1,2)$, symmetric positive definite matrices H_a, K_a, V_{lab} $(l=1,2)$, R, $M_1 \in \mathcal{R}^{2n\times 2n}$, $M_2 \in \mathcal{R}^{n\times n}$ and $Q_l \in \mathcal{R}^{2n\times 2n}$ $(l=1,2)$, positive definite diagonal matrices R_1, R_2, R_3 such that, for any $a \in \mathcal{C}$, the succeeding linear matrix inequalities are satisfied.

If $a \notin \mathcal{C}_k^a$,

$$H_b - H_a - V_{1ab} \le 0, \forall b \in \mathcal{C}_k^a, \tag{78}$$

$$K_b - K_a - V_{2ab} \le 0, \forall b \in \mathcal{C}_k^a, \tag{79}$$

$$H_b - H_a \le 0, \forall b \in \mathcal{C}_{uk}^a, b \ne a, \tag{80}$$

$$K_b - K_a \le 0, \forall b \in \mathcal{C}_{uk}^a, b \ne a, \tag{81}$$

$$\begin{bmatrix} \tilde{\Xi}(\mu_1,0) & \Sigma_1^T Y_2 \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{82}$$

$$\begin{bmatrix} \tilde{\Xi}(\mu_1,\tau) & \Sigma_2^T Y_1^T \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{83}$$

$$\begin{bmatrix} \tilde{\Xi}(\mu_2,0) & \Sigma_1^T Y_2 \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{84}$$

$$\begin{bmatrix} \tilde{\Xi}(\mu_2,\tau) & \Sigma_2^T Y_1^T \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{85}$$

$$\begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \ge 0, \begin{bmatrix} \hat{Q}_{3q} & X_2 \\ * & \hat{Q}_{4q} \end{bmatrix} \ge 0 \tag{86}$$

$$\vartheta_0 > 0,\ \vartheta_1 > 0,\ Q_{k,l} > 0,\ \digamma_{\sigma,l} > 0\ (k=3,4,\sigma, l=1,2), \tag{87}$$

where

$$\tilde{\Xi}(\mu_1,0) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=0} + \tilde{\Gamma}_t + \Gamma_{aa},$$
$$\tilde{\Xi}(\mu_2,0) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=0} + \tilde{\Gamma}_t + \Gamma_{aa},$$
$$\tilde{\Xi}(\mu_1,\tau) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=\tau} + \tilde{\Gamma}_t + \Gamma_{aa},$$
$$\tilde{\Xi}(\mu_2,\tau) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=\tau} + \tilde{\Gamma}_t + \Gamma_{aa},$$
$$\begin{aligned}\tilde{\Gamma}_t &= \mathrm{Sym}\{e_1^T H_a e_{14} + e_{13}^T K_a e_{15}\} + \phi_1^T Q_1 \phi_1 + e_1^T Z_1 e_1 \\ &- e_2^T Z_1 e_2 + e_2^T Z_2 e_2 - e_3^T Z_2 e_3 + \tau^2 e_{14}^T M_2 e_{14} - \phi_2^T Q_1 \phi_2 \\ &+ \phi_1^T Q_2 \phi_1 - \phi_3^T Q_2 \phi_3 + \mathrm{Sym}\{e_1^T \mathcal{F} e_{13} - e_1^T \mathcal{F} A e_1 - e_1^T \mathcal{F} e_{14} \\ &- \rho_1 e_{14}^T \mathcal{F} A e_1 + \rho_1 e_{14}^T \mathcal{F} e_{13} - \rho_1 e_{14}^T \mathcal{F} e_{14} - e_{13}^T \hat{\mathcal{F}} B_a e_{13} \\ &+ e_{13}^T \hat{\mathcal{F}} \bar{\alpha} W_a^1 e_4 + e_{13}^T \hat{\mathcal{F}} \bar{\beta} W_a^2 e_5 - e_{13}^T \hat{\mathcal{F}} e_{15} - e_{13}^T \hat{\mathcal{F}} C_a e_1 \\ &- \rho_2 e_{15}^T \hat{\mathcal{F}} B_a e_{13} + \rho_2 e_{15}^T \hat{\mathcal{F}} \bar{\alpha} W_a^1 e_4 + \rho_2 e_{15}^T \hat{\mathcal{F}} \bar{\beta} W_a^2 e_5 \\ &- \rho_2 e_{15}^T \hat{\mathcal{F}} e_{15} - \rho_2 e_{15}^T \hat{\mathcal{F}} C_a e_1 + e_1^T J_{1a} e_1 + \rho_1 e_{14}^T J_{1a} e_1 \\ &+ e_{13}^T J_{2a} e_{13} + \rho_2 e_{15}^T J_{2a} e_{13}\} + \tau \Pi_{11}^T M_1 \Pi_{11} + \phi_1^T \Gamma_1 \phi_1 \\ &+ \phi_2^T \Gamma_2 \phi_2 + \phi_3^T \Gamma_3 \phi_3,\end{aligned}$$

In matrices $\tilde{\Xi}(\mu_1,0)$, $\tilde{\Xi}(\mu_1,\tau)$, $\tilde{\Xi}(\mu_2,0)$ and $\tilde{\Xi}(\mu_2,\tau)$, only element $\tilde{\Gamma}_t$ is different from Γ_t in Theorem 1. The other elements are the same as the elements in Theorem 1.

If $a \in \mathcal{C}_k^a$,

$$H_b - H_a - V_{1ab} \le 0, \forall b \in \mathcal{C}, b \ne a, \tag{88}$$

$$K_b - K_a - V_{2ab} \le 0, \forall b \in \mathcal{C}, b \ne a, \tag{89}$$

$$\begin{bmatrix} \breve{\Xi}(\mu_1,0) & \Sigma_1^T Y_2 \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{90}$$

$$\begin{bmatrix} \breve{\Xi}(\mu_1,\tau) & \Sigma_2^T Y_1^T \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{91}$$

$$\begin{bmatrix} \breve{\Xi}(\mu_2,0) & \Sigma_1^T Y_2 \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{92}$$

$$\begin{bmatrix} \breve{\Xi}(\mu_2,\tau) & \Sigma_2^T Y_1^T \\ * & -\widehat{M}_2 \end{bmatrix} < 0, \tag{93}$$

$$\begin{bmatrix} P_2 & X_1 \\ * & P_3 \end{bmatrix} \ge 0, \begin{bmatrix} \hat{Q}_{3q} & X_2 \\ * & \hat{Q}_{4q} \end{bmatrix} \ge 0 \tag{94}$$

$$\vartheta_0 > 0,\ \vartheta_1 > 0,\ Q_{k,l} > 0,\ \digamma_{\sigma,l} > 0, (k=3,4,\sigma, l=1,2), \tag{95}$$

$$\breve{\Xi}(\mu_1,0) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=0} + \tilde{\Gamma}_t + \hat{\Gamma}_{aa},$$
$$\breve{\Xi}(\mu_2,0) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=0} + \tilde{\Gamma}_t + \hat{\Gamma}_{aa},$$
$$\breve{\Xi}(\mu_1,\tau) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_1,\tau(t)=\tau} + \tilde{\Gamma}_t + \hat{\Gamma}_{aa},$$
$$\breve{\Xi}(\mu_2,\tau) = \Gamma(\dot{\tau}(t),\tau(t))|_{\dot{\tau}(t)=\mu_2,\tau(t)=\tau} + \tilde{\Gamma}_t + \hat{\Gamma}_{aa},$$

In matrices $\breve{\Xi}(\mu_1,0)$, $\breve{\Xi}(\mu_1,\tau)$, $\breve{\Xi}(\mu_2,0)$, and $\breve{\Xi}(\mu_2,\tau)$, only element $\hat{\Gamma}_{aa}$ is different from Γ_{aa}. The other elements are the same as the elements in $\tilde{\Xi}(\mu_1,0)$, $\tilde{\Xi}(\mu_1,\tau)$, $\tilde{\Xi}(\mu_2,0)$, and $\tilde{\Xi}(\mu_2,\tau)$.

$$\begin{aligned}\hat{\Gamma}_{aa} &= e_1^T [\sum_{b\in\mathcal{C}_k^a, b\ne a}^{\mathcal{M}} [\underline{\lambda}_{ab}(H_b - H_a) + 2\varpi_{ab} V_{1ab}] \\ &+ \underline{\lambda}_a (H_l - H_a) + 2\hat{\Lambda}_a V_{1al}] e_1 \\ &+ e_{13}^T [\sum_{b\in\mathcal{C}_k^a, b\ne a}^{\mathcal{M}} [\underline{\lambda}_{ab}(K_b - K_a) + 2\varpi_{ab} V_{2ab}] \\ &+ \underline{\lambda}_a (K_l - K_a) + 2\hat{\Lambda}_a V_{2al}] e_{13}, \ (l \in \mathcal{C}_{uk}^a).\end{aligned}$$

Moreover, the desired controller gain matrices are given as follows

$$L_{1a} = \mathcal{F}^{-1} J_{1a} \tag{96}$$

$$L_{2a} = \hat{\mathcal{F}}^{-1} J_{2a} \tag{97}$$

Proof Assume $F_1 = \mathcal{F},\ F_2 = \rho_1 F_1,\ \hat{F}_1 = \hat{\mathcal{F}},\ \hat{F}_2 = \rho_2 \hat{F}_1$ and $\mathcal{F} L_{1a} = J_{1a},\ \hat{\mathcal{F}} L_{2a} = J_{2a}$. From Theorem 1, (78)-(97) hold. The proof is completed.

In Theorem 2, the nonlinear matrix inequalities of synchronization conditions in Theorem 1 are changed into linear matrix inequalities (LMIs). By adopting LMI toolbox in Matlab, the controllers of INNs system could be obtained from Theorem 2.

IV. Simulation

In this section, two examples are shown to demonstrate the effectiveness of the proposed method.

Example 1. Consider the INNs system as follows

$$\begin{aligned}
&\frac{d^2 u_1(t)}{dt^2} = -a_1(\imath_t)\frac{du_1(t)}{dt} - b_1(\imath_t)u_1(t) + \alpha_1(t) w_{11}^1(\imath_t) f_1(u_1(t)) \\
&+ \beta_1(t) w_{12}^1(\imath_t) f_2(u_2(t)) + \alpha(t) w_{11}^2(\imath_t) f_1(u_1(t-\tau(t))) + \beta(t) \\
&w_{12}^2(\imath_t) f_2(u_2(t-\tau(t))) + T_1 \\
&\frac{d^2 u_2(t)}{dt^2} = -a_2(\imath_t)\frac{du_2(t)}{dt} - b_2(\imath_t)u_2(t) + \alpha(t) w_{21}^1(\imath_t) f_1(u_1(t)) \\
&+ \beta(t) w_{22}^1(\imath_t) f_2(u_2(t)) + \alpha(t) w_{21}^2(\imath_t) f_1(u_1(t-\tau(t))) + \beta(t) \\
&w_{22}^2(\imath_t) f_2(u_2(t-\tau(t))) + T_2
\end{aligned} \tag{98}$$

The activation functions $f_k(u_k(t)) = tanh(u_k(t))\ (k = 1, 2)$ satisfy Assumption 1. We get $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$.

The system (98) is considered with the following parameters
$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
$B(\imath_t) = \begin{pmatrix} a_1(\imath_t) - \xi_1 & 0 \\ 0 & a_2(\imath_t) - \xi_2 \end{pmatrix}$, $C(\imath_t) = \begin{pmatrix} b_1(\imath_t) + \xi_1(\xi_1 - a_1(\imath_t)) & 0 \\ 0 & b_2(\imath_t) + \xi_2(\xi_2 - a_2(\imath_t)) \end{pmatrix}$,
$(\imath_t = 1, 2, 3)$, $B_1 = \begin{pmatrix} 1.0 & 0 \\ 0 & 1.5 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1.6 & 0 \\ 0 & 1.2 \end{pmatrix}$, $B_3 = \begin{pmatrix} 1.2 & 0 \\ 0 & 0.9 \end{pmatrix}$, $C_1 = \begin{pmatrix} -1.1 & 0 \\ 0 & -0.9 \end{pmatrix}$, $C_2 = \begin{pmatrix} -0.7 & 0 \\ 0 & -1.0 \end{pmatrix}$, $C_3 = \begin{pmatrix} -1.0 & 0 \\ 0 & -1.1 \end{pmatrix}$,
$W_1^1 = \begin{pmatrix} 0.9 & 0.2 \\ -0.5 & 2.7 \end{pmatrix}$, $W_2^1 = \begin{pmatrix} -1.0 & 1.7 \\ -1.0 & 1.3 \end{pmatrix}$, $W_3^1 = \begin{pmatrix} -1.0 & 1.4 \\ -1.0 & 1.0 \end{pmatrix}$,
$W_1^2 = \begin{pmatrix} 0.9 & 2.1 \\ -3.2 & 0.8 \end{pmatrix}$, $W_2^2 = \begin{pmatrix} -1.7 & 0.8 \\ -1.2 & 1.3 \end{pmatrix}$, $W_3^2 = \begin{pmatrix} -1.2 & 1.3 \\ -1.2 & 0.3 \end{pmatrix}$, $\mho_1 = \mho_2 = 1$, $\bar{\alpha} = 0.01$, $\bar{\beta} = 0.02$, $\nu_\alpha = 0.002$, $\nu_\beta = 0.004$, $\tau(t) = 0.8 + 0.2sin(t)$, $\tau = 1.0$, $\mu_1 = -0.2$, $\mu_2 = 0.2$. The transition matrix is given as
$\Omega = \begin{pmatrix} ? & ? & 2.32+\triangle_{13} \\ ? & -4.75+\triangle_{22} & 2.02+\triangle_{23} \\ 1.34+\triangle_{31} & ? & -5.05+\triangle_{33} \end{pmatrix}$,
where $\varpi_{13} = 0.1$, $\varpi_{22} = \varpi_{23} = 0.19$, $\varpi_{31} = \varpi_{33} = 0.18$.

Set the parameters $\rho_1 = \rho_2 = 1$. From Theorem 2, one gets the following feasible matrices.
$L_{11} = \begin{pmatrix} -5.9326 & -0.0033 \\ -0.0025 & -6.1008 \end{pmatrix}$, $L_{12} = \begin{pmatrix} -5.7784 & -0.0042 \\ -0.0023 & -6.1008 \end{pmatrix}$, $L_{13} = \begin{pmatrix} -5.9842 & -0.0033 \\ -0.0027 & -6.2532 \end{pmatrix}$,
$L_{21} = \begin{pmatrix} -3.4484 & 0.0030 \\ -0.0001 & -2.8959 \end{pmatrix}$, $L_{22} = \begin{pmatrix} -2.7404 & 0.0021 \\ -0.0003 & -3.1412 \end{pmatrix}$, $L_{23} = \begin{pmatrix} 0.0961 & -0.0006 \\ 0.0001 & -0.2162 \end{pmatrix}$.

The trajectories of error system with initial values $e_1(t) = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$, $e_2(t) = \begin{bmatrix} 0.18 \\ -0.22 \end{bmatrix}$ are given in Figs. 1-2.

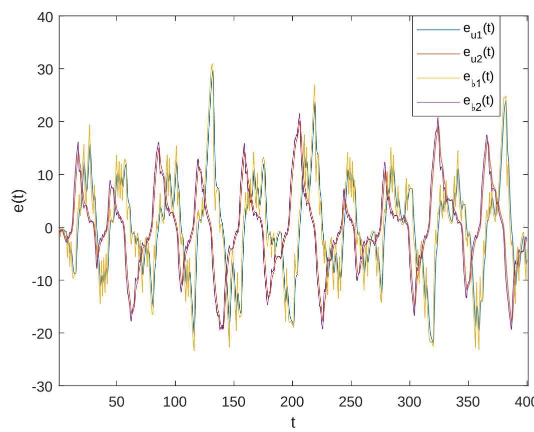


Fig. 1 State trajectories of error system without controllers.

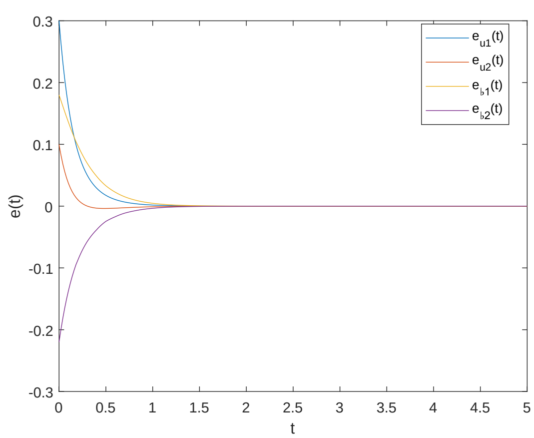


Fig. 2 State trajectories of error system with controllers.

Through the Theorem 2 and Figs. 1-2, one gets that the error system is asymptotically stable. So the driven system and response system could be synchronous under the designed controllers.

Example 2. Under the obtained results, the image with a size $L \times H \times 3$ could be encrypted. According to [44], the process of image encryption is shown as follows.

Input: A color image F_O with the size of $L \times H \times 3$.
Output: An encrypted image E with the size of $L \times H$.

S1: By separating color image F_O with red, green, and blue components, respectively, pixel series are obtained as $R(i,j)$, $G(i,j)$, $B(i,j)$ $(i \in \{1, \cdots, L\}$, $j \in \{1, \cdots, H\})$.

S2: Under uncontrolled system (6), by adopting fourth order Runge-Kutta method with step size 0.001 and the initial conditions $u(0)$, $\flat(0)$, $\hat{u}(0)$, $\hat{\flat}(0)$ with more than $I = L \times H$ times iterations, we could get a group of time series chaotic signals.

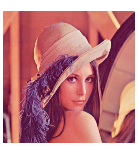
S3: Master system is iterated I times continuously. For each iteration, we get four values $u, \flat, \hat{u}, \hat{\flat}$. Hence, four floating-point number sequences of X_1, X_2, X_3, X_4 with the length of $I = L \times H$ are obtained as follows:

$$\begin{aligned} X_1 &= \{x_1(1), x_1(2), \cdots, x_1(I)\}, \\ X_2 &= \{x_2(1), x_2(2), \cdots, x_2(I)\}, \\ X_3 &= \{x_3(1), x_3(2), \cdots, x_3(I)\}, \\ X_4 &= \{x_4(1), x_4(2), \cdots, x_4(I)\}. \end{aligned}$$

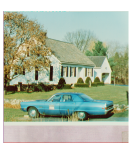
S4: From these sequences, X_R, X_G and X_B are generated from X_k $(k = 1, 2, 3, 4)$.

$$\begin{aligned} x_R(i) &= (|x_1(i)| - \lfloor |x_1(i)| \rfloor) \times 10^{14} \mod 256 \\ x_G(i) &= (|x_2(i)| - \lfloor |x_2(i)| \rfloor) \times 10^{14} \mod 256 \\ x_B(i) &= (|x_3(i) + x_4(i)| - \lfloor |x_3(i) + x_4(i)| \rfloor) \times 10^{14} \mod 256 \end{aligned}$$

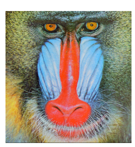
where $x_R(i) \in X_R$, $x_G(i) \in X_G$ and $x_B \in X_B$ $(i = 1, \cdots, I)$. $\lfloor \rfloor$ represents the values of nearest integer that is less than or equal to X. The mod$(\cdot,\cdot)$ represents the remainder after division. The size of each sequence (x_i) is I.



(a) Lena.



(b) House.



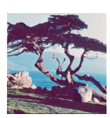
(c) Mandrill.

(d) Splash.

(e) Flowers.

(f) Jelly.

(g) Lady.



(h) Tree.

Fig. 3 The original images.

5: The RGB components of F_S are encrypted by X_R, X_B and X_G, and e_R, e_B and e_G of the encrypted image could be obtained as follows.

$$\begin{aligned} e_R(i) &= f_R(i) \oplus x_R(i) \\ e_G(i) &= f_G(i) \oplus x_G(i) \\ e_B(i) &= f_B(i) \oplus x_B(i) \end{aligned}$$

where $\oplus$ is the bitwise XOR operator. $f_R(i)$, $f_G(i)$ and $f_B(i)$ are the pixel sequence of the shuffled image. The encryption process of color image is shown as Figs. 3-9. The entropy values of original and encrypted images in Fig. 3 are shown in Table I. The obtained values of the encrypted images are very close to the theoretical value 8. According to [44], [54], [55], the encrypted image in this paper approaches a random source, and information leakage from the encryption algorithm is negligible.

Fig. 4 The original image of the tree.

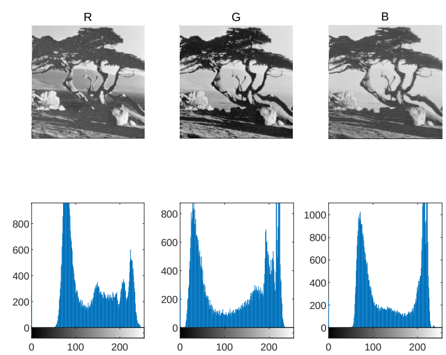


Fig. 5 Histogram of the RGB components of the tree.

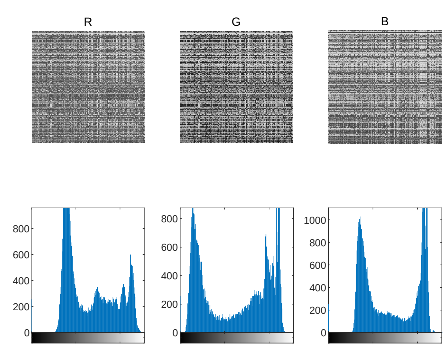


Fig. 6 Histogram of the RGB components of the shuffled image of the tree.

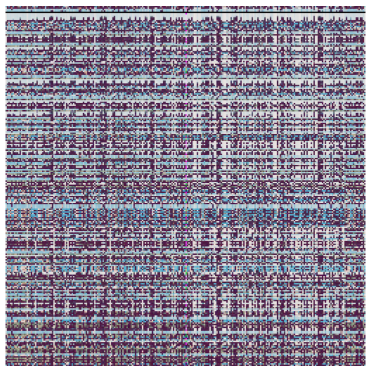


Fig. 7 The shuffled image of the tree.

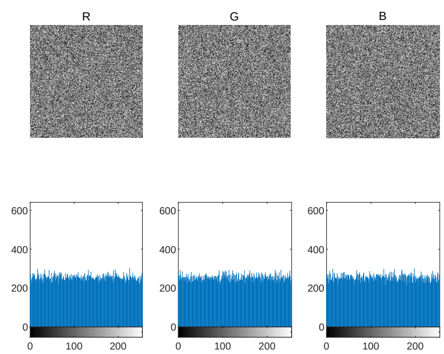


Fig. 8 Histogram of the RGB components of the encrypted image of the tree.

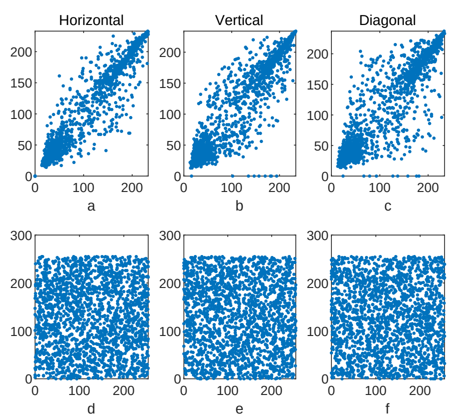


Fig. 9 Correlations of two adjacent pixels in horizontal, vertical, and diagonal directions of the red component of the tree. The figures a-c denote the original image, and d-f denote encrypted image.

CC is the correlations between original image and encrypted image in [44], [54], [55]. The correlation between two adjacent pixels is shown in Table II. According to the CC values, the proposed method in this paper has gained good performance. The original and encrypted images are significantly different because the CC values of the encrypted images in Table II approach to 0.

TABLE I
ENTROPY ANALYSIS OF RGB COMPONENTS

Image	Size	Original Image			Encrypted Image		
		Red	Green	Blue	Red	Green	Blue
Lena	512×512	7.2531	7.5940	6.9684	7.9992	7.9994	7.9993
House	512×512	7.4156	7.2295	7.4354	7.9993	7.9993	7.9992
Mandrill	512×512	7.7067	7.4744	7.7522	7.9992	7.9993	7.9994
Splash	512×512	6.9481	6.8845	6.1265	7.9992	7.9994	7.9993
Flowers	362×500	7.3824	7.2345	7.3641	7.9989	7.9990	7.9989
Jelly	256×256	5.2626	5.6947	6.5464	7.9973	7.9970	7.9913
Lady	256×256	6.4200	6.4457	6.3807	7.9972	7.9968	7.9973
Tree	256×256	7.2104	7.4136	6.9207	7.9976	7.9973	7.9969

TABLE II
CCs OF TWO ADJACENT PIXELS IN THE ORIGINAL/ENCRYPTED IMAGES

Image	Red			Green			Blue		
	H	V	D	H	V	D	H	V	D
Original Lena	0.9798	0.9893	0.9697	0.9327	0.9576	0.9183	0.9327	0.9576	0.9183
Encrypted Lena	0.0016	-0.0035	0.0019	0.0015	-0.0011	-0.0008	-0.0011	0.0000	0.0001
Original House	0.9536	0.9579	0.9224	0.9725	0.9686	0.9445	0.9391	0.9423	0.8901
Encrypted House	0.0028	-0.0004	-0.0024	-0.0008	0.0009	-0.0004	0.0011	0.0013	0.0023
Original Mandrill	0.9231	0.8660	0.8543	0.9073	0.8809	0.8399	0.8655	0.7650	0.7348
Encrypted Mandrill	0.0030	-0.0007	0.0029	-0.0005	0.0031	0.0014	0.0018	-0.0039	-0.0019
Original Splash	0.9936	0.9951	0.9894	0.9826	0.9789	0.9649	0.9812	0.9871	0.9711
Encrypted Splash	-0.0011	-0.0050	-0.0032	-0.0007	0.0038	-0.0017	-0.0021	0.0036	0.0017
Original Flowers	0.9718	0.9719	0.9551	0.9527	0.9527	0.9256	0.9510	0.9497	0.9218
Encrypted Flowers	0.0001	0.0004	-0.0020	-0.0005	0.0003	0.0026	0.0051	0.0012	-0.0066
Original Jelly	0.9745	0.9763	0.9537	0.9890	0.9880	0.9799	0.9757	0.9801	0.9603
Encrypted Jelly	0.0063	-0.0065	-0.0024	-0.0020	-0.0003	-0.0026	0.0009	0.0019	0.0045
Original Lady	0.9729	0.9622	0.9482	0.9584	0.9519	0.9377	0.9719	0.9647	0.9500
Encrypted Lady	-0.0010	-0.0046	0.0019	0.0002	0.0035	-0.0033	0.0027	-0.0049	0.0052
Original Tree	0.9590	0.9361	0.9159	0.9612	0.9406	0.9265	0.9687	0.9457	0.9318
Encrypted Tree	0.0011	0.0013	-0.0041	0.0028	0.0059	-0.0018	0.0017	0.0008	0.0001

V. Conclusion

In this paper, the synchronization problem of delayed INNs with generally uncertain Markovian jumping and random connection weight strengths is investigated. By implementing the DPTLKF and HOPRII, the DRDSC and corresponding controllers are obtained in this paper. Finally, two examples including image encryption application are shown to demonstrate the effectiveness of the theoretical results. Further studies include finite-time sampled-data synchronization and impulsive synchronization control of INNs with generally uncertain Markovian jumping and unbounded time delay.

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